MAS102 Calculus II

Ordinary Differential Equations: A Brief Summary

(I) Separable first-order

These are equations of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)g(y).$$

To solve, separate x and y by bringing the equation to the form $\frac{dy}{g(y)} = f(x)dx$ and then integrate. Remember that solutions may be lost when dividing through by $g(y)$.

(II) Homogeneous first-order

These equations are of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f\left(\frac{y}{x}\right).$$

To solve, make use of the substitution

$$y = xv, \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{dv}{dx},$$

which reduces the equation to a separable one.

(III) Equations reducible to homogeneous

These equations are of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{ax+by+c}{dx+ey+f}.$$

To solve, substitute

$$x = X + x_0, \qquad y = Y + y_0$$

and find x_0 and y_0 for which

$$\frac{ax+by+c}{dx+ey+f} = \frac{aX+bY}{dX+eY}.$$

This reduces the equation to a homogeneous one. If $ax+by = k(dx+ey)$ for some k (and hence x_0 and y_0 cannot be found) then substitute

$$z = x + y, \qquad \frac{dz}{dx} = 1 + \frac{\mathrm{d}y}{\mathrm{d}x}.$$

This substitution will reduce the equation to a separable one.

(IV) Linear first order

These are equations of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)$$

where P and Q are known functions. To solve, find integrating factor ($R(x)$):

$$R = e^{\int P(x)\,dx}.$$

Multiplying both sides of the equation by this factor reduces it to

$$\frac{d(Ry)}{dx} = QR$$

which can be readily integrated giving

$$y = \frac{1}{R(x)}\int Q(x)R(x)\ dx.$$

(V) Bernoulli

These are equations of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)y^n.$$

To solve, make use of the substitution

$$z = \frac{1}{y^{n-1}}$$

which reduces the equation to a linear equation in terms of z.

(VI) Linear ODEs with constant coefficients

These are equations of the form

$$a_n\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + a_{n-1}\frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + \ldots + a_1\frac{\mathrm{d}y}{\mathrm{d}x} + a_0 y = f(x)$$

where $f(x)$ is a known function. To solve, first solve the homogeneous part

$$a_n\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + a_{n-1}\frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + \ldots + a_1\frac{\mathrm{d}y}{\mathrm{d}x} + a_0 y = 0$$

to find the CF (y_h) and then find a particular solution (y_p) of the whole equation. The general solution is then $y = y_h + y_p$.

Complementary Function (CF)

To find the CF (y_h), find the roots $t_1, t_2, \ldots, t_n$ of the auxiliary equation

$$a_n t^n + a_{n-1} t^{n-1} + \ldots + a_1 t + a_0 = 0.$$

If there are no repeated roots among $t_1, t_2, \ldots t_n$ then

$$y_h = C_1 e^{t_1 x} + C_2 e^{t_2 x} + \ldots + C_n e^{t_n x}.$$

With repeated roots, we have repeated exponentials in the above sum. These have to be replaced by

$$(C_1 + C_2 x + \ldots + C_{r-1} x^{r-1}) e^{tx}$$

where r is the number of times root t is repeated.

Particular Integral

If the RHS $f(x)$ is of the form $f(x) = Q(x)e^{\beta x}$ with $Q(x) = \alpha_s x^s + \alpha_{s-1} x^{s-1} + \ldots \alpha_1 x + \alpha_0$, then a particular solution (y_h) can be found using trial functions. If $t = \beta$ is not a root of the auxiliary equation, try function

$$y_p = (b_s x^s + b_{s-1} x^{s-1} + \ldots b_1 x + b_0) e^{\beta x}$$

and if $t = \beta$ is a root of the auxiliary equation repeated r times, try

$$y_p = x^r (b_s x^s + b_{s-1} x^{s-1} + \ldots b_1 x + b_0) e^{\beta x}.$$

The coefficients $b_0, b_1, \ldots, b_s$ are to be determined by putting the trial function into the equation.

Remember that

$$e^{i\theta x} = \cos(\theta x) + \sin(\theta x).$$

Therefore, if $f(x)$ is of the form $f(x) = Q(x)\cos(\theta x)$ or $f(x) = Q(x)\sin(\theta x)$ then find a particular solution for the equation with $Q(x)e^{i\theta}$ in the RHS using the rule explained above and then take the real or the imaginary part respectively.

(VII) Linear ODEs of Euler type

These are equations of the form

$$a_n x^n \frac{\mathrm{d}^n y}{\mathrm{d}x^n} + a_{n-1} x^{n-1} \frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + \ldots + a_1 x \frac{\mathrm{d}y}{\mathrm{d}x} + a_0 y = f(x)$$

To solve, make use of the substitution

$$x = e^t, \quad \Rightarrow x\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t}, \quad x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{\mathrm{d}^2 y}{\mathrm{d}^2 t} - \frac{\mathrm{d}y}{\mathrm{d}t}, \text{ etc.,}$$

which reduces the equation to a linear ODE with constant coefficients, as in case (V).

(VIII) Simultaneous linear DEs with constant coefficients

To solve two first-order differential equations for two unknown functions x and y simultaneously, either eliminate x and obtain a second-order DE equation for y, or eliminate y and obtain a second-order DE equation for x.

(IX) Series solutions of ODEs

(a) Picard's method for solving $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$ subject to initial condition $y = y_0$ at $x = x_0$

Start with the function $y_0(x) = y_0$ and use successive iterations

$$y_{n+1}(x) = y_0 + \int_{x_0}^{x} f(t, y_n(t))\ dt.$$

(b) Frobenius method

First decide if $x = 0$ is an *ordinary* or a *regular singular* point. And then look for solutions of the types

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad \text{and} \quad y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

respectively.

General remark: When using changes of variables, remember to go back to original variables at the end.