Heat and wave equations

Introduction in 1d

18.303 Linear Partial Differential Equations: Analysis and Numerics

Operator adjoints, redux

We recall that the adjoint of an operator $\mathcal{L}$, $\mathcal{L}^*$, is defined using the inner product as

$$\langle \mathbf{f}, \mathcal{L}\mathbf{g} \rangle = \langle \mathcal{L}^* \mathbf{f}, \mathbf{g} \rangle$$

for all $\mathbf{f}, \mathbf{g} \in V$. If $\mathcal{L} = \mathcal{L}^*$ we say the operator is self-adjoint.

Example 1: Let us again think about functions $u : [0,1] \to \mathbb{R}$ with Dirichlet boundaries ($u(0) = u(1) = 0$). We will look at the Laplace operator $\Delta = \frac{\mathrm{d}^2}{\mathrm{d}x^2}$. Writing the inner product for some f and g gives

$$\langle f, \Delta g\rangle = \int_0^1 f(x)\frac{\mathrm{d}^2}{\mathrm{d}x^2}g(x)\mathrm{d}x.$$

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We can use integration by part giving

$$f(x)\frac{\mathrm{d}}{\mathrm{d}x}g(x)\Big|_0^1 - \int_0^1 \frac{\mathrm{d}}{\mathrm{d}x}f(x)\frac{\mathrm{d}}{\mathrm{d}x}g(x)\mathrm{d}x.$$

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We notice that the first term is zero due to the boundary conditions. Repeating integration by parts gives

$$\langle f, \Delta g \rangle = \int_0^1 \frac{\mathrm{d}^2}{\mathrm{d}x^2} f(x) g(x) \mathrm{d}x = \langle \Delta f, g \rangle$$

i.e. the Laplacian is self-adjoint with these boundaries.

Definition 1
The self-adjoint operator $\mathcal{L}$ is positive definite iff $\langle \mathbf{f}, \mathcal{L}\mathbf{f} \rangle > 0$ for all $0 \neq \mathbf{f} \in V$.

If the equality is not strict (i.e. $\geq$), the operator is called positive semidefinite.

Positive and negative definite operators

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Proof:

$$\langle u, \Delta u \rangle = \int_0^1 u(x) \frac{\mathrm{d}^2}{\mathrm{d}x^2} u(x) \mathrm{d}x = -\int_0^1 \frac{\mathrm{d}u(x)}{\mathrm{d}x} \frac{\mathrm{d}u(x)}{\mathrm{d}x} = -\left\| \frac{\mathrm{d}u}{\mathrm{d}x} \right\|^2 < 0.$$

Here we used integration by part as before. Why can't the norm be zero?

Superposition principle

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Boundary conditions give the necessary information for fixing all the coefficients in front of the individual solutions for the differential equation.

Heat equation

We look at functions $u : \mathbb{R}_+ \times [0,1] \to \mathbb{R}$. These are functions $u(t,x)$ defined at non-negative times and on the interval $x \in [0,1]$. Let us assume Dirichlet boundaries again.

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Heat equation with Dirichlet boundaries

$$\frac{\partial u(t,x)}{\partial t} = \frac{\partial^2 u(t,x)}{\partial x^2};$$
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Important! When the domain of the function is a Cartesian product of two spaces and the boundary conditions are defined *independently* for all variables, a linear differential equation is *separable*. This means that the individual solutions are of the form $a_n(t)\phi_n(x)$.

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$$\begin{aligned} \phi_n''(x) &= -\lambda_n^2 \phi_n, \\ \dot{a}_n(t) &= -\lambda_n^2 a_n(t). \end{aligned}$$

We know that the solution to the first one is $\phi_n(x) = \sin(n\pi x)$ (at any given time we have Dirichlet boundaries). This also gives $\lambda_n = n\pi$.

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$$\begin{aligned} \int_0^t \frac{\mathrm{d}\,\log(a_n(t'))}{\mathrm{d}t'}\mathrm{d}t' &= \int_0^t -\lambda_n \mathrm{d}t, \\ \log(a_n(t)) - \log(a_n(0)) &= -\lambda_n^2 t, \\ \log(a_n(t)/a_n(0)) &= -\lambda_n^2 t. \end{aligned}$$

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Solving for $a_n(t)$ gives

$$a_n(t) = a_n(0)e^{-\lambda_n^2 t}. \tag{1}$$

Now we have the general solution (according to the superposition principle) given as

Heat equation with Dirichlet boundaries, solution

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What we do is we express u_0 in the basis $\{\phi_n\}_{n=1}^{\infty}$ giving the coefficients $a_n(0)$. If $u_0 = \sum_{n=1}^{\infty} \hat{u}_0^{(n)}\phi_n$, then $a_n(0) = \hat{u}_0^{(n)}$.

Exercise 1

Solve the heat equation on the same domain with the same initial condition and the boundary condition

$$u(t,0) = 0,\ u'(t,1) = 0.$$

How is the solution different?

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$$u(0,x) = u_0(x),\ \dot{u}(0,x) = v_0(x),\ x \in [0,1].$$

Notice that we need more initial data since the order of the PDE in time is 2.

Again, the system can be separated and we get the eigenvalue problems

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The general single (real solution) for the latter equation is

$$a_n(t) = \alpha_n \cos(\lambda_n t) + \beta_n \sin(\lambda_n t).$$

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We get the coefficients α_n and β_n from the initial condition:

$$u_0(x) = \sum_{n=1}^{\infty} \alpha_n \phi_n(x),$$

$$v_0(x) = \dot{u}(0,x) = \sum_{n=1}^{\infty} \beta_n \lambda_n \phi_n(x).$$

Let $\hat{u}_0^{(n)}$ and $\hat{v}_0^{(n)}$ be the sine series coefficients of u and v, respectively. We have

$$\alpha_n = \hat{u}_0^{(n)},\ \beta_n = \hat{v}_0^{(n)}/\lambda_n.$$

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- Notice how solving the time dependent part of the equation reduced to solving an ordinary (simple) differential equation
- The same methodology carries over to higher dimensional complicated structures
- Often we have to solve the eigenvalue of the Laplacian in a complicated domain (on a computer) but after this, solving for the time evolution of the spatial eigenmodes (here ϕ_n) is pretty much trivial
- This idea can be generalized to other dynamical systems: this property follows from the fact that time is an *independent* parameter and doesn't mix with the spatial part of the boundary value problem

Notation

Some notation for differential operators

Name	Notation	Alternative notation	Cartesian coordinates
Total differential	$\mathrm{d}f/\mathrm{d}x$	$f'(x)$ (1d)	-
Partial differential	$\partial f/\partial x$	$\partial_x f$, f_x	-
Time derivative	$\mathrm{d}f/\mathrm{d}t$	$\dot{f}$	-
Gradient	$\nabla \cdot f$	$\mathrm{grad}(f)$	$(\partial_x f, \partial_y f)$
Divergence	$\nabla \cdot \mathbf{f}$	$\mathrm{div}(\mathbf{f})$	$\partial_x f_x + \partial_y f_y$
Curl (2d)	$\nabla \times \mathbf{f}$	$\nabla \wedge \mathbf{f}$	$\partial_x f_y - \partial_y f_x$
Laplacian	Δf	$\nabla^2 f$	$(\partial_x^2 + \partial_y^2)f$