Signal Processing

Fourier Transformation

Refresher Course

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Topics

The Fourier Transformation and Its Properties
- Fourier Transform
- Sampling Theorem
- Convolution Theorem
- Symmetry Property
- Other Properties

Further Readings

Fourier Transform

Definition

complex

The discrete ***Fourier transform*** of a 1-D signal $f : \{0, ..., N-1\} \longrightarrow \mathbb{C}$ is defined by:

xi ?

$$F(\xi) = \sum_{n=0}^{N-1} f(n) e^{\frac{-2\pi i n \xi}{N}}.$$

this is a composit of sin and cosin at frequency xi

this funktion basicly tells you how much a given frequency contributes to the signal

what frequencys are important to me? -> sampling theory

Sampling Theorem

Theorem

Given a continuous function $f : \mathbb{R} \longrightarrow \mathbb{R}$ with a limited bandwidth $B \in \mathbb{R}$, i. e., the Fourier transform F of f satisfies

$$F(\xi) = 0 \quad \textit{for} \quad \xi > B.$$

The interpolation using discrete sampling points allows the exact reconstruction of f if the function is sampled with a sampling rate that is larger than twice the bandwidth B, i. e.,

$$f_{sampling} > 2B.$$

Convolution Theorem

Theorem

***Convolution theorem**: The convolution of two signals f, h in the time domain corresponds to a multiplication of the respective Fourier transforms F, H in the frequency domain:*

convolution turns into multiplication in the frequency domain
and the other way around

$$g(n) = \sum_{k=0}^{N-1} f(k)h(n-k) = (f * h)(n),$$

$$G(\xi) = F(\xi)H(\xi),$$

where $G = \mathrm{FT}(g)$.

Convolution Theorem

the proove

furier

furier plus convolution

$$FT(f*h)(\xi) = \sum_{n=0}^{N-1} (f*h)(n) e^{\frac{-2\pi i n \xi}{N}} = \sum_{n=0}^{N-1}\sum_{k=0}^{N-1} f(k)h(n-k)e^{\frac{-2\pi i n \xi}{N}}$$

$$= \sum_{k=0}^{N-1} f(k) \sum_{n=0}^{N-1} h(n-k) e^{\frac{-2\pi i n \xi}{N}}$$

$$= \sum_{k=0}^{N-1} f(k) \sum_{n'=0}^{N-1} h(n') e^{\frac{-2\pi i (n'+k)\xi}{N}} \qquad (n' \to n-k,\ h \text{ periodic})$$

$$= \sum_{k=0}^{N-1} f(k) e^{\frac{-2\pi i k \xi}{N}} \sum_{n'=0}^{N-1} h(n') e^{\frac{-2\pi i n' \xi}{N}}$$

turns into multiplication of two furier transforms

$$= F(\xi)H(\xi) = G(\xi)$$

Symmetry Property for Real Valued Signals

Theorem

If $f(n)$ is a real valued discrete signal of length N, its discrete Fourier transform $F(\xi)$ fulfills the following symmetry property:

$$F(\xi) = \overline{F}(N - \xi),$$

where $\overline{F}$ denotes the complex conjugate of F.

Symmetry Property for Real Valued Signals

Using the property that for real numbers $a \in \mathbb{R} \subset \mathbb{C}$ they are identical with their complex conjugate $a = \overline{a}$, we have $f(n) = \overline{f}(n)$ for all sampling points n, and thus:

$$\overline{F}(N-\xi) = \overline{\sum_{k=0}^{N-1} f(k) e^{\frac{-2\pi i k(N-\xi)}{N}}} = \sum_{k=0}^{N-1} \overline{f}(k) e^{\frac{2\pi i k(N-\xi)}{N}} = \sum_{k=0}^{N-1} f(k) e^{\frac{-2\pi i k(\xi-N)}{N}}.$$

The N-th unit roots further fulfill

$$e^{\frac{2\pi i k \xi}{N}} = e^{\frac{2\pi i k(\xi+mN)}{N}},$$

where $m \in \mathbb{Z}$, so we finally get:

$$\overline{F}(N-\xi) = F(\xi).$$

Some Useful Properties of the Fourier Transform

	spatial domain	frequency domain
scaling	$f(at)$	$\frac{1}{\lvert a\rvert} F\left(\frac{\xi}{a}\right)$
shifting	$f(t-t_0)$	$e^{-i\xi t_0} F(\xi)$
symmetry	$-\frac{1}{2\pi} F(t)$	$f(-\xi)$
differentiation	$\frac{\mathrm{d}^n}{\mathrm{d}t^n} f(t)$	$(i\xi)^n F(\xi)$

Table 1: Summary of important properties of the Fourier transform

Topics

Further Readings

This topic is so important and well-known that further information can easily be found searching on the internet, or in certain calculus textbooks.

A concise mathematical overview is given in the rather old book:

R. D. Stuart. *An Introduction to Fourier Analysis*. Methuen's Monographs on Physical Subjects. London: Methuen & Co Ltd., 1961

It can be found online here (November 2016).