The Heath, Jarrow and Morton Model

In this lecture. . .

- the Heath, Jarrow & Morton forward rate model
- the relationship between HJM and spot rate models
- the advantages and disadvantages of the HJM approach
- how to decompose the random movements of the forward rate curve into its principal components

By the end of this lecture you will

- understand the HJM yield curve model
- know how to price derivatives using the model
- be able to analyze yield curve data to get a good model

Introduction

The **Heath, Jarrow & Morton** approach to the modeling of the whole forward rate curve was a major breakthrough in the pricing of fixed-income products.

They built up a framework that encompassed all of the models we have seen so far (and many that we haven't).

Instead of modeling a short-term interest rate and deriving the forward rates (or, equivalently, the yield curve) from that model, they start with a model for the whole forward rate curve.

Since the forward rates are known today, the matter of yield-curve fitting is contained naturally within their model, it does not appear as an afterthought.

The forward rate equation

The key concept in the HJM model is that we model the evolution of the whole forward rate curve, not just the short end.

Write $F(t;T)$ for the forward rate curve at time t. Thus the price of a zero-coupon bond at time t and maturing at time T, when it pays \$1, is

- $$Z(t;T) = e^{-\int_t^T F(t;s)ds}. \tag{1}$$

Let us assume that all zero-coupon bonds evolve according to

- $$dZ(t;T) = \mu(t,T)Z(t;T)dt + \sigma(t,T)Z(t;T)dX. \qquad (2)$$

This is not much of an assumption, other than to say that it is a one-factor model, and we will generalize that later.

In this $d\cdot$ means that time t evolves but the maturity date T is fixed.

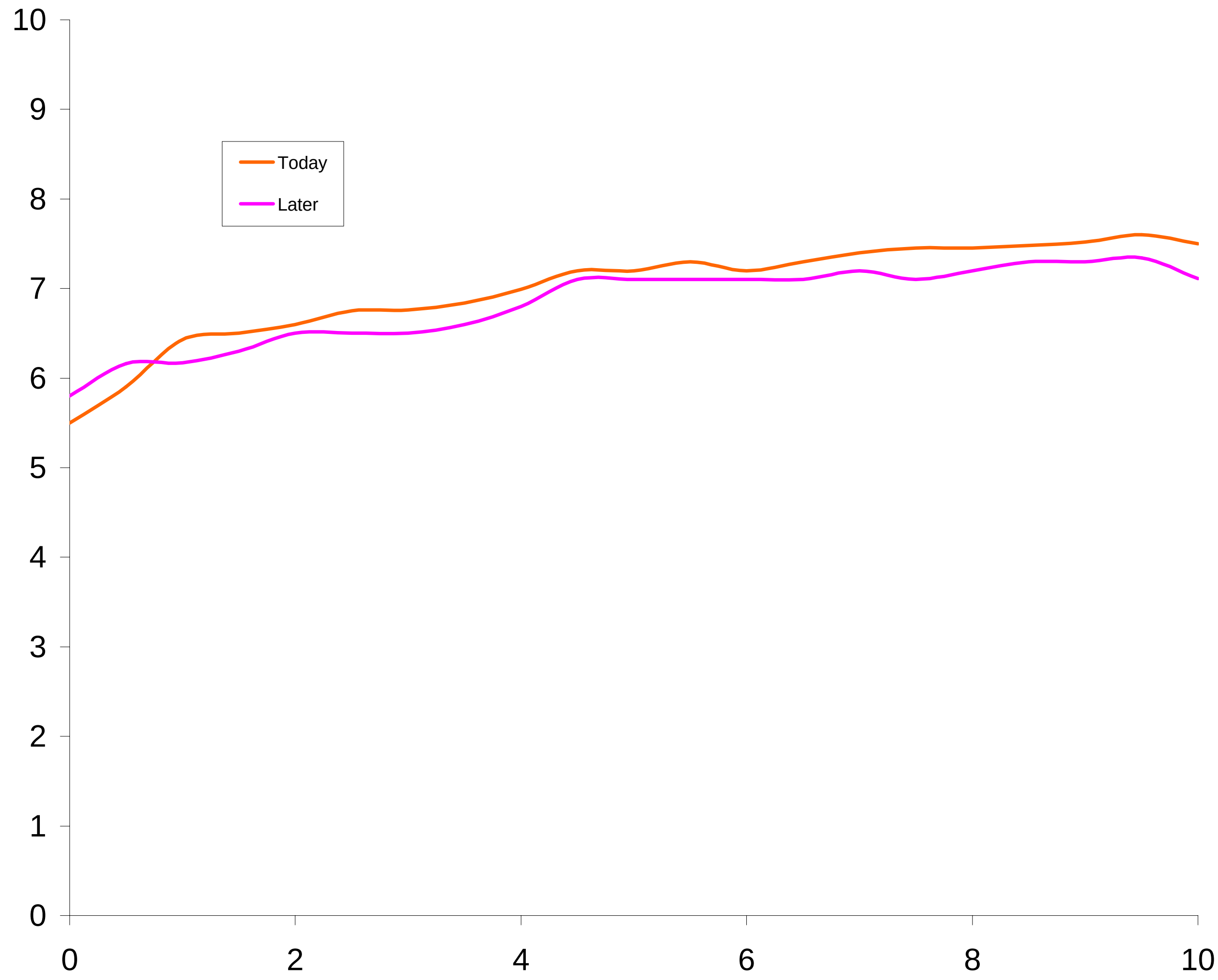
Note that since $Z(t;t) = 1$ we must have $\sigma(t,t) = 0$.

From (1) we have

- $$F(t;T) = -\frac{\partial}{\partial T}\log Z(t;T).$$

Differentiating this with respect to t and substituting from (2) results in an equation for the evolution of the forward curve:

- $$dF(t;T) = \frac{\partial}{\partial T}\left(\tfrac{1}{2}\sigma^2(t,T) - \mu(t,T)\right)\,dt - \frac{\partial}{\partial T}\sigma(t,T)\,dX. \tag{3}$$



The forward rate curve today and a few days later.

Where has this got us?

We have an expression for the drift of the forward rates in terms of the volatility of the forward rates.

- There is also a μ term, the drift of the bond.

  We have seen many times before how such drift terms disappear when we come to pricing derivatives, to be replaced by the risk-free interest rate r.

  Exactly the same will happen here.

At the moment we are in the real world.

- To price derivatives we need to move over to the risk-neutral world.

The first step in this direction is to see what happens when we hold a hedged portfolio.

The market price of risk

In the one-factor HJM model all stochastic movements of the forward rate curve are perfectly correlated.

We can therefore hedge one bond with another of a different maturity.

Such a hedged portfolio is

$$\Pi = Z(t;T_1) - \Delta Z(t;T_2).$$

The change in this portfolio is given by

$$
\begin{aligned}
d\Pi &= dZ(t;T_1) - \Delta dZ(t;T_2) \\
&= Z(t;T_1)\left(\mu(t,T_1)dt + \sigma(t,T_1)dX\right) \\
&\qquad\qquad -\Delta Z(t;T_2)\left(\mu(t,T_2)dt + \sigma(t,T_2)dX\right).
\end{aligned}
$$

If we choose

$$
\Delta = \frac{\sigma(t,T_1)Z(t;T_1)}{\sigma(t,T_2)Z(t;T_2)}
$$

then our portfolio is hedged, is risk free.

Setting its return equal to the risk-free rate $r(t)$ and rearranging we find that

$$\frac{\mu(t, T_1) - r(t)}{\sigma(t, T_1)} = \frac{\mu(t, T_2) - r(t)}{\sigma(t, T_2)}.$$

The left-hand side is a function of T_1 and the right-hand side is a function of T_2.

This is only possible if both sides are independent of the maturity date T:

- $$\mu(t, T) = r(t) + \lambda(t)\sigma(t, T).$$

As before, $\lambda(t)$ is the market price of risk (associated with the one factor).

Real and risk neutral

We are almost ready to price derivatives using the HJM model.

But first we must discuss the real and risk-neutral worlds.

All of the variables introduced above have been *real* variables. But when we come to pricing derivatives we must do so in the risk-neutral world.

- In the present HJM context, risk-neutral 'means' $\mu(t,T) = r(t)$.

  This means that in the risk-neutral world the expected return on any traded investment is simply $r(t)$.

  In particular, the expected return on the zero-coupon bond must be r.

The risk-neutral zero-coupon bond price satisfies

- $$dZ(t;T) = r(t)Z(t;T)dt + \sigma(t,T)Z(t;T)dX.$$

The deterministic part of this equation represents exponential growth of the bond at the risk-free rate.

The relationship between the risk-neutral forward rate drift and volatility

Let's write the stochastic differential equation for the *risk-neutral* forward rate curve as

$$dF(t;T) = m(t,T)dt + \nu(t,T)dX.$$

From (3)

$$\nu(t,T) = -\frac{\partial}{\partial T}\sigma(t,T),$$

is the forward rate volatility.

From (3), the drift of the forward rate is given by

$$\frac{\partial}{\partial T}\left(\tfrac{1}{2}\sigma^2(t,T) - \mu(t,T)\right) = \nu(t,T)\int_t^T \nu(t,s)ds - \frac{\partial}{\partial T}\mu(t,T),$$

where we have used $\sigma(t,t) = 0$.

In the risk-neutral world we have $\mu(t,T) = r(t)$, and so the drift of the risk-neutral forward rate curve is related to its volatility by

- $$m(t,T) = \nu(t,T)\int_t^T \nu(t,s)ds.$$

Pricing derivatives

Pricing derivatives is all about finding the expected present value of all cashflows in a risk-neutral framework.

Because of the non-Markov nature of HJM in general a partial differential equation approach is infeasible. This leaves us with two choices.

- One is to estimate directly the necessary expectations by simulating the random evolution of, in this case, the risk-neutral forward rates.

- The other is to build up a tree structure.

Simulations

If we want to use a Monte Carlo method, then we must simulate the evolution of the whole forward rate curve, calculate the value of all cashflows under each evolution and then calculate the present value of these cashflows by *discounting at the realized spot rate* $r(t)$.

Assume that we have chosen a model for the forward rate volatility, $\nu(t,T)$. Today is t^* when we know the forward rate curve $F(t^*;T)$.

To price a derivative using a Monte Carlo simulation perform the following steps.

1. Simulate a realized evolution of the whole risk-neutral forward rate curve for the necessary length of time, until T^*, say. This requires a simulation of

$$dF(t;T) = m(t,T)dt + \nu(t,T)dX,$$

where

$$m(t,T) = \nu(t,T)\int_t^T \nu(t,s)ds.$$

After this simulation we will have a realization of $F(t;T)$ for $t^* \leq t \leq T^*$ and $T \geq t$. We will have a realization of the whole forward rate path.

2. At the end of the simulation we will have the realized prices of all maturity zero-coupon bonds at every time up to T^*.

3. Using this forward rate path calculate the value of all the cashflows that would have occured.

4. Using the realized path for the spot interest rate $r(t)$ calculate the present value of these cashflows. Note that we discount at the continously compounded risk-free rate, not at any other rate. In the risk-neutral world all assets have an expected return of $r(t)$.

5. Return to Step 1 to perform another realization, and continue until you have a sufficiently large number of realizations to calculate the expected present value as accurately as required.

The disadvantage of the HJM model is that a Monte Carlo simulation such as this can be very slow.

On the plus side, because the whole forward rate curve is calculated the bond prices at all maturities are trivial to find during this simulation.

Trees

- If we are to build up a tree for a non-Markov model then we find ourselves with the unfortunate result that the forward curve after an up move followed by a down is *not* the same as the curve after a down followed by an up.

The equivalence of these two paths in the Markov world is what makes the binomial method so powerful and efficient.

- In the non-Markov world our tree structure becomes 'bushy,' and grows *exponentially* in size with the addition of new time steps.

The Musiela parameterization

Often in practice the model for the volatility structure of the forward rate curve will be of the form

$$\nu(t, T) = \bar{\nu}(t, T - t),$$

meaning that we will model the volatility of the forward rate at each maturity, one, two, three years, and not at each maturity date, 2000, 2001, 2002.

If we write τ for the maturity period $T - t$ then it is a simple matter to find that $\bar{F}(t;\tau) = F(t, t+\tau)$ satisfies

$$d\bar{F}(t;\tau) = \bar{m}(t,\tau)dt + \bar{\nu}(t,\tau)dX,$$

where

- $$\bar{m}(t,\tau) = \bar{\nu}(t,\tau)\int_0^\tau \bar{\nu}(t,s)ds + \frac{\partial}{\partial\tau}\bar{F}(t,\tau).$$

Multi-factor HJM

Often a single-factor model does not capture the subtleties of the yield curve that are important for particular contracts.

The obvious example is the spread option, that pays off the difference between rates at two different maturities.

We then require a multi-factor model.

If the risk-neutral forward rate curve satisfies the N-dimensional stochastic differential equation

$$dF(t,T) = m(t,T)dt + \sum_{i=1}^{N} \nu_i(t,T)dX_i,$$

where the dX_i are uncorrelated, then

- $$m(t,T) = \sum_{i=1}^{N} \nu_i(t,T) \int_t^T \nu_i(t,s)ds.$$

There is a Musiela parameterization version of the multi-factor HJM.

The spot rate process: Comparison with other models

The spot interest rate is simply given by the forward rate for a maturity equal to the current date i.e.

$$r(t) = F(t;t).$$

In this section we are going to manipulate this expression to derive the stochastic differential equation for the spot rate.

Suppose today is t^* and that we know the whole forward rate curve today, $F(t^*;T)$.

We can write the spot rate for *any* time t in the future as

$$r(t) = F(t;t) = F(t^*;t) + \int_{t^*}^{t} dF(s;t).$$

From our earlier expression (3) for the forward rate process for F we have

$$r(t) = F(t^*; t) + \int_{t^*}^{t} \left(\sigma(s,t) \frac{\partial \sigma(s,t)}{\partial t} - \frac{\partial \mu(s,t)}{\partial t} \right) ds - \int_{t^*}^{t} \frac{\partial \sigma(s,t)}{\partial t} dX(s).$$

Differentiating this with respect to time t we arrive at the stochastic differential equation for r

$$dr = \left(\frac{\partial F(t^*;t)}{\partial t} - \frac{\partial \mu(t,s)}{\partial s}\Big|_{s=t} \right.$$

$$+ \int_{t^*}^{t} \left(\sigma(s,t) \frac{\partial^2 \sigma(s,t)}{\partial t^2} + \left(\frac{\partial \sigma(s,t)}{\partial t} \right)^2 - \frac{\partial^2 \mu(s,t)}{\partial t^2} \right) ds$$

$$\left. - \underline{\int_{t^*}^{t} \frac{\partial^2 \sigma(s,t)}{\partial t^2} dX(s)} \right) dt - \frac{\partial \sigma(t,s)}{\partial s}\Big|_{s=t} dX.$$

The non-Markov nature of HJM

The details of this expression are not important.

But observe one point.

Compare this stochastic differential equation for the spot rate with that for any of the classical models.

- In particular, note that the term underlined depends on the history of σ from the date t^* to the future date t, and *it depends on the history of the stochastic increments* dX.

This term is thus highly path dependent.

Moreover, for a general HJM model it makes the motion of the spot rate **non-Markov**.

In a **Markov process** it is only the present state of a variable that determines the possible future (albeit random) state.

A simple one-factor example: Ho & Lee

One of the key points about the HJM approach is that the yield curve is fitted, by default.

The simplest yield-curve fitting spot rate model is Ho & Lee, so we draw a comparison between this and HJM.

In Ho & Lee the risk-neutral spot rate satisfies

$$dr = \eta(t)dt + c\,dX,$$

for a constant c. The prices of zero-coupon bonds, $Z(r,t;T)$, in this model satisfy

$$\frac{\partial Z}{\partial t} + \tfrac{1}{2}c^2\frac{\partial^2 Z}{\partial r^2} + \eta(t)\frac{\partial Z}{\partial r} - rZ = 0,$$

$$Z(r,T;T) = 1.$$

The solution is easily found to be

$$Z(r,t;T) = \exp\left(\tfrac{1}{6}c^2(T-t)^3 - \int_t^T \eta(s)(T-s)ds - (T-t)r\right).$$

In the Ho & Lee model $\eta(t)$ is chosen to fit the yield curve at time t^*. In forward rate terms this means that

$$F(t^*;T) = r(t^*) - \tfrac{1}{2}c^2(T-t^*)^2 + \int_{t^*}^T \eta(s)ds,$$

and so

$$\eta(t) = \frac{\partial F(t^*;t)}{\partial t} + c^2(t-t^*).$$

At any time later than t^*

$$F(t;T) = r(t) - \tfrac{1}{2}c^2(T-t)^2 + \int_t^T \eta(s)ds.$$

From this we find that

$$dF(t;T) = c^2(T-t)dt + c\,dX.$$

In our earlier notation, $\sigma(t,T) = -c(T-t)$ and $\nu(t,T) = c$. This is the evolution equation for the risk-neutral forward rates.

Most of the popular models have HJM representations.

Principal Component Analysis

There are two main ways to use HJM. One is to choose the volatility structure $\nu_i(t,T)$ to be ‘nice’ to make a tractable model, one that is Markov. This often leads us back to the ‘classical’ popular spot-rate models.

- The other way is to choose the volatility structure to match data.

  This is where **Principal Component Analysis** (PCA) comes in.

In analyzing the volatility of the forward rate curve one usually assumes that the volatility structure depends only on the time to maturity i.e.

$$\nu = \bar{\nu}(T - t).$$

We will examine the more general multi-factor model:

$$dF(t;T) = m(t,T)dt + \sum_{i=1}^{N} \bar{\nu}_i(T - t)dX_i.$$

- If we have forward rate time series data going back a few years we can calculate the covariances between the *changes* in the rates of different maturities.

We may have, for example, the one-, three-, six-month, one-, two, three-, five-, seven-, 10- and 30-year rates.

The covariance matrix would then be a ten $\times$ ten symmetric matrix with the variances of the rates along the diagonal and the covariances between rates off the diagonal.

PCA is a technique for finding common movements in the rates, for essentially finding eigenvalues and eigenvectors of the matrix.

- We expect to find, for example, that a large part of the movement of the forward rate curve is common between rates, that a parallel shift in the rates is the largest component of the movement of the curve in general.

- The next most important movement would be a twisting of the curve,

- The third most important would be bending.

Suppose that we have found the covariance matrix, **M** for the changes in the rates mentioned above.

This ten by ten matrix will have ten eigenvalues, λ_i, and eigenvectors, $\mathbf{v}_i$ satisfying

$$\mathbf{M}\mathbf{v}_i = \lambda_i \mathbf{v}_i;$$

$\mathbf{v}_i$ is a column vector.

Once we have found the eigenvalues and eigenvectors we must relate them to the original HJM volatility factors.

The eigenvector associated with the largest eigenvalue is the first principal component.

It gives the dominant part in the movement of the forward rate curve.

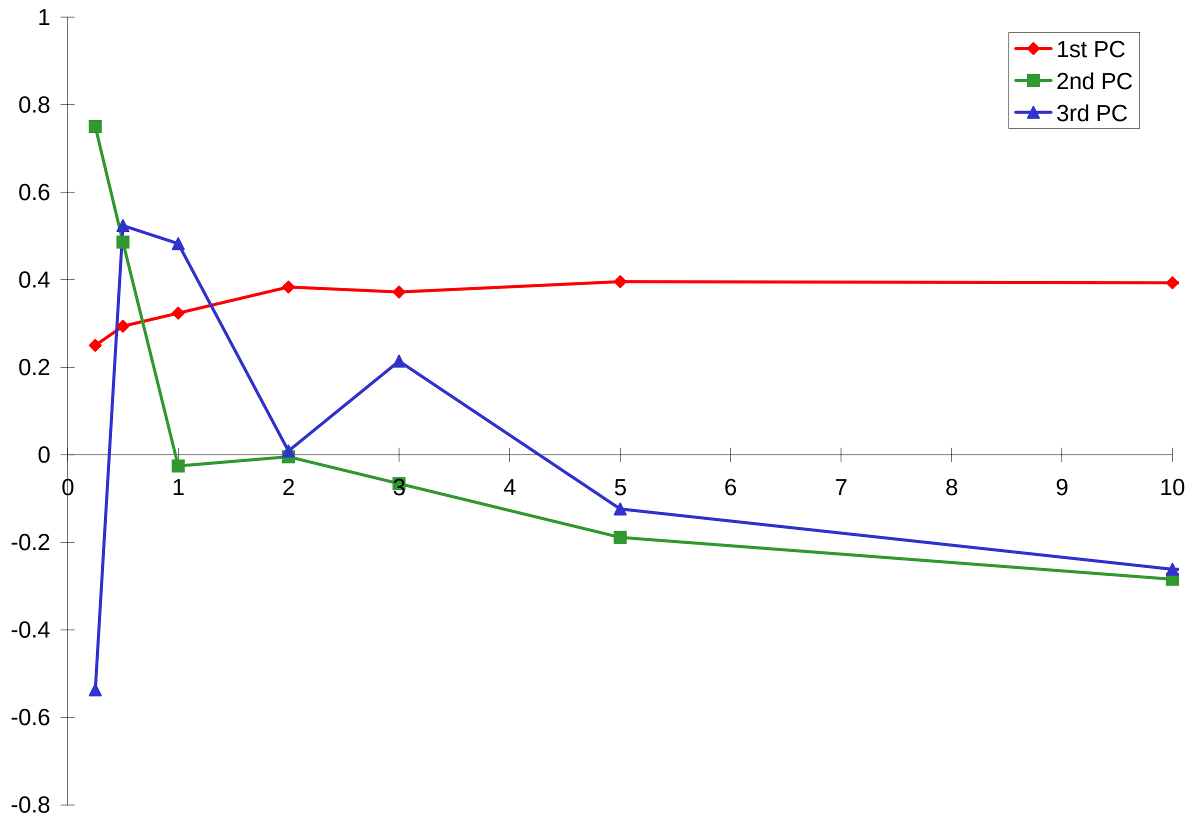
Its first entry represents the movement of the one-month rate, the second entry is the three-month rate etc.

Its eigenvalue is the variance of these movements.

The volatility factors are given by

$$\bar{\nu}_i(\tau_j) = \sqrt{\lambda_i}(\mathbf{v}_i)_j.$$

Here τ_j is the maturity i.e. 1/12, 1/4 etc. and $(\mathbf{v}_i)_j$ is the jth entry in the vector $\mathbf{v}_i$.



The first three components of the US forward rate curve.

The eigenvalues are also useful in telling you how much of the dynamics of the forward curve is explained by each factor.

The first factor explains

$$\frac{\lambda_1}{\lambda_1 + \lambda_2 + \cdots + \lambda_N}.$$

The first two factors together explain

$$\frac{\lambda_1 + \lambda_2}{\lambda_1 + \lambda_2 + \cdots + \lambda_N}.$$

Etc.

(In practice, as much as 95% of the dynamics can be explained by just two factors!)

Summary

Please take away the following important ideas

- The HJM model represents the movement of the whole yield curve
- In general the model is non Markov
- PCA is a method for determining the dominant structure within sets of random numbers