Simulating and Manipulating Stochastic Differential Equations

In this lecture. . .

- Using Ito's lemma to manipulate stochastic differential equations
- Continuous-time stochastic differential equations as discrete-time processes
- Simple ways of generating random numbers in Excel
- Correlated random walks

By the end of this lecture you will be able to

$$\frac{\partial p}{\partial t} = c^2 \frac{\partial^2 p}{\partial s^2}$$

1
- manipulate stochastic differential equations

3
- find transition probability density functions for arbitrary stochastic differential equations

2
- simulate stochastic differential equations

$$dy = \underbrace{A(s,t)\,dt} + \boxed{B(s,t)\,dW}$$

Introduction

In order to become comfortable with the kind of models commonly used in quantitative finance you must be able to manipulate stochastic differential equations and generate random walks numerically.

G- diffusion Process

Manipulating stochastic differential equations

An equation of the form

stochastic process

random cpt.

G_t

$$dG = \underbrace{a\,(G,t)\,dt}_{\text{determ.}} + b\,(G,t)\,dX$$

is called a Stochastic Differential Equation (SDE) for G (or random walk for dG) and consists of two components:

1. $a\,(G,t)\,dt$ is deterministic – coefficient of dt is known as the **drift** or **growth**

2. $b\,(G,t)\,dX$ is random – coefficient of dX is known as the **diffusion** or **volatility**

and we say G evolves according to (or follows) this process.

So if for example we have a random walk $\frac{dS}{S} = \mu dt + \sigma dX$

$$\boxed{dS = \mu S dt + \sigma S dX} \tag{1}$$

then the drift is $a(S,t) = \mu S$ and the diffusion is $b(S,t) = \sigma S$.

The process (1) is also called **Geometric Brownian Motion** (GMB) ~~or **Exponential Brownian motion**~~ (EMB) and is a popular model for a wide class of asset prices.

$$\boxed{dG = \mu G dt + \sigma G dX}$$

We have previously considered Ito's lemma to obtain the change in a function $f(X)$ when $X \to X + dX$, where X is a standard Brownian motion.

This jump $df = f(X + dX) - f(X)$ is given by

$f = f(X)$

$$df = \frac{df}{dX}dX + \frac{1}{2}\frac{d^2 f}{dX^2}dt \qquad (2)$$

using the result

$f(X,t)$

$$\lim_{dt \to 0} dX^2 = dt.$$

Suppose we now wish to extend the result (2) to consider the change in an option price $V(S)$ where the underlying variable S follows a geometric Brownian motion.

(Of course, you are not supposed to know anything about options yet. Just think of manipulating functions.)

$V(S)$

If we rewrite (1) as

$$\boxed{\frac{dS}{S} = \mu\,dt + \sigma\,dX}$$

then dS represents the change in asset price S in a small time interval dt.

This expression is the return on the asset.

μ is the average growth rate of the asset and σ the associated volatility (standard deviation) of the returns.

dX is an increment of a Brownian Motion, known as a Wiener process and is a Normally distributed random variable such that $dX \sim N\,(0, dt)$.

An obvious question we may ask is, what is the jump in $V(S+dS)$ when $S \to S+dS$?

We begin (again) by using a Taylor series as in (2), but for $V(S+dS)$ to get

$V(S+dS) = V + V' dS + \frac{1}{2} V'' dS^2$

$$dV = \frac{dV}{dS} dS + \frac{1}{2} \frac{d^2V}{dS^2} dS^2.$$

$V = V(S)$

S evolves according to G.B.M

We can proceed further now as we have an expression for dS (and hence dS^2). As dt is very small, any terms in $dt^{\frac{3}{2}}$ or dt^2 are insignificant in comparison and can be ignored. So working to $O(dt)$

$$dS^2 = \sigma^2 S^2 dt.$$

If we substitute this into the previous expression for dV we get Ito's lemma as applied to $V(S)$:

$$dV = \overbrace{\left(\mu S \frac{dV}{dS} + \frac{1}{2}\sigma^2 S^2 \frac{d^2V}{dS^2}\right)}^{\text{drift}} dt + \overbrace{\left(\sigma S \frac{dV}{dS}\right)}^{\text{diffusion}} dX. \tag{3}$$

Note that this is another stochastic differential equation!

It contains a predictable part and a random part.

Suppose that we had a formula for $V(S)$. Let's take a very special case, let's consider

$$\boxed{V(S) = \log S.}$$

Use (3) for dV

Differentiating this once gives

$$\frac{dV}{dS} = \frac{1}{S}.$$

Differentiating this again gives

$$\frac{d^2V}{dS^2} = -\frac{1}{S^2}.$$

Subst in (3)

Now from (3) we have

$$\boxed{d\left(\log S\right) = \left(\mu - \tfrac{1}{2}\sigma^2\right) dt + \sigma dX.}$$

Integrating both sides between $\boxed{0 \text{ and } t}$

X_τ

$$\cancel{\int_0^t} d\left(\log S\right) = \int_0^t \boxed{\left(\mu - \frac{1}{2}\sigma^2\right)} d\tau + \int_0^t \sigma\, dX \qquad (t > 0)$$

$$\log S_t - \log S_0 = \left(\mu - \tfrac{1}{2}\sigma^2\right) t + \sigma \underbrace{\left(X\left(t\right) - X\left(0\right)\right)} .$$

$\log \dfrac{S_t}{S_0}$

Therefore

$$\boxed{\log\left(\frac{S(t)}{S(0)}\right) = \left(\mu - \tfrac{1}{2}\sigma^2\right)t + \sigma\left(X(t) - \cancel{X}(0)\right)}$$

$\rightarrow 0$

S_0

Assuming $X(0) = 0$ and $S(0) = S_0$, the exact solution becomes

$\phi\sqrt{t}$

$$\boxed{S(t) = S_0 \exp\left(\left(\mu - \tfrac{1}{2}\sigma^2\right)t + \sigma X(t)\right).} \tag{4}$$

d(log) $\rightarrow$ G.B.M.

Another example:

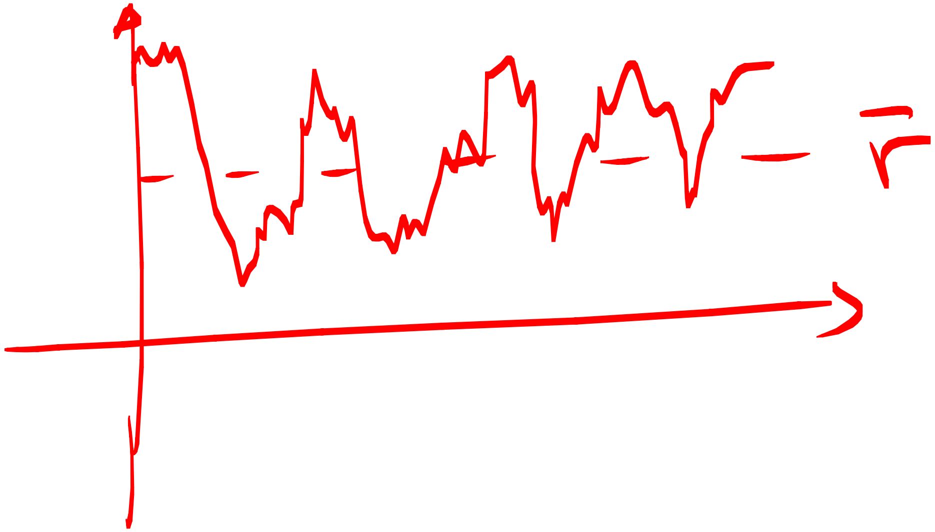
1977

Let's take a look at the Vasicek interest rate model for short-term interest rates, and try manipulating that.

$$dr = \overbrace{\gamma(\bar{r} - r)}^{\text{drift}}\, dt + \underbrace{\sigma}_{\text{diffusion}} dX.$$

γ refers to the **reversion rate** and $\bar{r}$ denotes the **mean rate**.

speed of reversion



By setting $u = r - \overline{r}$, u is a solution of

$r = u + \bar{r}$

$$\boxed{du = -\gamma u \, dt + \sigma dX.}$$

An analytic solution for this equation exists. To see, this write the equation as

$$\boxed{d\left(ue^{\gamma t}\right)} = \sigma e^{\gamma t} dX.$$

Integrating over from zero to t gives

$$u(t) = u(0)e^{-\gamma t} + \sigma \int_0^t e^{\gamma(s-t)} dX_s.$$

This can be **integrated by parts** to give

$$\boxed{u(t) = u(0)e^{-\gamma t} + \sigma \left(X(t) - \gamma \int_0^t X(s) e^{\gamma(s-t)} \, ds \right).}$$

$dy = B\,dx$

Transition probability density functions again

Let's look at the equations governing the probability distribution for an arbitrary random walk:

$$dy = \underbrace{A(y,t)\,dt} + \boxed{B(y,t)\,dX}$$

for the variable y.

Remember the **transition probability density function** $p(y,t;y',t')$ defined by

$$\text{Prob}(a < y' < b \text{ at time } t' | y \text{ at time } t) = \int_a^b p(y,t;y',t')\,dy'.$$

In words this is 'the probability that the random variable y lies between a and b at time t' in the future, given that it started out with value y at time t.'

Think of y and t as being current values with y' and t' being future values.

The transition probability density function can be used to answer questions such as

"What is the probability of the variable y being in a certain range at time t' given that it started out with value y at time t?"

The transition probability density function $p(y,t;y',t')$ satisfies two equations.

One involves derivatives with respect to the future state and time (y' and t') and is called the **forward equation.**

The other involves derivatives with respect to the current state and time (y and t) and is called the **backward equation.**

These can be derived by the same trinomial idea we used before (but the details are a lot messier for the general stochastic differential equation).

The forward equation

$dy = A(y,t)\,dt + B(y,t)\,dX$

Cutting to the chase, the transition probability density function satisfies the partial differential equation

$$\frac{\partial p}{\partial t'} = \frac{1}{2}\frac{\partial^2}{\partial y'^2}\left(\underbrace{B(y',t')^2 p}\right) - \frac{\partial}{\partial y'}\left(A(y',t')p\right)$$

new term due to the drift

This is the **Fokker–Planck** or **forward Kolmogorov equation**.

$p(y,t)$

y – RV

t – time.

The forward equation

$dy = A(y,t)\,dt + B(y,t)\,dX$

Cutting to the chase, the transition probability density function satisfies the partial differential equation

pde → ode

$$\frac{\partial p}{\partial t'} = \frac{1}{2}\frac{\partial^2}{\partial y'^2}\left(B(y',t')^2 p\right) - \frac{\partial}{\partial y'}\left(A(y',t')p\right)$$

→ 0

This is the **Fokker–Planck** or **forward Kolmogorov equation**.

$p(y,t)$

y – rv

t – time.

Example: The most important example to us is that of the distribution of equity prices in the future. If we have the random walk

$$dS = \overset{A}{\mu S}\,dt + \overset{B}{\sigma S}\,dX$$

then the forward equation becomes

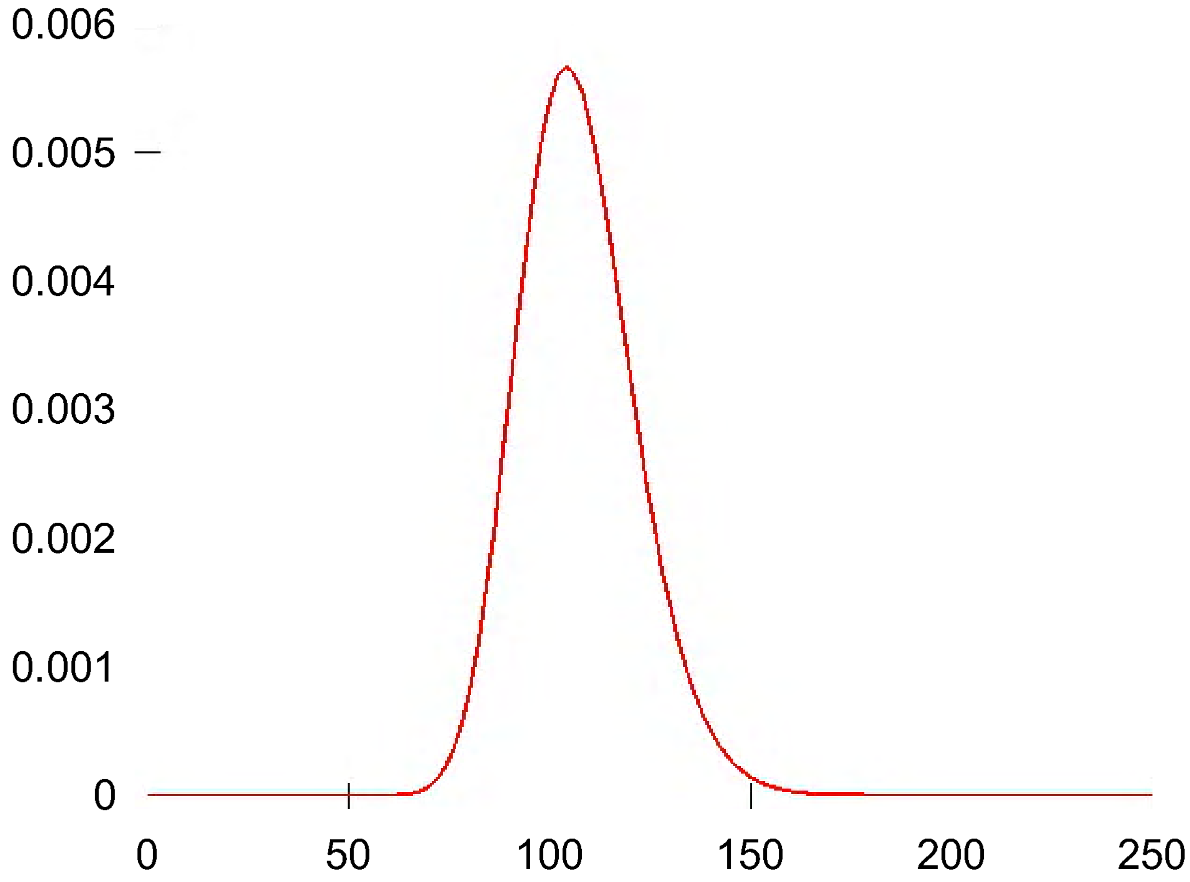
$$\frac{\partial p}{\partial t'} = \frac{1}{2}\frac{\partial^2}{\partial S'^2}\left(\overset{B}{\sigma^2 S'^2}p\right) - \frac{\partial}{\partial S'}\left(\overset{A}{\mu S'}p\right).$$

The solution of this representing a stock price starting at $S' = S$ at $t' = t$ is

$$p(S,t;S',t') = \frac{1}{\sigma S'\sqrt{2\pi(t'-t)}}e^{-\left(\log(S/S')+(\mu-\frac{1}{2}\sigma^2)(t'-t)\right)^2/2\sigma^2(t'-t)}$$

p(S)

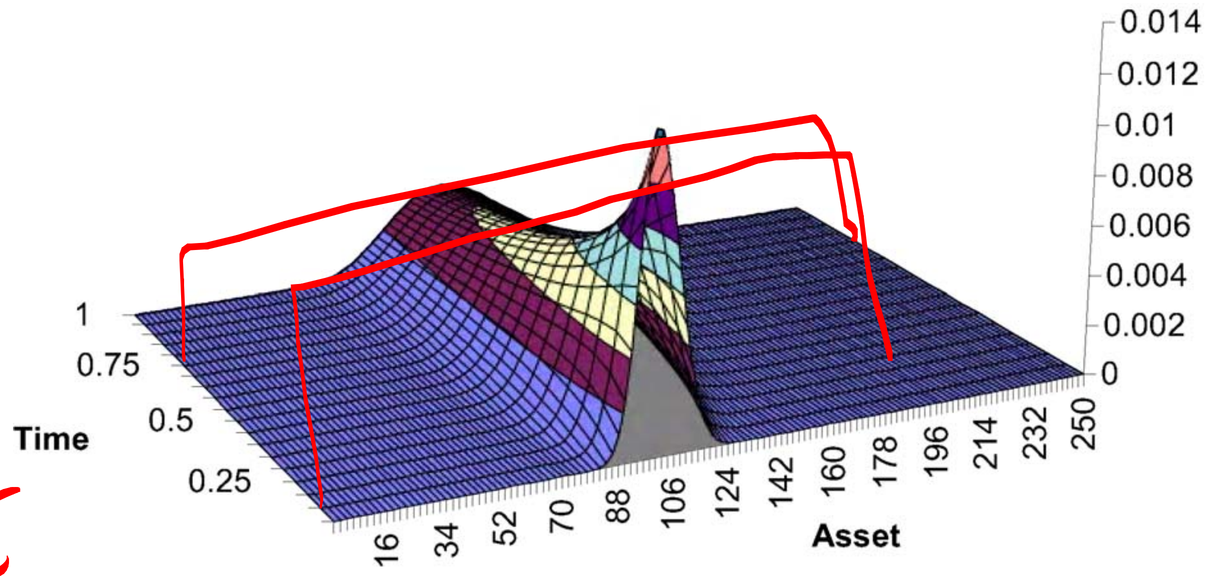
for a particular time



S

The probability density function for the lognormal random walk, after a certain time.

$p(S,t)$



pdf

p

t

S

The probability density function for the lognormal random walk evolving through time.

The steady-state distribution

Some random walks have a steady-state distribution.

That is, in the long run as $t' \to \infty$ the distribution $p(y, t; y', t')$ as a function of y' settles down to be independent of the starting state y and time t. Possible examples are stochastic differential equation models for interest rates, inflation, volatility.

Some random walks have no such steady state even though they have a time-independent equation. For example the lognormal random walk either grows without bound or decays to zero.

If there is a steady-state distribution $p_\infty(y')$ then it satisfies the *ordinary* differential equation

$$\frac{\partial p}{\partial t} \to 0$$

σ^2 $\quad (\bar{r}-r)\gamma$

$$\boxed{\frac{1}{2}\frac{d^2}{dy'^2}\left(B^2 p_\infty\right) - \frac{d}{dy'}\left(A p_\infty\right) = 0.}$$

$\frac{\partial}{\partial y} \to \frac{d}{dy}$

ODE

2nd order

steady eq^n

Example: The Vasicek model

$$dr = \overbrace{\gamma\left(\bar{r} - r\right)}^{A} dt + \overset{B}{\sigma}\, dX.$$

The steady-state distribution $p_\infty(r')$ satisfies

$$\boxed{\frac{1}{2}\sigma^2 \frac{d^2 p_\infty}{dr'^2} - \gamma\frac{d}{dr'}\left((\bar{r} - r')p_\infty\right) = 0.}$$

The solution is

$$\boxed{p_\infty = \frac{1}{\sigma}\sqrt{\frac{\gamma}{\pi}}\; e^{-\frac{2\gamma(\bar{r}-r')^2}{2\,\sigma^2}}.} = e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

In other words, the interest rate r is Normally distributed with mean $\bar{r}$ and standard deviation $\sigma/\sqrt{2\gamma}$.

$$\mu \equiv \bar{r}$$

$$e^{-\frac{1}{2}\frac{(r'-\bar{r})^2}{\sqrt{\sigma^2/2\gamma}}}$$

The backward equation

Now we come to the backward equation. This will be useful if we want to calculate probabilities of reaching a specified final state from various initial states.

The transition probability density function satisfies the **backward Kolmogorov equation**

extra term due to the drift

$$\frac{\partial p}{\partial t} + \tfrac{1}{2}B(y,t)^2\frac{\partial^2 p}{\partial y^2} + A(y,t)\frac{\partial p}{\partial y} = 0.$$

Simulating the lognormal random walk

The lognormal random walk model for assets can be written in continuous time as

$$dS = \mu S\, dt + \sigma S\, dX.$$

In discrete time this is

$$S_{i+1} - S_i = S_i \left(\mu\, \delta t + \sigma \phi\, \delta t^{1/2} \right).$$

To generate representative simulations of possible asset paths we must obviously work in discrete time.

The random walk on a spreadsheet

The random walk can be written as a 'recipe' for generating S_{i+1} from S_i:

$$S_{i+1} = S_i \left(1 + \mu \, \delta t + \sigma \phi \, \delta t^{1/2}\right).$$

$\phi \sim N(0,1)$

We can easily simulate the model using a spreadsheet.

The method is called the **Euler method**.

Start with an initial stock price, say, 100. $S_0 = 100$

And a couple of parameters, $\boxed{\mu = 0.1}$ and $\sigma = 0.2$, say, that best represent the asset in question.

Decide on a (small) time step, $\delta t = 0.01$, say. $\delta t = \frac{1}{100}$

Now start picking random numbers!

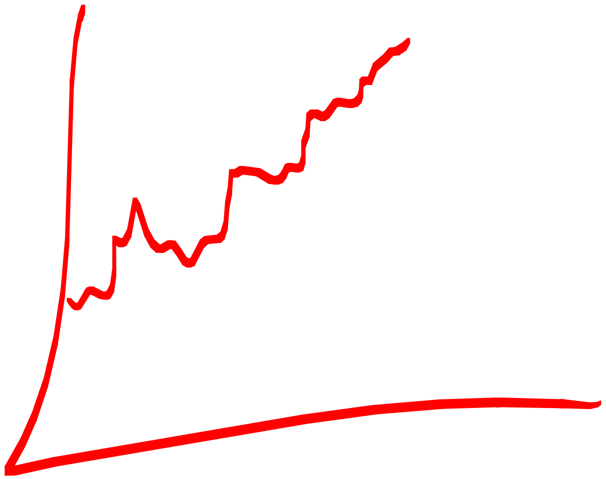
First time step: The random number is. . . 0.12. So

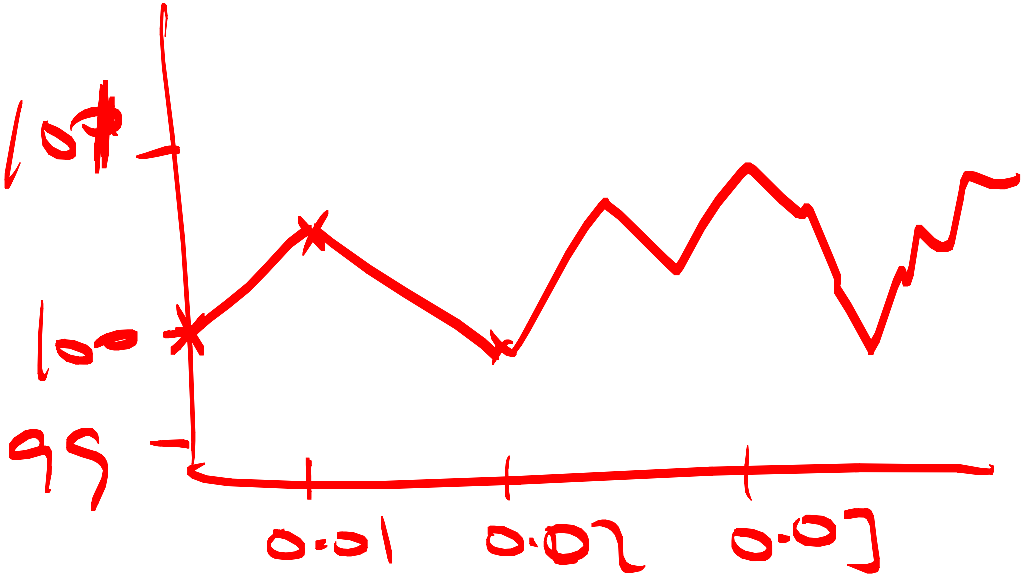
$$S_{i+1} = 100\,(1 + 0.1 \times 0.01 + 0.2 \times 0.1 \times 0.12) = 100.34.$$

Second time step: The random number is. . . -0.25. So

$$S_{i+1} = 100.34\,(1 + 0.1 \times 0.01 + 0.2 \times 0.1 \times (-0.25)) = 99.94.$$

And so on.





In this simulation there are several input parameters, which remain constant:

- a starting value for the asset $S_0 = 100$
- a time step δt $1/100$
- the drift rate μ 0.1
- the volatility σ 0.2
- the total number of time steps

Then, at each time step, we must choose a random number ϕ from a Normal distribution.

This can be done easily in Excel in several ways, we will see a couple now.

Slow but accurate

The Excel spreadsheet function `RAND()` gives a uniformly-distributed random variable.

This can be used, together with the inverse cumulative distribution function `NORMSINV` to give a genuinely Normally distributed number:

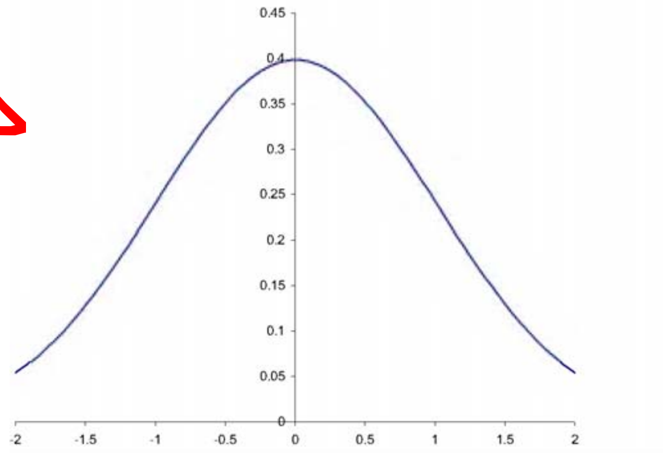
- `NORMSINV(RAND()).`

Unif[0,1]

Why does this work?

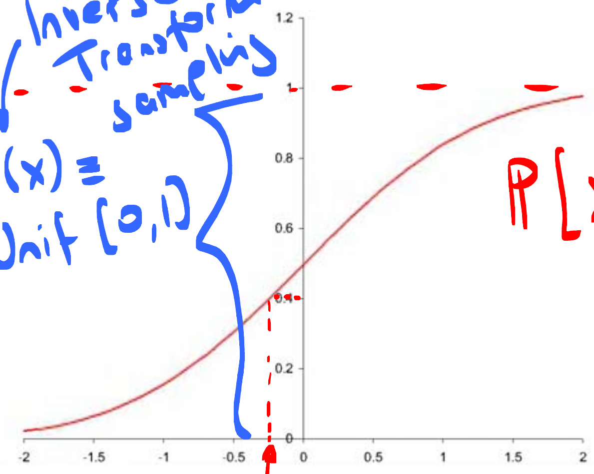
The pdf and cdf for the Normal distribution

$N(x)$



Inverse Transform sampling

$N(x) \equiv$ Unif $[0,1]$



-0.3

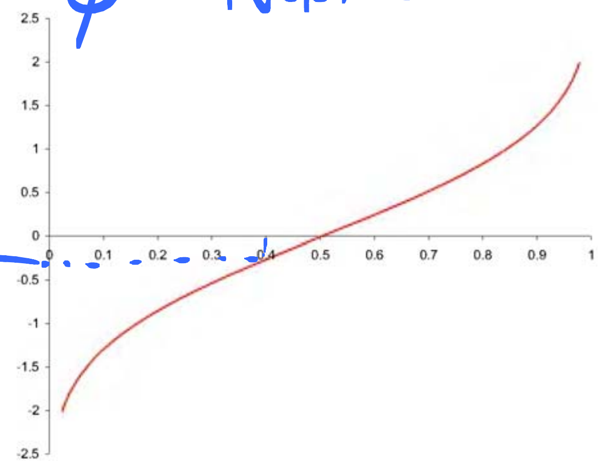
$\mathbb{P}[X \leq \underbrace{-0.3}] \approx 0.4$

$X \equiv \phi \sim N(0,1)$

The inverse cumulative distribution function

$N^{-1} \leftarrow \phi$ NORMSINV (RAND())

-0.3



$N(x) \equiv$ RAND()

Fast but inaccurate C.L.T.

An approximation to a Normal variable that is fast in a spreadsheet, and quite accurate, is simply to add up twelve random variables drawn from a uniform distribution over zero to one, and subtract six:

- $$\left(\sum_{i=1}^{12} \mathtt{RAND()}\right) - 6. \longrightarrow N(0,1)$$

Why 12?

Any 'large' number will do. The larger the number, the closer the end result will be to being normal, but the slower it is.

Why subtract off 6? ✓

The random number must have a mean of zero.

And the standard deviation?

Must be 1.

$$V\left[\sum_{i=1}^{12} R_i - 6\right] = \sum_{i=1}^{12} V(R_i)$$

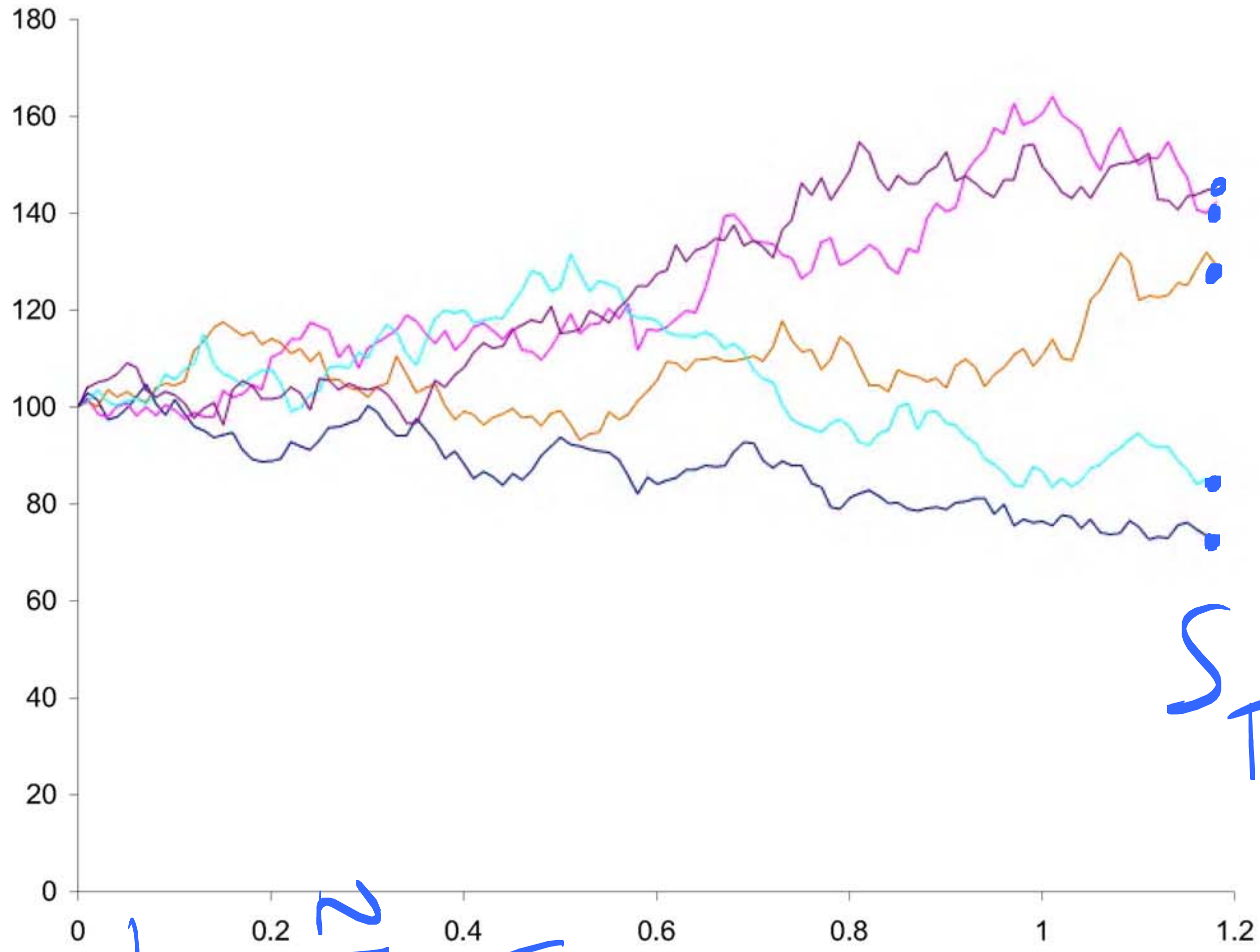
$$= 12 \cdot \frac{1}{12} = 1$$

	A	B	C	D	E	F	G
1	**Asset**	**100**		**Time**	**Asset**		
2	**Drift**	**0.15**		0	100		
3	**Volatility**	**0.25**		0.01	96.10692		
4	**Timestep**	**0.01**		0.02	96.99647		
5				0.03	94.76352		
6				0.04	91.46698		
7				0.05	88.83325		
8				0.06	88.42727		
9				0.07	90.62882		
10				0.08	88.80545		
11							
12							
13				0.11	84.93865		

=D4+$B$4

=E7*(1+$B$2*$B$4+$B$3*SQRT($B$4)*(RAND()+RAND()+RAND()+RAND()
+RAND()+RAND()+RAND()+RAND()+RAND()+RAND()+RAND()+RAND()-6))

Basis of MC Pricing



S_T

$$C = \frac{1}{N} \sum_{i=1}^{N} \left\{ e^{-rt} \, \mathrm{Max}[S_T - K, 0] \right\}$$

Simulating other random walks

This method is not restricted to the lognormal random walk.

Later in the course we will be modeling interest rates as stochastic differential equations.

The following is a stochastic differential equation model for an interest rate, that goes by the name of an **Ornstein-Uhlenbeck process** (an example of a mean-reverting random walk), or when used in an interest rate context the **Vasicek model**:

r_i

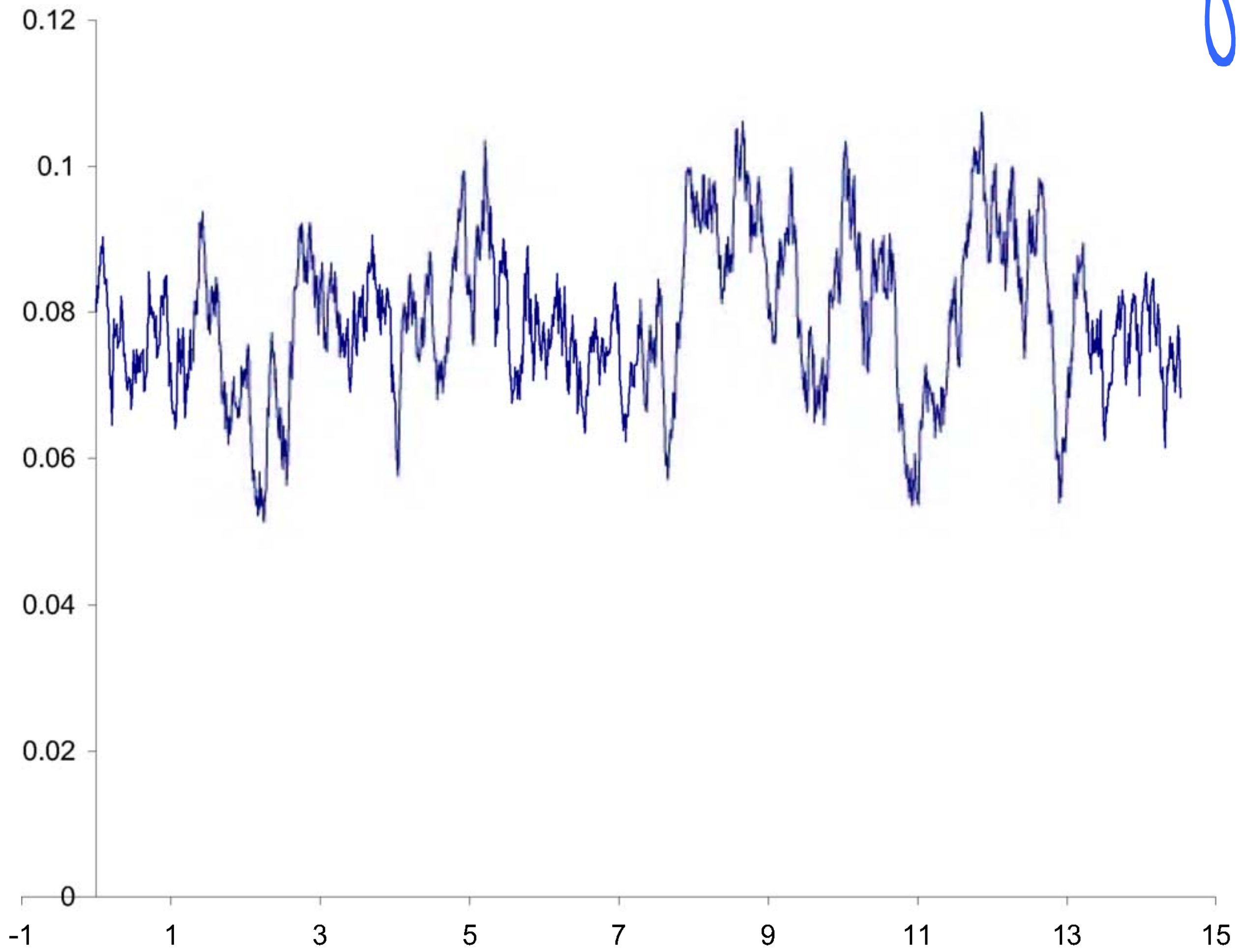
$r_{i+1} - r_i$

$$dr = \gamma\,(\overline{r} - r)\,dt + \sigma dX.$$

In discrete time we can approximate this by

$$\circledast \quad \boxed{r_{i+1} = r_i + \gamma\,(\overline{r} - r_i)\,dt + \sigma\phi\,\delta t^{1/2}.}$$

γ - high



Producing correlated random numbers

We will often want to simulate paths of correlated random walks.

We may want to examine the statistical properties of a portfolio of stocks, or value a convertible bond under the assumption of random asset price and random interest rates.

Example:

Assets S_1 and S_2 both follow lognormal random walks with correlation ρ.

$$\mathbb{E}(\phi_1 \phi_2) = \rho$$

In continuous time we write

$$dS_1 = \mu_1 S_1 \, dt + \sigma_1 S_1 \, dX_1,$$

$$dS_2 = \mu_2 S_2 \, dt + \sigma_2 S_2 \, dX_2,$$

with

$$E[dX_1 \, dX_2] = \rho \, dt.$$

$$\mathbb{E}\left(\phi_1 \sqrt{dt}\, \phi_2 \sqrt{dt}\right) = dt \, \underbrace{\mathbb{E}(\phi_1 \phi_2)}_{\rho}$$

In discrete time these become

$$S_{1_{i+1}} - S_{1_i} = S_{1_i}\left(\mu_1\,\delta t + \sigma_1 \phi_1\,\delta t^{1/2}\right)$$

and

$$S_{2_{i+1}} - S_{2_i} = S_{2_i}\left(\mu_2\,\delta t + \sigma_2 \phi_2\,\delta t^{1/2}\right)$$

with

$$E[\phi_1\,\phi_2] = \rho.$$

Want to correlate ϕ_1, ϕ_2

Q: How can we choose a ϕ_1 and a ϕ_2 which are both Normally distributed, both have mean zero and standard deviation of one, and with a correlation of ρ between them?

A: This can be done in two steps, first pick two *uncorrelated* Normally distributed random variables, and then combine them.

Step 1: Choose uncorrelated ϵ_1 and ϵ_2, both Normally distributed with zero means and standard deviations of one.

Step 2: Convert these independent Normal numbers into correlated Normals by taking a linear combination.

$$\phi_1 = \epsilon_1$$

$$\phi_2 = \rho\,\epsilon_1 + \sqrt{1-\rho^2}\,\epsilon_2.$$

Check:

$$E[\phi_1^2] = 1,$$

$$E\left[\phi_2^2\right] = E\left[\rho^2\epsilon_1^2 + 2\rho\sqrt{1-\rho^2}\epsilon_1\epsilon_2 + (1-\rho^2)\epsilon_2^2\right]$$

$$= \rho^2 + 0 + (1-\rho^2) = 1,$$

and

$$E\left[\phi_1\phi_2\right] = E\left[\rho\epsilon_1^2 + \sqrt{1-\rho^2}\epsilon_1\epsilon_2\right] = \rho.$$

And Normality?

Weighted sums of Normally distributed numbers are themselves Normally distributed!

If $X_i \sim N(\mu_i, \sigma_i^2)$ for $i = 1, \ldots, n$ then

$$\sum_{i=1}^{n} w_i X_i \sim N\left(\sum_{i=1}^{n} w_i \mu_i, \sum_{i=1}^{n} w_i^2 \sigma_i^2\right).$$

$w_1\mu_1 + w_2\mu_2 + \cdots, \quad w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + \cdots$

Summary

Please take away the following important ideas

- With the right tool (Ito's lemma) you can examine functions of stochastic variables
- Partial differential equations can be used for finding probability density functions for arbitrary random walks
- Simulating random walks can be very easy indeed