Projection Models and Homogeneous Coordinates

Homogeneous Coordinates

Refresher Course
Andreas Maier, Joachim Hornegger, Markus Kowarschik, Frank Schebesch
Pattern Recognition Lab (CS 5)

FRIEDRICH-ALEXANDER UNIVERSITÄT ERLANGEN-NÜRNBERG
TECHNISCHE FAKULTÄT

Topics

Homogeneous Coordinates

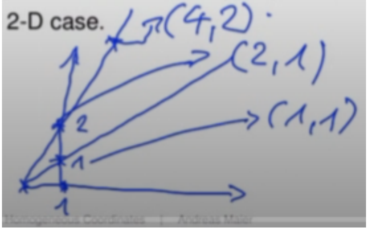
regular space -> projective space (plus 1 dim) -> regular space

Using a simple trick, we can extend 2-D or 3-D vectors by an additional component that allows us to write

- affine mappings as linear mappings, and
- the perspective projection as a linear mapping.

points now live on a line from origin throu them
we can go back to original coordinate by divining throu
second number (2,1) = (4,2) = 2

Let us first consider the 2-D case.



can be used to describe perspective
projections

second point 0 -> all lines merge :(
not allowed

Homogeneous Coordinates

We extend $\mathbb{R}^2$ by a third coordinate to create the projective space $\mathbb{P}^2$:

Definition

A two-dimensional point in Cartesian coordinates $\boldsymbol{p} = (x, y)^{\mathrm{T}} \in \mathbb{R}^2$ is represented by $\widetilde{\boldsymbol{p}} = (wx, wy, w)^{\mathrm{T}} \in \mathbb{P}^2$ in ***homogeneous coordinates***, where $w \in \mathbb{R} \setminus \{0\}$ is an arbitrary real value.

if we define w as one, we just have to add a one as 3rd coordinate

Homogeneous Coordinates

We extend $\mathbb{R}^2$ by a third coordinate to create the projective space $\mathbb{P}^2$:

Definition

A two-dimensional point in Cartesian coordinates $\boldsymbol{p} = (x, y)^{\mathrm{T}} \in \mathbb{R}^2$ is represented by $\widetilde{\boldsymbol{p}} = (wx, wy, w)^{\mathrm{T}} \in \mathbb{P}^2$ in ***homogeneous coordinates***, where $w \in \mathbb{R} \setminus \{0\}$ is an arbitrary real value.

Note: A vector $(\widetilde{x}, \widetilde{y}, \widetilde{z})^{\mathrm{T}}$ in homogeneous coordinates can be transformed into a 2-D vector by dividing the first two components $\widetilde{x}$ and $\widetilde{y}$ with the third component $\widetilde{z} \neq 0$:

projecting back:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \widetilde{x}/\widetilde{z} \\ \widetilde{y}/\widetilde{z} \end{pmatrix}.$$

Homogeneous Coordinates

- A 2-D point $(x, y)^{\mathrm{T}}$ in Cartesian coordinates corresponds to a line in 3-D:

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \left\{ w \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \;\middle|\; w \in \mathbb{R} \right\}.$$

- There exists an infinite number of homogeneous points that correspond to one and the same 2-D point.
- The representation in homogeneous coordinates has a singularity for $w \to 0$.

FRIEDRICH-ALEXANDER UNIVERSITÄT ERLANGEN-NÜRNBERG
TECHNISCHE FAKULTÄT

Homogeneous Coordinates

We now define an equivalence relation:

Definition

We call two homogeneous points $\widetilde{\boldsymbol{p}}$ and $\widetilde{\boldsymbol{q}}$ equivalent, if $\widetilde{\boldsymbol{p}} = \lambda \widetilde{\boldsymbol{q}}$ where $\lambda \in \mathbb{R} \setminus \{0\}$. This equivalence is denoted by $\widetilde{\boldsymbol{p}} \cong \widetilde{\boldsymbol{q}}$.

"identiccal up to scale" different as homogenious points, but get mapped to the same point in regular space. e.g. (2,1) and (4,2)

Homogeneous Coordinates

We now define an equivalence relation:

Definition

We call two homogeneous points $\widetilde{\boldsymbol{p}}$ and $\widetilde{\boldsymbol{q}}$ equivalent, if $\widetilde{\boldsymbol{p}} = \lambda \widetilde{\boldsymbol{q}}$ where $\lambda \in \mathbb{R} \setminus \{0\}$. This equivalence is denoted by $\widetilde{\boldsymbol{p}} \cong \widetilde{\boldsymbol{q}}$.

Example

The homogeneous points $\widetilde{\boldsymbol{p}} = (2,3,1)^{\mathrm{T}}$ and $\widetilde{\boldsymbol{q}} = (4,6,2)^{\mathrm{T}}$ are equivalent by $\widetilde{\boldsymbol{p}} \cong \widetilde{\boldsymbol{q}}$ as both project to the same point which is $(2,3)^{\mathrm{T}} \in \mathbb{R}^2$.
They are not equal considered as vectors in $\mathbb{R}^3$, i. e., $\widetilde{\boldsymbol{p}} \neq \widetilde{\boldsymbol{q}}$.
Note: It is $\widetilde{\boldsymbol{p}} \ncong (4,6,1)^{\mathrm{T}}$.

Homogeneous Coordinates

Let us now consider lines in 2-D.

- A line in $\mathbb{R}^2$ is fully determined by the equation

the line we drew earlyer, but 3D

no, this is 2D ?

$$ax + by + c = 0, \quad \text{where} \quad a, b, c \in \mathbb{R}.$$

This equation can be multiplied by an arbitrary factor $w \in \mathbb{R} \setminus \{0\}$, and it still represents the same line.

Homogeneous Coordinates

Let us now consider lines in 2-D.

- A line in $\mathbb{R}^2$ is fully determined by the equation

$$ax + by + c = 0, \quad \text{where} \quad a, b, c \in \mathbb{R}.$$

  This equation can be multiplied by an arbitrary factor $w \in \mathbb{R} \setminus \{0\}$, and it still represents the same line.

- Each vector $(a, b, c)^{\mathrm{T}} \in \mathbb{R}^3$ represents a line, and

$$ax + by + c = (w \cdot a)x + (w \cdot b)y + (w \cdot c) = 0$$

  holds for each non-zero w.

Homogeneous Coordinates

Let us now consider lines in 2-D.

- A line in $\mathbb{R}^2$ is fully determined by the equation

  $$ax + by + c = 0, \quad \text{where} \quad a, b, c \in \mathbb{R}.$$

  This equation can be multiplied by an arbitrary factor $w \in \mathbb{R} \setminus \{0\}$, and it still represents the same line.

- Each vector $(a, b, c)^{\mathrm{T}} \in \mathbb{R}^3$ represents a line, and

  $$ax + by + c = (w \cdot a)x + (w \cdot b)y + (w \cdot c) = 0$$

  holds for each non-zero w.

- In terms of homogeneous coordinates we can state that each 2-D line can be represented by a corresponding vector $\boldsymbol{l} = (a, b, c)^{\mathrm{T}} \in \mathbb{R}^3$.

Homogeneous Coordinates

- A point $\widetilde{\boldsymbol{p}}$ (represented in homogeneous coordinates) lies on the line $\boldsymbol{l}$ if

$$\boldsymbol{l}^{\mathrm{T}}\widetilde{\boldsymbol{p}} = 0.$$

like distance fro decicion boundary for rosenblatt perceptron

- Intersection of lines: Two lines $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$ intersect in point $\widetilde{\boldsymbol{p}}$ if

$$\boldsymbol{l}_1^{\mathrm{T}}\widetilde{\boldsymbol{p}} = \boldsymbol{l}_2^{\mathrm{T}}\widetilde{\boldsymbol{p}} = 0,$$

if we find a p for whis this is true, the lines intersect

so we find

$$\widetilde{\boldsymbol{p}} = \boldsymbol{l}_1 \times \boldsymbol{l}_2.$$

intersection point of lines

Homogeneous Coordinates

Definition

The set of ***ideal points*** lies on the line at infinity $\boldsymbol{l}_\infty = (0,0,1)^\mathrm{T}$:

$$(0,0,1)^\mathrm{T}(x,y,0) = 0.$$

only points with a weight of zero are ideal point
but this is not a valide point anywere

Note: The tupel $(0,0,0)^\mathrm{T}$ describes no valid coordinate in $\mathbb{P}^2$.

in this space two parrallel lines eventually meet -> calc at home

Homogeneous Coordinates

Definition

The set of ***ideal points*** lies on the line at infinity $\boldsymbol{l}_\infty = (0,0,1)^\mathrm{T}$:

$$(0,0,1)^\mathrm{T}(x,y,0) = 0.$$

Note: The tupel $(0,0,0)^\mathrm{T}$ describes no valid coordinate in $\mathbb{P}^2$.

Exercise: Do parallel lines intersect in $\mathbb{P}^2$? Where?

The concept of homogeneous coordinates can be transferred to higher dimensional spaces. We will not continue to look into the details of this theory. Interested students are referred to the literature on perspective geometry (see for instance Hartley's book).

Orthographic Projection

We will now formulate projections from 3-D to 2-D using homogeneous coordinates:

The *orthographic projection* in homogeneous coordinates is defined by:

$$\widetilde{\boldsymbol{p}} = (x, y, z, 1)^{\mathrm{T}} \mapsto \widetilde{\boldsymbol{p}}' = (x, y, 1)^{\mathrm{T}}.$$

This mapping from $\mathbb{P}^3 \to \mathbb{P}^2$ can be simply written in matrix form as

$$\widetilde{\boldsymbol{p}}' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \widetilde{\boldsymbol{p}}.$$

zust ignore z

Weak Perspective Projection

The *weak perspective projection* in homogeneous coordinates is defined by:

$$\widetilde{\boldsymbol{p}} = (x, y, z, 1)^{\mathrm{T}} \mapsto \widetilde{\boldsymbol{p}}' = (kx, ky, 1)^{\mathrm{T}},$$

where $k \in \mathbb{R}$ is a scaling factor.
This mapping from $\mathbb{P}^3 \to \mathbb{P}^2$ can be simply written in matrix form as:

$$\widetilde{\boldsymbol{p}}' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1/k \end{pmatrix} \widetilde{\boldsymbol{p}}.$$

scale

$$\begin{pmatrix} k & 0 & 0 & 0 \\ 0 & k & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

or write like this

simmilar to weak projection in carthesion space
but we can already solve this

it doesnt matter if a matrix is scaled - it is still
the same !?

new part

Perspective Projection

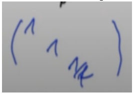
Using homogeneous coordinates, the *perspective projection* becomes a *linear* mapping:

$$\widetilde{\boldsymbol{p}} = (x, y, z, 1)^{\mathrm{T}} \mapsto \widetilde{\boldsymbol{p}}' = (fx, fy, z)^{\mathrm{T}} \cong (fx/z, fy/z, 1)^{\mathrm{T}}.$$

We get the following linear mapping from $\mathbb{P}^3 \to \mathbb{P}^2$:

$$\widetilde{\boldsymbol{p}}' = \underbrace{\begin{pmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}}_{\boldsymbol{P}} \widetilde{\boldsymbol{p}}.$$

just scale x and y ?



same

Topics

Take Home Messages

how to project?

thake point -> transform int high dim
calculate plane jou want to project onto
clculate intersection between planes -> intersection point
transform intersection point back -> coordinates point gets projected onto
?????????

- Points on a line through the origin in real vector space correspond to a single point in the projective plane.
- The nonlinear projective mapping in $\mathbb{R}^3$ can be written as a linear mapping using homogeneous coordinates.

Further Readings

For further details on geometric aspects of imaging see:

1. Richard Hartley and Andrew Zisserman. *Multiple View Geometry in Computer Vision*. 2nd ed. Cambridge: Cambridge University Press, 2004. DOI: 10.1017/CBO9780511811685
2. Olivier Faugeras. *Three-Dimensional Computer Vision: A Geometric Viewpoint*. MIT Press, Nov. 1993