CQF Exercise 5.2 Solution

1. (a) The default intensity is the probability that given this firm survive at time t and then default instantaneously.

   (b)

$$\begin{aligned}
P(t,T) * P(T,T+s) &= \Pr(\tau > T|\tau > t) * \Pr(\tau > T+s|\tau > T) \\
&= \frac{\Pr(\tau > T) * \Pr(\tau > T+s)}{\Pr(\tau > t) * \Pr(\tau > T)} \\
&= P(t,T+s)
\end{aligned}$$

   (c) if $s \to 0$, then $P(T,T+s) = 1 - \mu_T\, s$.

$$\begin{aligned}
P(t,T+s) &= P(t,T)(1-\mu_T\, s), \\
\frac{P(t,T+s) - P(t,T)}{s\, P(t,T)} &= -\mu_T, \\
\frac{\partial \log P(t,x)}{\partial x} &= -\mu_x.
\end{aligned}$$

   Integrate both side from t to T and using initial condition $P(t,t) = 1$, come up with

$$\log P(t,T) - \log P(t,t) = \log P(t,T) = -\int_t^T \mu_x dx$$

   Hence

$$P(t,T) = \exp\left\{-\int_t^T \mu_x dx\right\}.$$

   If μ is constant, the distribution of default waiting time is exponential distribution with intensity μ.

(d) By definition the PDF of $D(t,T)$ can be written as

$$\begin{aligned}\frac{\partial D(t,T)}{\partial T} &= \lim_{h\to 0^+}\frac{D(t,T+h)-D(t,T)}{h}\\ &= \lim_{h\to 0^+}\frac{P(t,T)-P(t,T+h)}{h}\\ &= \lim_{h\to 0^+}\frac{P(t,T)\left(1-P(T,T+h)\right)}{h}\\ &= \lim_{h\to 0^+}\frac{P(t,T)D(T,T+h)}{h}\\ &= P(t,T)\mu_T\end{aligned}$$

Integrate above from t to T end up with

$$D(t,T)=\int_t^T P(t,x)\mu_x\,dx.$$

2.

$$\begin{aligned}d\Pi &= dV-\Delta Z\\ &= (\frac{\partial V}{\partial t}+\frac{1}{2}w^2\frac{\partial^2 V}{\partial r^2})dt+\frac{\partial V}{\partial r}dr-\Delta\left((\frac{\partial Z}{\partial t}+\frac{1}{2}w^2\frac{\partial^2 Z}{\partial r^2})dt+\frac{\partial Z}{\partial r}dr\right)\\ &= \left(\frac{\partial V}{\partial t}+\frac{1}{2}w^2\frac{\partial^2 V}{\partial r^2}-\Delta(\frac{\partial Z}{\partial t}+\frac{1}{2}w^2\frac{\partial^2 Z}{\partial r^2})\right)dt+(\frac{\partial V}{\partial r}-\Delta\frac{\partial Z}{\partial r})dr\end{aligned}$$

Choose $\Delta=\frac{\partial V}{\partial r}/\frac{\partial Z}{\partial r}$ to eliminate risk. The value of hedging portfolio will suddenly jump $-V$ if default occurs with probability μdt, to sum up

$$d\Pi=\left(\frac{\partial V}{\partial t}+\frac{1}{2}w^2\frac{\partial^2 V}{\partial r^2}-\Delta(\frac{\partial Z}{\partial t}+\frac{1}{2}w^2\frac{\partial^2 Z}{\partial r^2})\right)dt-\mu V\,dt$$

Set $d\Pi=r\Pi dt$, leads to

$$\frac{\partial V}{\partial t}+\frac{1}{2}w^2\frac{\partial^2 V}{\partial r^2}-(r+p)V=\frac{\partial V}{\partial r}/\frac{\partial Z}{\partial r}(\frac{\partial Z}{\partial t}+\frac{1}{2}w^2\frac{\partial^2 Z}{\partial r^2}-rZ).$$

Which is

$$\frac{\frac{\partial V}{\partial t}+\frac{1}{2}w^2\frac{\partial^2 V}{\partial r^2}-(r+p)V}{\frac{\partial V}{\partial r}}=\frac{\frac{\partial Z}{\partial t}+\frac{1}{2}w^2\frac{\partial^2 Z}{\partial r^2}-rZ}{\frac{\partial V}{\partial r}}.$$

The only way this equation holds is that both sides are independent of V and Z(similar argument in interest rate model), and suppose both sides equal to a function $a(r,t) = w(r,t)\lambda(r,t) - u(r,t)$.Thus

$$\frac{\partial V}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 V}{\partial r^2} + (u - \lambda w)\frac{\partial V}{\partial r} - (r+\mu)V = 0.$$

3. (a) Denote the transition matrix over a short time period dt by $\mathbf{P}$, where

$$\mathbf{P} = \begin{pmatrix} 1-\mu dt & \mu dt \\ 0 & 1 \end{pmatrix}.$$

By definition the intensity matrix can be written as

$$\mathbf{Q}\, dt = \mathbf{I} - \mathbf{P}.$$

Thus

$$\mathbf{Q} = \begin{pmatrix} \mu & -\mu \\ 0 & 0 \end{pmatrix}.$$

(b) The system of ordinary linear DE to be solved are

$$\begin{aligned} \frac{\partial V}{dt} &= (r+\mu)V - \mu V_1 \\ \frac{\partial V_1}{dt} &= rV_1 \end{aligned}$$

with final condition $V(T) = 1$ and $V_1(T) = \theta$. One can first solve the 2nd ordinary differential equation for V_1 and then plug it into the 1st one and solve for V.

$$\begin{aligned} d\log V_1(t) &= rdt \\ \log V_1(T) - \log V_1(t) &= r(T-t) \end{aligned}$$

Hence

$$V_1(t) = \theta e^{-r(T-t)}.$$

Plug above into 1st DE to have

$$\frac{\partial V}{dt} - (r+\mu)V = -\mu\theta e^{-r(T-t)}$$

Integration factor is $e^{-(r+\mu)}$. Multiply both sides by IF come up with

$$\frac{d}{dt}\left(V(t)e^{-(r+\mu)t}\right) = -\mu\theta e^{-rT-\mu t}.$$

Integrate both side from $t$to$T$,

$$\begin{aligned} V(T)e^{-(r+\mu)T} - V(t)e^{-(r+\mu)t} &= -\mu\theta e^{-rT}\int_0^T e^{-\mu s}ds \\ e^{-(r+\mu)T} - V(t)e^{-(r+\mu)t} &= \theta\left(e^{-T(r+\mu)} - e^{-(rT+\mu t)}\right) \\ V(t) &= (1-\theta)e^{-(r+\mu)(T-t)} + \theta e^{-r(T-t)} \end{aligned}$$

(c) The bond pricing formula can be written as

$$\begin{aligned} BondPrice &= LGD \times \text{riskfree bond price} \times (1-PD) \\ &+ Recovery \times \text{riskfree bond price} \end{aligned}$$

By simply words, for every unit, the bond pays out θ(recovery) no matter what happen, plus if default doesn't occur it pays the rest of $1-\theta$.