Module 2.3 Exercises   Value at Risk

1. Using real data, make a spreadsheet that calculates an exponentially weighted moving average volatility (with arbitrary input weighting factor).

Solution: The file containing asset prices can be used - follow the method used in

class by PW, based on the technique given in the lecture notes.

2. Assuming a Normal distribution, what percentage of returns are outside the negative two standard deviations from the mean? What is the mean of returns falling in this tail?

Solution: PDF for $N\left(\mu,\, \sigma^2\right)$ is

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-(x-\mu)^2/2\sigma^2\right)$$

The percentage of returns outside -2 standard deviations is then

$$\int_{-\infty}^{\mu-2} f(x)dx = \int_{-\infty}^{\mu-2} \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-(x-\mu)^2/2\sigma^2\right) = N(-2/\sigma).$$

The mean of returns given that they fall in this tail is

$$\begin{aligned}
\frac{\int_{-\infty}^{\mu-2} xf(x)dx}{\int_{-\infty}^{\mu-2} f(x)dx} &= \frac{1}{N(-2/\sigma)} \int_{-\infty}^{\mu-2} \frac{x}{\sqrt{2\pi}\,\sigma} \exp\left(-(x-\mu)^2/2\sigma^2\right) dx \\
&= \frac{1}{N(-2/\sigma)} \left(-\frac{\sigma}{\sqrt{2\pi}} e^{-2/\sigma^2} + \mu N(-2/\sigma) \right) \\
&= -\frac{\sigma}{\sqrt{2\pi}\,N(-2/\sigma)} e^{-2/\sigma^2} + \mu.
\end{aligned}$$

As an example, for a standard Normal distribution, we find that the percentage of returns outside -2 standard deviations is

$$\int_{-\infty}^{-2} \frac{1}{\sqrt{2\pi}\,\sigma} \exp(-x^2/2) = N(-2) = 0.0228,$$

i.e. 2.28%. The mean of returns falling in this tail is

$$\frac{1}{N(-2)} \int_{-\infty}^{-2} \frac{x}{\sqrt{2\pi}} e^{-x^2/2} dx = \frac{-1}{\sqrt{2\pi}\,N(-2)} e^{-2} = -2.3684.$$

3. What criticisms of Value at Risk (discussed in the lecture) can you think of? Consider distributions other than Normal ans discontinuous paths.

Solution: The method given in the lecture for calculating VaR is

$$-\alpha(1-c)\delta t^{1/2} \sqrt{\sum_{j=1}^{M} \sum_{i=1}^{M} \Delta_i \Delta_j \sigma_i \sigma_j \rho_{ij}}\,,$$

where we used a delta approximation to value derivatives.
To derive this formula, we make assumptions which can be criticised:
(a) We assume the asset follows the random walk $dS = \mu S dt + \sigma S dX,$ where dX is drawn from a normal distribution. If we study asset price data, we see that the distribution for dX has fatter tails and a higher peak than a normal distribution.

(b) Another assumption is that all relevant volatilities and correlations are known - which in practice are notoriously difficult to calculate.

(c) We assume that we can approximate the sensitivity of the portfolio to the change in the underlying of an option using some linear model (for instance, dependent on the delta of the option).

The second method is the use of simulations. The two main approaches are Monte Carlo, based on the generation of Normally distributed random numbers, and boot-strapping, using actual asset price movements taken from historical data.

Criticisms of the MC approach include that we again assume the asset follows a lognormal random walk and that the method can be very slow since we must run tens of thousands of simulations and for each one, we may have to solve a multi-factor pde.

The main criticism of the bootstrapping approach is that the method requires a large amount of historical data and this may entail including data that corresponds to completely different economic circumstances.

4. Consider a portfolio consisting of a £5 million position in XYZ, and suppose the daily volatility of XYZ is 1% (approximately 16% per year). Calculate the standard deviation of the change in portfolio value per 10 days.
Assuming the change is normally distributed, what is the 10 day 99% VaR?

$\boxed{\text{Solution:}}$ Π_{XYZ} =£5 million $\quad \sigma_{\text{XYZ}}^{\text{daily}} = 1\% \Rightarrow \sigma_{\text{XYZ}}^{\text{year}} = \sqrt{252} \times 1\% = 16\%$
Now standard deviation of change in Π_{XYZ} per 10 days is given by

$$\sigma_{\Pi} = \sigma_{\text{XYZ}}^{\text{daily}} \sqrt{\text{days}}\, \Pi_{\text{XYZ}} = 1\% \times \sqrt{10} \times 5 \text{ million} = 158113.88$$

If this change in portfolio value is Normally distributed then the 10 day 99% VaR is

$$2.33\sigma_{\Pi} = 368405.34$$

5. Consider a position consisting of a £100,000 investment in asset X and another similar investment in asset Y. Assume that the daily volatilities of both assets are 1% and that the correlation between their returns is 0.3. What is the 5-day 99% value at risk for the portfolio?

$\boxed{\text{Solution:}}$ The standard deviation of the daily change in the investment in each asset is $\sigma_X = \sigma_Y$ = £1000 (which is 1%). The variance of the portfolio's daily change is

$$\sigma_\Pi^2 = (1000)^2 + (1000)^2 + 2(0.3)(1000)(1000) = 2.6 \times 10^6.$$

The standard deviation of the portfolio's daily change is the square root of this, which is $\sigma_\Pi = \pounds 1612.45$.

The standard deviation of the 5-day change is

$$1612.45 \times \sqrt{5} = \pounds 3605.55$$

From the Normal distribution tables for $N(x)$ we have $N(-2.33) = 0.01$. This implies that 1% of a normal distribution lies more than 2.33 standard deviations below the mean. The 5-day 99% value at risk becomes

$$2.33 \times 3605.55 = \pounds 8401.$$