

Medical Image Processing for Diagnostic Applications

Singular Value Decomposition

Online Course – Unit 4

Andreas Maier, Joachim Hornegger, Markus Kowarschik, Frank Schebesch
Pattern Recognition Lab (CS 5)

Topics

Singular Value Decomposition (SVD) – Part 1

General Remarks

On the Geometry of Linear Mappings

Normal Form of Matrices: SVD

Summary

Take Home Messages

Further Readings

Singular Value Decomposition

- Powerful normal form for matrices that allows for a simple solution of most linear problems in imaging and image processing.
- SVD is a method from linear algebra ...
 - ... invented in the 19th century.
 - ... rediscovered and pushed for practical applications by [Gene Golub](#).
 - ... established in computer vision by [Carlo Tomasi](#)'s famous factorization algorithm to compute structure and camera motion from image sequences.
 - ... which is extremely robust and simple to use.

Singular Value Decomposition

SVD is a perfect tool, e. g., for

- the computation of singular values,
- the computation of the null space,
- the computation of the (pseudo-) inverse,
- the solution of overdetermined linear equations,
- the computation of condition numbers,
- enforcing a rank criterion (numerical rank),
- and other applications of matrices.

On the Geometry of Linear Mappings

From linear algebra, we know that a matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ maps the unit vectors $\mathbf{e}_i \in \mathbb{R}^n$ of the standard base to the corresponding column vectors $\mathbf{a}_i \in \mathbb{R}^m$ of the matrix \mathbf{A} , $i = 1, \dots, n$.

Example

here we write a Matrix as a set of Vektors

Matrix times Vek

=

set of Vektors times Vek

$$\mathbf{A} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = (\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_6) \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \mathbf{a}_4$$

we extract column number 4

result will be a linear
kombination of the column
vectors in the set

$$\vec{x} = \vec{a}_1 x_1 + \vec{a}_2 x_2 + \dots + \vec{a}_n x_n$$

result is a vektor $N \times 6 * 6 \times 1 = N \times 1$

On the Geometry of Linear Mappings

In the example we have made use of the following notation:

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & \ddots & & \\ \vdots & & & \\ a_{m1} & & & a_{mn} \end{pmatrix} = (\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n).$$

We can write:

$$\mathbf{Ax} = \mathbf{a}_1 x_1 + \mathbf{a}_2 x_2 + \dots + \mathbf{a}_n x_n$$

and for the first two unit vectors $\mathbf{e}_1 = (1, 0, 0, \dots, 0)^T$, $\mathbf{e}_2 = (0, 1, 0, \dots, 0)^T$ find:

$$\mathbf{Ae}_1 = \mathbf{a}_1, \quad \mathbf{Ae}_2 = \mathbf{a}_2.$$

On the Geometry of Linear Mappings

Rotate vector by multiplying with rotation matrix

Example

Compute the orthogonal matrix R , i. e., $R^{-1} = R^T$, that rotates points in the 2-D image plane by the angle θ .

Solution:

The base vectors are mapped as follows:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix},$$

and thus the 2-D rotation matrix is:

$$R = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}.$$

-> eigentlich logisch, kann man
sich herleiten

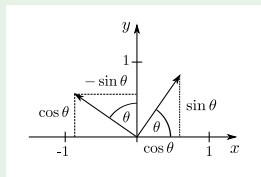
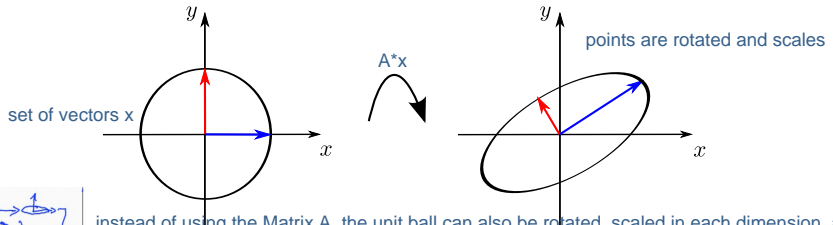


Figure 1: Rotation of 2-D unit vectors

On the Geometry of Linear Mappings

If \mathbf{A} is a real $m \times n$ -matrix of rank r , then \mathbf{A} maps the unit hyper-sphere in the n -dimensional space to an r -dimensional hyperellipsoid in the m -dimensional space.



instead of using the Matrix \mathbf{A} , the unit ball can also be rotated, scaled in each dimension, and then rotated back \rightarrow this will give the same result as multiplying by $\mathbf{A} \rightarrow$ it is a decomposition of \mathbf{A} .

Figure 2: A rank 2-matrix \mathbf{A} maps the 2-D unit sphere to a 2-D ellipse.

\mathbf{A} is rotation, scaling, rotation

$$\mathbf{A} = \mathbf{R}_1 \Sigma \mathbf{R}_2$$

Normal Form of Matrices: SVD

ortogonal -> rechtwinklige achsen ->
rotationsmatrix

Theorem

If \mathbf{A} is a real $m \times n$ -matrix, then there exist orthogonal matrices $\mathbf{U} \in \mathbb{R}^{m \times m}$ and $\mathbf{V} \in \mathbb{R}^{n \times n}$ such that

$$\mathbf{A} = \mathbf{U} \mathbf{\Sigma} \mathbf{V}^T, \text{ U and V are the rotations}$$

where

$$\mathbf{\Sigma} = \text{diag}(\sigma_1, \sigma_2, \dots, \sigma_p) \in \mathbb{R}^{m \times n} \text{ singular values -> usually ordered by size}$$

with $p = \min\{m, n\}$. The diagonal elements σ_i are the **singular values** that fulfill

$$\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_p \geq 0.$$

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- SVD is a useful tool to solve a multitude of problems.
- We studied the effect of a matrix on unit vectors and the unit sphere.
- An arbitrary real matrix \mathbf{A} can be decomposed by $\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T$.

Further Readings

Read the original:

Gene H. Golub and Charles F. Van Loan. *Matrix Computations*. 3rd ed. Johns Hopkins Studies in the Mathematical Sciences. Baltimore: The Johns Hopkins University Press, Oct. 1996

A very detailed and easy to follow introduction of the SVD can be found in:

Carlo Tomasi's class notes, chapter 3 (a **must-read**).

The theory is described in an easy to read format in:

Lloyd N. Trefethen and David Bau III. *Numerical Linear Algebra*. Philadelphia: SIAM, 1997

For details about the numerical computation of SVD see:

William H. Press et al. *Numerical Recipes – The Art of Scientific Computing*. 3rd ed. Cambridge University Press, 2007. Get at <http://numerical.recipes/> (August 2016).

Finally, have a look at:

Kaare Brandt Petersen and Michael Syskind Pedersen. *The Matrix Cookbook*. Online. Accessed: 25. April 2017. Technical University of Denmark, Nov. 2012. URL: <http://www2.imm.dtu.dk/pubdb/p.php?3274>