CQF Module 1.1 Exercises: Random Behaviour of Assets & Maths Revision

1. A stock price has an expected return of 12% per annum (with continuous compounding) and a volatility of 20% per annum. Changes in the share price satisfy $dS = \mu S dt + \sigma S dX$. What is the distribution of the price increase for the share movement?
   The price is Normally distributed . The mean is given by

   $$\mu = \text{expected return p.a } \times \text{time step} = 0.12 \times \delta t$$

   and variance is given by

   $$\sigma^2 = (\text{volatility p.a.})^2 \times \text{time step} = 0.2^2 \times \delta t$$

   If (e.g.) we take time step to be weekly then $\delta t = 1/52$, which gives

   $$\begin{aligned} \mu &= 0.0023 \\ \sigma^2 &= 0.0008 \end{aligned}$$

2. Using a different set of stock prices, repeat the Excel based computational exercise conducted in class.

3. Differentiate $y = (x^x)^x$.

   $$\ln y = x^2 \ln x$$

   now implicit differentiation and product rule

   $$\begin{aligned} \frac{1}{y}\frac{dy}{dx} &= x^2 \frac{1}{x} + 2x \ln x = x + 2x \ln x \\ y' &= (x^x)^x (x + 2x \ln x) . \end{aligned}$$

4. Express the complex number $\dfrac{7-2i}{5+3i}$ in the form $a + ib$. Hence find its modulus.

   $$\frac{7-2i}{5+3i} \times \frac{5-3i}{5-3i} = \frac{(7-2i)(5-3i)}{25+9} = \frac{35-21i-10i-6}{34} = \frac{29-31i}{24} = \frac{29}{34} - \frac{31}{34} i$$

   The modulus of $z = a + ib$ is given by

   $$|z| = \sqrt{a^2 + b^2}$$

   hence

   $$\left| \frac{29}{34} - \frac{31}{34} i \right| = \sqrt{\left(\frac{29}{34}\right)^2 + \left(\frac{31}{34}\right)^2} = \frac{1}{34}\sqrt{(29)^2 + (31)^2} \approx 1.25$$

5. Calculate the following indefinite integrals

$$\int \frac{x+2}{x^2+4x-5}dx$$

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}dx$$

$$\int xe^{2x}dx$$

$$I = \int \frac{x+2}{x^2+4x-5}dx$$

Use partial fractions here

$$\frac{x+2}{x^2+4x-5} = \frac{x+2}{(x+5)(x-1)} \equiv \frac{A}{(x+5)} + \frac{B}{(x-1)} \longrightarrow A = \tfrac{1}{2}, \quad B = \tfrac{1}{2}$$

$$I = \tfrac{1}{2}\int \left(\frac{1}{(x+5)} + \frac{1}{(x-1)}\right) dx = \tfrac{1}{2}\left(\ln|x+5| + \ln|x-1|\right) + A = \tfrac{1}{2}\ln|x+5|\,|x-1| + A$$

Alternatively you can also spot that this integrand is of type $\dfrac{f'(x)}{f(x)}$, i.e.

$$\frac{x+2}{x^2+4x-5} \equiv \frac{1}{2}\frac{2(x+2)}{x^2+4x-5}$$

so the solution is simply

$$I = \tfrac{1}{2}\ln\left|x^2+4x-5\right| + A$$

$$I = \int \frac{e^{\sqrt{x}}}{\sqrt{x}}dx = \int e^{\sqrt{x}}x^{-1/2}dx$$

Use integration by substitution, put $u = x^{1/2} \longrightarrow 2du = x^{-1/2}dx$

$$I = 2\int e^u du = 2e^u + C = 2e^{\sqrt{x}} + C$$

$$I = \int xe^{2x}dx$$

To be done by parts

$$\int vdu = uv - \int udv$$

$v = x \Rightarrow dv = dx$ and $du = e^{2x}dx \Rightarrow u = \frac{1}{2}e^{2x}$

$$\begin{aligned} I &= \frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x}dx + c \\ &= \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + c. \end{aligned}$$

6. Use the transformation $x = e^t$ to convert the Cauchy-Euler equation

$$x^2 y'' - 4xy' + 6y = 3x^4$$

to a constant coefficient differential equation and then solve this to obtain a solution of the original equation. If a and b are arbitrary constants, show that this solution is

$$y = ax^2 + bx^3 + \tfrac{3}{2}x^4.$$

Under this transformation, the derivative terms becomes

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{1}{x}\frac{dy}{dt}$$

$$\begin{aligned} \frac{d^{\,2}y}{dx^2} &= \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dt}\right) = \frac{1}{x}\frac{d}{dx}\frac{dy}{dt} - \frac{1}{x^2}\frac{dy}{dt} \\ &= \frac{1}{x}\frac{d}{dt}\frac{dt}{dx}\frac{dy}{dt} - \frac{1}{x^2}\frac{dy}{dt} = \frac{1}{x^2}\frac{d^2y}{dt^2} - \frac{1}{x^2}\frac{dy}{dt} \end{aligned}$$

$\therefore$ the Euler equation becomes

$$\begin{aligned} x^2\left(\frac{1}{x^2}\frac{d^2y}{dt^2} - \frac{1}{x^2}\frac{dy}{dt}\right) - 4x\left(\frac{1}{x}\frac{dy}{dt}\right) + 6y &= 3e^{4t} \longrightarrow \\ y'' - 5y' + 6y &= 3e^{4t} \end{aligned}$$

where $y = y(t)$. Solve the homogeneous equation $y'' - 5y' + 6y = 0$. The A.E is

$$\lambda^2 - 5\lambda + 6 = 0 \Rightarrow (\lambda - 2)(\lambda - 3) = 0 \Rightarrow \lambda = 2,\ 3$$

the C.F is $y_c = ae^{2t} + be^{3t}$. For the Particular Integral (P.I) look for a solution of the form $y_p = Ce^{4t}$, hence substituting in the ode we have

$$Ce^{4t}[16 - 20 + 6] = 3e^{4t} \longrightarrow C = \tfrac{3}{2}$$

and the general solution becomes

$$y(t) = ae^{2t} + be^{3t} + \tfrac{3}{2}e^{4t}.$$

Now to get $y(x)$ we use $x = e^t$, i.e. $t = \ln x$ and hence the solution to the original Cauchy problem becomes

$$y = ax^2 + bx^3 + \tfrac{3}{2}x^4.$$

7. Consider the real vector space $\mathbb{R}^4$ and vectors

$$\mathbf{u}_1 = (-1, 1, 1, 1), \mathbf{u}_2 = (1, -1, 1, 1), \mathbf{u}_3 = (1, 1, -1, 1), \mathbf{u}_4 = (1, 1, 1, -1).$$

Test these for linear independence. Repeat this test for the set

$$\mathbf{v}_1 = (-1, -1, 1, 1)\,, \mathbf{v}_2 = (1, -1, -1, 1)\,, \mathbf{v}_3 = (1, 1, -1, -1)\,, \mathbf{v}_4 = (-1, 1, 1, -1)\,.$$

Put

$$\lambda_1 \mathbf{u}_1 + \lambda_2 \mathbf{u}_2 + \lambda_3 \mathbf{u}_3 + \lambda_4 \mathbf{u}_4 = \mathbf{0}$$

then we can express this in augmented matrix form

$$\begin{pmatrix} -1 & 1 & 1 & 1 & 0 \\ 1 & -1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 1 & 0 \\ 1 & 1 & 1 & -1 & 0 \end{pmatrix}$$

and row reduction gives us the only choice for the scalars as

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \mathbf{0}$$

so the set $\mathbf{u}_i$ is a linearly independent one. For the other set we can repeat the method. However we can write

$$1.\mathbf{v}_1 + 0.\mathbf{v}_2 + 1.\mathbf{v}_3 + 0.\mathbf{v}_4 = \mathbf{0}$$

so not all scalars are zero hence $\mathbf{v}_i$ is a linearly dependent set.