CQF Module 2, Session 4: Martingales I Exercises

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1. Let's say we want to price a given financial instrument deriving its value $V(t, S_1, S_2, S_3)$ from 3 stochastic processes S_1, S_2, S_3, where

$$\begin{aligned} dS_i &= f_i(t, S_k, k = 1, \ldots, 3)dt \\ &\quad + g_i(t, S_k, k = 1, \ldots, 3)dX_i, \\ &\qquad i = 1, \ldots, 3 \end{aligned}$$

   and where

$$dX_i dXj = \rho_{ij} dt, \quad i, j = 1, \ldots, 3, \; i < j$$

   For simplicity, we will write

$$dS_i = f_i dt + g_i dX_i, \quad i = 1, \ldots, 3$$

   Let $V(t, S_1(t), S_2(t), S_3(t))$ be a function on $[0, T]$ with $V(0, S_1(0), S_2(0), S_3(0)) = v$. Using Itô, compute the SDE for dV and deduce the stochastic integral for $V(T)$.

2. **The Heston Model.** The Heston Model (1993) is a popular stochastic volatility model used for option valuation. In this model, the stock price dynamics follows a GBM in which the stock variance v is itself stochastic and follows a square root process [1]. The stock price dynamics is:

$$dS_t = \mu S_t dt + \sqrt{v_t} S_t dX_1(t) \tag{1}$$

   the dynamics of the stock variance is

$$dv_t = -\lambda(v_t - \bar{v})dt + \eta\sqrt{v_t} dX_2(t) \tag{2}$$

   and the two processes have correlation ρ correlated, i.e.

$$dX_1(t)dX_2(t) = \rho dt \tag{3}$$

   Let $F(t, S_t, v_t)$ be a function on $[0, T]$ with $F(0, S_0, v_0) = f$. Using Itô, compute the SDE for dV and deduce the stochastic integral for $F(T)$.

[1] In the fixed income world, the square root process is called a Cox-Ingersoll-Ross process and is used to model short-term interest

3. Let $Y_t = X_t^4$ where X_t is a Brownian motion. Using Itô's lemma, express the SDE for Y_t. Then, deduce the stochastic integral for Y_t over $[0, T]$. Finally, deduce from the stochastic integral an expression for $\mathbf{E}[Y_t]$.

4. **Discrete Time Martingale**: Let $Y_1, \ldots, Y_n$ be a sequence of independent random variables such that $\mathbf{E}[Y_i] = 0$ for $i = 1, \ldots, n$. Let $\mathcal{F}_n$ be the filtration generated by the sequence $Y_1, \ldots, Y_n$. Consider the random variable $S_n = \sum_{i=1}^{n} Y_i$. Prove that S_n is a martingale for all n.

   Reminder - *proving that a process S_n is a martingale involves proving that $\mathbf{E}[|S_n|] < \infty$ and that $\mathbf{E}[S_{n+1}|\mathcal{F}_n] = S_n$*