$$dS = \mu S\, dt + \sigma S\, \underbrace{dX}$$

B.M | W.P.

$$W_t \longleftarrow \equiv W(t)$$

$$B_t$$

$$\boxed{X_t}$$

Ω – sample space

ω – sample pt.

$\omega \in \Omega$

$$\boxed{X : \omega \in \Omega \longrightarrow \mathbb{R}}$$

$\mathbb{P}(+1) = \frac{1}{2}$

$\mathbb{P}(-1) = 1/2$

$H \to +1$

$T \to -1$

$$\mathbb{E}[X] = \sum x_i f_i = \mu$$

$$\sigma^2 = \mathbb{E}[X^2] - \underbrace{\mathbb{E}^2[X]}_{=0}$$

$$\boxed{\sigma^2 = \mathbb{E}[X^2]} \text{ when } \mu = 0$$

$$\mathbb{E}[XY] = \underbrace{\mathbb{E}(X)}_{=0}\ \underbrace{\mathbb{E}(Y)}_{=0}$$

$$V^n = \sum |f_i - f_{i-1}|^n$$

n^{th} variation

$n=1$ variation – sum of abs changes

$n=2$ quadratic variation – sum of squared changes

Variation

$$V^n = \sum_{i=1}^{n} |X_{t_i} - X_{t_{i-1}}|^n$$

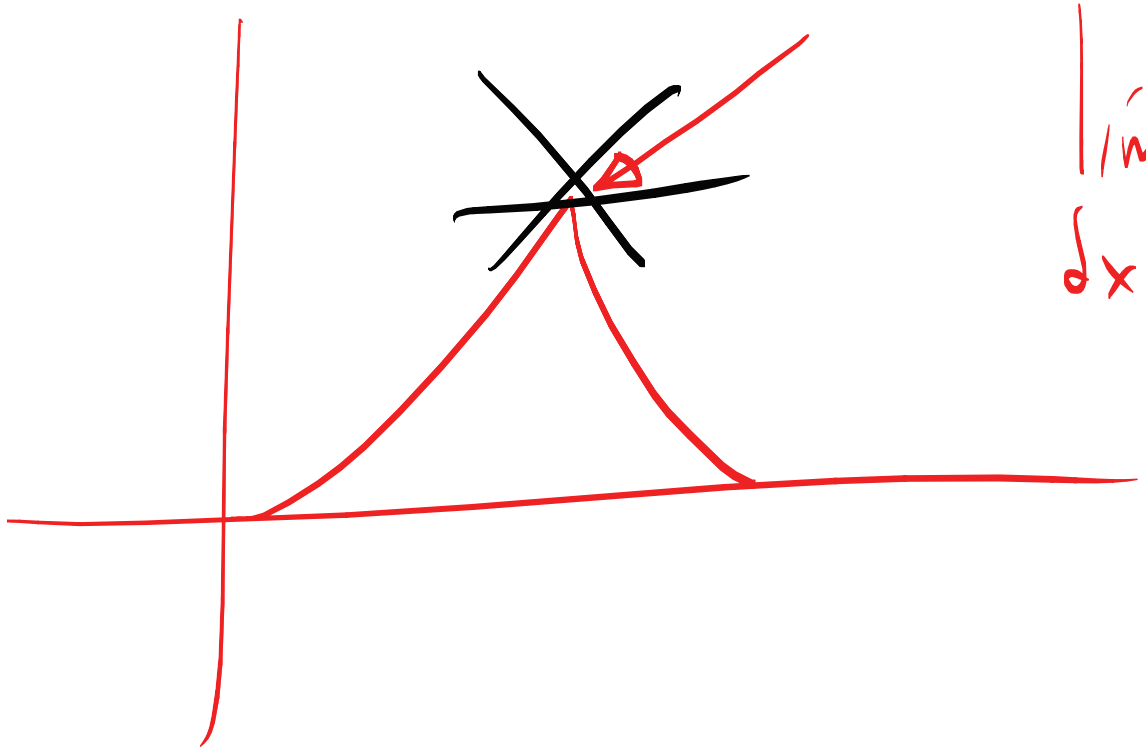
$n = 1, 2$ important

for B.M unbounded variation

$n=1$ $\quad V = \sum_{i=1}^{n} |X_{t_i} - X_{t_{i-1}}|$ Variation of the trajectory (sum of abs changes)

$n=2$ quadratic variation sum of squared change,

$$V^2 = \sum_{i=1}^{n} \left| X_{t_i} - X_{t_{i-1}} \right|^2$$



$$\lim_{dx \to 0} \frac{f(x+dx)-f(x)}{dx}$$

$$X \sim N(\mu, \sigma^2)$$

$$W_t \sim N(0, |t-s|)$$

$$p(x) = \frac{1}{\sqrt{|t-s|}\sqrt{2\pi}} e^{-\frac{1}{2}\frac{x^2}{|t-s|}}$$

$$\lim_{dt \to 0} dW^2 \longrightarrow dt$$

time steps:

discrete	Δt $\quad$ δt
cts	dt

$$\underbrace{\Delta t \to 0}_{dt}$$

$$\mathbb{E}[X^n] = \int_{\mathbb{R}} x^n \, p(x) \, dx$$

Γ

$$dW \sim N(0, dt)$$

$$W \sim N(0, t)$$

$$\lim_{dt \to 0} dW^2 \longrightarrow dt$$

If $F = f(x)$

correction term

$$\boxed{dF = \frac{dF}{dx}\,dX + \frac{1}{2}\frac{d^2F}{dX^2}\,dt}$$

C

Itô's Lemma

$$F = e^X \qquad F' = e^X \rightarrow F'' = e^X$$

$$dF = F'\,dX + \frac{1}{2} F''\,dt$$

$$dF = e^X\,dX + \frac{1}{2} e^X\,dt$$

$$\frac{dF}{F} = \frac{1}{2}dt + dX$$

$F = F(x)$	$F = f(t, x)$
x^n	tx^n
$\sin x$	$e^{ntx} + \ln x$
$e^x + x^2$	$t^2 + x^2 - t \ln x$

Take ⊛ from p. 38

$$dF = \left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial W^2}\right)dt + \underbrace{\frac{\partial f}{\partial W}\, dW}$$

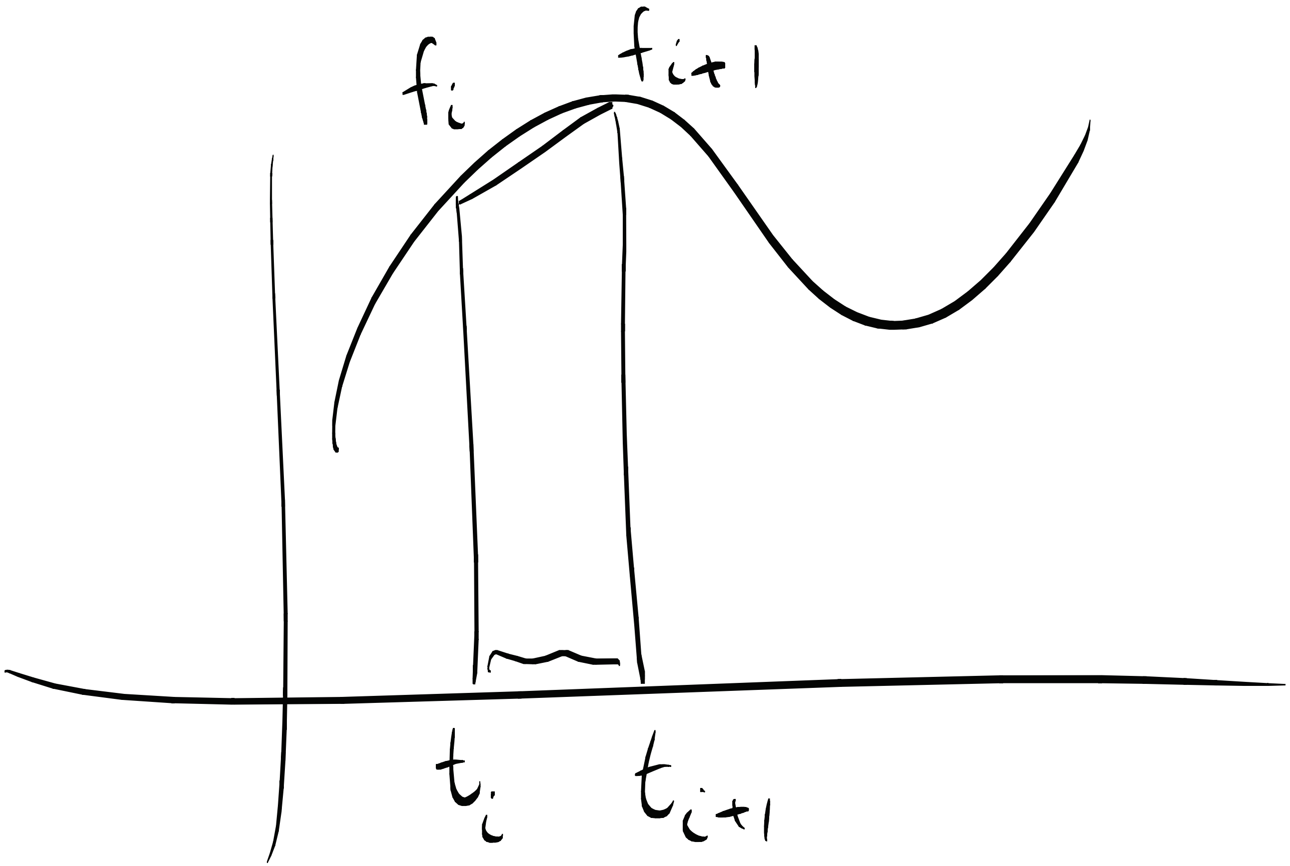
$$\frac{\partial f}{\partial W}\, dW = dF - \left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial W^2}\right)dt$$

Now integrate throughout over $[0, t]$

$$\int_0^t \frac{\partial f}{\partial W}\, dW = \int_0^t dF - \int_0^t \left(\frac{\partial f}{\partial \tau} + \frac{1}{2}\frac{\partial^2 f}{\partial W^2}\right)d\tau$$

$$F(t, W_t) - F(0, W_0)$$

$$\therefore \quad F = F(t, W(t))$$


f_i
f_{i+1}
t_i
t_{i+1}

discrete　　　　　　　　　　cts

$$\sum \longleftarrow \int$$

$$\int_0^T \underbrace{\frac{dF}{dW}}_{} dW = F(W) - F(0) - \frac{1}{2}\int_0^T \frac{d^2F}{dW^2}\,dt$$

$$\frac{dF}{dW} = W^2 \longrightarrow F = \frac{W^3}{3}$$

$$\downarrow$$

$$\frac{d^2F}{dW^2} = 2W$$

$$3\int_0^T W^2\,dW = \frac{W(T)^3}{\cancel{3}} - \frac{W(0)^3}{\cancel{3}} - \frac{1\,3}{\cancel{2}}\int_0^T \cancel{2}W\,dt$$

$$dS = (\nu - \mu S)\,dt + \sigma\, dW$$

$$= \mu\Big(\underbrace{\nu/\mu}_{\eta} - S\Big)\,dt + \sigma\, dW$$

$$\boxed{dS = \mu(\eta - S)\,dt + \sigma\, dW}$$ Vasicek

CIR $\quad dS = \mu(\eta - S)\,dt + \underbrace{\sigma S^{1/2}\,dW}_{\to 0}$

as $S \to 0$

$\mathbb{E}[\Sigma \quad]$

Explicit

t $W(t)$

$t \to t + dt$

$W \to W + dW$

Implicit

$W(t)$

$W \to W + dW$

$$\mathbb{E}^2(X) \equiv \{\mathbb{E}(X)\}^2$$

$\mathbb{E}(X^2)$ 2nd moment.

$\sigma^2 = \mathbb{E}(X^2)$ if $\mu = 0$

$$\sigma^2 = \mathbb{E}[X^2] - \mathbb{E}^2[X]$$

2nd Mom   square of mean

$= 0$

$$\mathbb{V}[X] = \mathbb{E}[X^2]$$

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