CQF Exercises 3.1 Black Scholes Model

Throughout this exercise you may use assume (where appropriate) the following results without proof

$$
\begin{aligned}
d_1 &= \frac{\log(S/E) + \left(r - D + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}, \\
d_2 &= \frac{\log(S/E) + \left(r - D - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \quad \text{and} \\
N(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp(-\phi^{\ 2}/2) d\phi
\end{aligned}
$$

where $S \geq 0$ is the spot price, $t \leq T$ is the time, $E > 0$ is the strike, $T > 0$

the expiry date, $r \geq 0$ the interest rate, D is the dividend yield and σ is the volatility of S.

1. The Black–Scholes formula for a European call option $C(S,t)$ is given by

$$C(S,t) = S\exp(-D(T-t))N(d_1) - E\exp(-r(T-t))N(d_2).$$

By differentiating with respect to S and σ show that the delta and vega are given by

$$\Delta = \exp(-D(T-t))N(d_1), \quad \text{and} \quad \upsilon = \sqrt{\frac{T-t}{2\pi}} S\exp(-D(T-t))\exp(-{d_1}^2/2).$$

You may find the following relationship useful:

$$Se^{(-D(T-t))}\exp\left(-\frac{{d_1}^2}{2}\right) = Ee^{(-r(T-t))}\exp\left(-\frac{{d_2}^2}{2}\right)$$

(It is quite messy to prove).

2. The Black–Scholes Equation (BSE) in the presence of a continuous dividend yield D, is

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 V}{\partial S^2}+(r-D)\,S\frac{\partial V}{\partial S}-rV=0.$$

Find all separable solutions of the form $V(S,t)=\Phi(S)\,\Psi(t)$.

3. The Black–Scholes formula for a European call option $C(S,t)$ is

$$C(S,t)=S\exp(-D(T-t))N(d_1)-E\exp(-r(T-t))N(d_2)$$

From this expression, find the Black–Scholes value of the call option in the following limits:

(a) (time tends to expiry) $t\to T^{\,-}$, $\sigma>0$ (*this depends on* S/E);

(b) (volatility tends to zero) $\sigma\to 0^+$, $t<T$; (*this depends on* $S\exp(-D(T-t))/E\exp(-r(T-t))$)

(c) (volatility tends to infinity) $\sigma\to\infty$, $t<T$;

(d) (expiry tends to infinity) $T\ \to\infty$ (*2 cases:* $D=0,\ D\neq 0$)

(e) (dividends yield tends to infinity) $D\to\infty$, $t<T$, $\sigma>0$ and finite

4. Suppose S evolves according to the stochastic differential equation (SDE)

$$dS=\mu S dt+S^{\alpha}dX$$

where μ and α are positive constants. Derive the corresponding Black–Scholes partial differential equation (PDE) for the option based upon this asset S (you are not required to solve any equation). Write this PDE in terms of the greeks.

5. An asset pasy a continuous dividend yield, D. Find the put-call parity relationship for European options on this underlying. (You may use the exlicit option value formulas given in the lecture notes.)

6. The value of an option $V\left(S,t\right)$ satisfies the Black–Scholes equation. Write the option value in the form

$$V(S,t) = \exp(-r(T-t))q(S,t).$$

Show that the function $q(S,t)$ satisfies the equation

$$\frac{\partial q}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 q}{\partial S^2} + (r-D)S\frac{\partial q}{\partial S} = 0.$$

This is the backward Kolmogorov equation, used for calculating the expected value of stochastic quantities.

7. Consider an option with value $V\left(S,t\right)$, which has payoff at time T. Reduce the Black-Scholes equation, with final and boundary conditions, to the diffusion equation, using the following transformations:

$$S = Ee^x, \quad t = T - \frac{2\tau}{\sigma^2}, \quad V\left(S,t\right) = Ev\left(x,\tau\right)$$

$$v = \exp\left(\alpha x + \beta\tau\right) u\left(x,\tau\right),$$

for some α and β. What is the transformed payoff? What are the new initial and boundary conditions? Illustrate with a European call option (ignore dividends).