Reduced Form Model

Siyi Zhou

Lecture 5.2
CQF Jan 2009

The main topics covered in this lecture are...

- modeling default by Poisson Process,
- derivation of pricing PDE for a risky bond with deterministic and stochastic intensity,
- fundamental pricing formula for general contingent claims subject to default risk,
- Affine intensity models.

By the end of this lecture you will be able to

- understand what are reduced form(intensity based) model,
- its pros and cons relative to structural model,
- derive risky bond pricing equation when both interest rate and hazard rate(intensity) are stochastic,
- solve simple affine intensity models,
- calibrate default probability on bond spreads.

Introduction

In previous lecture we have studied structural approach to modeling default risk. In this lecture I will introduce a different approach to modeling the default risk of a bond or a general contingent claim that is subject to default risk. These models are named as "reduced-form" or "intensity based" models, in which we treat default as an unpredictable event governed by an exogenous hazard-rate process.

The hazard rate is connected to the likelihood of default which is assumed to be exogenous. In addition to that, the hazard rate can be modeled in the same way as that of short rate which conveniently leads us to existing term structure models. The reduced form models, therefore, are simpler to use and are the most popular type of credit risk models.

A simple model for default event

Before embark on valuation problem, we should have a model for default first. Like many other discrete and countable events, such as the number of buses will arrive in the next 10 min, default events can be modeled as Poisson process.

Definition of Poisson Process

A Poisson Process with intensity λ is a stochastic process

$$N_t : t \geq 0$$

taking values in $S = \{0, 1, 2, \cdots\}$ such that

1. $N_0 = 0$

2. if $s < t$, then $N_s \leq N_t$

3. if $s < t$, then the increment $N_t - N_s$ is independent of what happened during $[0, s]$

4. let $h \to 0^+$

$$Pr(N_{t+h} = n + m | N_t = n) = \begin{cases} \lambda h + o(h), & m = 1 \\ o(h), & m > 1 \\ 1 - \lambda h + o(h), & m = 0 \end{cases}$$

which means within an infinitesimal time interval h, maximum only one event can happen.

Distribution function of N_t

N_t has Poisson distribution with parameter λt

$$Pr(N_t = i) = \frac{(\lambda t)^i}{i!} e^{-\lambda t}$$

where

$$i = 0, 1, 2, \cdots$$

N_t is the number of events occurred by time t. It is called "counting process", and it is the simplest markov chain(we will see more complicated ones later). We can use Poisson process as a model for default event.

Construction of Poisson Process

Define T_n to be the time of the nth event arrival, i.e.,

$$T_n = \inf\{t : N_t = n\}, \quad T_0 = 0,$$

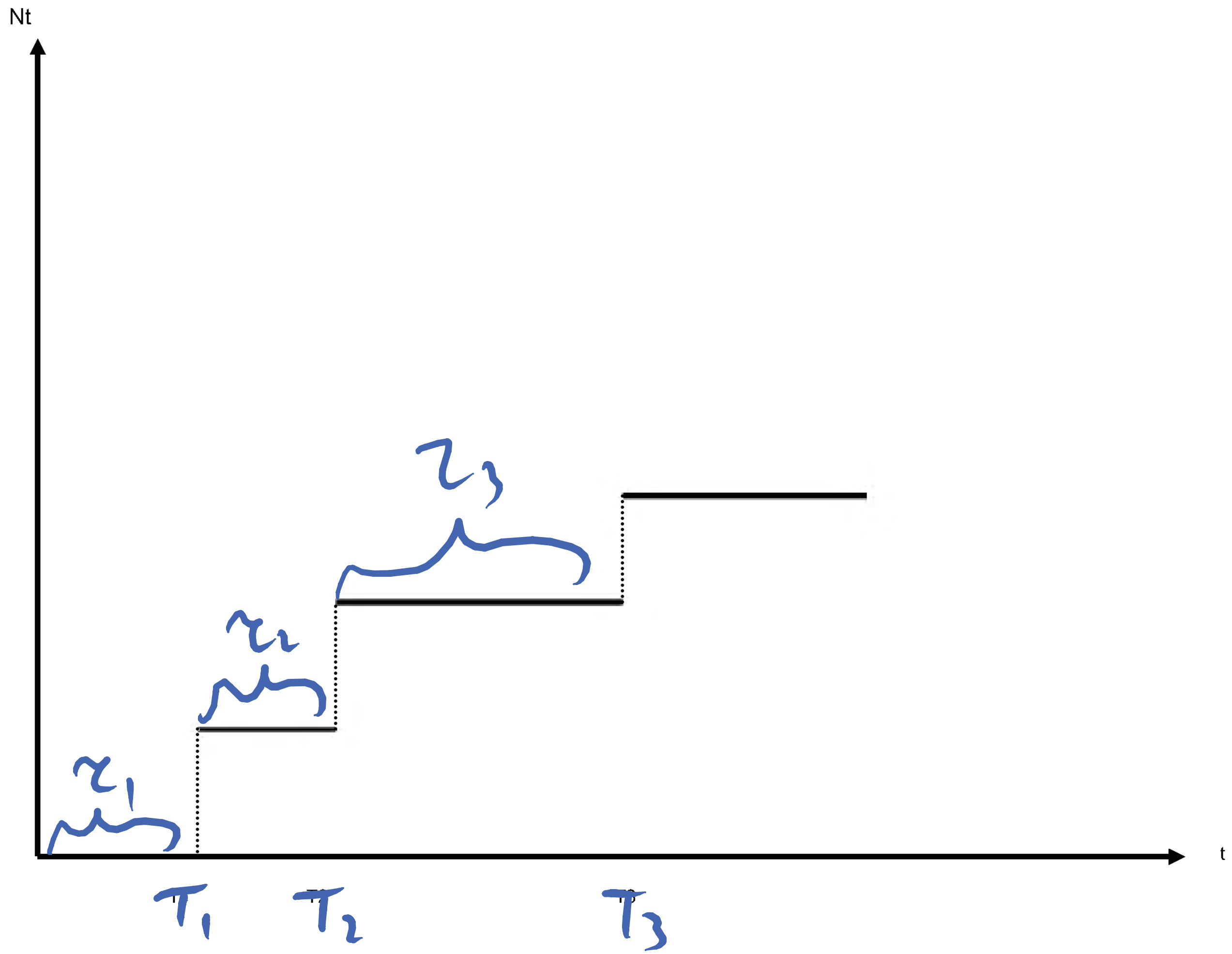
and τ_n is the inter arrival time, which is given by

$$\tau_n = T_n - T_{n-1}.$$

So T_n can be written as

$$\boxed{T_n = \sum_{i=1}^{n} \tau_i}$$

Poisson plot



Distribution of τ

Since Poisson Process has independent increment, the associated inter arrival times are also independent, i.e., $\forall i \neq j$, τ_i and τ_j are independent.

Further more, every inter arrival time has exponential distribution with intensity λ. To summarize τ_i are i.i.d. $\exp(\lambda)$.

To see that we now need to formally introduce the definition of intensity or hazard rate if we are modeling default.

Definition of Intensity

Denote $F(\tau)$ the CDF of τ_1(the very first arrival), so the survival condition is

$$S(\tau) = 1 - F(\tau).$$

According to the definition of Poisson process (4) when $h \to 0^+$

$$Pr(t < \tau \leq t + h | \tau > t) = \lambda h + o(h).$$

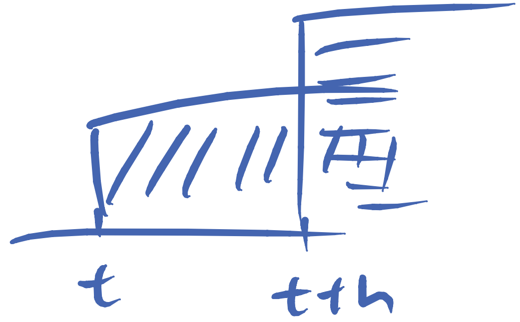
So

$$\lambda = \lim_{h \to 0^+} \frac{Pr(t < \tau \leq t + h | \tau > t)}{h}$$

$$= \lim_{h \to 0^+} \frac{Pr(\tau < t < t+h)}{h \, Pr(\tau > t)}$$

Finding distribution of τ

$$
\begin{aligned}
\lambda &= \lim_{h \to 0^+} \frac{Pr(t < \tau \le t + h)}{h Pr(\tau > t)} \\
&= \lim_{h \to 0^+} \frac{S(t) - S(t+h)}{h S(t)} \\
&= -\frac{d \log S(t)}{dt}
\end{aligned}
$$



$$\frac{S'(t)}{S(t)}$$

Solve above ODE for S with boundary condition $S(0) = 1$, we get

$$S(t) = e^{-\lambda t},$$

which implies

$$S(t) \sim \exp(\lambda).$$

$$\int_0^t \lambda \, dt = -\int_0^t \frac{d \log S(\tau)}{d\tau}$$

$$\lambda t = -\log S(t) - \log S(0)$$

$$S(t) = e^{-\lambda t}$$

$$Pr(\tau > t) = e^{-\lambda t}$$

Simulating Poisson Process

So far we have distribution for the 1st inter arrival time, what about the rest of them? By independence we know

$$S_2(t) = Pr(\tau_2 > t|\tau_1 = s) = Pr(\tau_2 = t).$$

Then follow the same argument we have

$$\boxed{\tau_2 \sim \exp(\lambda),}$$

and so on for τ_i.

As a consequence of this results, simulating Poisson process is equivalent to simulating i.i.d. exponential random variables. We will see how to do this in lecture 5.5 by using copula function.

Inhomogenous Poisson process

What if intensity isn't a constant but deterministic? Then the counting process will become inhomogenous Poisson process. The analysis are almost the same as before, I will leave it as an exercise to you. The the survival function in this case will be

$$S(t) = \exp\left(-\int_0^t \lambda_s \, ds\right).$$

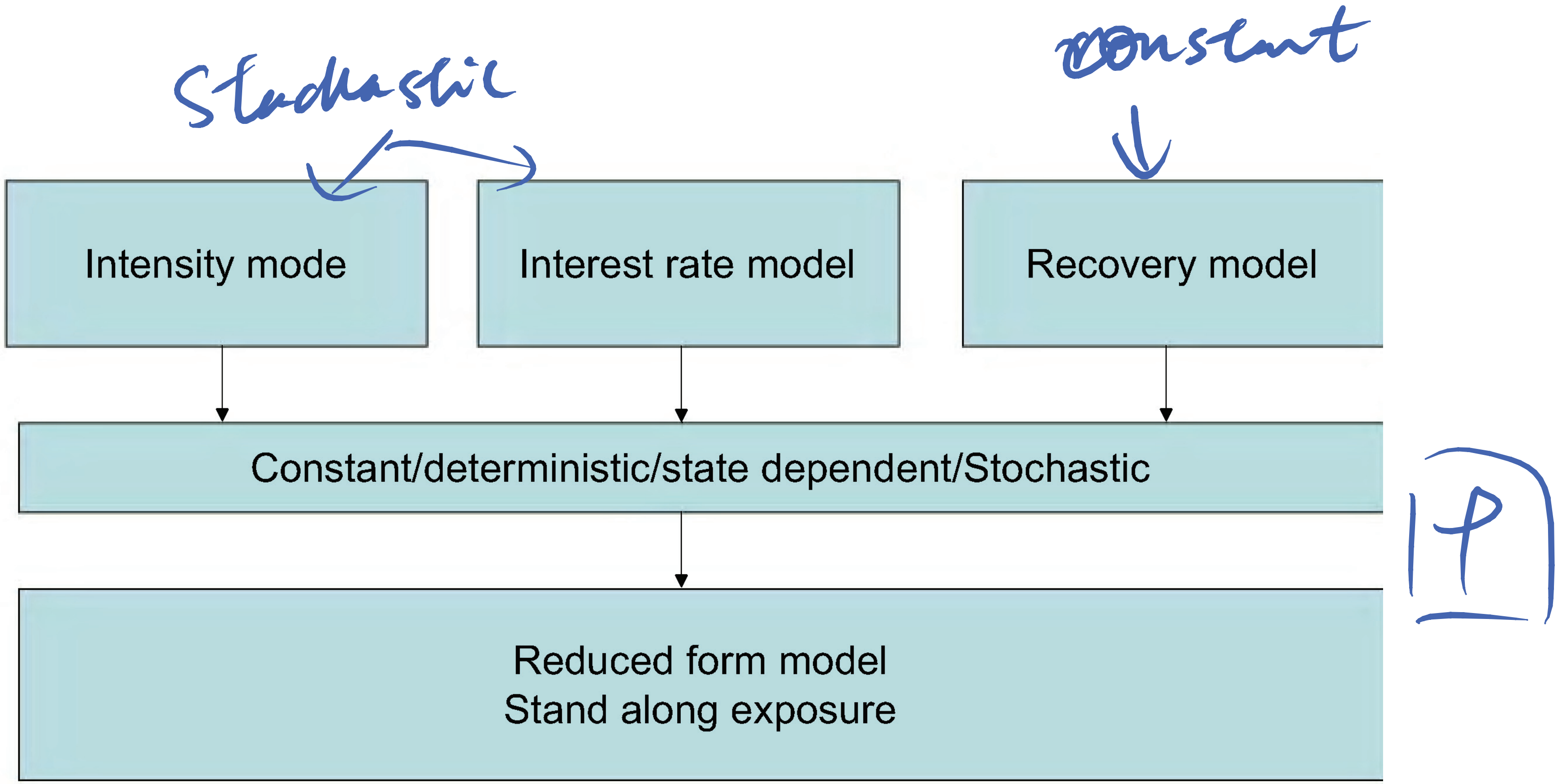
$= e^{-\lambda t}$ if λ_t is constant

Cox Process

Later on I will present a model in which intensity is not deterministic but a stochastic process. The analysis is more complicated and we don't provide proof in this lecture. The general ideal is to use conditional expectation, i.e., conditional on the path of intensity the survival function is known. The survival function in this case is

$$S(t) = \mathbb{E} \exp\left(- \int_0^t \lambda_s \, ds \right).$$

Inp



Stochastic
constant
Intensity mode
Interest rate model
Recovery model
Constant/deterministic/state dependent/Stochastic
1P
Reduced form model
Stand along exposure

Plan for risky bond pricing model

In the next a few slides, I will derive several different BPEs for a risky bond based on different assumptions made on hazard rate, hedging and recovery.

In order of complexity there are

1. constant hazard rate and zero recovery without hedging of default risk

2. stochastic hazard rate and zero recovery with hedging of default risk

3. stochastic hazard rate and positive recovery with hedging of default risk.

We will always assume interest rate is stochastic.

Risky bond with constant hazard rate

Suppose a corporate's default follows a homogenous Poisson process with intensity p, and a ZCB with maturity T is issued by this company, the value of the bond is denoted by $V(t, r, p)$.

Like usual suppose $Z(t, r)$ is the value of a riskless ZCB with same maturity as the risky bond, where

$$d\,r = u(r, t)\,dt + w(r, t)dX.$$

For simplicity we will assume that there is no correlation between the diffusive change in the spot interest rate and the Poisson process.

Hedging interest rate risk

Now we follow our convention when pricing fixed income product, construct a 'hedged' portfolio:

we don't hedge

$\boxed{P}$

$$\Pi = V(r, p, t) - \Delta Z(r, t).$$

Note here only interest rate risk is hedged.

No default occurs

There is a probability of $(1 - p\,dt)$ that the bond does not default. Then the change in the value of the portfolio during an infinitesimal time step is

$1 - p\,dt$

$$d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2} w^2 \frac{\partial^2 V}{\partial r^2} \right) dt + \frac{\partial V}{\partial r} dr$$

$$-\Delta \left(\left(\frac{\partial Z}{\partial t} + \frac{1}{2} w^2 \frac{\partial^2 Z}{\partial r^2} \right) dt + \frac{\partial Z}{\partial r} dr \right).$$

- Choose Δ to eliminate the risky dr term.

Default occurs

On the other hand, if the bond defaults, with a probability of $p\,dt$, then the change in the value of the portfolio is

$$d\Pi = -V + O(dt^{1/2}).$$

$p\,dt$

This is due to the sudden loss of the risky bond, the other terms are small in comparison.

Bond pricing PDE

Taking expectations and using the bond-pricing equation for the riskless bond, we find that the value of the risky bond satisfies

$$\frac{\partial V}{\partial t} + \frac{1}{2} w^2 \frac{\partial^2 V}{\partial r^2} + (u - \lambda w) \frac{\partial V}{\partial r} - (r + p)V = 0.$$

Feynman Kac:1st call

$$V(t,T) = \mathbb{E}\left(e^{-\int_t^T (r_s+p)\,ds} | \mathcal{F}_t\right)$$

Very similar analysis can be carried out with deterministic hazard rate, the result will be almost identical apart from changing p to p_t, so that

$$V(t,T) = \mathbb{E}\left(e^{-\int_t^T (r_s+p_s)\,ds} | \mathcal{F}_t\right)$$

$\leftarrow$ p is deterministic

$$= e^{-\int_t^T p_s\,ds}\, E\left(e^{-\int_t^T r_s\,ds} / \mathcal{F}_t\right)$$

$$= S(t)\, Z(t,T) \Rightarrow S(t) = \frac{V(t,T)}{Z(0,T)}$$

$$V(t,T) = 0\, F(t) + S(t)\, Z(0,t)$$

Yield Spread

The yield to maturity on this bond is now given by

spread.

$$y = -\frac{\log(Z(t,T)S_t(T))}{T-t} = y_f + \frac{1}{T-t}\int_t^T p_s ds,$$

where y_f is the yield to maturity of a risk free bond with the same maturity as the risky bond.

Thus the effect of the risk of default on the yield is to add a spread on riskless yield. In this simple model, the spread will be the average of the hazard rate from t to T.

Forward spread

If one calculates the forward rate implied by the risky bond

$$-\frac{\partial}{\partial T}\log(V(t,T)) = f(t,T) + p_T.$$

The spread in this case will simply be the hazard rate.

Implied default probability:no recovery

Given term structure of risk free bond and risky bond, one can extract implied default probability by using

$$S_t(T) = \frac{V(t,T)}{Z(t,T)} = \exp(-(T-t)(y-y_f)).$$

One can also calculate implied hazard rate by using forward spread.

PD

Year	Riskfree zero rate	Risky bond zero rate	Cummulative PD	Marginal PD
1	5%	5.25%	0.2497%	0.2497%
2	5%	5.50%	0.9950%	0.7453%
3	5%	5.70%	2.0781%	1.0831%
4	5%	5.85%	3.3428%	1.2647%
5	5%	5.95%	4.6390%	1.2961%

$$1 - e^{-(5.25\% - 5\%)}$$

$$1 - e^{-2(5.5\% - 5\%)}$$

To find the risky bond value:

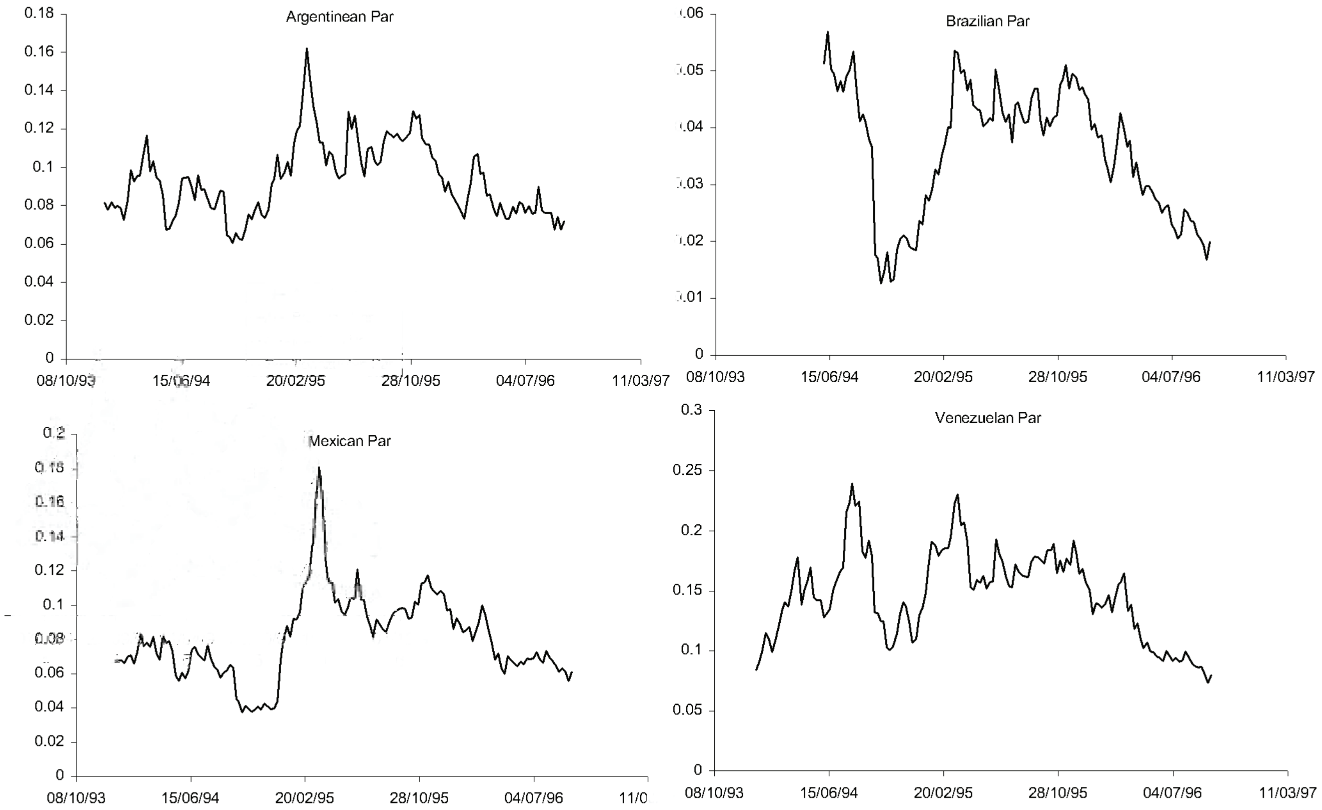
1. Find the risk-free yield for the maturity of each cashflow in the risky bond;

2. Add a constant spread, p, to each of these yields;

3. Use this new yield to calculate the present value of each cashflow;

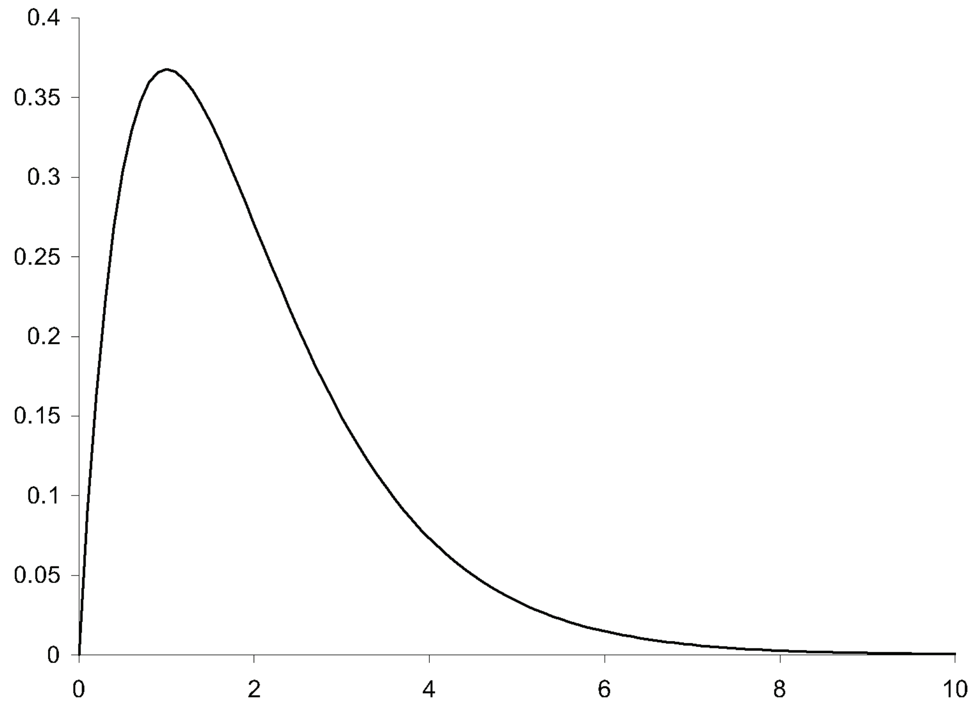
4. Sum all the present values.

Time series of implied default rate



Implied rates of default for four Latin American Brady bonds.

Implied hazard rate plot



A plausible structure for a time-dependent hazard rate.

Stochastic risk of default

Now consider a model in which the default intensity is itself random:

$$dp = \gamma(r, p, t)dt + \delta(r, p, t)dX_1,$$

with interest rates still given by

$$dr = u(r, t)dt + w(r, t)dX_2,$$

where

$$dX_1\, dX_2 = \rho\, dt.$$

Hedging default intensity

In the previous model we used riskless bonds to hedge the random movements in the spot interest rate.

Can we introduce another risky bond or bonds into the portfolio to help with hedging the default risk?

To do this we must assume that default in one bond automatically means default in the other.

Hedged portfolio

To value our risky zero-coupon bond we construct a portfolio with one of the risky bond, with value $V(r,p,t)$, and delta hedged by shorting Δ of a riskless bond, with value $Z(r,t)$, and Δ_1 of another risky bond issued by the same company, with value $V_1(r,p,t)$:

$$\Pi = V(r,p,t) - \Delta Z(r,t) - \Delta_1 V_1(r,p,t).$$

No default

Suppose that the bond does not default, the change in the value of the portfolio during an infinitesimal time step is

$$d\Pi = dV - \Delta\, dZ - \Delta_1 dV_1 .$$

$1 - p_t\, dt$

By using Itô's lemma, above can be written as

$$d\Pi = \left(\mathcal{L}'(V) - \Delta \mathcal{L}(Z) - \Delta_1 \mathcal{L}'(V_1)\right) dt$$

$$+ \left(\frac{\partial V}{\partial r} - \Delta \frac{\partial Z}{\partial r} - \Delta_1 \frac{\partial V_1}{\partial r} r\right) dr$$

$$+ \left(\frac{\partial V}{\partial p} - \Delta_1 \frac{\partial V_1}{\partial p}\right) dp$$

where

$$\mathcal{L}'(V) = \frac{\partial V}{\partial t} + \tfrac{1}{2}w^2\frac{\partial^2 V}{\partial r^2} + \rho w \delta \frac{\partial^2 V}{\partial r \partial p} + \tfrac{1}{2}\delta^2\frac{\partial^2 V}{\partial p^2}$$

$$\mathcal{L}(Z) = \frac{\partial Z}{\partial t} + \tfrac{1}{2}w^2\frac{\partial^2 Z}{\partial r^2}$$

and ρ is the correlation between dX_1 and dX_2.

Choose Δ to eliminate the risky terms.

$$\Delta_1 = \frac{\partial V}{\partial p} \Big/ \frac{\partial V_1}{\partial p}$$

and

$$\Delta = \frac{\frac{\partial V}{\partial r} - \Delta_1 \frac{\partial V_1}{\partial r}}{\frac{\partial Z}{\partial r}}$$

If defaults

If the bond defaults then the change in the value of the portfolio is

$$d\Pi = -V + \Delta_1 V_1 + O(dt^{1/2}).$$

$p_t \, dt$

Taking expectations and using the bond-pricing equation for the riskless bond, we find that the value of the risky bond satisfies

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}w^2\frac{\partial^2 V}{\partial r^2} + \rho w \delta \frac{\partial^2 V}{\partial r \partial p} + \tfrac{1}{2}\delta^2\frac{\partial^2 V}{\partial p^2} +$$

$$(u - \lambda w)\frac{\partial V}{\partial r} + (\gamma - \lambda' \delta)\frac{\partial V}{\partial p} - (r + p)V = 0.$$

Feynman Kac:2nd call

Similar to interest rate risk, λ' is called market price of default intensity risk. So the fundamental pricing formula for the risky bond under risk neutral measure is

$$V(t,T) = \overset{Q}{\mathbb{E}}\left(e^{-\int_t^T (r_s+p_s)ds} | \mathcal{F}_t\right).$$

Positive recovery

In default there is usually *some* payment, not all of the money is lost. In the table are shown the mean and standard deviations for recovery according to the seniority of the debt. This emphasizes the fact that the rate of recovery is itself very uncertain.

Class	Mean (%)	Std Dev. (%)
Senior secured	53.80	26.86
Senior unsecured	51.13	25.45
Senior subordinated	38.52	23.81
Subordinated	32.74	20.18
Junior subordinated	17.09	10.90

Rate of recovery. Source: Moody's.

There is also a statistical relationship between rate of recovery and default rates. (Years with low default rates have higher recovery when there is default.)

Recovery of market value

Suppose that on default we know that we will loss l percent of pre-default value. This will change the partial differential equation.

recovery = $1-l$

On default we have

$$d\Pi = -lV + l\Delta_1 V_1 + O(dt^{1/2});$$

we lose the bond but recover $1-l$. The pricing equation will be

$$\frac{\partial V}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 V}{\partial r^2} + \rho w\delta\frac{\partial^2 V}{\partial r\partial p} + \frac{1}{2}\delta^2\frac{\partial^2 V}{\partial p^2}$$

$$+(u-\lambda w)\frac{\partial V}{\partial r} + (\gamma - \lambda'\delta)\frac{\partial V}{\partial p} - (r+lp)V = 0.$$

Recovery of treasury

Although the assumption of recovery on market value is convenient for the purpose of mathematical modeling and makes economic sense since it measures the loss in value associated with default, it is impossible to give immediate expression for implied default probability.

Recovery on treasury assumes that, if a corporate bond defaults, its value will be replaced by a treasury bond with the same maturity. Under this assumption and with the independence of interest rate and hazard rate, the bond price will be

$$V(0,t) = (1 - F(t))\, Z(0,t) + F(t)\theta Z(0,t).$$

$$e^{-(y-y_f)t} = \frac{V(0,t)}{Z(0,t)} = 1 - (1-\theta)F(t)$$

$$F(t) = \frac{1 - e^{-(y-y_f)t}}{1-\theta}$$

Calibration

Year	Riskfree zero rate	Risky bond zero rate	Cummulative PD	Marginal PD
1	5%	5.25%	0.6242%	0.6242%
2	5%	5.50%	2.4875%	1.8633%
3	5%	5.70%	5.1953%	2.7077%
4	5%	5.85%	8.3571%	3.1619%
5	5%	5.95%	11.5974%	3.2403%
	Recovery	40%		

p $\quad$ p

$F(t)$

Feynman Kac:3rd call

$$V(t,T) = \mathbb{E}\left(e^{-\int_t^T (r_s + lp_s)ds} | \mathcal{F}_t\right).$$

It can be rewritten as

$$V(t,T) = \mathbb{E}\left(e^{-\int_t^T R_s ds} | \mathcal{F}_t\right).$$

Where

$$R_t = r_t + l\, p_t,$$

is called risk adjusted discount rate.

Affine intensity model

Some spot interest rate models lead to explicit solutions for bond prices (e.g. Vasicek, CIR and the general affine model).

- We can find simpler equations if we choose the functions $u - \lambda w$, w, $\gamma - \lambda'\delta$, δ and ρ carefully.

- We must choose $u - \lambda w$ and w^2 to be linear in state variables, similarly for $\gamma - \lambda'\delta$ and δ^2.

- The form of the correlation coefficient is assumed to be constant.

With these choices for the functions in the two stochastic differential equations we find that the solution with $V(r, p, T) = 1$ is

$$V = \exp\left(A(t, T) - B(t, T)r - C(t, T)p\right)$$

where A, B and C satisfy non-linear first-order ordinary differential equations.

Note: If we allow the spot interest rate model to have some simple time dependence then we have the freedom to fit the yield curve.

Similarly, if there is time dependence in the model for the intensity, and the model is sufficiently tractable, then you can also fit risky bond term structure.

Example:Vasicek intensity model

Suppose there are two state variables $\overset{X}{X_1}$ and $\overset{Y}{X_2}$ whose dynamics can be written as

$$
\begin{aligned}
dX &= (a_1 + b_{11}X - b_{12}Y)dt + \sigma_1 dW_1 \\
dY &= (a_2 + b_{21}X - b_{22}Y)dt + \sigma_2 dW_2
\end{aligned}
$$

where

$$dW_1 dW_2 = \rho \; dt$$

The risk adjusted discount rate is

$$\underline{R = g_0 + g_1 X + g_2 Y.}$$

Modified Vasicek

To simplify the parameterization we work with canonical form

$$
\begin{aligned}
dX &= -a\,X dt + \sigma dW_1 \\
dY &= -b\,Y dt + \eta dW_2
\end{aligned}
$$

and

$$R(t) = \phi(t) + \underset{r(t)}{X(t)} + \underset{sp}{Y(t)}.$$

$$V(t,T) = E\left[e^{-\int_t^T R_s ds} \,\middle|\, \mathcal{F}_t\right]$$

Note in order to calibrate to risky bond yield we add in a time dependent parameter into risk adjusted rate. Here we correspond state variable X to short rate r and Y to spread $l\,p$.

$$-\int_t^T R_s ds \sim N(?\,,\,?)$$

Solution for risky bond price

$$V(t,T) = \exp\left\{-\int_t^T \phi(s)ds - \frac{1-e^{-a(T-t)}}{a}X(t) - \frac{1-e^{-b(T-t)}}{b}Y(t) + \tfrac{1}{2}M(t,T)\right\}$$

where

$$\begin{aligned} M(t,T) &= \frac{\sigma^2}{a^2}\left[T-t+\frac{2}{a}e^{-a(T-t)} - \frac{1}{2a}e^{-2a(T-t)} - \frac{3}{2a}\right] \\ &+ \frac{\eta^2}{b^2}\left[T-t+\frac{2}{b}e^{-b(T-t)} - \frac{1}{2b}e^{-2b(T-t)} - \frac{3}{2b}\right] \\ &+ 2\rho\frac{\sigma\eta}{ab}\left[T-t-\frac{1-e^{-a(T-t)}}{a} - \frac{1-e^{-b(T-t)}}{b} - \frac{e^{-(a+b)(T-t)-1}}{a+b}\right] \end{aligned}$$

Compare to structural approach

Structural models assume that the modeler has the same information set as the firms manager-complete knowledge of all the firms assets and liabilities. In most situations, this knowledge leads to a predictable default time. In contrast, reduced form models assume that the modeler has the same information set as the market-incomplete knowledge of the firms condition. In most cases, this imperfect knowledge leads to an inaccessible default time. As such, we argue that the key distinction between structural and reduced form models is not whether the default time is predictable or inaccessible, but whether the information set is observed by the market or not. Consequently, for pricing and hedging, reduced form models are the preferred model.

Summary

Please take away the following important ideas

- Poisson process can be used to model default if intensity is constant

- stochastic default intensity leads us to higher dimension bond pricing PDE.

- Risky bond pricing PDE is consistent with fundamental pricing formula by Feynman Kac.

- Reduced-form models can be tractable in affine term structure model.