1. **Set theory** is a branch of [mathematical logic](https://en.wikipedia.org/wiki/Mathematical_logic) that studies [sets](https://en.wikipedia.org/wiki/Set_(mathematics)), which informally are collections of objects. Although any type of object can be collected into a set, set theory is applied most often to objects that are relevant to mathematics. The language of set theory can be used to define nearly all [mathematical objects](https://en.wikipedia.org/wiki/Mathematical_object).

The modern study of set theory was initiated by [Georg Cantor](https://en.wikipedia.org/wiki/Georg_Cantor) and [Richard Dedekind](https://en.wikipedia.org/wiki/Richard_Dedekind) in the 1870s. After the discovery of [paradoxes](https://en.wikipedia.org/wiki/Paradoxes_of_set_theory) in [naive set theory](https://en.wikipedia.org/wiki/Naive_set_theory), such as [Russell's paradox](https://en.wikipedia.org/wiki/Russell's_paradox), numerous [axiom systems](https://en.wikipedia.org/wiki/Axiomatic_system) were proposed in the early twentieth century, of which the [Zermelo–Fraenkel axioms](https://en.wikipedia.org/wiki/Zermelo–Fraenkel_set_theory), with or without the [axiom of choice](https://en.wikipedia.org/wiki/Axiom_of_choice), are the best-known.

Set theory is commonly employed as a [foundational system for mathematics](https://en.wikipedia.org/wiki/Foundations_of_mathematics), particularly in the form of Zermelo–Fraenkel set theory with the axiom of choice.[[1]](https://en.wikipedia.org/wiki/Set_theory" \l "cite_note-FOOTNOTEKunen1980xi-1) Beyond its foundational role, set theory is a branch of [mathematics](https://en.wikipedia.org/wiki/Mathematics) in its own right, with an active research community. Contemporary research into set theory includes a diverse collection of topics, ranging from the structure of the [real number](https://en.wikipedia.org/wiki/Real_number) line to the study of the [consistency](https://en.wikipedia.org/wiki/Consistency) of [large cardinals](https://en.wikipedia.org/wiki/Large_cardinal).

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## History



[Georg Cantor](https://en.wikipedia.org/wiki/Georg_Cantor).

Mathematical topics typically emerge and evolve through interactions among many researchers. Set theory, however, was founded by a single paper in 1874 by [Georg Cantor](https://en.wikipedia.org/wiki/Georg_Cantor): "[On a Property of the Collection of All Real Algebraic Numbers](https://en.wikipedia.org/wiki/On_a_Property_of_the_Collection_of_All_Real_Algebraic_Numbers)".[[2]](https://en.wikipedia.org/wiki/Set_theory" \l "cite_note-cantor1874-2)[[3]](https://en.wikipedia.org/wiki/Set_theory" \l "cite_note-3)

Since the 5th century BC, beginning with [Greek](https://en.wikipedia.org/wiki/Greek_mathematics) mathematician [Zeno of Elea](https://en.wikipedia.org/wiki/Zeno_of_Elea) in the West and early [Indian mathematicians](https://en.wikipedia.org/wiki/Indian_mathematics) in the East, mathematicians had struggled with the concept of [infinity](https://en.wikipedia.org/wiki/Infinity). Especially notable is the work of [Bernard Bolzano](https://en.wikipedia.org/wiki/Bernard_Bolzano) in the first half of the 19th century.[[4]](https://en.wikipedia.org/wiki/Set_theory" \l "cite_note-4) Modern understanding of infinity began in 1870–1874 and was motivated by Cantor's work in [real analysis](https://en.wikipedia.org/wiki/Real_analysis).[[5]](https://en.wikipedia.org/wiki/Set_theory" \l "cite_note-5) An 1872 meeting between Cantor and [Richard Dedekind](https://en.wikipedia.org/wiki/Richard_Dedekind) influenced Cantor's thinking and culminated in Cantor's 1874 paper.

Cantor's work initially polarized the mathematicians of his day. While [Karl Weierstrass](https://en.wikipedia.org/wiki/Karl_Weierstrass) and Dedekind supported Cantor, [Leopold Kronecker](https://en.wikipedia.org/wiki/Leopold_Kronecker), now seen as a founder of [mathematical constructivism](https://en.wikipedia.org/wiki/Mathematical_constructivism), did not. Cantorian set theory eventually became widespread, due to the utility of Cantorian concepts, such as [one-to-one correspondence](https://en.wikipedia.org/wiki/One-to-one_correspondence) among sets, his proof that there are more [real numbers](https://en.wikipedia.org/wiki/Real_number) than integers, and the "infinity of infinities" ("[Cantor's paradise](https://en.wikipedia.org/wiki/Cantor's_paradise)") resulting from the [power set](https://en.wikipedia.org/wiki/Power_set) operation. This utility of set theory led to the article "Mengenlehre" contributed in 1898 by [Arthur Schoenflies](https://en.wikipedia.org/wiki/Arthur_Schoenflies) to [Klein's encyclopedia](https://en.wikipedia.org/wiki/Klein's_encyclopedia).

The next wave of excitement in set theory came around 1900, when it was discovered that some interpretations of Cantorian set theory gave rise to several contradictions, called [antinomies](https://en.wikipedia.org/wiki/Antinomy) or [paradoxes](https://en.wikipedia.org/wiki/Logical_paradox). [Bertrand Russell](https://en.wikipedia.org/wiki/Bertrand_Russell) and [Ernst Zermelo](https://en.wikipedia.org/wiki/Ernst_Zermelo) independently found the simplest and best known paradox, now called [Russell's paradox](https://en.wikipedia.org/wiki/Russell's_paradox): consider "the set of all sets that are not members of themselves", which leads to a contradiction since it must be a member of itself and not a member of itself. In 1899 Cantor had himself posed the question "What is the [cardinal number](https://en.wikipedia.org/wiki/Cardinal_number) of the set of all sets?", and obtained a related paradox. Russell used his paradox as a theme in his 1903 review of continental mathematics in his [*The Principles of Mathematics*](https://en.wikipedia.org/wiki/The_Principles_of_Mathematics).

In 1906 English readers gained the book *Theory of Sets of Points*[[6]](https://en.wikipedia.org/wiki/Set_theory" \l "cite_note-6) by husband and wife [William Henry Young](https://en.wikipedia.org/wiki/William_Henry_Young) and [Grace Chisholm Young](https://en.wikipedia.org/wiki/Grace_Chisholm_Young), published by [Cambridge University Press](https://en.wikipedia.org/wiki/Cambridge_University_Press).

The momentum of set theory was such that debate on the paradoxes did not lead to its abandonment. The work of [Zermelo](https://en.wikipedia.org/wiki/Zermelo) in 1908 and the work of [Abraham Fraenkel](https://en.wikipedia.org/wiki/Abraham_Fraenkel) and [Thoralf Skolem](https://en.wikipedia.org/wiki/Thoralf_Skolem) in 1922 resulted in the set of axioms [ZFC](https://en.wikipedia.org/wiki/ZFC), which became the most commonly used set of axioms for set theory. The work of [analysts](https://en.wikipedia.org/wiki/Real_analysis) such as [Henri Lebesgue](https://en.wikipedia.org/wiki/Henri_Lebesgue) demonstrated the great mathematical utility of set theory, which has since become woven into the fabric of modern mathematics. Set theory is commonly used as a foundational system, although in some areas—such as algebraic geometry and algebraic topology—[category theory](https://en.wikipedia.org/wiki/Category_theory) is thought to be a preferred foundation.

## Basic concepts and notation

Main articles: [Set (mathematics)](https://en.wikipedia.org/wiki/Set_(mathematics)) and [Algebra of sets](https://en.wikipedia.org/wiki/Algebra_of_sets)

Set theory begins with a fundamental [binary relation](https://en.wikipedia.org/wiki/Binary_relation) between an object *o* and a set *A*. If *o* is a [**member**](https://en.wikipedia.org/wiki/Set_membership) (or **element**) of *A*, the notation *o* ∈ *A* is used. A set is described by listing elements separated by commas, or by a characterizing property of its elements, within braces { }. Since sets are objects, the membership relation can relate sets as well.

A derived binary relation between two sets is the subset relation, also called **set inclusion**. If all the members of set *A* are also members of set *B*, then *A* is a [**subset**](https://en.wikipedia.org/wiki/Subset) of *B*, denoted *A* ⊆ *B*. For example, {1, 2} is a subset of {1, 2, 3} , and so is {2} but {1, 4} is not. As insinuated from this definition, a set is a subset of itself. For cases where this possibility is unsuitable or would make sense to be rejected, the term [**proper subset**](https://en.wikipedia.org/wiki/Proper_subset) is defined. *A* is called a **proper subset** of *B* if and only if *A* is a subset of *B*, but *A* is not equal to *B*. Also 1, 2, and 3 are members (elements) of the set {1, 2, 3} but are not subsets of it; and in turn, the subsets, such as {1}, are not members of the set {1, 2, 3}.

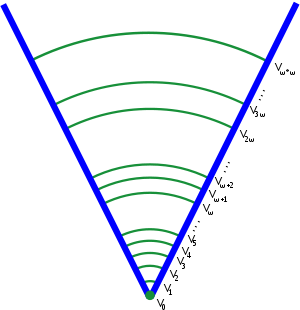
Just as [arithmetic](https://en.wikipedia.org/wiki/Arithmetic) features [binary operations](https://en.wikipedia.org/wiki/Binary_operation) on [numbers](https://en.wikipedia.org/wiki/Number), set theory features binary operations on sets.[[7]](https://en.wikipedia.org/wiki/Set_theory" \l "cite_note-7) The:

* [**Union**](https://en.wikipedia.org/wiki/Union_(set_theory)) of the sets *A* and *B*, denoted *A* ∪ *B*, is the set of all objects that are a member of *A*, or *B*, or both. The union of {1, 2, 3} and {2, 3, 4} is the set {1, 2, 3, 4} .
* [**Intersection**](https://en.wikipedia.org/wiki/Intersection_(set_theory)) of the sets *A* and *B*, denoted *A* ∩ *B*, is the set of all objects that are members of both *A* and *B*. The intersection of {1, 2, 3} and {2, 3, 4} is the set {2, 3} .
* [**Set difference**](https://en.wikipedia.org/wiki/Set_difference) of *U* and *A*, denoted *U* \ *A*, is the set of all members of *U* that are not members of *A*. The set difference {1, 2, 3} \ {2, 3, 4} is {1} , while, conversely, the set difference {2, 3, 4} \ {1, 2, 3} is {4} . When *A* is a subset of *U*, the set difference *U* \ *A* is also called the [**complement**](https://en.wikipedia.org/wiki/Complement_(set_theory)) of *A* in *U*. In this case, if the choice of *U* is clear from the context, the notation *Ac* is sometimes used instead of *U* \ *A*, particularly if *U* is a [universal set](https://en.wikipedia.org/wiki/Universal_set) as in the study of [Venn diagrams](https://en.wikipedia.org/wiki/Venn_diagram).
* [**Symmetric difference**](https://en.wikipedia.org/wiki/Symmetric_difference) of sets *A* and *B*, denoted *A* △ *B* or *A* ⊖ *B*, is the set of all objects that are a member of exactly one of *A* and *B* (elements which are in one of the sets, but not in both). For instance, for the sets {1, 2, 3} and {2, 3, 4} , the symmetric difference set is {1, 4} . It is the set difference of the union and the intersection, (*A* ∪ *B*) \ (*A* ∩ *B*) or (*A* \ *B*) ∪ (*B* \ *A*).
* [**Cartesian product**](https://en.wikipedia.org/wiki/Cartesian_product) of *A* and *B*, denoted *A* × *B*, is the set whose members are all possible [ordered pairs](https://en.wikipedia.org/wiki/Ordered_pair) (*a*, *b*) where *a* is a member of *A* and *b* is a member of *B*. The cartesian product of {1, 2} and {red, white} is {(1, red), (1, white), (2, red), (2, white)}.
* [**Power set**](https://en.wikipedia.org/wiki/Power_set) of a set *A* is the set whose members are all of the possible subsets of *A*. For example, the power set of {1, 2} is { {}, {1}, {2}, {1, 2} } .

Some basic sets of central importance are the [empty set](https://en.wikipedia.org/wiki/Empty_set) (the unique set containing no elements; occasionally called the *null set* though this name is ambiguous), the set of [natural numbers](https://en.wikipedia.org/wiki/Natural_number), and the set of [real numbers](https://en.wikipedia.org/wiki/Real_number).

## Some ontology

Main article: [von Neumann universe](https://en.wikipedia.org/wiki/Von_Neumann_universe)



An initial segment of the von Neumann hierarchy.

A set is [pure](https://en.wikipedia.org/wiki/Pure_set) if all of its members are sets, all members of its members are sets, and so on. For example, the set {{}} containing only the empty set is a nonempty pure set. In modern set theory, it is common to restrict attention to the [**von Neumann universe**](https://en.wikipedia.org/wiki/Von_Neumann_universe) of pure sets, and many systems of axiomatic set theory are designed to axiomatize the pure sets only. There are many technical advantages to this restriction, and little generality is lost, because essentially all mathematical concepts can be modeled by pure sets. Sets in the von Neumann universe are organized into a [cumulative hierarchy](https://en.wikipedia.org/wiki/Cumulative_hierarchy), based on how deeply their members, members of members, etc. are nested. Each set in this hierarchy is assigned (by [transfinite recursion](https://en.wikipedia.org/wiki/Transfinite_recursion)) an [ordinal number](https://en.wikipedia.org/wiki/Ordinal_number) , known as its **rank**. The rank of a pure set  is defined to be the [least upper bound](https://en.wikipedia.org/wiki/Least_upper_bound) of all [successors](https://en.wikipedia.org/wiki/Successor_ordinal) of ranks of members of . For example, the empty set is assigned rank 0, while the set {{}} containing only the empty set is assigned rank 1. For each ordinal , the set  is defined to consist of all pure sets with rank less than . The entire von Neumann universe is denoted  .

## Axiomatic set theory

Elementary set theory can be studied informally and intuitively, and so can be taught in primary schools using [Venn diagrams](https://en.wikipedia.org/wiki/Venn_diagram). The intuitive approach tacitly assumes that a set may be formed from the class of all objects satisfying any particular defining condition. This assumption gives rise to paradoxes, the simplest and best known of which are [Russell's paradox](https://en.wikipedia.org/wiki/Russell's_paradox) and the [Burali-Forti paradox](https://en.wikipedia.org/wiki/Burali-Forti_paradox). Axiomatic set theory was originally devised to rid set theory of such paradoxes.[[note 1]](https://en.wikipedia.org/wiki/Set_theory" \l "cite_note-8)

The most widely studied systems of axiomatic set theory imply that all sets form a [cumulative hierarchy](https://en.wikipedia.org/wiki/Cumulative_hierarchy). Such systems come in two flavors, those whose [ontology](https://en.wikipedia.org/wiki/Ontology) consists of:

* *Sets alone*. This includes the most common axiomatic set theory, [**Z**ermelo–**F**raenkel set theory](https://en.wikipedia.org/wiki/Zermelo–Fraenkel_set_theory) with the [Axiom of **C**hoice](https://en.wikipedia.org/wiki/Axiom_of_Choice) (**ZFC**). Fragments of ZFC include:
  + [Zermelo set theory](https://en.wikipedia.org/wiki/Zermelo_set_theory), which replaces the [axiom schema of replacement](https://en.wikipedia.org/wiki/Axiom_schema_of_replacement) with that of [separation](https://en.wikipedia.org/wiki/Axiom_schema_of_separation);
  + [General set theory](https://en.wikipedia.org/wiki/General_set_theory), a small fragment of [Zermelo set theory](https://en.wikipedia.org/wiki/Zermelo_set_theory) sufficient for the [Peano axioms](https://en.wikipedia.org/wiki/Peano_axioms) and [finite sets](https://en.wikipedia.org/wiki/Finite_set);
  + [Kripke–Platek set theory](https://en.wikipedia.org/wiki/Kripke–Platek_set_theory), which omits the axioms of infinity, [powerset](https://en.wikipedia.org/wiki/Axiom_of_power_set), and [choice](https://en.wikipedia.org/wiki/Axiom_of_choice), and weakens the axiom schemata of [separation](https://en.wikipedia.org/wiki/Axiom_schema_of_separation) and [replacement](https://en.wikipedia.org/wiki/Axiom_schema_of_replacement).
* *Sets and* [*proper classes*](https://en.wikipedia.org/wiki/Proper_class). These include [Von Neumann–Bernays–Gödel set theory](https://en.wikipedia.org/wiki/Von_Neumann–Bernays–Gödel_set_theory), which has the same strength as [ZFC](https://en.wikipedia.org/wiki/ZFC) for theorems about sets alone, and [Morse–Kelley set theory](https://en.wikipedia.org/wiki/Morse–Kelley_set_theory) and [Tarski–Grothendieck set theory](https://en.wikipedia.org/wiki/Tarski–Grothendieck_set_theory), both of which are stronger than ZFC.

The above systems can be modified to allow [**urelements**](https://en.wikipedia.org/wiki/Urelement), objects that can be members of sets but that are not themselves sets and do not have any members.

The [**New Foundations**](https://en.wikipedia.org/wiki/New_Foundations) systems of **NFU** (allowing [urelements](https://en.wikipedia.org/wiki/Urelement)) and **NF** (lacking them) are not based on a cumulative hierarchy. NF and NFU include a "set of everything, " relative to which every set has a complement. In these systems urelements matter, because NF, but not NFU, produces sets for which the [axiom of choice](https://en.wikipedia.org/wiki/Axiom_of_choice) does not hold.

Systems of [constructive set theory](https://en.wikipedia.org/wiki/Constructive_set_theory), such as CST, CZF, and IZF, embed their set axioms in [intuitionistic](https://en.wikipedia.org/wiki/Intuitionistic_logic) instead of [classical logic](https://en.wikipedia.org/wiki/Classical_logic). Yet other systems accept classical logic but feature a nonstandard membership relation. These include [rough set theory](https://en.wikipedia.org/wiki/Rough_set) and [fuzzy set theory](https://en.wikipedia.org/wiki/Fuzzy_set_theory), in which the value of an [atomic formula](https://en.wikipedia.org/wiki/Atomic_formula) embodying the membership relation is not simply **True** or **False**. The [Boolean-valued models](https://en.wikipedia.org/wiki/Boolean-valued_model) of [ZFC](https://en.wikipedia.org/wiki/ZFC) are a related subject.

An enrichment of [ZFC](https://en.wikipedia.org/wiki/ZFC) called [internal set theory](https://en.wikipedia.org/wiki/Internal_set_theory) was proposed by [Edward Nelson](https://en.wikipedia.org/wiki/Edward_Nelson) in 1977.

## Applications

Many mathematical concepts can be defined precisely using only set theoretic concepts. For example, mathematical structures as diverse as [graphs](https://en.wikipedia.org/wiki/Graph_(discrete_mathematics)), [manifolds](https://en.wikipedia.org/wiki/Manifolds), [rings](https://en.wikipedia.org/wiki/Ring_(mathematics)), and [vector spaces](https://en.wikipedia.org/wiki/Vector_space) can all be defined as sets satisfying various (axiomatic) properties. [Equivalence](https://en.wikipedia.org/wiki/Equivalence_relation) and [order relations](https://en.wikipedia.org/wiki/Order_relation) are ubiquitous in mathematics, and the theory of mathematical [relations](https://en.wikipedia.org/wiki/Relation_(mathematics)) can be described in set theory.

Set theory is also a promising foundational system for much of mathematics. Since the publication of the first volume of [*Principia Mathematica*](https://en.wikipedia.org/wiki/Principia_Mathematica), it has been claimed that most or even all mathematical theorems can be derived using an aptly designed set of axioms for set theory, augmented with many definitions, using [first](https://en.wikipedia.org/wiki/First-order_logic) or [second-order logic](https://en.wikipedia.org/wiki/Second-order_logic). For example, properties of the [natural](https://en.wikipedia.org/wiki/Natural_number) and [real numbers](https://en.wikipedia.org/wiki/Real_number) can be derived within set theory, as each number system can be identified with a set of [equivalence classes](https://en.wikipedia.org/wiki/Equivalence_class) under a suitable [equivalence relation](https://en.wikipedia.org/wiki/Equivalence_relation) whose field is some [infinite set](https://en.wikipedia.org/wiki/Infinite_set).

Set theory as a foundation for [mathematical analysis](https://en.wikipedia.org/wiki/Mathematical_analysis), [topology](https://en.wikipedia.org/wiki/Topology), [abstract algebra](https://en.wikipedia.org/wiki/Abstract_algebra), and [discrete mathematics](https://en.wikipedia.org/wiki/Discrete_mathematics) is likewise uncontroversial; mathematicians accept that (in principle) theorems in these areas can be derived from the relevant definitions and the axioms of set theory. Few full derivations of complex mathematical theorems from set theory have been formally verified, however, because such formal derivations are often much longer than the natural language proofs mathematicians commonly present. One verification project, [Metamath](https://en.wikipedia.org/wiki/Metamath), includes human-written, computer-verified derivations of more than 12,000 theorems starting from [ZFC](https://en.wikipedia.org/wiki/ZFC) set theory, [first-order logic](https://en.wikipedia.org/wiki/First-order_logic) and [propositional logic](https://en.wikipedia.org/wiki/Propositional_logic).

## Areas of study

Set theory is a major area of research in mathematics, with many interrelated subfields.

### Combinatorial set theory

Main article: [Infinitary combinatorics](https://en.wikipedia.org/wiki/Infinitary_combinatorics)

**Combinatorial set theory** concerns extensions of finite [combinatorics](https://en.wikipedia.org/wiki/Combinatorics) to infinite sets. This includes the study of [cardinal arithmetic](https://en.wikipedia.org/wiki/Cardinal_arithmetic) and the study of extensions of [Ramsey's theorem](https://en.wikipedia.org/wiki/Ramsey's_theorem) such as the [Erdős–Rado theorem](https://en.wikipedia.org/wiki/Erdős–Rado_theorem).

### Descriptive set theory

Main article: [Descriptive set theory](https://en.wikipedia.org/wiki/Descriptive_set_theory)

**Descriptive set theory** is the study of subsets of the [real line](https://en.wikipedia.org/wiki/Real_line) and, more generally, subsets of [Polish spaces](https://en.wikipedia.org/wiki/Polish_space). It begins with the study of [pointclasses](https://en.wikipedia.org/wiki/Pointclass) in the [Borel hierarchy](https://en.wikipedia.org/wiki/Borel_hierarchy) and extends to the study of more complex hierarchies such as the [projective hierarchy](https://en.wikipedia.org/wiki/Projective_hierarchy) and the [Wadge hierarchy](https://en.wikipedia.org/wiki/Wadge_hierarchy). Many properties of [Borel sets](https://en.wikipedia.org/wiki/Borel_set) can be established in ZFC, but proving these properties hold for more complicated sets requires additional axioms related to determinacy and large cardinals.

The field of [effective descriptive set theory](https://en.wikipedia.org/wiki/Effective_descriptive_set_theory) is between set theory and [recursion theory](https://en.wikipedia.org/wiki/Recursion_theory). It includes the study of [lightface pointclasses](https://en.wikipedia.org/wiki/Lightface_pointclass), and is closely related to [hyperarithmetical theory](https://en.wikipedia.org/wiki/Hyperarithmetical_theory). In many cases, results of classical descriptive set theory have effective versions; in some cases, new results are obtained by proving the effective version first and then extending ("relativizing") it to make it more broadly applicable.

A recent area of research concerns [Borel equivalence relations](https://en.wikipedia.org/wiki/Borel_equivalence_relation) and more complicated definable [equivalence relations](https://en.wikipedia.org/wiki/Equivalence_relation). This has important applications to the study of [invariants](https://en.wikipedia.org/wiki/Invariant_(mathematics)) in many fields of mathematics.

### Fuzzy set theory

Main article: [Fuzzy set theory](https://en.wikipedia.org/wiki/Fuzzy_set_theory)

In set theory as [Cantor](https://en.wikipedia.org/wiki/Georg_Cantor) defined and [Zermelo](https://en.wikipedia.org/wiki/Zermelo) and [Fraenkel](https://en.wikipedia.org/wiki/Fraenkel) axiomatized, an object is either a member of a set or not. In [fuzzy set theory](https://en.wikipedia.org/wiki/Fuzzy_set_theory) this condition was relaxed by [Lotfi A. Zadeh](https://en.wikipedia.org/wiki/Lotfi_A._Zadeh) so an object has a *degree of membership* in a set, a number between 0 and 1. For example, the degree of membership of a person in the set of "tall people" is more flexible than a simple yes or no answer and can be a real number such as 0.75.

### Inner model theory

Main article: [Inner model theory](https://en.wikipedia.org/wiki/Inner_model_theory)

An **inner model** of Zermelo–Fraenkel set theory (ZF) is a transitive [class](https://en.wikipedia.org/wiki/Proper_class) that includes all the ordinals and satisfies all the axioms of ZF. The canonical example is the [constructible universe](https://en.wikipedia.org/wiki/Constructible_universe) *L* developed by Gödel. One reason that the study of inner models is of interest is that it can be used to prove consistency results. For example, it can be shown that regardless of whether a model *V* of ZF satisfies the [continuum hypothesis](https://en.wikipedia.org/wiki/Continuum_hypothesis) or the [axiom of choice](https://en.wikipedia.org/wiki/Axiom_of_choice), the inner model *L* constructed inside the original model will satisfy both the generalized continuum hypothesis and the axiom of choice. Thus the assumption that ZF is consistent (has at least one model) implies that ZF together with these two principles is consistent.

The study of inner models is common in the study of [determinacy](https://en.wikipedia.org/wiki/Axiom_of_determinacy) and [large cardinals](https://en.wikipedia.org/wiki/Large_cardinal), especially when considering axioms such as the axiom of determinacy that contradict the axiom of choice. Even if a fixed model of set theory satisfies the axiom of choice, it is possible for an inner model to fail to satisfy the axiom of choice. For example, the existence of sufficiently large cardinals implies that there is an inner model satisfying the axiom of determinacy (and thus not satisfying the axiom of choice).[[8]](https://en.wikipedia.org/wiki/Set_theory" \l "cite_note-9)

### Large cardinals

Main article: [Large cardinal property](https://en.wikipedia.org/wiki/Large_cardinal_property)

A **large cardinal** is a cardinal number with an extra property. Many such properties are studied, including [inaccessible cardinals](https://en.wikipedia.org/wiki/Inaccessible_cardinal), [measurable cardinals](https://en.wikipedia.org/wiki/Measurable_cardinal), and many more. These properties typically imply the cardinal number must be very large, with the existence of a cardinal with the specified property unprovable in Zermelo-Fraenkel set theory.

### Determinacy

Main article: [Determinacy](https://en.wikipedia.org/wiki/Determinacy)

**Determinacy** refers to the fact that, under appropriate assumptions, certain two-player games of perfect information are determined from the start in the sense that one player must have a winning strategy. The existence of these strategies has important consequences in descriptive set theory, as the assumption that a broader class of games is determined often implies that a broader class of sets will have a topological property. The [axiom of determinacy](https://en.wikipedia.org/wiki/Axiom_of_determinacy) (AD) is an important object of study; although incompatible with the axiom of choice, AD implies that all subsets of the real line are well behaved (in particular, measurable and with the perfect set property). AD can be used to prove that the [Wadge degrees](https://en.wikipedia.org/wiki/Wadge_degree) have an elegant structure.

### Forcing

Main article: [Forcing (mathematics)](https://en.wikipedia.org/wiki/Forcing_(mathematics))

[Paul Cohen](https://en.wikipedia.org/wiki/Paul_Cohen_(mathematician)) invented the method of [forcing](https://en.wikipedia.org/wiki/Forcing_(mathematics)) while searching for a [model](https://en.wikipedia.org/wiki/Model_theory) of [ZFC](https://en.wikipedia.org/wiki/ZFC) in which the [continuum hypothesis](https://en.wikipedia.org/wiki/Continuum_hypothesis) fails, or a model of ZF in which the [axiom of choice](https://en.wikipedia.org/wiki/Axiom_of_choice) fails. Forcing adjoins to some given model of set theory additional sets in order to create a larger model with properties determined (i.e. "forced") by the construction and the original model. For example, Cohen's construction adjoins additional subsets of the [natural numbers](https://en.wikipedia.org/wiki/Natural_number) without changing any of the [cardinal numbers](https://en.wikipedia.org/wiki/Cardinal_number) of the original model. Forcing is also one of two methods for proving [relative consistency](https://en.wikipedia.org/wiki/Consistency_(mathematical_logic)) by finitistic methods, the other method being [Boolean-valued models](https://en.wikipedia.org/wiki/Boolean-valued_model).

### Cardinal invariants

Main article: [Cardinal characteristics of the continuum](https://en.wikipedia.org/wiki/Cardinal_characteristics_of_the_continuum)

A **cardinal invariant** is a property of the real line measured by a cardinal number. For example, a well-studied invariant is the smallest cardinality of a collection of [meagre sets](https://en.wikipedia.org/wiki/Meagre_set) of reals whose union is the entire real line. These are invariants in the sense that any two isomorphic models of set theory must give the same cardinal for each invariant. Many cardinal invariants have been studied, and the relationships between them are often complex and related to axioms of set theory.

### Set-theoretic topology

Main article: [Set-theoretic topology](https://en.wikipedia.org/wiki/Set-theoretic_topology)

**Set-theoretic topology** studies questions of [general topology](https://en.wikipedia.org/wiki/General_topology) that are set-theoretic in nature or that require advanced methods of set theory for their solution. Many of these theorems are independent of ZFC, requiring stronger axioms for their proof. A famous problem is the [normal Moore space question](https://en.wikipedia.org/wiki/Moore_space_(topology)), a question in general topology that was the subject of intense research. The answer to the normal Moore space question was eventually proved to be independent of ZFC.

## Objections to set theory as a foundation for mathematics

From set theory's inception, some mathematicians [have objected to it](https://en.wikipedia.org/wiki/Controversy_over_Cantor's_theory) as a [foundation for mathematics](https://en.wikipedia.org/wiki/Foundations_of_mathematics). The most common objection to set theory, one [Kronecker](https://en.wikipedia.org/wiki/Leopold_Kronecker) voiced in set theory's earliest years, starts from the [constructivist](https://en.wikipedia.org/wiki/Mathematical_constructivism) view that mathematics is loosely related to computation. If this view is granted, then the treatment of infinite sets, both in [naive](https://en.wikipedia.org/wiki/Naive_set_theory) and in axiomatic set theory, introduces into mathematics methods and objects that are not computable even in principle. The feasibility of constructivism as a substitute foundation for mathematics was greatly increased by [Errett Bishop](https://en.wikipedia.org/wiki/Errett_Bishop)'s influential book *Foundations of Constructive Analysis*.[[9]](https://en.wikipedia.org/wiki/Set_theory" \l "cite_note-10)

A different objection put forth by [Henri Poincaré](https://en.wikipedia.org/wiki/Henri_Poincaré) is that defining sets using the axiom schemas of [specification](https://en.wikipedia.org/wiki/Axiom_schema_of_specification) and [replacement](https://en.wikipedia.org/wiki/Axiom_schema_of_replacement), as well as the axiom of [power set](https://en.wikipedia.org/wiki/Axiom_of_power_set), introduces [impredicativity](https://en.wikipedia.org/wiki/Impredicativity), a type of [circularity](https://en.wikipedia.org/wiki/Circular_definition), into the definitions of mathematical objects. The scope of predicatively founded mathematics, while less than that of the commonly accepted Zermelo-Fraenkel theory, is much greater than that of constructive mathematics, to the point that [Solomon Feferman](https://en.wikipedia.org/wiki/Solomon_Feferman) has said that "all of scientifically applicable analysis can be developed [using predicative methods]".[[10]](https://en.wikipedia.org/wiki/Set_theory" \l "cite_note-11)

[Ludwig Wittgenstein](https://en.wikipedia.org/wiki/Ludwig_Wittgenstein) condemned set theory philosophically for its connotations of [Mathematical platonism](https://en.wikipedia.org/wiki/Mathematical_platonism).[[11]](https://en.wikipedia.org/wiki/Set_theory" \l "cite_note-12) He wrote that "set theory is wrong", since it builds on the "nonsense" of fictitious symbolism, has "pernicious idioms", and that it is nonsensical to talk about "all numbers".[[12]](https://en.wikipedia.org/wiki/Set_theory" \l "cite_note-13) Wittgenstein identified mathematics with algorithmic human deduction;[[13]](https://en.wikipedia.org/wiki/Set_theory" \l "cite_note-FOOTNOTERodych2018§2.1-14) the need for a secure foundation for mathematics seemed, to him, nonsensical.[[14]](https://en.wikipedia.org/wiki/Set_theory" \l "cite_note-FOOTNOTERodych2018§3.4-15) Moreover, since human effort is necessarily finite, Wittgenstein's philosophy required an ontological commitment to radical [constructivism](https://en.wikipedia.org/wiki/Constructivism_(math)) and [finitism](https://en.wikipedia.org/wiki/Finitism). Meta-mathematical statements — which, for Wittgenstein, included any statement quantifying over infinite domains, and thus almost all modern set theory — are not mathematics.[[15]](https://en.wikipedia.org/wiki/Set_theory" \l "cite_note-FOOTNOTERodych2018§2.2-16) Few modern philosophers have adopted Wittgenstein's views after a spectacular blunder in [*Remarks on the Foundations of Mathematics*](https://en.wikipedia.org/wiki/Remarks_on_the_Foundations_of_Mathematics): Wittgenstein attempted to refute [Gödel's incompleteness theorems](https://en.wikipedia.org/wiki/Gödel's_incompleteness_theorems) after having only read the abstract. As reviewers [Kreisel](https://en.wikipedia.org/wiki/Georg_Kreisel), [Bernays](https://en.wikipedia.org/wiki/Paul_Bernays), [Dummett](https://en.wikipedia.org/wiki/Michael_Dummett), and [Goodstein](https://en.wikipedia.org/wiki/R._L._Goodstein) all pointed out, many of his critiques did not apply to the paper in full. Only recently have philosophers such as [Crispin Wright](https://en.wikipedia.org/wiki/Crispin_Wright) begun to rehabilitate Wittgenstein's arguments.[[16]](https://en.wikipedia.org/wiki/Set_theory" \l "cite_note-FOOTNOTERodych2018§3.6-17)

[Category theorists](https://en.wikipedia.org/wiki/Category_theory) have proposed [topos theory](https://en.wikipedia.org/wiki/Topos_theory) as an alternative to traditional axiomatic set theory. Topos theory can interpret various alternatives to that theory, such as [constructivism](https://en.wikipedia.org/wiki/Mathematical_constructivism), finite set theory, and [computable](https://en.wikipedia.org/wiki/Turing_Machine) set theory.[[17]](https://en.wikipedia.org/wiki/Set_theory" \l "cite_note-18)[[18]](https://en.wikipedia.org/wiki/Set_theory" \l "cite_note-19) Topoi also give a natural setting for forcing and discussions of the independence of choice from ZF, as well as providing the framework for [pointless topology](https://en.wikipedia.org/wiki/Pointless_topology) and [Stone spaces](https://en.wikipedia.org/wiki/Stone_space).[[19]](https://en.wikipedia.org/wiki/Set_theory" \l "cite_note-20)

An active area of research is the [univalent foundations](https://en.wikipedia.org/wiki/Univalent_foundations) and related to it [homotopy type theory](https://en.wikipedia.org/wiki/Homotopy_type_theory). Within homotopy type theory, a set may be regarded as a homotopy 0-type, with [universal properties](https://en.wikipedia.org/wiki/Universal_properties) of sets arising from the inductive and recursive properties of [higher inductive types](https://en.wikipedia.org/wiki/Higher_inductive_type). Principles such as the [axiom of choice](https://en.wikipedia.org/wiki/Axiom_of_choice) and the [law of the excluded middle](https://en.wikipedia.org/wiki/Law_of_the_excluded_middle) can be formulated in a manner corresponding to the classical formulation in set theory or perhaps in a spectrum of distinct ways unique to type theory. Some of these principles may be proven to be a consequence of other principles. The variety of formulations of these axiomatic principles allows for a detailed analysis of the formulations required in order to derive various mathematical results.[[20]](https://en.wikipedia.org/wiki/Set_theory" \l "cite_note-21)[[21]](https://en.wikipedia.org/wiki/Set_theory" \l "cite_note-22)

## Set theory in mathematical education

As set theory gained popularity as a foundation for modern mathematics, there has been support for the idea of introducing basic theory, or [naive set theory](https://en.wikipedia.org/wiki/Naive_set_theory), early in [mathematics education](https://en.wikipedia.org/wiki/Mathematics_education).

In the US in the 1960s, the [New Math](https://en.wikipedia.org/wiki/New_Math) experiment aimed to teach basic set theory, among other abstract concepts, to primary grade students, but was met with much criticism. The math syllabus in European schools followed this trend, and currently includes the subject at different levels in all grades.

Set theory is used to introduce students to [logical operators](https://en.wikipedia.org/wiki/Logical_operators) (NOT, AND, OR) and semantic or rule description (technically [intensional definition](https://en.wikipedia.org/wiki/Intensional_definition)[[22]](https://en.wikipedia.org/wiki/Set_theory" \l "cite_note-Ruda2011-23)) of sets, (e.g. "months starting with the letter *A*"). This may be useful when learning [computer programming](https://en.wikipedia.org/wiki/Computer_programming), as sets and [boolean logic](https://en.wikipedia.org/wiki/Boolean_logic) are basic building blocks of many programming languages.

Sets are commonly referred to when teaching about different types of numbers ([**N**](https://en.wikipedia.org/wiki/Natural_number), [**Z**](https://en.wikipedia.org/wiki/Integer), [**R**](https://en.wikipedia.org/wiki/Real_number), ...) and when defining [mathematical functions](https://en.wikipedia.org/wiki/Mathematical_functions) as a relationship between two sets.

## See also

* [Mathematics portal](https://en.wikipedia.org/wiki/Portal:Mathematics)
* [Glossary of set theory](https://en.wikipedia.org/wiki/Glossary_of_set_theory)
* [Category theory](https://en.wikipedia.org/wiki/Category_theory)
* [List of set theory topics](https://en.wikipedia.org/wiki/List_of_set_theory_topics)
* [Relational model](https://en.wikipedia.org/wiki/Relational_model) – borrows from set theory

## Notes

2. In his 1925, [John von Neumann](https://en.wikipedia.org/wiki/John_von_Neumann) observed that "set theory in its first, "naive" version, due to Cantor, led to contradictions. These are the well-known [antinomies](https://en.wikipedia.org/wiki/Antinomy) of the set of all sets that do not contain themselves (Russell), of the set of all transfinite ordinal numbers (Burali-Forti), and the set of all finitely definable real numbers (Richard)." He goes on to observe that two "tendencies" were attempting to "rehabilitate" set theory. Of the first effort, exemplified by [Bertrand Russell](https://en.wikipedia.org/wiki/Bertrand_Russell), [Julius König](https://en.wikipedia.org/wiki/Julius_König), [Hermann Weyl](https://en.wikipedia.org/wiki/Hermann_Weyl) and [L. E. J. Brouwer](https://en.wikipedia.org/wiki/L._E._J._Brouwer), von Neumann called the "overall effect of their activity . . . devastating". With regards to the axiomatic method employed by second group composed of [Zermelo](https://en.wikipedia.org/wiki/Zermelo), [Abraham Fraenkel](https://en.wikipedia.org/wiki/Abraham_Fraenkel) and [Arthur Moritz Schoenflies](https://en.wikipedia.org/wiki/Arthur_Moritz_Schoenflies), von Neumann worried that "We see only that the known modes of inference leading to the antinomies fail, but who knows where there are not others?" and he set to the task, "in the spirit of the second group", to "produce, by means of a finite number of purely formal operations . . . all the sets that we want to see formed" but not allow for the antinomies. (All quotes from von Neumann 1925 reprinted in van Heijenoort, Jean (1967, third printing 1976), *From Frege to Gödel: A Source Book in Mathematical Logic, 1879–1931*, Harvard University Press, Cambridge MA, [ISBN](https://en.wikipedia.org/wiki/ISBN_(identifier)) [0-674-32449-8](https://en.wikipedia.org/wiki/Special:BookSources/0-674-32449-8) (pbk). A synopsis of the history, written by van Heijenoort, can be found in the comments that precede von Neumann's 1925.

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