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A BASIC MATHEMATICS

① $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$

a) $f(x) = \frac{x}{(-(i-x)(i+x))^2} = \frac{x}{(i-x)^2(1+x)^2} =$

$= \frac{x}{i^2(1-\frac{x}{i})^2(1+\frac{x}{i})^2 i^2} = \frac{x}{(1-(\frac{x}{i}))^2(1+(\frac{x}{i}))^2}$

$x \to \frac{x}{i}$ - using

we take

$\frac{1}{1-(\frac{x}{i})} = 1 + \frac{x}{i} + (\frac{x}{i})^2 + (\frac{x}{i})^3 + \dots =$

$= 1 - ix - x^2 + ix^3 + \dots$

$x \to -\frac{x}{i}$

we derive:

$\frac{1}{1+(\frac{x}{i})} = 1 - \frac{x}{i} + (-\frac{x}{i})^2 + (-\frac{x}{i})^3 + \dots =$

$= 1 + ix - x^2 - ix^3 + \dots$

$f(x) = x \cdot (1 - ix - x^2 + ix^3 + \dots)^2 (1 + ix - x^2 - ix^3 + \dots) =$

$f(x) = x - 2x^3 - x^5$

b) $\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots$

If $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$

THEN

$$\int \frac{1}{1-x}\,dx = \int (1 + x + x^2 + x^3 + \dots)\,dx$$

$$-\log(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \dots \quad \times -1$$

$$+\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

(2) $\dfrac{a + ib}{c + id}$

$$a + ib = \sqrt{a^2+b^2}\, e^{i\theta_1}$$

$$\theta_1 = \arctan \left| \frac{b}{a} \right|$$

$$c + id = \sqrt{c^2+d^2}\, e^{i\theta_2}$$

$$\theta_2 = \arctan \frac{d}{c}$$

$$\frac{a+ib}{c+id} = \sqrt{\frac{a^2+b^2}{c^2+d^2}} \cdot e^{i(\theta_1 + \theta_2)}$$

AND $\sqrt{\dfrac{a^2+b^2}{c^2+d^2}}$, $\theta_1 - \theta_2$

③ $D = \begin{vmatrix} y-z & z-x & x-y \\ z-x & x-y & y-z \\ x-y & y-z & z-x \end{vmatrix}$

WE KEEP FIRST AND SECOND ROWS AS THEY ARE:
WE MULTIPLY THE FIRST ROW BY 1 AND WE ADD IT IN THE THIRD.
WE MULTIPLY THE SECOND ROW BY 1 AND WE ADD IT IN THE THIRD.

SO:

$$D = \begin{vmatrix} y-z & z-x & x-y \\ z-x & x-y & y-z \\ (x-y+y-z+z-x) & (y-z+z-x+x-y) & (z-x+x-y+z-x) \end{vmatrix} =$$

$$= \begin{vmatrix} y-z & z-x & x-y \\ z-x & x-y & y-z \\ 0 & 0 & 0 \end{vmatrix} \Rightarrow D = 0$$

4/1

4

$$A = \begin{vmatrix} 3 & 3 & 3 \\ 3 & -1 & 1 \\ 3 & 1 & -1 \end{vmatrix}$$

$$\det(A - \lambda I) = 0 \Rightarrow \begin{vmatrix} 3-\lambda & 3 & 3 \\ 3 & -1-\lambda & 1 \\ 3 & 1 & -1-\lambda \end{vmatrix} = 0$$

columns

$\{-C_3 + C_2 = C_2\}$

$$\det = \begin{vmatrix} 3-\lambda & 0 & 3 \\ 3 & -1 & 1 \\ 3 & 2+\lambda & -1-\lambda \end{vmatrix} = 0 \Rightarrow$$

$$\Rightarrow (2+\lambda)\begin{vmatrix} 3-\lambda & 0 & 3 \\ 3 & -1 & 1 \\ 3 & 1 & -1-\lambda \end{vmatrix} = 0 \Rightarrow$$

Rows

$\{R_3 + R_2 \to R_2\}$

$$\Rightarrow (2+\lambda)\begin{vmatrix} 3-\lambda & 0 & 3 \\ 6 & 0 & -\lambda \\ 3 & 1 & -1-\lambda \end{vmatrix} = 0 \Rightarrow$$

$$\Rightarrow (2+\lambda)(-1)\cdot 1 \begin{vmatrix} 3-\lambda & 3 \\ 6 & -\lambda \end{vmatrix} = 0 \Rightarrow$$

$$(2+\lambda)\big((3-\lambda)(-\lambda) - 6\cdot 3\big) = 0$$

$$(\lambda+2)(\lambda^2 - 3\lambda - 18) = 0 \Rightarrow (\lambda+2)(\lambda+3)(\lambda-6) = 0$$

4/2

$$\left.\begin{array}{l} \lambda_1 = -2 \\ \lambda_2 = -3 \\ \lambda_3 = +6 \end{array}\right\} \text{ eigenvalues}$$

for even eigenvalue

$$u = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad Au = \lambda u$$

$$(A - \lambda I)\, u = 0$$

$$\lambda = \lambda_1 = -2$$

$$\begin{pmatrix} 5 & 3 & 3 \\ 3 & 1 & 1 \\ 3 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 5 & 3 & 3 & 0 \\ 3 & 1 & 1 & 0 \\ 3 & 1 & 1 & 0 \end{array}\right) \sim \begin{array}{l} \frac{1}{5} R_1 \to R_1 \\ -R_2 + R_3 \to R_3 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & \frac{3}{5} & \frac{3}{5} & 0 \\ 3 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \sim -3R_1 + R_2 \to R_2$$

$$\left(\begin{array}{ccc|c} 1 & \frac{3}{5} & \frac{3}{5} & 0 \\ 0 & -\frac{4}{5} & -\frac{4}{5} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \sim -\frac{5}{4} R_2 \to R_2$$

4/3

$$\left(\begin{array}{ccc|c} 1 & \frac{3}{5} & \frac{3}{5} & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \sim -\frac{3}{5}R_2 + R_1 \to R_1$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{array}{l} x = 0 \\ y + z = 0 \end{array}$$

A good approach could be $u_1 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$

$\lambda = \lambda_2 = -3$

$$\begin{pmatrix} 6 & 3 & 3 \\ 3 & 2 & 1 \\ 3 & 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 6 & 3 & 3 & 0 \\ 3 & 2 & 1 & 0 \\ 3 & 1 & 2 & 0 \end{array}\right) \sim \frac{1}{6}R_1 \to R_1$$

$$\left(\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 3 & 2 & 1 & 0 \\ 3 & 1 & 2 & 0 \end{array}\right) \sim \begin{array}{l} -3R_1 + R_2 \to R_2 \\ -3R_1 + R_3 \to R_3 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \end{array}\right) \sim 2R_2 \to R_2$$

4/4

$$\left(\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \end{array}\right) \sim \begin{array}{l} -\frac{1}{2}R_2 + R_1 \to R_1 \\ \frac{1}{2}R_1 + R_3 \to R_3 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{array}{l} x + z = 0 \\ y - z = 0 \end{array}$$

a good approach could be $u_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

$\lambda = \lambda_3 = 6$

$$\begin{pmatrix} -3 & 3 & 3 \\ 3 & -7 & 1 \\ 3 & 1 & -7 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} -3 & 3 & 3 & 0 \\ 3 & -7 & 1 & 0 \\ 3 & 1 & -7 & 0 \end{array}\right) \sim -\frac{1}{3}R_1 \to R_1$$

$$\left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 3 & -7 & 1 & 0 \\ 3 & 1 & -7 & 0 \end{array}\right) \sim \begin{array}{l} -\frac{1}{4}R_2 \to R_2 \\ \frac{1}{4}R_3 \to R_3 \end{array} \quad -R_2 + R_3 \to R_3$$

$$\left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \sim R_2 + R_1 \to R_1$$

4/5

$$\left(\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{array}{l} x - 2z = 0 \\ y - z = 0 \end{array}$$

A good approach could be $u_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

IT IS POSSIBLE TO DIAGONALIZE

$A = PDP$

$$P = (u_1\ u_2\ u_3) = \begin{pmatrix} 0 & -1 & 2 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \Rightarrow$$

$$P^{-1} = \begin{pmatrix} 0 & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{6} \end{pmatrix} \qquad D = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

We observe that

$\langle u_1, u_2 \rangle = 0.(-1) + (-1).1 + 1.1 = 0$

$\Rightarrow u_1 \perp u_2$

$\langle u_1, u_3 \rangle = 0.2 + (-1).1 + 1.1 = 0 \Rightarrow$

$\Rightarrow u_1 \perp u_3$

$\langle u_2, u_3 \rangle = (-1).2 + 1.1 + 1.1 = 0 \Rightarrow$

$\Rightarrow u_2 \perp u_3$

1.

B PROBABILITY

1)

a) It holds that

$$\int_0^{\infty} p(x,\lambda)\,dx = 1 \Rightarrow \int_0^{+\infty} A\lambda x e^{-\lambda x^2}\,dx = 1$$

$$-\frac{A}{2}\int e^{-\lambda x^2}(-\lambda x^2)'\,dx = 1 \Rightarrow$$

$$-\frac{A}{2}\, e^{-\lambda x^2}\Big|_0^{+\infty} = 1 \Rightarrow -\frac{A}{2}(0-1) = 1 \Rightarrow$$

$$\frac{A}{2} = 1 \Rightarrow A = 2$$

Consequently, $p(x,\lambda) = 2\lambda x e^{-\lambda x^2}, x>0, \lambda>0$

b) it is

$$E(X^{2n}) = \int_0^{\infty} x^{2n} p(x,\lambda)\,dx = \int_0^{\infty} x^{2n} 2\lambda x e^{-\lambda x^2}\,dx =$$

$$= -\int_0^{+\infty} x^{2n}\left(e^{-\lambda x^2}\right)'\,dx = -x^{2n} e^{-\lambda x^2}\Big|_0^{+\infty} +$$

$$+\int_0^{+\infty} 2n x^{2n-1} e^{-\lambda x^2}\,dx = -0+0+\frac{n}{\lambda}\int_0^{\infty} x^{2(n-1)} 2\lambda x e^{-\lambda x^2}\,dx$$

$$= \frac{n}{\lambda} E(X^{2(n-1)})$$ It is $E(X^0) = E(1) = 1$

$$E(X^{2n}) = \frac{n}{\lambda}E(X^{2(n-1)}) = \frac{n(n-1)}{\lambda^2}E(X^{2(n-2)}) = \frac{n(n-1)\cdots 2}{\lambda^n}E(X^0) =$$

$$= \frac{n!}{\lambda^n},\quad n = 0,1,\ldots$$

2) USUALLY, IT IS RAND() = $U(0,1)$

If there is a random variable

$X_i \sim U(0,1)$, then $E(X_i) = \frac{1}{2}$, $V(X_i) = \frac{1}{12}$

$i = 1, \dots 12$

Consequently from the Central Limited Theorem, it is

$$\sum_{i=1}^{12} X_i \sim N\left(12 \cdot \frac{1}{2} = 6,\ 12 \cdot \frac{1}{12} = 1\right) \text{ approximately}$$

That means $$\frac{\sum_{i=1}^{12} RAND - 6}{1} \sim N(0,1)$$

In the same way RAND(N) $\equiv N \cdot U(0,1)$

If $X_i \sim RAND(N)$, then $E(X_i) = \frac{N}{2}$ and

$V(X_i) = \frac{N^2}{12}$, $i = 1, \dots 12$

Consequently from the Central Limited theorem, it is

$$\sum_{i=1}^{12} X_i \sim N\left(12 \frac{N}{2} = 6N,\ 12 \cdot \frac{N^2}{12} = N^2\right)$$

This leads to the conclusion that

$$\sum_{i=1}^{12} \frac{RAND(N) - 6N}{N} \sim N(0,1)$$

C ~~S~~TOCHASTIC CALCULUS

(1) $df = A\,dt + B\,dX$

a) $f(X) = \ln |X^n| = n \ln |X|$

$f(x) = n \ln|x| = \ln|x^n|$

$f'(x) = \frac{1}{x^n}$ $\qquad f'' = -\frac{1}{x^{2n}}$

$df = \frac{1}{2} f''(x)\,dt + f'(x)\,dX$

$df = \frac{1}{x^n}\,dX - \underbrace{\frac{1}{2}\cdot\frac{1}{x^{2n}}}_{\text{growth rate}}\,dt$

b) $f(x) = e^{nx}$

$f'(x) = n e^{nx}$ $\qquad f'' = nn e^{nx}$

$df = \underbrace{\frac{1}{2} n^2 e^{nx}}_{\text{growth rate}}\,dt + n e^{nx}\,dt$

c) $f(x) = a^X \quad a>1$

$f'(x) = a^x \ln a$ $\qquad f'(x) = a^x \ln a$

$df = \underbrace{\frac{1}{2} a^x \ln a}_{\text{growth rate}}\,dt + a^x \ln a\,dX$

C STOCHASTIC CALCULUS

2/

$$\frac{\partial p}{\partial t'} = \frac{1}{2}\frac{\partial^2}{\partial y^2}\left(B(y',t')^2 p\right) - \frac{\partial}{\partial y'}\left(A(y',t')p\right)$$

If we have random walk with
$$dr = -ar\,dt + b\,dX$$

then the forward equation becomes

$$\frac{\partial p}{\partial t'} = \frac{1}{2}\frac{\partial^2}{\partial r'^2}\left(b^2 p\right) - \frac{\partial}{\partial r'}\left(a p\right)$$

The solution of this representing a spot rate starting at $r' = r$ at $t' = t$ is

$$p(r,t,r',t') = \frac{1}{b r'\sqrt{2\pi(t'-t)}}\, e^{-\frac{\left[\log\frac{r}{r'} + \left(a-\frac{b}{2}\right)(t'-t)\right]^2}{2b^2(t'-t)}}$$

If there is steady-state distribution $p_\infty(y')$ then it satisfies the ordinary differential equation

$$\frac{1}{2}\frac{d^2}{dy'^2}\left(b^2 p_\infty\right) - \frac{d}{dy'}\left(a p_\infty\right) = 0 \implies$$

$$\frac{1}{2} b^2 \frac{d^2 p_\infty}{dr'^2} - a\frac{d}{dr'} p_\infty = 0$$

Integrate both side

$$\frac{1}{2}\sigma\int\frac{d}{dr^2}(p) = a\int\frac{d}{dr}(p)$$

$$\frac{1}{2}\sigma\, p'_\infty = a\, p_\infty + C$$

when $a \to \infty$ $\left.\begin{array}{l} p_\infty \to 0 \\ p'_\infty \to 0 \end{array}\right\} \Rightarrow C = 0$

$$\frac{dp}{dr} = -\frac{2a}{\sigma^2}p \Rightarrow \frac{dp}{dr} = -\frac{2a}{\sigma^2}\int d\tau$$

$$\therefore\ p_\infty = \frac{1}{\sigma}\sqrt{\frac{a}{\pi}}\, e^{-\frac{2a\tau^2}{2\sigma^2}}$$

C STOCHASTIC CALCULUS

3/ a)

$$G = e^{(t + a e^{X_t})}$$

We use ITÔ lemma on function $G(X(t), t)$

$$dG_{(t)} = \frac{\partial G}{\partial X} dx + \left(\frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial X^2} \right) dt$$

$$G_{(t)} = e^{(t + a e^{X_t})} \qquad | \ln$$

$$\ln G = \ln e^t + \ln e^{a e^{X(t)}} = t + a e^{X(t)}$$

$$a e^{X(t)} = \ln G - t$$

$$dG_{(t)} = a e^{X(t)} dx + \left(1 + \frac{1}{2} a e^{X(t)} + \frac{1}{2} (a e^{X_t})^2 \right) dt$$

$$\frac{dG_t}{G} = a e^{X_t} dX + \left(1 + \frac{1}{2} a e^{X_t} + \frac{1}{2} (a e^{X_t})^2 \right) dt$$

$$dG_t = G [\ln(G - t)] dx + G \left[1 + \frac{1}{2} \ln(G - t) + \frac{1}{2} \ln(G - t)^2 \right] dt$$

D Further Mathematical Methods

2) 1

$$xy' = y + \sqrt{x^2+y^2}$$

$$y = vx \qquad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$x\left(v + x\frac{dv}{dx}\right) = vx + \sqrt{x^2+v^2x^2}$$

$$xv + x^2\frac{dv}{dx} = vx + x\sqrt{1+v^2}$$

$$xv + \frac{x^2\,dv}{dx} = x\left(v + \sqrt{1+v^2}\right)$$

$$x^2\frac{dv}{dx} = x\left(v+\sqrt{1+v^2}\right) - xv$$

$$\frac{x^2\,dv}{dx} = x\left(\left(v+\sqrt{1+v^2}\right) - v\right)$$

$$\frac{x^2}{x} \frac{dv}{dx} = \sqrt{1+v^2}$$

$$\int \frac{dv}{\sqrt{1+v^2}} = \int \frac{dx}{x}$$

$$\ln|v + \sqrt{1+v^2} = \ln|x| + \ln C$$

$$v = \frac{y}{x}$$

$$\ln\left|\frac{v + \sqrt{1+v^2}}{x}\right| = \ln c_1 \cdot c_1$$

$$\frac{v + \sqrt{1+v^2}}{x} = c$$

$$\frac{v}{x} + \frac{1}{x}\sqrt{1+v^2} = c$$

$$\frac{1}{x}\left(v + \sqrt{1+v^2}\right) = c$$

$$\frac{1}{x}\left(\frac{y}{x} + \sqrt{1+\frac{y^2}{x^2}}\right) = c$$

$$\frac{1}{x}\left(\frac{y}{x} + \frac{\sqrt{x^2+y^2}}{x}\right) = c$$

$$\frac{1}{x}\,\frac{1}{x}\left(y + \sqrt{x^2+y^2}\right) = c$$

$$\frac{1}{x^2}\left(y + \sqrt{x^2+y^2}\right) = c$$

2/11

$$y' = \frac{2x+9y-20}{6x+2y-10}$$

$$x = X + h \; ; \; y = Y + k$$

$$\frac{dy}{dx} = \frac{dY}{dX}$$

$$\frac{dY}{dX} = \frac{2(X+h)+9(Y+k)-20}{6(X+h)+2(Y+k)-10}$$

$$\frac{dY}{dX} = \frac{2X+9Y+\overbrace{(2h+9k-20)}^{=0}}{6X+2Y+\underbrace{(6h+2k-10)}_{=0}}$$

$$2h+9k-20=0 \quad / \cdot (-3)$$
$$6h+2k-10=0$$
$$-6h-27k+60+6h+2k-10=0$$
$$-25k+50=0$$
$$k = \frac{-50}{-25} = 2 \qquad k=2$$
$$2h+18-20=0$$
$$2h=2$$
$$h = \frac{2}{2} = 1 \qquad h=1$$

$$x = X+h = X+1 \Rightarrow \boxed{X = x-1}$$
$$y = Y+k = Y+2 \Rightarrow \boxed{Y = y-2}$$

$$Y = uX$$

$$u + X\frac{du}{dX} = \frac{2X+9uX}{6X+2uX} = \frac{\cancel{X}(2+9u)}{\cancel{X}(6+2u)}$$

$$X\frac{du}{dX} = \frac{2+9u}{6+2u} - u = \frac{2+9u-u(6+2u)}{6+2u}$$

$$X\frac{du}{dX} = \frac{2+9u-6u-2u^2}{6+2u} = \frac{-2u^2+3u+2}{6+2u}$$

2/11

$$\frac{2v+6}{-2v^2+3v+2}\,dv=\frac{1}{x}\,dx$$

$$v_{1/2}=\frac{-3\pm\sqrt{9+4\cdot2\cdot2}}{-2\cdot2}\quad 9+16=\sqrt{25}$$

$$\frac{-3+5}{-4}=\frac{+2}{-4}=-\frac{1}{2}$$

$$\frac{-3-5}{-4}=\frac{-8}{-4}=2$$

$$v_1=-\frac{1}{2}\qquad v_2=2$$

$$-2(v-2)\left(v+\frac{1}{2}\right)=-2v^2+3v+2$$

$$\frac{2v+6}{-2v^2+3v+2}=\frac{2v+6}{-2(v-2)\left(v+\frac{1}{2}\right)}$$

$$\frac{2v+6}{-2(v-2)\left(v+\frac{1}{2}\right)}=-\frac{1}{2}\left(\frac{A}{v-2}+\frac{B}{\left(v+\frac{1}{2}\right)}\right)$$

$$2v+6=A\left(v+\frac{1}{2}\right)+B(v-2)$$

$$v=-\frac{1}{2}\quad B=\frac{2\cdot-\frac{1}{2}+6}{-\frac{1}{2}-\frac{4}{2}}=5\cdot-\frac{2}{5}=-2$$

$$v=2\quad A=\frac{2\cdot2+6}{\frac{4}{2}+\frac{1}{2}}=10\cdot\frac{2}{5}=4$$

$$-\frac{1}{2}\left(\frac{4}{v-2}-\frac{2}{v+\frac{1}{2}}\right)=\frac{dx}{x}$$

$$\frac{1\,dv}{v+\frac{1}{2}}-\frac{2\,dv}{v-2}=\frac{dx}{x}$$

$$\int\frac{1}{2v+1}\,dv-2\int\frac{1}{v-2}\,dv=\int\frac{1}{x}\,dx$$

$2/_{11}$

$$\ln|2u+1| - 2\ln|u-2| = \ln x + \ln c$$

$$\ln \frac{2u+1}{(u-2)^2} = \ln x.c \qquad |^{\,}e$$

$$\frac{2u+1}{(u-2)^2} = x.c \qquad u = \frac{y}{x}$$

$$\frac{\frac{2y+x}{x}}{\frac{(y-2x)^2}{x^2}} = x.c$$

$$\frac{(2y+x)}{\cancel{x}} \cdot \frac{x^{\cancel{2}}}{(y-2x)^2} = x.c \quad \Big| \cdot \frac{1}{x}$$

$$\frac{2y+x}{(y-2x)^2} = c \qquad \begin{array}{l} Y = y-2 \\ X = x-1 \end{array}$$

$$\frac{\left(2(y-2)+(x-1)\right)}{\left(y-2-2(x-1)\right)^2} = c$$

$$\frac{(2y-4+x-1)}{(y-\cancel{2}-2x+\cancel{2})^2} = c$$

$$\frac{2y+x-5}{(y-2x)^2} = c$$

2/111

$$y' = \frac{3x-4y-2}{3x-4y-3} \qquad a + b\frac{dy}{dx} = \frac{dv}{dx}$$

$$\frac{dy}{dx} = \frac{3x-4y-2}{3x-4y-3} \qquad v = 3x-4y$$

$$3 - 4\frac{dy}{dx} = \frac{dv}{dx}$$

(" -1)

$$-4\frac{dy}{dx} = \frac{dv}{dx} + 3$$

$$\frac{dy}{dx} = -\frac{1}{4}\left(\frac{dv}{dx} + 3\right)$$

$$-\frac{1}{4}\left(\frac{dv}{dx}+3\right) = \frac{v-2}{v-3}$$

$$-\left(\frac{dv}{dx}+3\right) = \frac{4v-8}{v-3}$$

$$\frac{dv}{dx} = \frac{4v-8}{v-3} - 3 = \frac{4v-8-3(v-3)}{v-3}$$

$$\frac{dv}{dx} = \frac{4v-8-3v+9}{v-3} = \frac{v+1}{v-3}$$

$$\frac{v-3}{v+1}dv = dx$$

$$-\frac{dv}{dx} = \frac{4v-8}{v-3} - 3$$

$$-\frac{dv}{dx} = \frac{4v-8-3v+9}{v-3} = \frac{v+1}{v-3}$$

$$-\left(\frac{v-3}{v+1}\right) dv = d\cancel{x} \quad \int -1$$

$$\frac{v-3}{v+1}\, dv = -\,dx$$

$$\frac{v+1-4}{v+1}\, dv = -\,dx$$

$$\left(1 - \frac{4}{v+1}\right) dv = -\,dx$$

$$\int dv - 4\int \frac{1}{v+1}\, dv = -\int dx$$

$$v - 4\ln(v+1) = -x + C$$

$$3x - 4y - 4\ln(3x - 4y + 1) = -x + C$$

$$4x - 4y - 4\ln(3x - 4y + 1) = C \qquad \frac{1}{4}$$

$$x - y - \ln(3x - 4y + 1) = C \qquad \int -1$$

$$\ln(3x - 4y + 1) - x + y = C_1$$

2/IV

$$2y' + y = (x-1)\, y^3 \qquad | \cdot \frac{1}{y^3}$$

$$\frac{2}{y^3} y' + \frac{\cancel{y}}{y^{\cancel{3}2}} = \frac{(x-1)\cancel{y^3}}{\cancel{y^3}}$$

$$\frac{2}{y^3} y' + \frac{1}{y^2} = (x-1) \quad | \cdot -1 \qquad u = \frac{1}{y^2}$$

$$u' = \frac{y^2 \cdot 1' - 2y \cdot 1}{y^4} \cdot y' \qquad (1' = 0)$$

$$-\frac{2}{y^3} y' - \frac{1}{y^2} = -(x-1)$$

$$u' = -\frac{2\cancel{y}}{y^{\cancel{4}3}} \cdot y' = -\frac{2}{y^3} y'$$

$$u' - u = 1 - x$$

$$P(x) = -1 \qquad Q(x) = 1-x$$

$$R(x) = e^{-\int 1\, dx} = e^{-x}$$

$$e^{-x}(u' - u) = (1-x)e^{-x}$$

$$\upsilon = x \qquad u' = e^{-x}$$

$$u' = 1 \qquad u = -e^{-x}$$

$$\frac{d}{dx}(e^{-x}u) = e^{-x} - xe^{-x}$$

$$\int \frac{d}{dx}(e^{-x}u) = \int e^{-x}\,dx - \int \overset{u}{x}\, \overset{u'}{e^{-x}}\,dx$$

$$u e^{-x} = -e^{-x} - \left(-xe^{-x} + \int -e^{-x} \cdot 1\, dx\right) + c$$

$$u e^{-x} = -e^{-x} - \left(-xe^{-x} + \int e^{-x}\,dx\right) + c$$

$$u e^{-x} = -e^{-x} - \left(-xe^{-x} - e^{-x}\right) + c$$

$$u e^{-x} = \cancel{-e^{-x}} + xe^{-x} \cancel{+e^{-x}} + c$$

$$u e^{-x} = xe^{-x} + c \qquad e^{x}$$

$$u = x + Ce^{x} \implies \frac{1}{y^2} = x + Ce^{x}$$

2/V

$$(x+3y-1)\,dx + (3x-2y+4)\,dy = 0$$

$$P(x,y)\,dx + Q(x,y)\,dy = 0$$

$$\frac{\partial P(x,y)}{\partial y} = \frac{\partial P(x+3y-1)}{\partial y} = 3$$

$$\frac{\partial Q(x,y)}{\partial x} = \frac{\partial Q(3x-2y+4)}{\partial x} = 3$$

$$\frac{\partial P(x,y)}{\partial y} = \frac{\partial Q(x,y)}{\partial x} = 3$$

$$\int P(x,y)\,dx + \int\left[Q(x,y) - \int \frac{\partial P(x,y)}{\partial y}\,dx\right]dy = C$$

$$\int(x+3y-1)\,dx + \int\left[3x-2y+4-\int 3\,dx\right]dy = C$$

$$\frac{x^2}{2}+3yx-x+\int\left[3x-2y+4-(3x+C_1)\right]dy = C$$

$$\frac{x^2}{2}+3yx-x+\int\left[\cancel{3x}-2y+4\,\cancel{-3x}-C_1)\right]dy = C_3$$

$$\frac{x^2}{2}+3yx-x+\int\left[4-2y-C_1\right]dy = C_3$$

$$\frac{x^2}{2}+3yx-x-\int\left[2y\,\cancel{+}4+C_1\right]dy = C_3$$

$$\frac{x^2}{2}+3yx-x-\left(\frac{2}{2}y^2-4y+C_1y\right) = C_3$$

$$\frac{x^2}{2}+3yx-x\,\cancel{-}\,y^2+4y-C_1y = C_3$$

0

3/1 D FUTHER MATHEMATICAL METHODS

We find $|z| = a + ib$

$z = \sqrt{3} + i$ AND

$$\rho = |z| = \sqrt{(\sqrt{3})^2 + 1^2} = \sqrt{4} = 2$$

Thus we have

$$\cos\theta = \frac{x}{\rho} = \frac{\sqrt{3}}{2}$$

$$\sin\theta = \frac{y}{\rho} = \frac{1}{2}$$

From above we conclude that:

$$\left.\begin{aligned} \cos\theta &= \cos\frac{\pi}{6} \\ \sin\theta &= \sin\frac{\pi}{6} \end{aligned}\right\} \quad \theta = \frac{\pi}{6} + 2k\pi$$

for $k \in \mathbb{Z}$ because $0 \le \theta < 2\pi$

we have: $\theta = \frac{\pi}{6}$

which leads: $z = \sqrt{3} + i = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

By applying the Moivre theorem

$$z^{25} = (\sqrt{3} + i)^{25} = 2^{25}\left(\cos\frac{25\pi}{6} + i\sin\frac{25\pi}{6}\right) =$$

$$= 2^{25}\left[\cos\left(4\pi + \frac{\pi}{6}\right) + i\sin\left(4\pi + \frac{\pi}{6}\right)\right] =$$

$$= 2^{25}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) =$$

$$= 2^{25}\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right) = 2^{24}(\sqrt{3} + i)$$

3/11

According the Moivre theorem we have;

(1) $(\cos\theta + i\sin\theta)^5 = \cos(5\theta) + i\sin(5\theta)$

By developing the first part of (1) we get after some calculations;

$$(\cos\theta + i\sin\theta)^5 = \cos^5\theta + 5\cos^4\theta(i\sin\theta) +$$
$$+ 20\cos^3\theta(i\sin\theta)^2 + 10.\cos^2\theta.(i\sin\theta)^3 +$$
$$+ 5\cos\theta(i\sin\theta)^4 + (i\sin\theta)^5 =$$
$$= \cos^5\theta + 5i.\cos^4\sin\theta - 10\cos^3\theta\sin^2\theta - 10i\cos^2\theta\sin^3\theta +$$
$$+ 5\cos\theta\sin^4\theta + i\sin^5\theta =$$
$$= (\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta) +$$
$$+ i(5\cos^4\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta)$$

That mean that from (1)

$$\cos(5\theta) + i\sin(5\theta) = (\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta) +$$
$$+ \cdot i(5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta)$$

Thus, setting equal the real and imaginary parts:

$$\cos(5\theta) = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta =$$
$$= \cos^5\theta - 10\cos^3\theta(1-\cos^2\theta) + 5\cos\theta(1-\cos^2\theta)^2 =$$
$$= \ldots = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$$

$$\sin(5\theta) = 5\cos^4\theta\sin\theta - 20\cos^2\theta\sin^3\theta + 5\sin^5\theta =$$
$$= 5(1-\sin^2\theta)^2\sin\theta - 10(1-\sin^2\theta)\sin^3\theta + \sin^5\theta =$$
$$= \ldots = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$$

3/111

BECAUSE $\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$

THEN $2^5\cos^5\theta = \left(z + \frac{1}{z}\right)^5$

THE SAME FOR SIN

$$\sin\theta = \frac{1}{2}\left(z - \frac{1}{z}\right) \Rightarrow$$

$$2^5\sin^5\theta = \left(z - \frac{1}{z}\right)^5$$

$$2^5\cos^5\theta = z^5 + 5z^4\frac{1}{z} + 10z^3\left(\frac{1}{z}\right)^2 + 10z^2\left(\frac{1}{z}\right)^3 + 5z\left(\frac{1}{z^4}\right) + \frac{1}{z^5} =$$

$$= z^5 + 5z^3 + 10z + 10\frac{1}{z} + \frac{5}{z^3} + \frac{1}{z^5} \Rightarrow$$

$$2^5\cos^5\theta = \left(z^5 + \frac{1}{z^5}\right) + 5\left(z^3 + \frac{1}{z^3}\right) + 10\left(z + \frac{1}{z}\right)$$

$$2^5\sin^5\theta = \ldots = z^5 - 5z^3 + 10z - \frac{10}{z} + \frac{5}{z^3} - \frac{1}{z^5} \Rightarrow$$

$$2^5\sin^5\theta = \left(z^5 - \frac{1}{z^5}\right) - 5\left(z^3 - \frac{1}{z^3}\right) + 10\left(z - \frac{1}{z}\right)$$

$$2^5\cos^5\theta = 2\cdot\frac{1}{2}\left(z^5 + \frac{1}{z^5}\right) + 5\cdot2\cdot\frac{1}{2}\left(z^3 + \frac{1}{z^3}\right) + 10\cdot2\frac{1}{2}\left(z + \frac{1}{z}\right)$$

$$2^5\cos^5\theta = 2\cos 5\theta + 10\cos 3\theta + 20\cos\theta$$

$$2^5\sin^5\theta = 2\cdot\frac{1}{2}\left(z^5 - \frac{1}{z^5}\right) - 5\cdot2\frac{1}{2}\left(z^3 - \frac{1}{z^3}\right) + 10\cdot2\frac{1}{2}\left(z - \frac{1}{z}\right)$$

$$2^5\cos^5\theta = 2\sin 5\theta - 10\sin 3\theta + 20\sin\theta$$

$$\boxed{\begin{aligned}\cos^5\theta &= \frac{1}{16}\left(\cos 5\theta + 5\cos 3\theta + 10\cos\theta\right)\\ \sin^5\theta &= \frac{1}{16}\left(\sin 5\theta - 5\sin 3\theta + 10\sin\theta\right)\end{aligned}}$$

3/IV

If $1 = \cos\theta + i\sin\theta$ then

$$x = \sqrt[6]{1} = \cos\frac{2k\pi}{6} + i\sin\left(\frac{2k\pi}{6}\right)$$

$k = 0, 1, \ldots 5$

$$x_0 = \cos\theta + i\sin\theta = 1$$

$$x_1 = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \frac{1}{2} + i\frac{\sqrt{3}}{2} = \frac{1}{2}(1 + i\sqrt{3})$$

$$x_2 = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2} = \frac{1}{2}(-1 + i\sqrt{5})$$

$$x_3 = \cos\pi + i\sin\pi = -1 + i.0 = -1$$

$$x_4 = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = -\frac{1}{2} - i\frac{\sqrt{3}}{2} = -\frac{1}{2}(1 + \sqrt{5})$$

$$x_5 = \cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3} = \frac{1}{2} - i\frac{\sqrt{3}}{2} = \frac{1}{2}(1 - i\sqrt{5})$$

The equation $z^4 = \alpha$ with $\alpha \in \mathbb{C}$ and $\alpha \neq 0$ has roots which are given from:

$$z_k = \sqrt[n]{\rho}\left[\cos\frac{\theta + 2k\pi}{n} + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right]$$

$k = 0, 1, 2, 3 \ldots, n-1$ ρ, θ modulus, arguments of α

D Further Mathematical Methods

4/1

$z = x + iy$ where $\cos z = 4$

$\Downarrow$

$$\frac{e^{iz} + e^{-iz}}{2} = 4 \Rightarrow$$

$$e^{iz} + \frac{1}{e^{iz}} = 8 \Rightarrow e^{2iz} + 1 = 8e^{iz} \Rightarrow$$

$$e^{2iz} - 8e^{iz} + 1 = 0, \quad (\Delta = 64 - 4 = 60)$$

$$e^{iz} = \frac{8 \pm \sqrt{60}}{2} \Rightarrow e^{iz} = 4 \pm \sqrt{15} \quad (1)$$

If in equation (1) we substitud z with $z = x + iy$

$$e^{i(x+iy)} = 4 \pm \sqrt{15} \Rightarrow e^{ix} \cdot e^{-y} = 4 \pm \sqrt{15}$$

$$e^{-y}(\cos x + i \sin x) = 4 \pm \sqrt{15} \Rightarrow$$

$$e^{-y}\cos x + i e^{-y} \sin x = 4 \pm \sqrt{15} + 0i \Rightarrow$$

$$e^{-y} \sin x = 0 \quad (2)$$
$$e^{-y} \cos x = 4 \pm \sqrt{15} \quad (3)$$

equation (2) gives $\sin x = 0 \Rightarrow x = k\pi, k \in \mathbb{Z}$
equation (3) for $x = k\pi$ gives
$e^{-y} \cos(k\pi) = 4 \pm \sqrt{15}$ (4) but
because $4 \pm \sqrt{15} > 0$ and $e^{-y} > 0$
it must be $\cos(k\pi) > 0$
where k even, thus equation (4) gives
$e^{-y} = 4 \pm \sqrt{15} \Rightarrow y = -\ln|4 \pm \sqrt{15}|$

$$\boxed{z = x + iy = k\pi - i\ln|4 \pm \sqrt{15}|}$$