Question 21

The least numnber of queries is $N+1$.

$$RMSE(h) = 0 \rightarrow \sum_{n=1}^{N}(y_n - h(x_n))^2 = 0$$

Let $k \in \mathbb{R}^N$ and $query(k) = \sum_{n=1}^{N}(y_n - k_n)^2 = d$.

Start with $k_0 = \{0\}^N$, then $query(k_0) = \sum_{n=1}^{N} y_n^2 = d_0$

Counstruct a query $query(k_i)$ then substract from $query(k_0)$, we get :

$$\begin{aligned} query(k_0) - query(k_i) &= \sum_{n=1}^{N} y_n^2 - \sum_{n=1}^{N}(y_n - k_{in})^2 \\ &= \sum_{n=1}^{N}(2k_{in}y_n - k_{in}^2) \\ &= d_i \end{aligned}$$

which is a linear equation about y. If we want to solve y, we need at least N such linear equations about y. e.g. Consider a resonable K given below:

$$K = \begin{bmatrix} k_1 \\ \vdots \\ k_i \\ \vdots \\ k_N \end{bmatrix} = \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 1 \end{bmatrix}$$

$$query(K) = \begin{bmatrix} query(k_1) \\ \vdots \\ query(k_i) \\ \vdots \\ query(k_N) \end{bmatrix}$$

$$query(k_0) - query(K) = \begin{bmatrix} query(k_0) - query(k_1) \\ \vdots \\ query(k_0) - query(k_i) \\ \vdots \\ query(k_0) - query(k_N) \end{bmatrix} = \begin{bmatrix} 2y_1 - 1 \\ \vdots \\ 2y_i - 1 \\ \vdots \\ 2y_N - 1 \end{bmatrix} = \begin{bmatrix} d_1 \\ \vdots \\ d_i \\ \vdots \\ d_N \end{bmatrix}$$

Then we can solve it easily. So the total number of queries we need is at least $N+1$.

Question 23

$$\min_{w_1, w_2, \ldots, w_K} RMSE(H) \rightarrow \min_{w_1, w_2, \ldots, w_K} \sum_{n=1}^{N}(y_n - H(x_n))^2$$

Let

$$\begin{aligned}
f(w) &= \sum_{n=1}^{N}(y_n - H(x_n))^2 \\
&= \sum_{n=1}^{N} y_n^2 - 2\sum_{n=1}^{N} y_n H(x_n) + \sum_{n=1}^{N} H^2(x_n)
\end{aligned}$$

Where $H(x_n) = \sum_{k=1}^{K} w_k h_k(x_n)$ If f is minimized, the partial derivative of all w_i should be 0.

$$\begin{aligned}
\frac{\partial f}{\partial w_i} &= -2\sum_{n=1}^{N} y_n h_i(x_n) + 2\sum_{n=1}^{N} H(x_n) h_i(x_n) \\
&= -2h_i^T y + 2\sum_{n=1}^{N}(h_i(x_n) \cdot \sum_{k=1}^{K} w_k h_k(x_n)) \\
&= -2h_i^T y + 2\sum_{k=1}^{K}(\sum_{n=1}^{N} h_i(x_n) h_k(x_n)) \cdot w_k \\
&= -2h_i^T y + 2\sum_{k=1}^{K} h_i^T h_k w_k \\
&= 0
\end{aligned}$$

That means we should solve such linear equations:

$$\begin{aligned}
h_1^T h_1 w_1 + h_1^T h_2 w_2 + \cdots + h_1^T h_K w_K &= h_1^T y \\
h_2^T h_1 w_1 + h_2^T h_2 w_2 + \cdots + h_2^T h_K w_K &= h_2^T y \\
&\vdots \\
h_K^T h_1 w_1 + h_K^T h_2 w_2 + \cdots + h_K^T h_K w_K &= h_K^T y
\end{aligned}$$

Followed by question 22, as we have already known $h_i^T h_j$, the least number of queries to get $h_1^T y \sim h_K^T y$ is $K+1$, so the least number of queries would be $K+1$ to solve $\underset{w_1, w_2, \ldots, w_K}{min} RMSE(H)$.