Delta Hedging :

* Create a portfolio $\Pi = V - \Delta S$

shares

⇒ Objective : to eliminate risk .

Π is riskless if its value is unchanged .
whatever terminal state we are in

$$\Pi_u = \Pi_d \iff V_u - \Delta S_u = V_d - \Delta S_d = \overline{\Pi}$$

$$\Rightarrow \Delta = \frac{V_u - V_d}{(u-d) S_0}$$

Since the portfolio is riskless $\Rightarrow$ it must earn the risk-free rate.

$$V_u - \Delta S_u = V_d - \Delta S_d = B\left(V_0 - \Delta S_0\right)$$

$$= \frac{1}{D}\left(V_0 - \Delta S_0\right) \qquad D = \frac{1}{B}$$

$$\boxed{\frac{1}{D} V_0 = \frac{1}{D}\,\frac{V_u - V_d}{u-d} + \frac{uV_d - dV_u}{u-d}}$$

option value at time 0

Risk - Neutral Valuation : $p^* \to$ RN proba.

$\to$ in the RN world $\Rightarrow$ everything priced based on expectation.

$\to$ "On average" the stock price must return the risk-free rate.

$$S_0 = D\,\mathbb{E}^{RN}[S_T]$$

S_0: stock price at time 0

S_T: stock price at time $T = \Delta t$

$$= D\left(p^* u S_u + (1-p^*)\, d\, S_d\right)$$

$$\Rightarrow \boxed{p^* = \frac{\frac{1}{D} - d}{u - d}}$$

$$\boxed{\begin{aligned} V_0 &= D\,\mathbb{E}^{RN}(V_T) \\ &= D\,(p^* V_u + (1-p^*)\,V_d) \end{aligned}}$$

RN valuation equation.

Equation (1) for Δ-hedging

$$\frac{1}{D} V_0 = \frac{1}{D}\,\frac{V_u - V_d}{u-d} + \frac{u V_d - d V_u}{u-d}$$

$$V_0 = D\left[\underbrace{\frac{\frac{1}{D} - d}{u-d}}_{p^*} V_u + \underbrace{\frac{u - \frac{1}{D}}{u-d}}_{1-p^*} V_d\right]$$

$$= D\left(p^* V_u - (1-p^*) V_d\right)$$

$$= D\,\mathbb{E}^{RN}(V_T)$$

RN

Δ - hedging

RN valuation

w/ others as well

- Rewrite the Binomial model in a more "probabilistic" way.
- In the binomial model.

$$\begin{cases} \omega_u = \text{up-move in stock} \\ \omega_d = \text{down-move in stock.} \end{cases}$$

$$\Omega = \{\omega_u, \omega_d\}$$

physical / actual probability measure

$\mathbb{P}$ $\rightarrow$ is uniquely defined by our choice of parameter $p \in (0,1)$.

$$\begin{cases} \mathbb{P}[\omega_u] = p \\ \mathbb{P}[\omega_d] = 1-p = 1 - \mathbb{P}[\omega_u] \end{cases}$$

$\rightarrow$ if you were to take p_1, p_2 $\quad p_1 \neq p_2$

$\mathbb{P}_1[\omega_u] \neq \mathbb{P}_2[\omega_u]$

$\rightarrow$ $\mathbb{P}_1$ and $\mathbb{P}_2$ are naturally equivalent

Slide 17-18 :

-> try to implement the martingale approach (L 3.3) by finding an equivalent martingale measure Q.

-> Q is uniquely defined by the probability of an up-move q:

$$\begin{cases} Q[\omega_u] = q \\ Q[\omega_d] = 1-q \end{cases}$$

How do we find q?

under $\mathbb{Q}$, the discounted stock price DS is a martingale.

$$\Rightarrow \boxed{S_0 = \mathbb{E}^{\mathbb{Q}}[D S_T]}$$

$$\Rightarrow S_0 = q(DS_u) + (1-q)(DS_d)$$

$$\Rightarrow q = \frac{\cancel{S_0}/D - S_d}{S_u - S_d} = \frac{\frac{1}{D} - d}{u - d} = p^*$$

RN measure $\mathbb{P}^*$ is the EMM $\mathbb{Q}$

$$V_0 = \mathbb{E}^{Q}\left[D \max(S_T - E, 0) \right]$$

3.3 Fundamental Asset Pricing Formula

$$= D\left(q^* V_u + (1 - q^*) V_d \right)$$

CRR

instantaneous volatility

$$\begin{cases} u = e^{\sigma\sqrt{\delta t}} \\ d = e^{-\sigma\sqrt{\delta t}} \\ D = e^{-r\sqrt{\delta t}} \end{cases}$$

instantaneous risk-free rate

$$ud = du = 1$$