Homework 3
MATH 6610-001, Fall 2019

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Problem assignment

Lecture 20 #20.1

Let $A \in {}^{m\times m}$ be nonsingular. Let's show that A has an LU factorization if and only if for each $1 \le k \le m$, the upper-left $k \times k$ block $A_{1:k,1:k}$ is nonsingular.

Solution

$\Longrightarrow$) Let $A \in \mathbb{C}^{m\times m}$ such that has an LU factorization.

$$\begin{bmatrix} A_{1:k,1:k} & A_{1:k,k+1:m} \\ A_{k+1:m,1:k} & A_{k+1:m,k+1:m} \end{bmatrix} = \begin{bmatrix} L_{1:k,1:k} & 0 \\ L_{k+1:m,1:k} & L_{k+1:m,k+1:m} \end{bmatrix} * \begin{bmatrix} U_{1:k,1:k} & U_{1:k,k+1:m} \\ 0 & U_{k+1:m,k+1:m} \end{bmatrix}$$

So, $A_{1:k,1:k} = L_{1:k,1:k} * U_{1:k,1:k}$

$\Longrightarrow$ $\det(A_{1:k,1:k}) = \det(L_{1:k,1:k}) * \det(U_{1:k,1:k}) = \prod_{j=1}^{k} L_{jj}U_{jj} = \prod_{j=1}^{k} U_{jj}$ because the diagonal elements of L are equal to 1.

U is upper-triangular with nonzero diagonal elements.Therefore, $\det(A_{1:k,1:k}) = \prod_{j=1}^{k} U_{jj} \neq 0$

$\Longrightarrow$ $\det(A_{1:k,1:k}) \neq 0$ and $A_{1:k,1:K}$ is nonsingular.

$\Longleftarrow$) Let's suppose that $\forall k, 1 \le k \le m, A_{1:k,1:k}$ is nonsingular and let's show that A has an LU factorization.

We will do a proof by induction.

For $\dim(A) = 1$, $A = (a_{11})$ nonsingular means $a_{11} \neq 0$. A has an LU factorization $A = 1.a_{11}$ where $L = 1$ and $U = a_{11}$.

Let's suppose that for $\dim(A) \le m-1$, A has an LU factorization and let's try to find L, l^T, U, v and w such that

$$\tilde{A} = \begin{bmatrix} A & b \\ c^T & d \end{bmatrix} = \begin{bmatrix} L & 0 \\ l^T & 1 \end{bmatrix} * \begin{bmatrix} U & v \\ 0 & w \end{bmatrix} \Longrightarrow$$

$$\begin{cases} A = LU & (1) \\ b = Lv & (2) \\ l^T U = c^T & (3) \\ l^T v + w = d & (4) \end{cases}$$

We have $\dim(A) = m-1 \implies A = LU$. A invertible $\implies L$ and U invertible; this implies that we can solve(2), (3) and (4) for v, l^t and w. Meaning $\tilde{A}$ has an LU factorization.

Proof that the LU factorization is unique.
Let's suppose that A is nonsingular with 2 LU factorizations: $A = L_1U_1$ and $A = L_2U_2$.

$L_1U_1 = L_2U_2 \implies L_2^{-1}L_1U_1 = U_2 \implies L_2^{-1}L_1 = U_2^{-1}U_1^{-1}$

$L_2^{-1}L_1$ is lower-triangular because the inverse of a lower-triangular matrix is lower-triangular and the product of 2 lower-triangular matrices is a lower-triangular matrix.
Also, $U_2U_1^{-1}$ is upper- triangular because the inverse of an upper-triangular matrix is upper-triangular and the product of 2 upper-triangular matrices is an upper-triangular matrix.
Therefore, $L_2^{-1}L_1$ and $U_2U_1^{-1}$ are diagonal matrices.
$L_2^{-1}L_1$ diagonal matrix $\implies L_2^{-1}L_1 = I_{m\times m} \implies L_2 = L_1$. Also, $U_2U_1^{-1} = I_{m\times m} \implies U_2 = U_1$.
In conclusion, $L_1U_1 = L_2U_2$. If A is nonsingular, A has an unique LU factorization.

Lecture 21 #21.6

Suppose $A \in \mathbb{C}^{m\times m}$ is strictly column diagonally dominant, which means for each k,

$$|a_{kk}| > \sum_{j\neq k} |a_{jk}|$$

Let's show that if Gaussian elimination with partial pivoting is applied to A, no row interchanges take place.
<u>**Solution**</u>
For the first column of A, A strictly column dominant $\implies |a_{11}| > \sum_{j\neq 1} |a_{j1}| \implies |a_{11}| = max_{1\leq i\leq n}|a_{i1}|$; so there is no need to pivot the first row.
Let $A^{(k)}$ be the $(n-k)\times(n-k)$ matrix in the lower left corner obtained after the k^{th} round of Gaussian elimination.
It suffices to prove that all the $A^{(k)}$ are diagonally dominant.
Let's show that $A^{(1)}$ is diagonally dominant.
Let

$$A = \begin{bmatrix} \alpha & w \\ v & B \end{bmatrix}$$

One step Gaussian elimination yields:

$$A = \begin{bmatrix} 1 & w \\ \frac{v}{\alpha} & I \end{bmatrix} \begin{bmatrix} \alpha & w \\ 0 & B - \frac{vw}{\alpha} \end{bmatrix}$$

$A^{(1)} = B - \frac{vw}{\alpha}$.

Let $a_{ij}^{(1)}$ be the (i, j) entry of $A^{(1)}$.

$$
\begin{aligned}
\sum_{i\geq 2,\ i\neq j}^{n} \left|a_{ij}^{(1)}\right| &= \sum_{i\geq 2,\ i\neq j}^{n} \left|b_{ij} - \frac{v_i w_j}{\alpha}\right| \\
&\leq \sum_{i\geq 2,\ i\neq j}^{n} \left|b_{ij}\right| + \sum_{i\geq 2,\ i\neq j}^{n} \left|\frac{v_i w_j}{\alpha}\right| \\
&< \left|b_{jj}\right| - \left|w_j\right| + \frac{\left|w_j\right|}{\alpha}(|\alpha| - \left|v_j\right|) \text{ because A is diagonally dominant} \\
&= \left|b_{jj}\right| - \frac{\left|w_j\right|}{\alpha}\left|v_j\right| \\
&\leq \left|b_{jj} - \frac{w_j v_j}{\alpha}\right| = a_{jj}^{(1)}
\end{aligned}
$$

This means that A diagonally dominant $\Longrightarrow$ $A^{(1)}$ diagonally dominant $\Longrightarrow$ $A^{(2)}$ diagonally dominant$\cdots \Longrightarrow$ $A^{(k)}$ diagonally dominant.
So if the Gaussian elimination is applied to A, diagonally dominant, no row interchanges take place.

Lecture 22 #22.3

Reproduction of the figures of the 22^{nd} lecture of our textbook, approximately if not in full detail, but based on random matrices with entries uniformly distributed in [-1,1] rather that normally distributed.
<u>Solution</u>

(a) Growth factor versus matrix dimension.

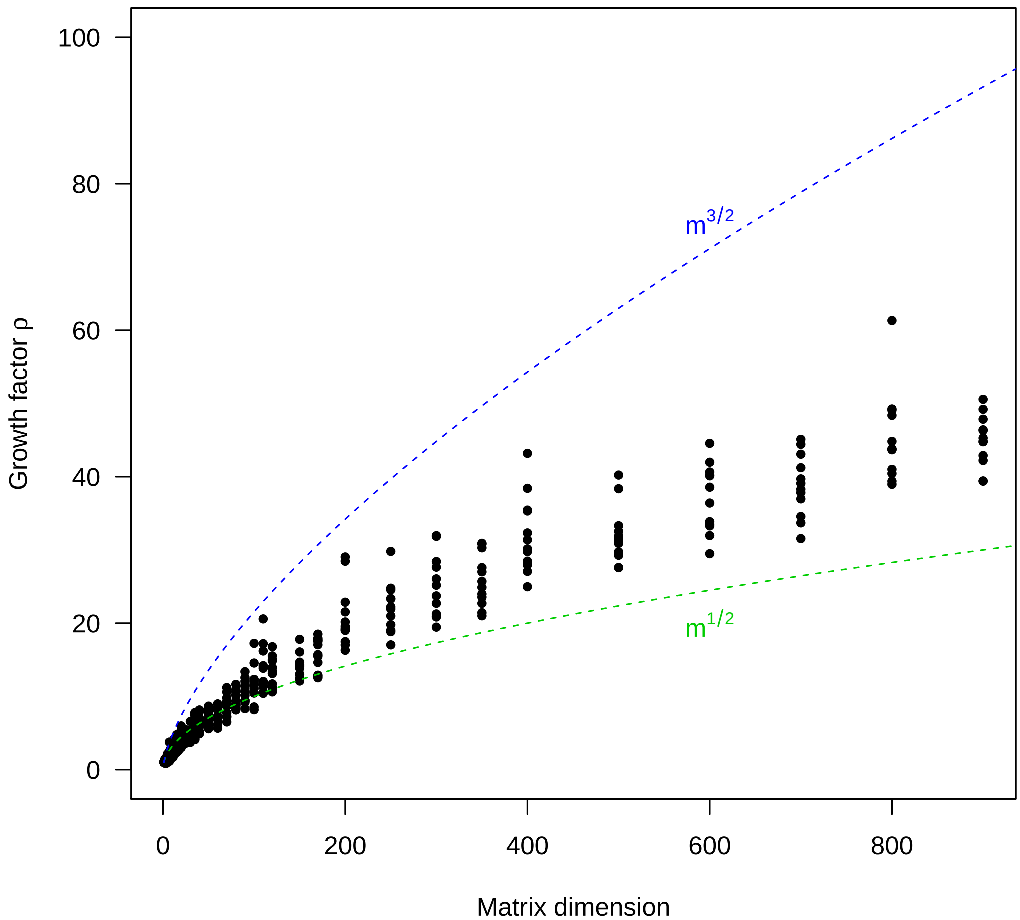


Figure 1: Growth factors for Gaussian elimination with partial pivoting applied to 468 random matrices (independent, entries uniformly distributed on [-1,1])

Figure 1 shows that when Gaussian elimination with partial pivoting is applied to matrices with entries from random uniform distributions on [-1, 1], the typical size of ρ is between $m^{1/2}$ and $m^{3/2}$. This is different from what appears in our textbook when the entries of the matrices are chosen from random normal distributions.

(b) Probability density function for growth factors of random matrices

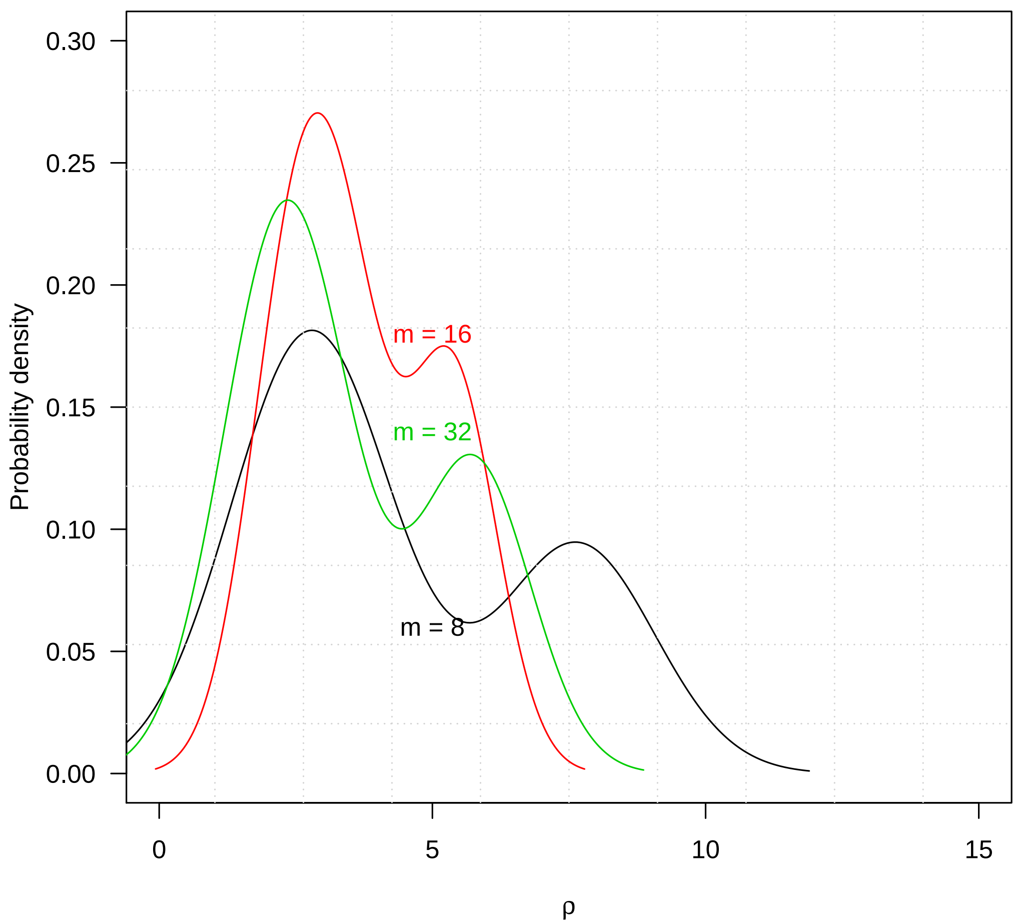
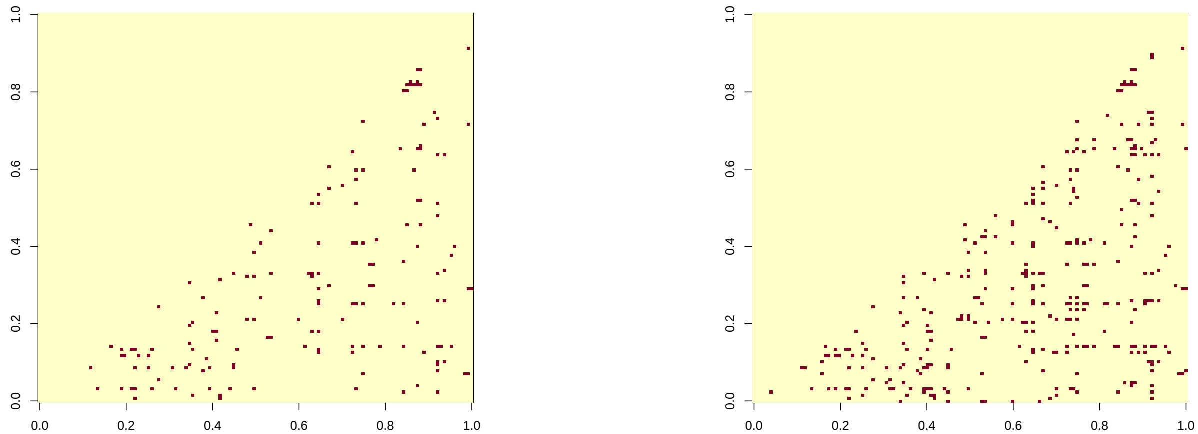


Figure 2: Probability density distributions for growth factors of random matrices of dimensions m = 8, 16, 32, based on samples sizes of one million for each dimension

The probability density plots show that in each case (m =8, 16, 32), the growth factor decreases exponentially with ρ. The peaks reflect the fact that the distribution is uniform and not normal.

(c) $PA = LU$ factorization



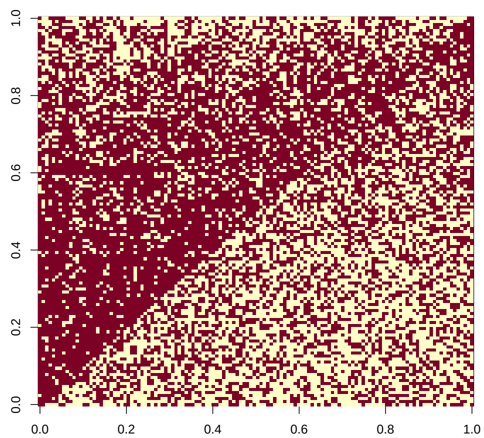
(a) random A $max_{i,j} \left|(L^{-1})_{ij}\right| = 2.49$

(b) random $\tilde{L} \max_{i,j} \left|(\tilde{L}^{-1})_{ij}\right| = 2.49$

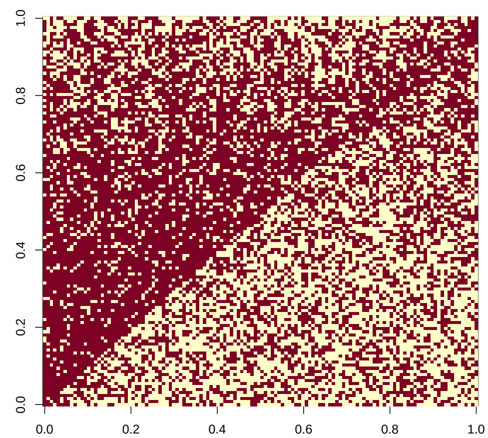
Figure 3: Let A be a random 128×128 matrix with factorization $PA = LU$. On the left, L^{-1} is shown: the dots represent entries with magnitude ≥ 1. On the right, a similar picture for $\tilde{L}^{-1}$, where $\tilde{L}$ is the same as L except that the signs of its subdiagonal entries have been randomized

The plots on figure 3 show that Gaussian elimination tends to produce matrices L that are extraordinarily well-conditioned.

(d) Q portraits of the same two matrices



(a) random A



(b) random $\tilde{L}$

Figure 4: On the left, the random matrix A after permutation of the form PA, or equivalently, the factor L. On the right, the matrix $\tilde{L}$ with randomized signs.

The graphs on figure 4 show that the column spaces of $\tilde{L}$ are skewed in manner exponentially unlikely to arise in typical classes of random matrices.

The results shown by figures 3 and 4 are similar to those obtained when the distribution is normal.

Lecture 23 #23.1

Let $A \in \mathbb{C}^{m\times m}$ be a nonsingular square matrix and let $A = QR$ and $A^*A = U^*U$ be QR and Cholesky factorizations, respectively, with the usual normalizations $r_{jj}, u_{jj} > 0$. Is it true or false that $R = U$?

Solution

A nonsingular $\implies$ A has a unique QR factorization with $r_{jj} > 0$

$$\implies A^*A = R^*Q^*QR = R^*R \qquad (*)$$

By hypothesis, $A^*A = U^*U = AA^*$, meaning A^*A is symmetric.

Let $x \in \mathbb{C}^m$, $x^*A^*Ax = (Ax)^*Ax = \sum_{j=1}^{m} (Ax)^2 > 0$. Hence, A^* is positive-definite.

A^*A is symmetric and positive-definite, therefore A^*A has a unique LU type decomposition

$$A = U^*U \qquad (**)$$

(*) and (**) $\implies$ $R^*R = U^*U$. Thus, $U = R$.

In conclusion, it is true that $R = U$.

Lecture 24 #24.1

For each of the following statements, proof that it is true or an example to show it is false. Throughout, $A \in \mathbb{C}^{m\times m}$ unless otherwise indicated, and "ew" stands for eigenvalue. (This comes from "Eigenwert". The corresponding abbreviation for eigenvector is "ev", from "Eigenvektor".)

(a) If λ is an ew of A and $\mu \in \mathbb{C}$, then $\lambda - \mu$ is an ew of $A - \lambda I$.
Solution: TRUE
λ ew of A $\implies \exists v \in \mathbb{C}^m - \{0\}$ such that $\lambda v = Av$
$\implies \lambda v - \mu v = Av - \mu v \;\; \forall \mu \in \mathbb{C} \implies (\lambda - \mu)v = (A - \mu I)v$
So, $\lambda - \mu$ is a eigenvalue of $A - \mu I$ associated to v.

(b) If A is real and λ is an ew of A, then so is $-\lambda$.
Solution: FALSE

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\lambda = 1$ is an eigenvalue of A but -1 is not an eigenvalue of A.

(c) If A is real and λ is an ew of A, then so is $\bar{\lambda}$.
Solution: TRUE
λ ew of A $\implies \exists v \in \mathbb{C}^m - \{0\}$ such that $\lambda v = Av$
$\implies \bar{\lambda}\bar{v} = \bar{A}\bar{v} \implies \bar{\lambda}\bar{v} = A\bar{v}$ because A is real
$\implies \bar{\lambda}$ ew of A associated to $\bar{v}$.

(d) If λ is an ew of A and A is nonsingular, then λ^{-1} is an ew of A^{-1}.
Solution: TRUE
λ ew of A $\implies \exists v \in \mathbb{C}^m - \{0\}$ such that $\lambda v = Av (*)$
$(*) \implies v = \lambda^{-1}Av \; (\lambda \neq 0)$ (**)
Also (*) $\implies v = A^{-1}\lambda v$ (***)
(**) and (***) $\implies \lambda^{-1}Av = A^{-1}\lambda v \implies \lambda^{-1}w = A^{-1}w$ where $w = Av = \lambda v \implies \lambda^{-1}$ is an eigenvalue of A^{-1} associated to w.

(e) If all ew's of A are zero, then $A = 0$.
Solution: FALSE

Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

The eigenvalues of A are $\lambda = (0,0)$ but A is not 0.

(f) If A is hermitian and λ is an ew of A, then $|\lambda|$ is a singular value of A.
Solution: TRUE
A hermitian $\implies A$ normal $\implies A$ has an unitary diagonalization $A = Q\Lambda Q^* = QSign(\Lambda)\,|\Lambda|\,Q^*$ where Q is an unitary matrix, $|\Lambda|$ a diagonal matrix whose entries are the absolute values of the eigenvalues λ_j of A and $Sign(\Lambda)$ is the diagonal matrix whose entries are the signs of the eigenvalues λ_j of A.
In order to have the values of Λ sorted in a non-increasing order. We can insert the suitable permutation matrices P_1 and P_2 such that:

$$A = P_1 QSign(\tilde{\Lambda})\left|\tilde{\Lambda}\right| Q^* P_2. \tag{5}$$

Where $\tilde{\Lambda}$ is the diagonal matrix with the entries of Λ sorted in a non-increasing order.

P_1, Q and $Sign(\tilde{\Lambda})$ are unitary matrices, so $P_1 Q Sign(\tilde{\Lambda})$ is also a unitary matrix. Also, P_2^* and Q^* are unitary matrices, so $P_2^* Q$ is also an unitary matrix.
This means that (5) is an SVD decomposition of A. Therefore, the absolute values of the eigen values of A are the singular values of A.

(g) If A is diagonalizable and all its ew's are equal, then A is diagonal.
Solution: TRUE
Let $A \in \mathbb{C}^{m\times m}$. By Jordan decomposition, $\exists V$ invertible and J block diagonal such that: $A = VJV^-1$ and $J = \text{diag}(J_1, J_2, \cdots, J_n)$
A diagonalizable $\iff$ all the J_k are of order 1; $J_k = [\lambda_k]$.
Meaning $n = m$ and $J = \text{diag}(\lambda_1, \lambda_2, \cdots \lambda_n)$
All the eigenvalues of A are equal $\implies J = \lambda \text{diag}(1, 1, \cdots, 1) = \lambda I_{m\times m}$
So, $A = V\lambda I_{m\times m} V^{-1} = \lambda V V^{-1} = \lambda I_{m\times m}$
$\implies A$ diagonal.

Lecture 24 #24.4

For an arbitrary $A \in \mathbb{C}^{m\times m}$ and norm $\|.\|$, proof using Theorem 24.9 or our textbook:

(a) $\lim_{n\to\infty} \|A^n\| = 0 \iff \rho(A) < 1$, where ρ is the spectral radius (Exercise 3.2).
Solution
Let $A \in \mathbb{C}^{m\times m}$ such that $\rho(A) < 1$
$\exists T$ upper-triangular and Q unitary such that $A = QTQ^*$ (Schur complement).
Let's define $\epsilon_j \in \mathbb{C}$ such that $\forall i \neq j \;\; T_{ii} + \epsilon_i \neq T_{jj} + \epsilon_j$ and $max_{1\le i\le n} |T_{ii}| + |\epsilon_i| < 1$

$$\text{Let } T' = T + \begin{bmatrix} \epsilon_1 & 0 & 0 & \cdots & 0 \\ 0 & \epsilon_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & 0 \\ 0 & 0 & 0 & \cdots & \epsilon_m \end{bmatrix}$$

$T - T'$ is diagonal and

$$\begin{aligned} A = QTQ^* &= Q(T - T' + T')Q^* \\ &= Q(T - T')Q^* + QT'Q^* \\ &= T - T' + QT'Q^* \end{aligned}$$

T' is diagonalizable means $\exists X$ and D such that $T' = XDX^{-1}$ and we also have $\rho(T') < 1$.

$$\begin{aligned}
\|A^n\| &= \sum_{k=0}^{n} \binom{n}{k} (T - T')^{n-k} Q X D^k X^{-1} Q^* \\
&= \sum_{k=0}^{n} \binom{n}{k} \|T - T'\|^{n-k} \|X\| \|X^{-1}\| \|D\|^k \\
&= \sum_{k=0}^{n} \binom{n}{k} \|T - T'\|^{n-k} K(X) \|D\|^k \\
&= K(X) \sum_{k=0}^{n} \binom{n}{k} \|T - T'\|^{n-k} \|D\|^k \\
&\leq K(X) \sum_{k=0}^{n} \binom{n}{k} max_{1\leq i\leq m} |\epsilon_i|^{n-k} * max_{1\leq i\leq m} |D_{ii}|^k \\
&\leq K(X)(max_{1\leq i\leq m}|\epsilon_i| + max_{1\leq i\leq m}|D_{ii}|)^n \\
&\leq K(X)(max_{1\leq i\leq m}|\epsilon_i| + \rho(T'))^n
\end{aligned}$$

Thus, $\lim_{n\to\infty} \|A^n\| \leq \lim_{n\to\infty} K(X)(max_{1\leq i\leq m}|\epsilon_i| + \rho(T'))^n = 0$

Therefore, $\lim_{n\to\infty} \|A^n\| = 0$

(b) $\lim_{n\to\infty} \|e^{tA}\| = 0 \iff \alpha(A) < 0$, where α is the spectral abcissa.

Solution

By the Schur factorization, there exist $T \in \mathbb{C}^{m\times m}$, upper-triangular and $Q \in \mathbb{C}^{m\times m}$ unitary such that $A = QTQ^*$

Let $\epsilon_j \in \mathbb{C}$ such that $\forall i \neq j\ \ T_{ii} + \epsilon_i \neq T_{jj} + \epsilon_j$ and $Re(T_{ii} + \epsilon_i) < 0,\ \ Re(\epsilon_i) < 0$.

Let $T' = T + \begin{bmatrix} \epsilon_1 & 0 & 0 & \cdots & 0 \\ 0 & \epsilon_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & 0 \\ 0 & 0 & 0 & \cdots & \epsilon_m \end{bmatrix}$

$$\begin{aligned}
\|\exp(tA)\| &= \|\exp(tQTQ^*)\| \\
&= \|\exp(tQ(T - T' + T')Q^*)\| \\
&= \left\| I + tQ(T - T' + T')Q^* + \frac{1}{2} t^2 (Q(T - T' + T')Q)^2 + \cdots \right\| \\
&\leq \|\exp(\text{t(T - T')})\| * \|\exp tT'\|
\end{aligned}$$

$T - T'$i s diagonal andT' is diagonalizable, therefore;

$\lim_{t\to\infty} \|\exp(\text{tA})\| \leq \lim_{t\to\infty} \|\exp(t(T - T'))\| * \|\exp(tT')\|$

$\leq \lim_{t\to\infty} max_{1\leq i\leq m} \exp(-t|\epsilon_i|).max_{1\leq i\leq m} \exp(-t|T_{ii} + \epsilon_i|) = 0$

Therefore, $\lim_{t\to\infty} \|\exp(\text{tA})\| = 0$

Problem assignment

P1

Let $P \in \mathbb{R}^{n\times n}$ be a permutation matrix.

(a) Proof that P^T is also a permutation matrix
Solution
P is a permutation matrix means that P has precisely one nonzero entry in each row and and in each column and all those entries take the value 1.
P^T is such that the i^{th} row of P is the i^{th} column of P^T and the j^{th} column of P is the j^{th} row of P^T. Therefore, P^T has also one nonzero entry (of value 1) on each row and on each column. Meaning P^T is also a permutation matrix.

(b) Proof that $P^T = P^{-1}$
Solution

$$(PP^T)_{ij} = \sum_{k=1}^{n} P_{ik}P^T_{kj} = \sum_{k=1}^{n} P_{ik}P_{jk}$$

$$\exists! l \text{ such that } P_{ik} = \begin{cases} 1 & \text{if } i = l \\ 0 & \text{if } i \neq l \end{cases}$$

So,

$$(PP^T)_{ij} = P_{il}P_{jl} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} = I$$

$$\implies PP^T = I \implies P^T = P^{-1}$$

(c) Proof that if P1 and P2 are both permutation matrices, then P1P2 is also a permutation matrix.
Solution
Let $P_1 = (X_{ij})_{1\leq i,j\leq n}$ and $P_2 = (Y_{ij})_{1\leq i,j\leq n}$. The i^{th} row of P_1P_2 is:

$$(\sum_{j=1}^{n} X_{ij}Y_{j1}, \sum_{j=1}^{n} X_{ij}Y_{j2}, \cdots, \sum_{j=1}^{n} X_{ij}Y_{jn})$$

Let i^*be the column of the nonzero entry of the i^{th} row; $X_{ii^*} = 1$

This implies that

$$(\sum_{j=1}^{n} X_{ij}Y_{j1}, \sum_{j=1}^{n} X_{ij}Y_{j2}, \cdots, \sum_{j=1}^{n} X_{ij}Y_{jn}) = (Y_{i^*1}, Y_{i^*2}, \cdots, Y_{i^*1})$$

which is the i^{*th} row of P_2. This means that P_1P_2 is a shuffling of the rows of P_2. P_2 is a permutation, the shuffling of the rows of a permutation is a permutation; therefore P_1P_2 is a permutation.

(d) Is it true in general that $P^2 = P$?

Solution

No in general, $P^2 \neq P$. As an example,

$$P^2 = PP = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} * \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \neq P$$

P2

Let $A \in \mathbb{C}^{n\times n}$ be invertible. Proof that the "LU" decomposition algorithm with partial pivoting always successfully computes $PA = LU$

Solution

Let $A \in \mathbb{C}^{n\times n}$ invertible. Let's suppose that the LU decomposition algorithm with partial pivoting does not successfully compute $PA = LU$. It means that $\exists k, 1 \leq k \leq n$ such that the Gaussian elimination with partial pivoting breaks down at the k^{th} stage.
Let's $A^{(k)} = (a^{(k)})_{ij}$ the matrix obtained at the k^{th} stage. It means that $a_{ij}^{(k)} = 0 \ \ \forall i \geq k$ and $j \leq k$.
In other words, the k^{th} column of $A^{(k)}$ is a linear combination of the first $k-1$ columns of $A^{(k)}$; hence $A^{(k)}$ is singular.
Also, $A^{(k)} = L^{(k-1)}P^{(k-1)}L^{(k-2)}P^{(k-2)}\cdots L^{(1)}P^{(1)}A$; where $L^{(j)}$ and $P^{(j)}$, $1 \leq j \leq k-1$ are respectively lower triangular and permutation matrices.
$L^{(j)}$ and $P^{(j)}$, $1 \leq j \leq k-1$ are nonsingular, thus the only possibility for $A^{(k)}$ to be nonsingular is that A is singular. This is a contradiction, since A is invertible.

In conclusion, if A is invertible, the LU decomposition algorithm with partial pivoting always successfully computes $PA = LU$.

P3

Given $A \in \mathbb{C}^{m\times n}$, consider a *column-pivoted* QR decomposition, i.e., a factorization of the form,

$$AP = QR$$

where P is a permutaion matrix that is chosen in the following way: At step j in the the orthogonalization process (say step j of Gram-Schmidt), the columns $, j, j+1, \cdots, n$ are permuted/pivoted so that r_{jj} will be as large as possible.
Note that the vector p defined as

$$p := P^T \begin{pmatrix} 1 \\ 2 \\ \vdots \\ n \end{pmatrix}$$

has entries that identify the *column pivots*, i.e., the ordered columns indices of A chosen by the pivoting process.

(a) (Column-pivoted QR decompositions are rank-revealing, in a sense). Proof that the number of nonzero diagonal entries in R equals the rank of A.

Solution

Let $s = \min(n, m)$ and $A^{(s)}$ be the matrix obtained at the step s.
$\forall j, 1 \le j \le s$ such that $r_{jj} = 0$. By gram-Schimdt, we have $r_{jj} = 0 \implies \left\| a_j^{(s)} - \sum_{k=1}^{j-1} \right\| = 0 \implies$ $a_j^{(s)} = \sum_{k=1}^{j-1}$ which means that $a_j^{(s)}$ belongs to the span of $a_1, a_2, \cdots, a_{j-1}$. Meaning $a_j^{(s)}$ is a linear combination of the other $(j-1)$ columns of $A^{(s)}$.
Therefore, for all j such that $r_{jj} = 0$, $a_j^{(s)}$ can be represented as a linear combination of the other columns of $A^{(s)}$. The rank of $A^{(s)}$ is the number of linearly independent columns of $A^{(s)}$, thus the rank of $A^{(s)}$ is the number of nonzero diagonal entries in R.
Also, $A^{(s)} = AP_1P_2 \cdots P_s$ and rank of $P_1, P_2, \cdots P_s = n$. This implies that $\text{rank}(A^{(s)}) = \text{rank}(A)$.
So $\text{rank}(A)$ is the number of nonzero diagonal entries in R.

(b) (Column-pivoted QR is greedy determinant maximization). Let $r = \text{rank}(A)$. For S any subset of $1, 2, \cdots, n$, let A_S denote the $m \times |S|$ submatrix of A formed by selected column indices in S. Furthermore, let, $G_S \in \mathbb{C}^{|S| \times |S|}$ be defined as

$$G_S = (A_S)^* A_S$$

Consider the following iterative, greedy, determinant maximization:

$$s_j = \text{ argmax}_{k \in \{1, \cdots, n\}} \det G_{S_{j-1} \cup \{k\}}, \qquad S_j := S_{j-1} \cup s_j$$

Where $S_0 := \{\}$.
If each maximization yields a unique s_j, let's show that $p_j = s_j$ for $j = 1, \cdots, r$.
Solution
$G_S = (A_S)^* (A_S) = R^* Q^* QR = R^* R$
So, $\det G_S = det R^* R = \prod_{i=1}^{|S|} r_{ii}^2$ because R upper-triangular.

$$\begin{aligned}
s_1 &= \text{ argmax}_{k \in \{1, \cdots, n\}} \det G_{S_0 \cup \{k\}} \\
&= \text{ argmax}_{k \in \{1, \cdots, n\}} \det G_{\{k\}} \\
&= \text{ argmax}_{k \in \{1, \cdots, n\}} r_{kk}^2 \\
&= p_1 \quad \text{ because by part (a), } \quad r_{p_1 p_1} > r_{kk} \;\; \forall 1 \le k \le n
\end{aligned}$$

Also,

$$\begin{aligned}
s_2 &= \text{ argmax}_{k \in \{1, \cdots, n\}} \det G_{S_1 \cup \{k\}} \\
&= \text{ argmax}_{k \in \{1, \cdots, n\}} r_{p_1 p_1}^2 r_{kk}^2 \\
&= p_2 \quad \text{ because by part (a), the largest product of 2 diagonal entries of R is } \quad r_{p_1 p_1} r_{p_2 p_2}
\end{aligned}$$

In general, we have $\forall j = 1, \cdots, r$:

$$\begin{aligned}
s_i &= \text{ argmax}_{k \in \{1, \cdots, n\}} \det G_{S_{j-1} \cup \{k\}} \\
&= \text{ argmax}_{k \in \{1, \cdots, n\}} r_{p_1 p_1}^2 r_{p_2 p_2}^2 \cdots r_{p_{j-1} p_{j-1}}^2 r_{kk}^2 \\
&= p_j \quad \text{ because by part (a), the largest product of j diagonal entries of R is } \quad r_{p_1 p_1} r_{p_2 p_2} \cdots r_{p_{j-1} p_{j-1}} r_{jj}
\end{aligned}$$

In conclusion, $p_j = s_j$ for $j = 1, \cdots, r$.

P4

Let $A \in \mathbb{C}^{m\times n}$, and let r = rank(A). Let's show that the LU factorization with partial row pivoting,

$$PA = LU,$$

selects pivots via another kind of greedy determinant maximization, I.e., with S as in the previous problem, let ${}_SA$ denote the $|S| \times n$ matrix formed by selecting the rows with indices S from A. Consider the optimization problem

$$s_j = \text{argmax}_{k\in\{1,\cdots,m\}} \left|\det_{S_{j-1}\cup\{k\}} A_{\{1,\cdots,j\}}\right|$$

for $j \geq 1$ where $S_0 = \{\}$.
If each maximization yields a unique s_j, let's show that for $j = 1, \cdots, r,\ \ s_j$ is the jth entry of the vector p defined by

$$p := P\begin{pmatrix} 1 \\ 2 \\ 3 \\ \vdots \\ m \end{pmatrix} \in \mathbb{R}^m$$

__**Solution**__

$\det A = \det LU = \prod_{i=1}^{r} u_{ii}$ because L and U are triangular and the diagonal element of L are equal to 1.

$$\begin{aligned} s_1 &= \text{argmax}_{k\in\{1,\cdots,m\}} \left|\det_{S_0\cup\{k\}} A_{\{1\}}\right| \\ &= \text{argmax}_{k\in\{1,\cdots,m\}} \left|\det_{S_0\cup\{k\}} A_{\{1\}}\right| \\ &= \text{argmax}_{k\in\{1,\cdots,m\}} |u_{k1}| \\ &= p_1 \text{ because by construction, } \left|u_{p_1 1}\right| \geq |u_{k1}| \ \ \forall 1 \leq k \leq m \end{aligned}$$

Also,

$$\begin{aligned} s_2 &= \text{argmax}_{k\in\{1,\cdots,m\}} \left|\det_{S_1\cup\{k\}} A_{\{1,2\}}\right| \\ &= \text{argmax}_{k\in\{1,\cdots,m\}} \left|\det_{p_1\cup\{k\}} A_{\{1,2\}}\right| \\ &= \text{argmax}_{k\in\{1,\cdots,m\}} \left|u_{p_1 1} u_{k2}\right| \\ &= p_2 \end{aligned}$$

In general, we have $\forall j = 1, \cdots, r$

$$\begin{aligned} s_j &= \text{argmax}_{k\in\{1,\cdots,m\}} \left|\det_{S_{j-1}\cup\{k\}} A_{\{1,2,\cdots j\}}\right| \\ &= \text{argmax}_{k\in\{1,\cdots,m\}} \left|u_{p_1 1} U_{p_2 2} \cdots U_{p_{j-1} j-1} U_{kj}\right| \\ &= p_j \end{aligned}$$

So, s_j is the jth entry of p such that:

$$p := P\begin{pmatrix} 1 \\ 2 \\ 3 \\ \vdots \\ m \end{pmatrix}$$