

Exercise for Session 4.5

CQF

Exercise 1:

The objective of the exercise is to check that the following fact is true:

Fact 1. If a process $Y(t)$ is a martingale under \mathbb{Q} and $\eta_t = \frac{d\mathbb{Q}}{d\mathbb{P}}$, then the process $M(t) = Y(t)\eta_t$ is a martingale under \mathbb{P} .

We will focus on the case where both $Y(t)$ and $\eta(t)$ are modelled as diffusions processes with respective dynamics

$$dY(t) = f(t, Y(t))dt + g(t, Y(t))dX(t)$$

and

$$\frac{d\eta(t)}{\eta(t)} = -\theta(t)dX(t)$$

where $X(t)$ is a standard Brownian motion under the \mathbb{P} measure.

Questions -

- (i). Knowing that $Y(t)$ is a martingale under \mathbb{Q}^θ , express the drift function $f(\cdot)$ in terms of the diffusion function $g(\cdot)$ and of the process $\theta(t)$.
- (ii). Apply the Itô product rule to show that $M(t) = Y(t)\eta_t$ is a martingale under \mathbb{P} .

Exercise 2: (Optional)

Derive formula (25) on slide 80

$$C(t) = B(t, U)N[d_1(B(t, U), t, T)] - KB(t, T)N[d_2(B(t, U), t, T)] \quad (1)$$

where

$$\begin{aligned} d_1(b, t, T) &= \frac{\ln\left(\frac{b}{K}\right) - \ln B(t, T) + \frac{1}{2}v_U(t, T)}{v_U(t, T)} \\ d_2(b, t, T) &= d_1 - v_U(t, T) \\ v_U^2(t, T) &= \int_t^T (b(s, U) - b(s, T))^2 ds \end{aligned}$$

Start from the forward asset pricing formula given in equation (24), on slide 79,

$$C(t) = B(t, T) \mathbf{E}^{\mathbb{P}^T} [(F_B(T, T, U) - K)^+ | \mathcal{F}_t] \quad (2)$$

where the dynamics of the forward price $F_B(t, T, U)$ is given in equations (22) and (23) on slide 78.

Hints:

1. you could use an approach similar to the derivation of the Black-Scholes formula presented in Section 3.3 of Lecture 3.3 (slides 63-75);
2. Note that the random variable $Y(T) = \int_t^T (b(s, U) - b(s, T)) dX^T(s)$ is Normally distributed with mean 0 and variance $v_U^2(t, T)$.

Solutions

1- Exercise 1

(i). By Girsanov, the \mathbb{Q} -Brownian motion $X^{\mathbb{Q}}(t)$ is defined as

$$X_t^{\mathbb{Q}} = X_t + \int_0^t \theta(s) ds, \quad t \in [0, T] \quad (3)$$

Hence, the dynamics of $Y(t)$ under the \mathbb{Q} -measure is given by

$$\begin{aligned} dY(t) &= f(t, Y(t))dt + g(t, Y(t))dX(t) \\ &= f(t, Y(t))dt + g(t, Y(t)) \left(dX^{\mathbb{Q}}(t) - \theta(t)dt \right) \\ &= [f(t, Y(t)) - g(t, Y(t))\theta(t)] dt + g(t, Y(t))dX^{\mathbb{Q}}(t) \end{aligned}$$

For $Y(t)$ to be a \mathbb{Q} -martingale, we need it to be driftless, which implies that

$$f(t, Y(t)) - g(t, Y(t))\theta(t) = 0$$

Therefore, we can express the drift function $f(t, Y(t))$ in terms of the diffusion function $g(\cdot)$ and of the process $\theta(t)$ as

$$f(t, Y(t)) = g(t, Y(t))\theta(t)$$

(ii). We apply the Itô product rule to derive the dynamics of $M(t)$ under \mathbb{P} :

$$\begin{aligned} dM(t) &= d(Y_t \eta_t) \\ &= dY_t \cdot \eta_t + Y_t \cdot d\eta_t - \theta(t)\eta(t)g(t, Y(t))dt \\ &= (f(t, Y(t))dt + g(t, Y(t))dX(t)) \eta(t) \\ &\quad - \theta(t)\eta(t)Y(t)dX(t) - \theta(t)\eta(t)g(t, Y(t))dt \\ &= (f(t, Y(t))\eta(t) - \theta(t)\eta(t)g(t, Y(t))) dt \\ &\quad + \eta(t) [g(t, Y(t)) - \theta(t)Y(t)] dX(t) \end{aligned}$$

Now, $Y(t)$ is \mathbb{Q} -martingale, which implies that $f(t, Y(t)) = g(t, Y(t))\theta(t)$. Substituting in the previous equation, we find that

$$dM(t) = (g(t, Y(t)) - \theta(t)Y(t)) \eta(t) dX(t)$$

The dynamics of $M(t)$ is driftless. Therefore $M(t)$ is a martingale under \mathbb{P} .

2- Derivation of Formula (24) on Slide 79

We start from the forward asset pricing formula

$$C(t) = B(t, T) \mathbf{E}^{\mathbb{P}_T} [(F_B(T, T, U) - K)^+ | \mathcal{F}_t] \quad (4)$$

which we rewrite in the (now) usual way as

$$\begin{aligned} C(t) &= B(t, T) \left(\mathbf{E}^{\mathbb{P}_T} [F_B(T, T, U) \mathbf{1}_{\{F_B(T, T, U) \geq K\}} | \mathcal{F}_t] \right. \\ &\quad \left. - K \mathbf{E}^{\mathbb{P}_T} [\mathbf{1}_{\{F_B(T, T, U) \geq K\}} | \mathcal{F}_t] \right) \end{aligned} \quad (5)$$

Step 1: Evaluating the Second Expectation: $\mathbf{E}^{\mathbb{P}_T} [\mathbf{1}_{\{F_B(T, T, U) \geq K\}} | \mathcal{F}_t]$

$$\begin{aligned} &\mathbf{E}^{\mathbb{P}_T} [\mathbf{1}_{\{F_B(T, T, U) \geq K\}} | \mathcal{F}_t] \\ &= \mathbb{P}_T [F_B(T, T, U) \geq K | \mathcal{F}_t] \\ &= \mathbb{P}_T \left[F_B(t, T, U) \exp \left\{ \int_t^T (b(s, U) - b(s, T)) dX^T(s) - \frac{1}{2} v_U^2(t, T) \right\} \geq K \right] \\ &= \mathbb{P}_T \left[\int_t^T (b(s, U) - b(s, T)) dX^T(s) \geq \ln \frac{K}{F_B(t, T, U)} + \frac{1}{2} v_U^2(t, T) \right] \end{aligned}$$

Note that the random variable $Y(T) = \int_t^T (b(s, U) - b(s, T)) dX^T(s)$ is Normally distributed with mean 0 and variance $v_U^2(t, T)$.

We can then define a standard Normal random variable $Z \sim N(0, 1)$ as $Z = \frac{Y}{v_U(t, T)}$ and express the expectation as

$$\begin{aligned} &\mathbf{E}^{\mathbb{P}_T} [\mathbf{1}_{\{F_B(T, T, U) \geq K\}} | \mathcal{F}_t] \\ &= \mathbb{P}_T \left[Z \geq \frac{\ln \frac{K}{F_B(t, T, U)} + \frac{1}{2} v_U^2(t, T)}{v_U(t, T)} \right] \end{aligned}$$

By symmetry of the Normal distribution, we conclude that:

$$\begin{aligned} &\mathbf{E}^{\mathbb{P}_T} [\mathbf{1}_{\{F_B(T, T, U) \geq K\}} | \mathcal{F}_t] \\ &= \mathbb{P}_T \left[Z \leq \frac{\ln \frac{F_B(t, T, U)}{K} - \frac{1}{2} v_U^2(t, T)}{v_U(t, T)} \right] \\ &= N[d_2(B(t, U), t, T)] \end{aligned}$$

where

$$d_2(b, t, T) = \frac{\ln\left(\frac{b}{K}\right) - \ln B(t, T) - \frac{1}{2}v_U(t, T)}{v_U(t, T)}$$

Step 2: Evaluating the First Expectation: $\mathbf{E}^{\mathbb{P}^T} [F_B(T, T, U) \mathbf{1}_{\{F_B(T, T, U) \geq K\}} | \mathcal{F}_t]$

$$\begin{aligned} & \mathbf{E}^{\mathbb{P}^T} [F_B(T, T, U) \mathbf{1}_{\{F_B(T, T, U) \geq K\}} | \mathcal{F}_t] \\ = & F_B(t, T, U) \mathbf{E}^{\mathbb{P}^T} \left[\exp \left\{ \int_t^T (b(s, U) - b(s, T)) dX^T(s) - \frac{1}{2}v_U^2(t, T) \right\} \mathbf{1}_{\{F_B(T, T, U) \geq K\}} \right] \end{aligned}$$

The process

$$\Lambda(t) = \exp \left\{ \int_t^T (b(s, U) - b(s, T)) dX^T(s) - \frac{1}{2}v_U^2(t, T) \right\}$$

is an exponential martingale which we can use to define a new measure \mathbb{Z} via the Radon-Nikod'ym derivative:

$$\frac{d\mathbb{Z}}{d\mathbb{P}^T} = \Lambda(t) \quad (6)$$

Under the \mathbb{Z} -measure,

$$X^{\mathbb{Z}}(t) = X^T(t) - \int_t^T (b(s, U) - b(s, T)) ds$$

is a standard Brownian motion and

$$F_B(T, T, U) = F_B(t, T, U) \exp \left\{ \int_t^T (b(s, U) - b(s, T)) dX^T(s) + \frac{1}{2}v_U^2(t, T) \right\}$$

Therefore,

$$\begin{aligned} & \mathbf{E}^{\mathbb{P}^T} [F_B(T, T, U) \mathbf{1}_{\{F_B(T, T, U) \geq K\}} | \mathcal{F}_t] \\ = & F_B(t, T, U) \mathbf{E}^{\mathbb{Z}} [\mathbf{1}_{\{F_B(T, T, U) \geq K\}}] \\ = & F_B(t, T, U) \mathbb{Z} [F_B(T, T, U) \geq K] \\ = & F_B(t, T, U) \mathbb{Z} \left[\int_t^T (b(s, U) - b(s, T)) dX^{\mathbb{Z}}(s) \geq \ln \frac{K}{F_B(t, T, U)} - \frac{1}{2}v_U^2(t, T) \right] \end{aligned}$$

After a few additional manipulations similar to what was done in **Step 1**, we obtain:

$$\begin{aligned}
& \mathbf{E}^{\mathbb{P}^T} [F_B(T, T, U) \mathbf{1}_{\{F_B(T, T, U) \geq K\}} | \mathcal{F}_t] \\
&= F_B(t, T, U) \mathbb{Z} \left[Z \leq \frac{\ln \frac{F_B(t, T, U)}{K} + \frac{1}{2} v_U^2(t, T)}{v_U(t, T)} \right] \\
&= F_B(t, T, U) N [d_1(B(t, U), t, T)]
\end{aligned}$$

where

$$d_1(b, t, T) = \frac{\ln \left(\frac{b}{K} \right) - \ln B(t, T) + \frac{1}{2} v_U(t, T)}{v_U(t, T)}$$

Step 3: Concluding

Putting it all together,

$$\begin{aligned}
C(t) &= B(t, T) \left(\mathbf{E}^{\mathbb{P}^T} [F_B(T, T, U) \mathbf{1}_{\{F_B(T, T, U) \geq K\}} | \mathcal{F}_t] - K \mathbf{E}^{\mathbb{P}^T} [\mathbf{1}_{\{F_B(T, T, U) \geq K\}} | \mathcal{F}_t] \right) \\
&= B(t, T) F_B(t, T, U) N [d_1(B(t, U), t, T)] - K N [d_2(B(t, U), t, T)] \\
&= B(t, U) N [d_1(B(t, U), t, T)] - K N [d_2(B(t, U), t, T)]
\end{aligned}$$

where in the last line, we have used the fact that $F_B(t, T, U) = \frac{B(t, U)}{B(t, T)}$ (see equation (19) on slide 74).