Mod. Fin. (derivatives) based on random movement of fin. assets

Simulating and Manipulating Stochastic Differential Equations

In this lecture. . .

- Using Itô's lemma to manipulate stochastic differential equations Continuation / extension
- Continuous-time stochastic differential equations as discrete-time processes Calculus for stoch. processes
- Simple ways of generating random numbers in Excel Look at RAND and NORMSINV
- Correlated random walks Some nice maths

By the end of this lecture you will be able to

- manipulate stochastic differential equations

- find transition probability density functions for arbitrary stochastic differential equations   Start with time indep. ones

- simulate stochastic differential equations

Introduction

In order to become comfortable with the kind of models commonly used in quantitative finance you must be able to manipulate stochastic differential equations and generate random walks numerically.

Before looking at finance models need to understand SDE's. At the heart of derivatives - equities, interest rates, volatility - different SDE for dynamics

$\underline{X(t)}$

Markov Process: Conditional dist^{n} of $X(t)$ up until $\tau < t$ depends only on τ

Martingale: Conditional expectation of $X(t)$ up until $\tau < t$ is equal to X_τ

$$\mathbb{E}[X_t \mid X_\tau] = X_\tau$$

$X(t)$

Markov Process: Conditional dist^{n} of $X(t)$ up until $\tau < t$ depends only on τ

Martingale: Conditional expectation of $X(t)$ up until $\tau < t$ is equal to X_τ

$$\mathbb{E}[X_t | X_\tau] = X_\tau$$

Manipulating stochastic differential equations

An equation of the form

Standard form

$\mathbb{E}[dG] = a\,dt$

$\mathbb{V}[dG] = b^2\,dt$

Stoch process

$$(*) \qquad dG = \underbrace{a(G,t)\,dt}_{\text{drift}} + \overbrace{b(G,t)}^{\text{diff/vol. (level of noise)}}\,\underbrace{dX}_{\text{increment of Brownian Motion}}$$

is called a Stochastic Differential Equation (SDE) for G (or random walk for dG) and consists of two components:

1. $a(G,t)\,dt$ is deterministic – coefficient of dt is known as the **drift** or **growth**

2. $b(G,t)\,dX$ is random – coefficient of dX is known as the **diffusion** or **volatility**

and we say G evolves according to (or follows) this process.

G is called a diffusion process if it can be written as $(*)$

Not all Stoch. processes are Diff. processes

Integral form of $*$

$$G(t) = G(0) + \int_0^t A(G,\tau)\,d\tau + \int_0^t B(G,\tau)\,dX(\tau)$$

diffusion process G is a Markov process

$dG = A(G, t-\tau)\,dt + B(G,t)\,dX$ is not Markov.

dG is a Markov process $\therefore$ $dG = G(t) - G(t-1)$

dG is not a Martingale unless $a(G,t) = 0$ $\Rightarrow$ zero drift process

So if for example we have a random walk

popular recipe for generating asset prices

$$dS = \mu S dt + \sigma S dX \quad (1)$$

then the drift is $a(S,t) = \mu S$ and the diffusion is $b(S,t) = \sigma S$.

The process (1) is also called **Geometric Brownian Motion** (GMB) or **Exponential Brownian motion** (EMB) and is a popular model for a wide class of asset prices.

We have previously considered Itô's lemma to obtain the change in a function $f(X)$ when $X \to X + dX$, where X is a standard Brownian motion. B.M

This jump $df = f(X + dX) - f(X)$ is given by

$$df = \frac{df}{dX}dX + \frac{1}{2}\frac{d^2 f}{dX^2}dt \qquad (2)$$

Basic form for Itô

using the result

$$\boxed{\lim_{dt \to 0} dX^2 = dt.}$$

Stoch. calculus without 'tears'

Taylor – Determ. Varis

Itô – function of a stoch. var.

Moving Forward

Suppose we now wish to extend the result (2) to consider the change in an option price $V(S)$ where the underlying variable S follows a geometric Brownian motion.

(Of course, you are not supposed to know anything about options yet. Just think of manipulating functions.)

Simple Option V where $V(S)$

Stoch. var./process

If we rewrite (1) as

$$\frac{dS}{S} = \mu\, dt + \sigma\, dX$$

then dS represents the change in asset price S in a small time interval dt.

This expression is the return on the asset.

μ is the average growth rate of the asset and σ the associated volatility (standard deviation) of the returns.

dX is an increment of a Brownian Motion, known as a Wiener process and is a Normally distributed random variable such that $dX \sim N(0, dt)$.

An obvious question we may ask is, what is the jump in $V(S + dS)$ when $S \rightarrow S + dS$?

We begin (again) by using a Taylor series as in (2), but for $V(S + dS)$ to get

$$V(S+dS) = V(S) + V' dS + \tfrac{1}{2} V'' dS^2 + O(dS^3)$$

$$\Longrightarrow \quad dV = \frac{dV}{dS}\, dS + \frac{1}{2}\, \frac{d^2V}{dS^2}\, dS^2. \qquad \text{(Not quite an SDE)}$$

We know $\quad dS = \mu S\, dt + \sigma S\, dX$

$$\Rightarrow \quad dS = \underbrace{\mu^2 S^2 dt^2}_{O(dt^2)} + \underbrace{\sigma^2 S^2 dX^2}_{O(dt)} + \underbrace{2\mu\sigma S^2 dt\, dX}_{O(dt^{3/2})}$$

$dX^2 = dt \Rightarrow$

$dX = \sqrt{dt}$

We can proceed further now as we have an expression for dS (and hence dS^2). As dt is very small, any terms in $dt^{\frac{3}{2}}$ or dt^2 are insignificant in comparison and can be ignored. So working to $O\left(dt\right)$

$$\boxed{dS^2 = \sigma^2 S^2 dt.}$$

If we substitute this into the previous expression for dV we get Itô's lemma as applied to $V(S)$:

$$dV = \overbrace{\left(\mu S \frac{dV}{dS} + \tfrac{1}{2}\,\sigma^2 S^2 \frac{d^2V}{dS^2}\right)}^{\text{drift}} dt + \overbrace{\left(\sigma S \frac{dV}{dS}\right)}^{\text{diffusion}} dX. \qquad (3)$$

Note that this is <u>another</u> <u>stochastic</u> <u>differential</u> <u>equation</u>!

It contains a predictable part and a random part.

Suppose that we had a formula for $V(S)$. Let's take a very special case, let's consider

$$V(S) = \log S.$$

Differentiating this once gives

$$\frac{dV}{dS} = \frac{1}{S}.$$

Differentiating this again gives

$$\frac{d^2V}{dS^2} = -\frac{1}{S^2}.$$ Now subst. in (3)

Now from (3) we have

$$d(\log S) = \left(\mu - \tfrac{1}{2}\sigma^2\right) dt + \sigma dX.$$

Integrating both sides between 0 and t

$$\int_0^t d(\log S) = \int_0^t \left(\mu - \frac{1}{2}\sigma^2\right) d\tau + \int_0^t \sigma \, dX \qquad (t > 0)$$

$$\underbrace{\log S \Big|_0^t}_{} = \left(\mu - \tfrac{1}{2}\sigma^2\right) t + \sigma \left(X(t) - X(0)\right).$$

$= \log S(t) - \log S(0)$ $\qquad \left[\log a - \log b = \log \frac{a}{b} \right]$

$= \ln \dfrac{S(t)}{S(0)}$

Therefore

$$\log\left(\frac{S(t)}{S(0)}\right) = \left(\mu - \tfrac{1}{2}\sigma^2\right)t + \sigma\left(X(t) - X(0)\right)$$

Assuming $X(0) = 0$ and $S(0) = S_0$, the exact solution becomes

$$S(t) = S_0 \exp\left(\left(\mu - \tfrac{1}{2}\sigma^2\right)t + \sigma X(t)\right). \tag{4}$$

This is why $\frac{dS}{S} = \mu dt + \sigma dX$ is used for stock prices – it will never be negative.

Now suppose $V = V(S, t)$ $\left.\begin{array}{l} S \to S + dS \\ t \to t + dt \end{array}\right\}$ What is $V(S+dS, t+dt)$

Again use Taylor

$$V(S+dS, t+dt) = V(S,t) + \frac{\partial V}{\partial S} dS + \frac{\partial V}{\partial t} dt + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} dS^2 + O(dt^2)$$

$$dV = V_S (\mu S\, dt + \sigma S\, dX) + V_t\, dt + \frac{1}{2}\sigma^2 S^2 V_{SS}\, dt$$

$$dV = \left(V_t + \mu S V_S + \frac{1}{2}\sigma^2 S^2 V_{SS}\right) dt + (\sigma S V_S)\, dX$$

<u>Ex:</u> $V = t^2 S^3$ $V_t = 2tS^3$ $V_S = 3t^2 S^2$ $V_{SS} = 6t^2 S$

Subst into dV above

$$dV = \underbrace{(2tS^3 + 3\mu t^2 S^3 + 3\sigma^2 t^2 S^3)}_{\text{drift}}\, dt + \underbrace{(3\sigma t^2 S^3)}_{\text{diff.}}\, dX$$

Another example: $dr = (\eta - \gamma r)\,dt + \sigma dX$

$= \gamma(\eta/\gamma - r)\,dt + \sigma dX$ $\qquad \eta/\gamma = \bar{r}$

Let's take a look at the Vasicek interest rate model for short-term interest rates, and try manipulating that.

$$dr = \gamma\,(\bar{r} - r)\,dt + \sigma dX.$$

Mean reverting Process

γ refers to the **reversion rate** and $\bar{r}$ denotes the **mean rate**.
(Speed)

γ – acts like a spring. Mean reversion means a process which when ↑ γ pulls it down to a mean level $\bar{r}$, and when r ↘ γ pulls it back up to $\bar{r}$. Look at dynamics later

Write $dr = -\gamma(r - \bar{r})dt + \sigma dX$

$$\int_0^t d(ue^{\gamma s}) = \sigma\int_0^t e^{\gamma s}\,dX \rightarrow u(t)e^{\gamma s}\Big|_0^t = \sigma\int_0^t e^{\gamma s}\,dX \rightarrow e^{\gamma t}u(t) - u(0) = \sigma\int_0^t e^{\gamma s}\,dX$$

$$\therefore\quad u(t) = u(0)e^{-\gamma t} + \sigma e^{-\gamma t}\int_0^t e^{\gamma s}\,dX = \alpha e^{-\gamma t} + \sigma\int_0^t e^{\gamma(s-t)}\,dX \qquad \alpha = u(0)$$

Now look at $\displaystyle\int_0^t \underbrace{e^{\gamma(s-t)}}_{v}\,\underbrace{dX}_{u}$ Use integ. by parts i.e

$$\int v\,du = uv - \int u\,dv$$

$$\therefore\quad u = X_s \Rightarrow du = dX_s$$

$$v = e^{\gamma(s-t)} \Rightarrow \frac{dv}{ds} = \gamma e^{\gamma(s-t)} \longrightarrow dv = \gamma e^{\gamma(s-t)}\,ds$$

Hence $\displaystyle X_s e^{\gamma(s-t)}\Big|_0^t - \gamma\int_0^t e^{\gamma(s-t)}X_s\,ds$ $\qquad X_0 = X(0) = 0$

$$= X(t) - \gamma\int_0^t e^{\gamma(s-t)}X_s\,ds$$

$d(r-\bar{r}) = dr - d\bar{r} = dr$

$\Rightarrow du = dr$

By setting $u = r - \overline{r}$, u is a solution of

$$du = -\gamma u \, dt + \sigma dX.$$

Ornstein - Uhlenbeck process

An analytic solution for this equation exists. To see, this write the equation as

$\frac{d}{dt}(e^{\gamma t}) = \gamma e^{\gamma t}$

$\therefore d(e^{\gamma t}) = \gamma e^{\gamma t} dt$

$$d\left(ue^{\gamma t}\right) = \sigma e^{\gamma t} dX.$$

$d(ue^{\gamma t}) = e^{\gamma t} du + u\gamma e^{\gamma t} dt$

$= e^{\gamma t}(-\gamma u dt + \sigma dX) + \gamma u e^{\gamma t} dt$

$= \sigma e^{\gamma t} dX$

Integrating over from zero to t gives

$$u(t) = u(0)e^{-\gamma t} + \sigma \int_0^t e^{\gamma(s-t)} dX_s.$$

This can be **integrated by parts** to give

$$u(t) = u(0)e^{-\gamma t} + \sigma \left(X(t) - \gamma \int_0^t X(s) e^{\gamma(s-t)} \, ds \right).$$

Transition probability density functions again

Let's look at the equations governing the probability distribution for an arbitrary random walk:

$$dy = A(y,t)\,dt + B(y,t)\,dX$$

for the variable y.

Remember the **transition probability density function** $p(y,t;y',t')$ defined by

$$\text{Prob}(a < y' < b \text{ at time } t' | y \text{ at time } t) = \int_a^b p(y,t;y',t')\,dy'.$$

In words this is 'the probability that the random variable y lies between a and b at time t' in the future, given that it started out with value y at time t.'

Think of y and t as being current values with y' and t' being future values.

The transition probability density function can be used to answer questions such as

"What is the probability of the variable y being in a certain range at time t' given that it started out with value y at time t?"

The transition probability density function $p(y, t; y', t')$ satisfies two equations.

One involves derivatives with respect to the future state and time (y' and t') and is called the **forward equation.**

The other involves derivatives with respect to the current state and time (y and t) and is called the **backward equation.** BSE

These can be derived by the same trinomial idea we used before (but the details are a lot messier for the general stochastic differential equation).

Let's go straight to the PDE's

The forward equation (Future states y', t')

Cutting to the chase, the transition probability density function satisfies the partial differential equation

extra term due to the drift

$$\frac{\partial p}{\partial t'} = \frac{1}{2}\frac{\partial^2}{\partial y'^2}\left(\underbrace{B(y',t')}_{\text{diff.}}{}^2 p\right) - \frac{\partial}{\partial y'}\left(\underbrace{A(y',t')}_{\text{drift}}p\right)$$

This is the **Fokker–Planck** or **forward Kolmogorov equation**.

for a SDE

Example: The most important example to us is that of the distribution of equity prices in the future. If we have the random walk

$$dS = \overset{A}{\mu S}\, dt + \underset{B}{\sigma S}\, dX$$

then the forward equation becomes

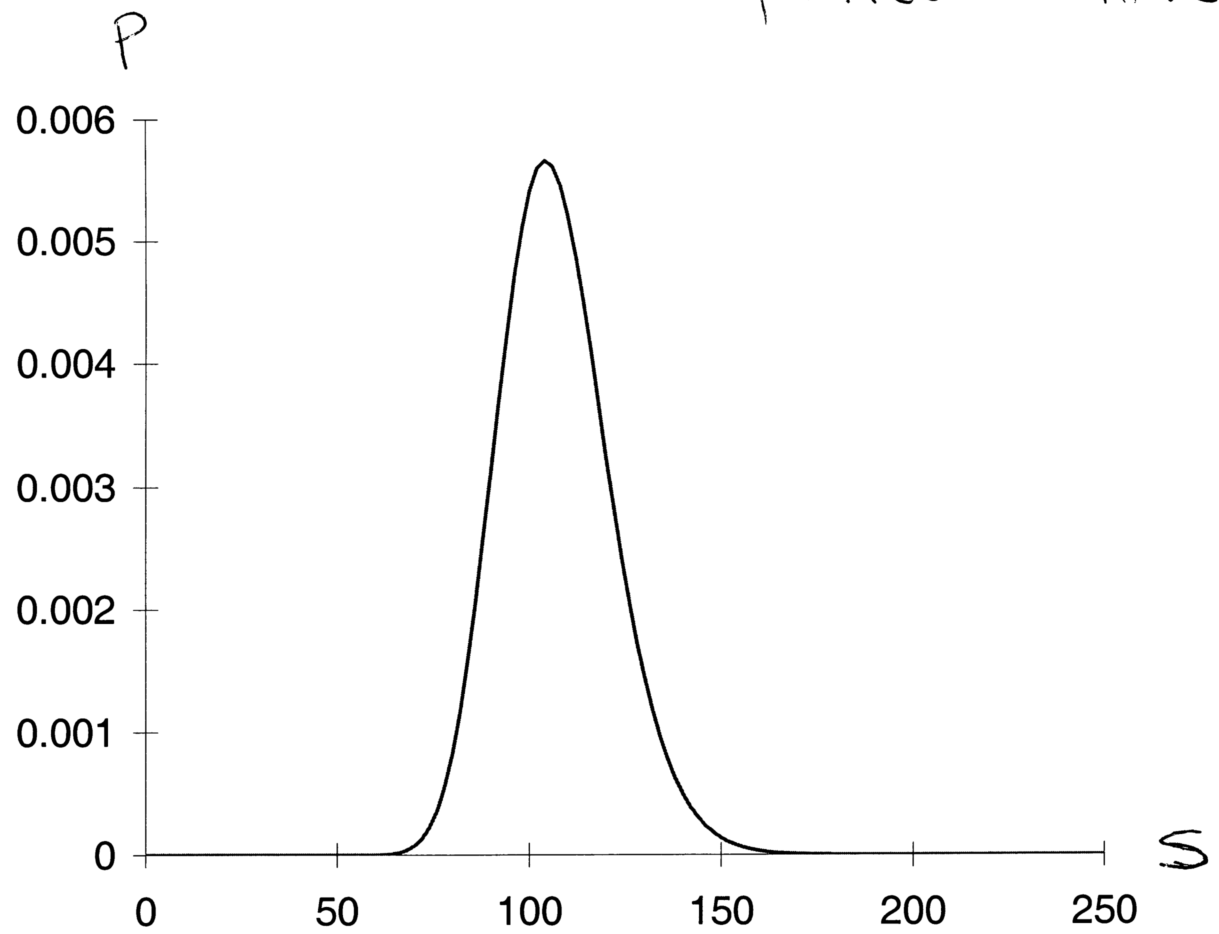
Simply

$$\frac{\partial p}{\partial t'} = \frac{1}{2}\frac{\partial^2}{\partial S'^2}\left(\overset{B^2}{\sigma^2 S'^2 p}\right) - \frac{\partial}{\partial S'}\left(\overset{A}{\mu S' p}\right).$$

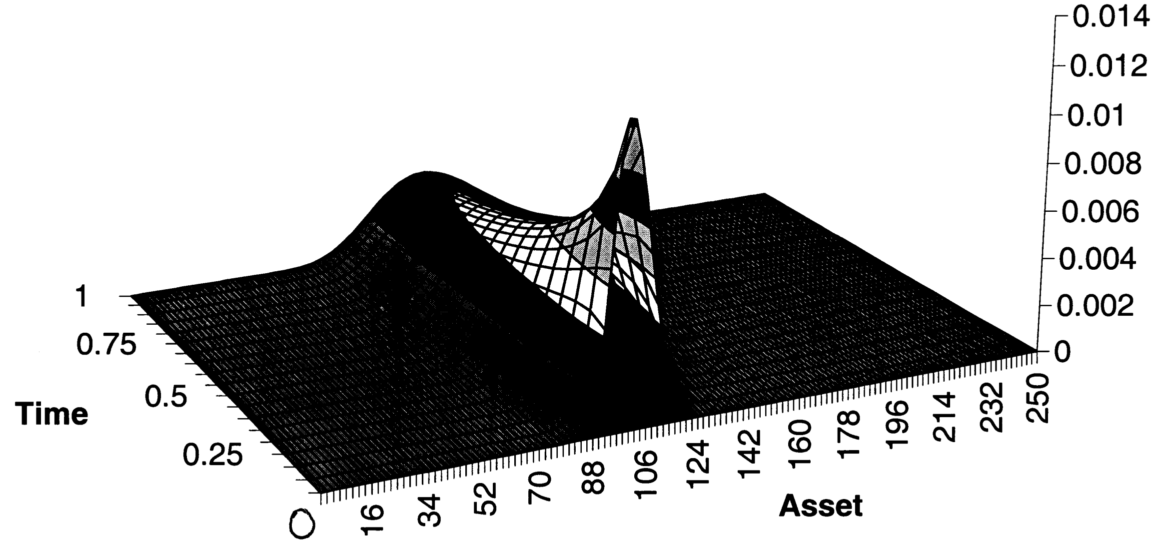
The solution of this representing a stock price starting at $S' = S$ at $t' = t$ is

$$p(S, t; S', t') = \frac{1}{\sigma S' \sqrt{2\pi(t' - t)}} e^{-\left(\log(S/S') + (\mu - \frac{1}{2}\sigma^2)(t' - t)\right)^2 / 2\sigma^2 (t' - t)}$$

For a particular time



The probability density function for the lognormal random walk, after a certain time.



p.d.f.

The probability density function for the lognormal random walk evolving through time.

The steady-state distribution

Some random walks have a steady-state distribution.

That is, in the long run as $t' \to \infty$ the distribution $p(y, t; y', t')$ as a function of y' settles down to be independent of the starting state y and time t. Possible examples are stochastic differential equation models for interest rates, inflation, volatility.

Some random walks have no such steady state even though they have a time-independent equation. For example the lognormal random walk either grows without bound or decays to zero.

$$p(y', t') \longrightarrow p(y')$$

$$\frac{1}{2}\sigma^2 p_\infty'' - \gamma \frac{d}{dr'}\left((\bar{r}-r')p_\infty\right) = 0$$

Integrate throughout

$$\frac{1}{2}\sigma^2 p_\infty' - \gamma(\bar{r}-r')p_\infty = C$$ C – const. of integ.

When $r' \to \infty$ $\begin{cases} p_\infty \to 0 \\ p_\infty' \to 0 \end{cases} \Rightarrow C = 0$

$\therefore$ $$\frac{1}{2}\sigma^2 p_\infty' = \gamma(\bar{r}-r')p_\infty$$ Var. sep. eqn

Solve and use condition $\int_{\mathbb{R}} p_\infty \, dr' = 1$ to obtain 2nd const. of integ.

If there is a steady-state distribution $p_\infty(y')$ then it satisfies the *ordinary* differential equation

Call p, p_∞ now as $t' \to \infty$

$P = p(y')$

PDE $\to$ ODE

$$\frac{1}{2}\frac{d^2}{dy'^2}\left(B^2 p_\infty\right) - \frac{d}{dy'}\left(A p_\infty\right) = 0.$$

$\frac{\partial p}{\partial t'} \to 0$

Example: The Vasicek model

$$dr = \underbrace{\gamma\left(\bar{r} - r\right)}_{A} dt + \underbrace{\sigma}_{B} dX.$$

The steady-state distribution $p_\infty(r')$ satisfies

$$\frac{1}{2}\sigma^2\frac{d^2 p_\infty}{dr'^2} - \gamma\frac{d}{dr'}\left(\overbrace{(\bar{r} - r')}^{A} p_\infty\right) = 0.$$

B

The solution is

$$p_\infty = \frac{1}{\sigma}\sqrt{\frac{\gamma}{\pi}}\; e^{-\frac{\gamma(\bar{r}-r')^2}{\sigma^2}}.$$

In other words, the interest rate r is Normally distributed with mean $\bar{r}$ and standard deviation $\sigma/\sqrt{2\gamma}$.

$$-\frac{(x-\mu)^2}{2\bar{\sigma}^2} \equiv -\frac{(\overset{\text{mean}}{r'-\bar{r}})^2}{2\underbrace{\left(\frac{1}{2}\sigma^2/\gamma\right)}_{\text{variance}}} \implies \text{Variance} = \sigma^2/2\gamma \quad \therefore \text{st. dev.} = \frac{\sigma}{\sqrt{2\gamma}}$$

$$\therefore \bar{\sigma} = \sqrt{\sigma^2/2\gamma} = \frac{\sigma}{\sqrt{2\gamma}} \quad \text{st. dev.}$$

The backward equation

Now we come to the backward equation. This will be useful if we want to calculate probabilities of reaching a specified final state from various initial states.

The transition probability density function satisfies the **backward Kolmogorov equation**

$$\frac{\partial p}{\partial t} + \tfrac{1}{2}B(y,t)^2\frac{\partial^2 p}{\partial y^2} + \underline{\underline{A(y,t)\frac{\partial p}{\partial y}}} = 0.$$

extra term due to drift.

Simulating the lognormal random walk

The lognormal random walk model for assets can be written in continuous time as

$$dS = \mu S\, dt + \sigma S\, dX.$$

$dS = S_{i+1} - S_i$ In discrete time $dt \to \delta t$

$dX \to \phi \sqrt{\delta t}$

In discrete time this is

$$S_{i+1} - S_i = S_i \left(\mu\, \delta t + \sigma \phi\, \delta t^{1/2} \right).$$

$\phi \sim N(0,1)$

To generate representative simulations of possible asset paths we must obviously work in discrete time.

The random walk on a spreadsheet

The random walk can be written as a 'recipe' for generating S_{i+1} from S_i:

$$S_{i+1} = S_i \left(1 + \mu\,\delta t + \sigma\phi\,\delta t^{1/2}\right).$$

We can easily simulate the model using a spreadsheet.

The method is called the **Euler method**. Accurate to $O(\delta t)$

Milstein Method is accurate to $O(\delta t^2)$

Start with an initial stock price, say, 100.

And a couple of parameters, $\mu = 0.1$ and $\sigma = 0.2$, say, that best represent the asset in question.

Decide on a (small) time step, $\delta t = 0.01$, say.

Now start picking random numbers!

First time step: The random number is. . . 0.12. So

$$S_{i+1} = 100\,(1 + 0.1 \times 0.01 + 0.2 \times 0.1 \times 0.12) = 100.34.$$

Second time step: The random number is. . . -0.25. So

$$S_{i+1} = 100.34\,(1 + 0.1 \times 0.01 + 0.2 \times 0.1 \times (-0.25)) = 99.94.$$

And so on.

In this simulation there are several input parameters, which remain constant:

- a starting value for the asset
- a time step δt
- the drift rate μ
- the volatility σ
- the total number of time steps

Basis of MC

Then, at each time step, we must choose a random number ϕ from a Normal distribution. $\phi \sim N(0,1)$

This can be done easily in Excel in several ways, we will see a couple now.

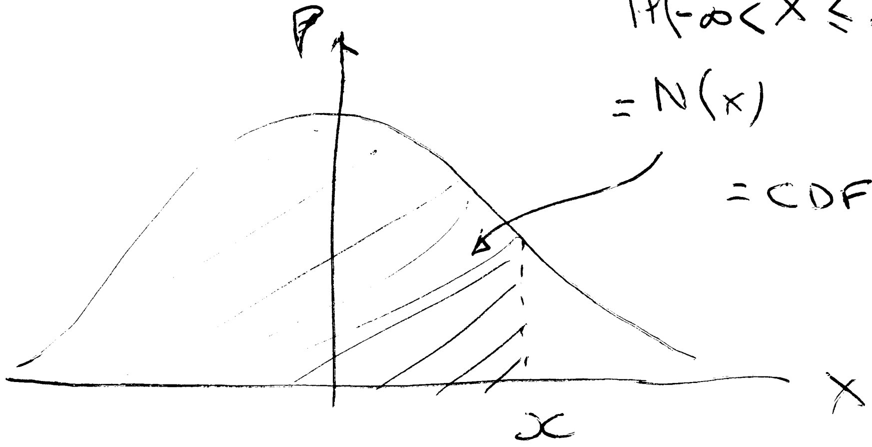
Slow but accurate

The Excel spreadsheet function `RAND()` gives a uniformly-distributed random variable. $\text{RAND} \sim U[0,1]$

This can be used, together with the inverse cumulative distribution function `NORMSINV` to give a genuinely Normally distributed number:

- `NORMSINV(RAND())`.

Why does this work?



$$\mathbb{P}(-\infty < X \leq x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}\phi^2} d\phi$$

$= N(x)$

$= \text{CDF}$

$$p(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx$$

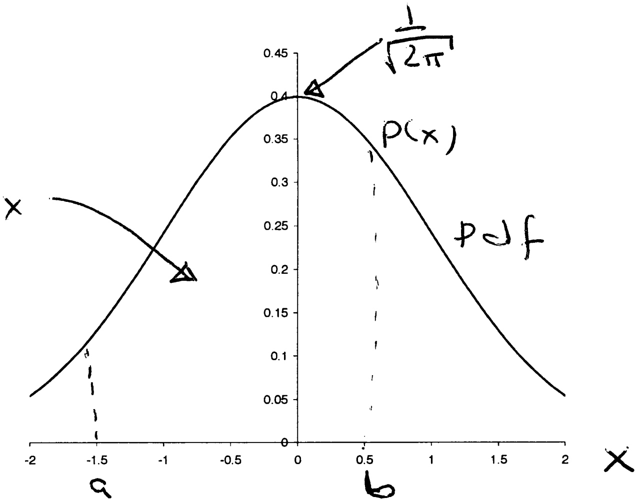
RAND( )

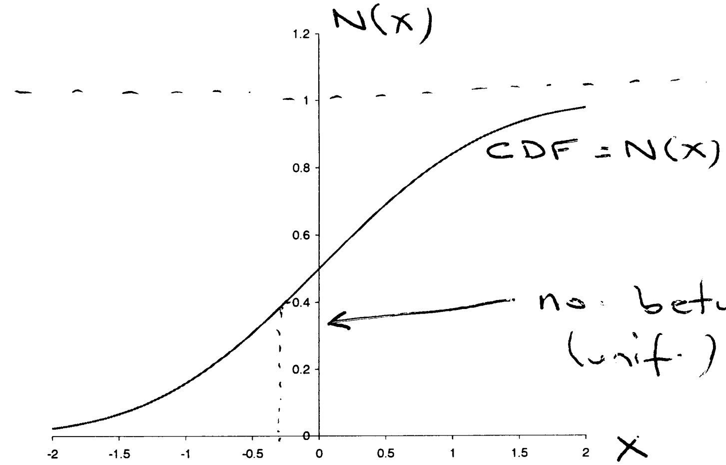
$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}\phi^2} d\phi$$

we want this

The pdf and cdf for the Normal distribution

$$\mathbb{P}[a \le X \le b] = \int_a^b p(x)\,dx$$





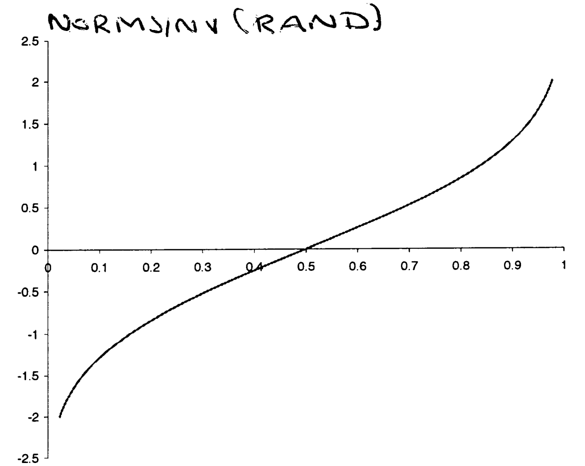
no. between 0 & 1 (unif.)

$\mathbb{P}[X \le -0.3] \approx 0.4$

we want this no. ϕ

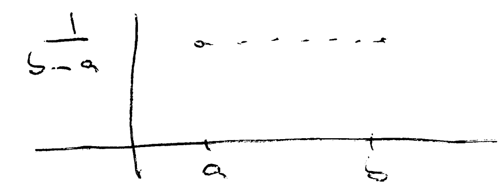
The inverse cumulative distribution function

Any random no. between 0 and 1 gives a $\phi \sim N(0,1)$



Uniform Dist^n

$$p(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$



RAND( ) : $p(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$

$$\mathbb{E}[RAND] = \mathbb{E}[x] = \int_0^1 x\, p(x)\, dx = \frac{1}{2}x^2\Big|_0^1 = \frac{1}{2} = \mu$$

$$\mathbb{V}(RAND) = \mathbb{E}[x^2] - \mathbb{E}^2[x] = \int_0^1 x^2 p(x)\, dx - \mu^2 = \frac{x^3}{3}\Big|_0^1 - \frac{1}{4} = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

We have put $\Psi = \sum_{i=1}^{12} RAND() - 6$

What is $\mathbb{E}\left[\sum_{i=1}^{12} RAND()\right] = \sum_{i=1}^{12} \mathbb{E}[RAND] = \sum_{i=1}^{12} \frac{1}{2} = 12 \cdot \frac{1}{2} = 6$

So substract off 6 which guarantees $\mathbb{E}[\Psi] = 0$ (want $\phi \sim N(0,1)$)

Now $\mathbb{V}[\Psi] = \mathbb{V}\left[\sum_{i=1}^{12} RAND()\right] - \mathbb{V}[6] = \sum_{i=1}^{12} \mathbb{V}(RAND] - 0$

$$= \sum_{i=1}^{12} \frac{1}{12} = 12 \cdot \frac{1}{12} = 1$$

Fast but inaccurate   Inverting integrals – very slow

An approximation to a Normal variable that is fast in a spreadsheet, and quite accurate, is simply to add up twelve random variables drawn from a uniform distribution over zero to one, and subtract six:

- $$\Psi = \left(\sum_{i=1}^{12} \mathtt{RAND}() \right) - 6. \longrightarrow \phi \sim N(0,1)$$

(see across)

$\mathbb{E}[\Psi] = 0$

$\mathbb{V}[\Psi] = 1$

Why 12?

Any 'large' number will do. The larger the number, the closer the end result will be to being normal, but the slower it is.

Why subtract off 6?

The random number must have a mean of zero.

And the standard deviation?

Must be 1.

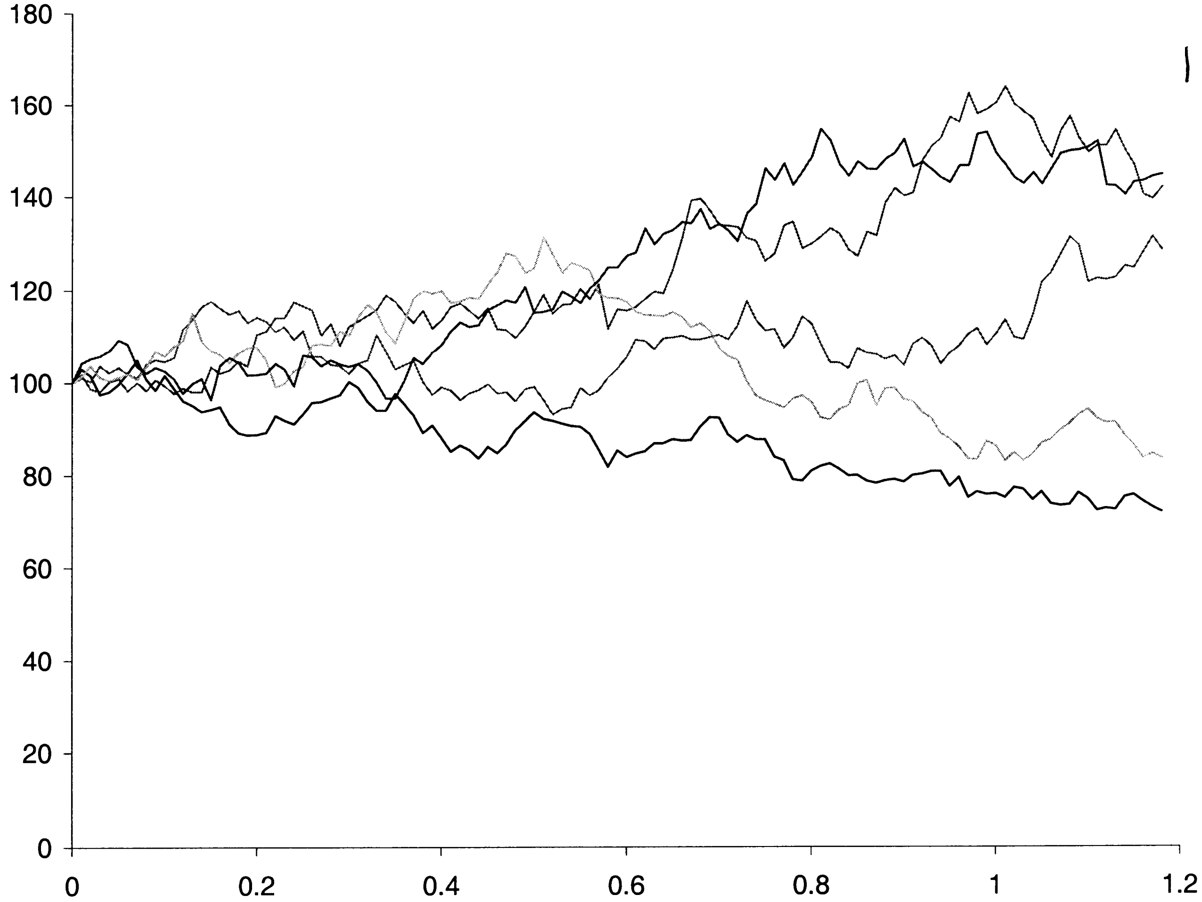
	A	B	C	D	E	F	G
1	**Asset**	**100**		**Time**	**Asset**		
2	**Drift**	**0.15**		0	100		
3	**Volatility**	**0.25**		0.01	96.10692		
4	**Timestep**	**0.01**		0.02	96.99647		
5				0.03	94.76352		
6				0.04	91.46698		
7				0.05	88.83325		
8				0.06	88.42727		
9				0.07	90.62882		
10				0.08	88.80545		
11							
12							
13				0.11	84.93865		

=D4+$B$4

S_i μ δt σ $\sqrt{\delta t}$ ϕ

=E7*(1+$B$2*$B$4+$B$3*SQRT($B$4)*(RAND()+RAND()+RAND()+RAND()
+RAND()+RAND()+RAND()+RAND()+RAND()+RAND()+RAND()+RAND()-6))

Use NORMSINV $\equiv \phi \sim N(0,1)$



Idea behind MC. Pricing

Simulating other random walks

This method is not restricted to the lognormal random walk.

Later in the course we will be modeling interest rates as stochastic differential equations.

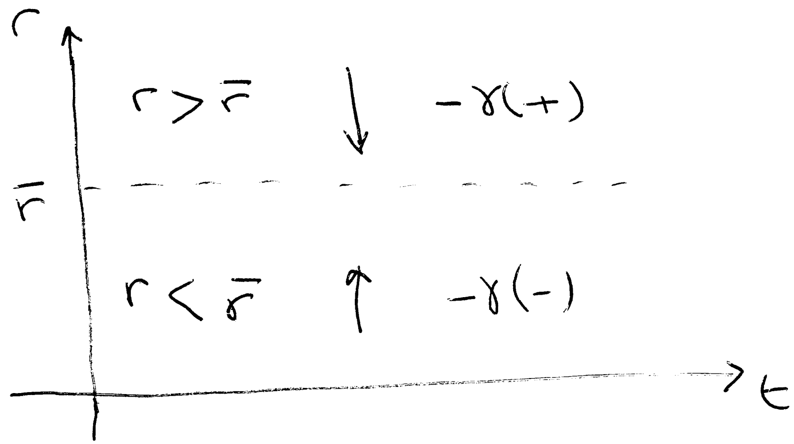
The following is a stochastic differential equation model for an interest rate, that goes by the name of an **Ornstein-Uhlenbeck process** (an example of a mean-reverting random walk), or when used in an interest rate context the **Vasicek model**:

$$dr = \gamma \left(\overline{r} - r \right) dt + \sigma dX.$$

In discrete time we can approximate this by

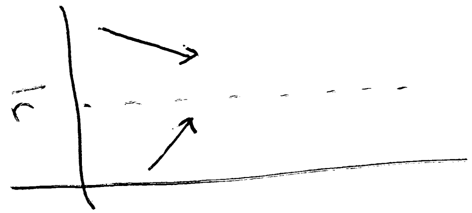
$$r_{i+1} = r_i + \gamma \left(\overline{r} - r_i \right) dt + \sigma \phi \, \delta t^{1/2}.$$

Dynamics of Vasicek: $dr = -\gamma(r-\bar{r})dt + \sigma dX$

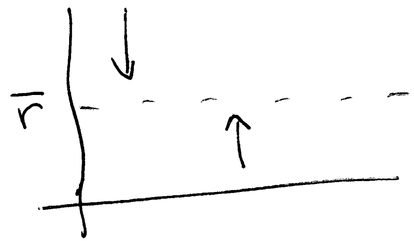


γ has dimensions of $1/\text{Time}$. It controls the speed with which the process returns to the mean $\bar{r}$.. Exponential decay

Small γ



large γ



To get an understanding look at dr in absence of diffusion

$\therefore \quad dr = -\gamma(r-\bar{r})dt$

$$\int \frac{dr}{r-\bar{r}} = -\gamma \int dt$$

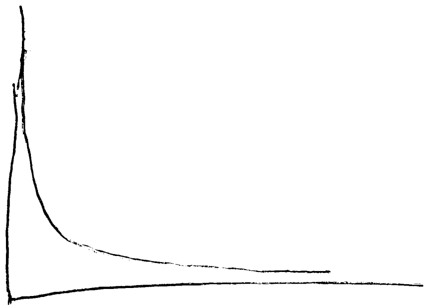
$$\ln|r-\bar{r}| = -\gamma t + c$$

$$r = \bar{r} + Ke^{-\gamma t}$$

γ controls rate of exp. decay

Small γ

large γ



In this example γ is high

Producing correlated random numbers

We will often want to simulate paths of correlated random walks.

We may want to examine the statistical properties of a portfolio of stocks, or value a convertible bond under the assumption of random asset price and random interest rates.

Suppose we wish to simulate asset prices with stoch int rates

$$dS = rS\,dt + \sigma S\,dX_1 \qquad dX_1 = \phi_1\sqrt{dt}$$

$$dr = -\gamma(r-\bar{r})\,dt + \sigma\,dX_2 \qquad dX_2 = \phi_2\sqrt{dt}$$

$$\mathbb{E}[\phi_1\phi_2] = \rho \qquad \mathbb{E}[dX_1\,dX_2] = \rho\,dt$$

So want correlated random variables

Example:

Assets S_1 and S_2 both follow lognormal random walks with correlation ρ.

In continuous time we write

$$dS_1 = \mu_1 S_1 \, dt + \sigma_1 S_1 \, dX_1,$$

$$dS_2 = \mu_2 S_2 \, dt + \sigma_2 S_2 \, dX_2,$$

with

$$E[dX_1 \, dX_2] = \rho \, dt.$$

In discrete time these become

$$S_{1_{i+1}} - S_{1_i} = S_{1_i} \left(\mu_1 \, \delta t + \sigma_1 \phi_1 \, \delta t^{1/2} \right)$$

and

$$S_{2_{i+1}} - S_{2_i} = S_{2_i} \left(\mu_2 \, \delta t + \sigma_2 \phi_2 \, \delta t^{1/2} \right)$$

with

$$E[\phi_1 \, \phi_2] = \rho.$$

$$\phi_2^2 = (\alpha \varepsilon_1 + \beta \varepsilon_2)^2 = \alpha^2 \varepsilon_1^2 + \beta^2 \varepsilon_2^2 + 2\alpha\beta\, \varepsilon_1 \varepsilon_2$$

$$\therefore \quad \mathbb{E}[\phi_2^2] = \alpha^2 \mathbb{E}[\varepsilon_1^2] + \beta^2 \mathbb{E}[\varepsilon_2^2] + 2\alpha\beta\, \mathbb{E}[\varepsilon_1 \varepsilon_2]$$

$$= \alpha^2 + \beta^2 + 2\alpha\beta \cdot 0 = 1$$

$$\Rightarrow \quad \alpha^2 + \beta^2 = 1 \qquad \therefore \quad \beta = \sqrt{1-\alpha^2}$$

$$\mathbb{E}[\phi_1 \phi_2] = \mathbb{E}[\varepsilon_1(\alpha \varepsilon_1 + \beta \varepsilon_2)] = \mathbb{E}[\alpha \varepsilon_1^2 + \alpha\beta\, \varepsilon_1 \varepsilon_2] = \rho$$

$$\alpha \mathbb{E}[\varepsilon_1^2] + \alpha\beta\, \mathbb{E}[\varepsilon_1 \varepsilon_2] = \rho \longrightarrow \alpha = \rho$$

$$\therefore \quad \beta = \sqrt{1-\rho^2}$$

$$\phi_1 = \varepsilon_1$$

$$\phi_2 = \rho\, \varepsilon_1 + \sqrt{1-\rho^2}\, \varepsilon_2$$

Q: How can we choose a ϕ_1 and a ϕ_2 which are both Normally distributed, both have mean zero and standard deviation of one, and with a correlation of ρ between them?

A: This can be done in two steps, first pick two *uncorrelated* Normally distributed random variables, and then combine them.

$\varepsilon_1, \varepsilon_2 \sim N(0,1)$ $\quad \mathbb{E}[\varepsilon_1 \varepsilon_2] = 0$

Want to produce $\phi_1, \phi_2 \sim N(0,1)$ with $\mathbb{E}[\phi_1 \phi_2] = \rho$

So set $\phi_1 = \varepsilon_1$ and l.c. for ϕ_2; $\phi_2 = \alpha \varepsilon_1 + \beta \varepsilon_2$

We know $\mathbb{E}[\phi_2^2] = 1$ $\quad (\because \mathbb{V}[\phi_2^2] = 1)$

$\mathbb{E}[\phi_1 \phi_2] = \rho$

Step 1: Choose uncorrelated ϵ_1 and ϵ_2, both Normally distributed with zero means and standard deviations of one.

Step 2: Convert these independent Normal numbers into correlated Normals by taking a linear combination.

$$\phi_1 = \epsilon_1$$

$$\phi_2 = \rho\,\epsilon_1 + \sqrt{1-\rho^2}\,\epsilon_2.$$

Check:

$$E[\phi_1^2] = 1,$$

$$E\left[\phi_2^2\right] = E\left[\rho^2\epsilon_1^2 + 2\rho\sqrt{1-\rho^2}\epsilon_1\epsilon_2 + (1-\rho^2)\epsilon_2^2\right]$$

$$= \rho^2 + 0 + (1-\rho^2) = 1,$$

and

$$E\left[\phi_1\phi_2\right] = E\left[\rho\epsilon_1^2 + \sqrt{1-\rho^2}\epsilon_1\epsilon_2\right] = \rho.$$

And Normality?

Weighted sums of Normally distributed numbers are themselves Normally distributed!

If $X_i \sim N(\mu_i, \sigma_i^2)$ for $i = 1, \ldots, n$ then

$$\sum_{i=1}^{n} w_i X_i \sim N\left(\underbrace{\sum_{i=1}^{n} w_i \mu_i}_{\text{mean}}, \overbrace{\sum_{i=1}^{n} w_i^2 \sigma_i^2}^{\text{Variance}}\right).$$

The result extends to weighted sums

$$w_1 X_1 + w_2 X_2 + \cdots + w_n X_n \sim N\left(w_1 \mu_1 + \cdots + w_n \mu_n, \; w_1^2 \sigma_1^2 + \cdots + w_n^2 \sigma_n^2\right)$$

Summary

Please take away the following important ideas

- With the right tool (Itô's lemma) you can examine functions of stochastic variables

- Partial differential equations can be used for finding probability density functions for arbitrary random walks

- Simulating random walks can be very easy indeed