CQF Module 1.4 Exercises

Stochastic Differential Equations and Itô's Lemma

Throughout this problem sheet, you may assume that X is a Brownian Motion (Weiner Process) and dX is its increment. $X(0) = 0$.

1. Use Itô's lemma to show that

$$d\cos(X(t)) = \alpha\cos(X(t))\,dt + \beta\sin(X(t))\,dX$$

&

$$d\sin(X(t)) = \alpha\sin(X(t))\,dt - \beta\cos(X(t))\,dX$$

and determine the constants α & β.

2. Consider the stochastic differential equation

$$dG(t) = a(G,t)\,dt + b(G,t)\,dX.$$

Find $a(G,t)$ and $b(G,t)$ where

(a) $G(t) = X^2(t)$

(b) $G(t) = 1 + t + \exp(X(t))$

(c) $G(t) = f(t)X(t)$, where f is a bounded and continuous function.

3. The change in a share price $S(t)$ satisfies

$$dS = A(S,t)\,dX + B(S,t)\,dt,$$

for some functions A and B. If $f = f(S,t)$, then Itô's lemma gives the following stochastic differential equation

$$df = \left(\frac{\partial f}{\partial t} + B\frac{\partial f}{\partial S} + \frac{1}{2}A^2\frac{\partial^2 f}{\partial S^2}\right)dt + A\frac{\partial f}{\partial S}dX.$$

Can A and B be chosen so that a function $g = g(S)$ has a change which has zero drift, but non-zero diffusion? State any appropriate conditions.

4. Show that $F = \arcsin(2aX(t) + \sin F_0)$ is a solution of the stochastic differential equation

$$dF = 2a^2(\tan F)\left(\sec^2 F\right)dt + 2a(\sec F)\,dX,$$

where $F_0 = F(0)$, $X(0) = 0$ and a is a constant. **Hint: you may find the following useful**

$$\frac{d}{dx}\arcsin ux = \frac{u}{\sqrt{1 - u^2x^2}}$$

5. Show that

$$\int_0^t X(\tau)\left(1 - e^{-X^2(\tau)}\right) dX(\tau) = \overline{F}(X(t)) + \int_0^t G(X(t))\, d\tau$$

where the functions $\overline{F}$ and G should be determined.