HRP with Expected Returns

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HRP

Stage 3 (see Section 16.4.3) splits a weight in inverse proportion to the subset's variance. We now prove that such allocation is optimal when the covariance matrix is diagonal. Consider the standard quadratic optimization problem of size N,

$$\min_{\omega} \omega' V \omega$$
$$\text{s.t. } \omega' a = 1$$

with solution $\omega = \frac{V^{-1}a}{a'V^{-1}a}$. For the characteristic vector $a = 1_N$, the solution is the minimum variance portfolio. If V is diagonal, $\omega_n = \frac{V_{n,n}^{-1}}{\sum_{i=1}^{N} V_{i,i}^{-1}}$. In the particular case of $N = 2$,

$$\omega_1 = \frac{1}{V_{1,1}}\left(\frac{1}{V_{1,1}} + \frac{1}{V_{2,2}}\right)^{-1} = 1 - \frac{V_{1,1}}{V_{1,1} + V_{2,2}},$$

which is how stage 3 splits a weight between two bisections of a subset.

Proof.

We consider the optimization problem:

$$\min_{\omega} \omega' V \omega$$
$$\text{s.t. } \omega' a = 1$$

where V is a symmetric positive-definite covariance matrix of size $N \times N$, ω is the vector of portfolio weights, and $a = 1_N$ is a vector of ones.

We can express the objective function as a *Lagrangian*:

$$\mathcal{L}(\omega, \lambda) = \omega' V \omega - \lambda(\omega' a - 1)$$

where λ is the Lagrange multiplier. Let's solve for w:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \omega} &= 0 \\
2V\omega - \lambda a &= 0 \\
V\omega &= \frac{\lambda}{2} a \\
\omega &= \frac{1}{2} V^{-1} a \lambda && V \text{ is invertible}
\end{aligned}$$

We apply the constraint $\omega' a = 1$ to solve for λ:

$$\begin{aligned}
\omega' a &= 1 \\
\left(\frac{1}{2}V^{-1}a\lambda\right)' a &= 1 \\
\frac{1}{2}\lambda a' V^{-1} a &= 1 \\
\lambda &= \frac{2}{a' V^{-1} a}
\end{aligned}$$

Substitute λ back into the expression for ω:

$$\begin{aligned}
\omega &= \frac{1}{2}V^{-1}a\frac{2}{a'V^{-1}a} \\
&= \frac{V^{-1}a}{a'V^{-1}a}
\end{aligned}$$

Special Case: Diagonal Covariance Matrix V:
If V is diagonal, then V^{-1} is also diagonal with entries $V_{n,n}^{-1} = \frac{1}{V_{n,n}}$. Therefore:

$$V^{-1}a = \begin{pmatrix} \frac{1}{V_{1,1}} \\ \frac{1}{V_{2,2}} \\ \vdots \\ \frac{1}{V_{N,N}} \end{pmatrix}$$

The denominator becomes:

$$a'V^{-1}a = \sum_{i=1}^{N} \frac{1}{V_{i,i}}$$

Hence, the optimal weights are:

$$\omega_n = \frac{\frac{1}{V_{n,n}}}{\sum_{i=1}^{N} \frac{1}{V_{i,i}}}$$

This shows that each weight ω_n is inversely proportional to its variance $V_{n,n}$.
Particular Case: $N = 2$:
For $N = 2$, we have:

$$\omega_1 = \frac{\frac{1}{V_{1,1}}}{\frac{1}{V_{1,1}} + \frac{1}{V_{2,2}}} = \frac{V_{2,2}}{V_{1,1} + V_{2,2}} = 1 - \frac{V_{1,1}}{V_{1,1} + V_{2,2}}$$

Similarly,

$$\omega_2 = \frac{\frac{1}{V_{2,2}}}{\frac{1}{V_{1,1}} + \frac{1}{V_{2,2}}} = \frac{V_{1,1}}{V_{1,1} + V_{2,2}}$$

This demonstrates that when the covariance matrix is diagonal, the weight allocated to each asset is inversely proportional to its variance.

HRP with Expected Returns, Long-Short, No Leverage

Now we want to solve the following optimization problem:

$$\max_{\omega} \omega'\mu - \frac{\gamma}{2}\omega' V \omega$$
$$\text{s.t. } |\omega'| 1_N = 1,$$

where μ is the vector of expected returns, and γ is the risk aversion parameter. The solution is given by:

We can express the objective function as a *Lagrangian*:

$$\mathcal{L}(\omega, \lambda) = \omega'\mu - \frac{\gamma}{2}\omega' V \omega - \lambda(|\omega'| 1_N - 1)$$

, let's solve for w:

$$\frac{\partial \mathcal{L}}{\partial \omega} = 0$$
$$\mu - \gamma V \omega - \text{sign}(\omega)\lambda = 0 \qquad \text{where sign}(\omega) \text{ is a } Nx1\text{vector of signs of } \omega$$
$$\omega = \frac{1}{\gamma} V^{-1}(\mu - \text{sign}(\omega)\lambda)$$