The Black–Scholes Model

In this lecture. . .

- the assumptions that go into the Black–Scholes model
- foundations of options theory: delta hedging and no arbitrage
- the Black–Scholes partial differential equation
- the Black–Scholes formulæ for calls, puts and simple digitals
- the meaning and importance of the 'greeks,' delta, gamma, theta, vega and rho
- American options and early exercise
- relationship between option values and expectations

By the end of this lecture you will be able to

- derive the Black–Scholes partial differential equation
- quote formulæ for simple contracts
- understand the meaning of the common greeks
- interpret the early-exercise feature mathematically and graphically

Introduction

The Black–Scholes equation was the biggest breakthrough in the pricing of options.

The theory is quite straightforward, using just the ideas from stochastic calculus that we have already seen.

The end result is a diffusion-type partial differential equation which can be used for the pricing of many different derivatives.

What determines the value of an option?

The value of an option is a function of the stock price S and time t.

The value of the option is also a function of parameters in the contract, such as the strike price E and the time to expiry $T - t$, T is the date of expiry.

The value will also depend on properties of the asset, such as its drift and its volatility, as well as the risk-free rate of interest:

$$V(S, t; \sigma, \mu; E, T; r).$$

Semi-colons separate different types of variables and parameters.

- S and t are variables;
- σ and μ are parameters associated with the asset price;
- E and T are parameters associated with the particular contract;
- r is a parameter associated with the currency.

For the moment just use $V(S,t)$ to denote the option value.

The Black–Scholes assumptions

- The underlying follows a lognormal random walk with known volatility
- The risk-free interest rate is a known function of time
- There are no dividends on the underlying
- Delta hedging is done continuously
- There are no transaction costs on the underlying
- There are no arbitrage opportunities

And more. . .

A very special portfolio

We assume that the asset evolves according to

$$dS = \mu S \, dt + \sigma S \, dX.$$

Then we imagine constructing a special portfolio.

Use Π to denote the value of a portfolio of one long option position and a short position in some quantity Δ, **delta**, of the underlying:

$$\Pi = V(S,t) - \Delta S. \tag{1}$$

Intuition: Think of moves in S and accompanying move in V, for a call.

How does the value of the portfolio change?

The change in the portfolio value is due partly to the change in the option value and partly to the change in the underlying:

$$d\Pi = dV - \Delta \, dS.$$

- Notice that Δ has not changed during the time step.

From Itô we have

$$dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}dt.$$

Thus the portfolio changes by

$$d\Pi = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}dt - \Delta\, dS. \qquad (2)$$

The right-hand side of (2) contains two types of terms, the deterministic and the random.

- The deterministic terms are those with the dt.

- The random terms are those with the dS. These random terms are the risk in our portfolio.

Elimination of risk: Delta hedging

Is there any way to reduce or even eliminate this risk? This can be done in theory by carefully choosing Δ.

If we choose

$$\Delta = \frac{\partial V}{\partial S} \tag{3}$$

then the randomness is reduced to zero.

- Any reduction in randomness is generally termed **hedging**. The perfect elimination of risk, by exploiting correlation between two instruments (in this case an option and its underlying) is generally called **Delta hedging**.

Delta hedging is an example of a **dynamic hedging** strategy. From one time step to the next the quantity $\frac{\partial V}{\partial S}$ changes, since it is, like V a function of the ever-changing variables S and t.

This means that the perfect hedge must be continually rebalanced.

No arbitrage

After choosing the quantity Δ as suggested above, we hold a portfolio whose value changes by the amount

$$d\Pi = \left(\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt. \tag{4}$$

- This change is completely *riskless*.

If we have a completely risk-free change $d\Pi$ in the value Π then it must be the same as the growth we would get if we put the equivalent amount of cash in a risk-free interest-bearing account:

$$d\Pi = r\Pi\, dt. \tag{5}$$

- This is an example of the **no arbitrage** principle.

The Black–Scholes equation

Substituting (1), (3) and (4) into (5) we find that

$$\left(\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt = r\left(V - S\frac{\partial V}{\partial S}\right) dt.$$

On dividing by dt and rearranging we get

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0. \tag{6}$$

This is the **Black–Scholes equation**.

Observations:

- The Black–Scholes equation equation is a **linear parabolic partial differential equation**

- The Black–Scholes equation contains all the obvious variables and parameters such as the underlying, time, and volatility, but there is no mention of the drift rate μ.

- This means that if two people agree on the volatility of an asset they will agree on the value of its derivatives *even if they have differing estimates of the drift*.

Replication

Another way of looking at the hedging argument is to ask what happens if we hold a portfolio consisting of just the stock, in a quantity Δ, and cash.

If Δ is the partial derivative of some option value then such a portfolio will yield an amount at expiry that is simply that option's payoff.

In other words, we can use the same Black–Scholes argument to **replicate** an option just by buying and selling the underlying asset.

- This leads to the idea of a **complete market**. In a complete market an option can be replicated with the underlying, thus making options redundant.

Final conditions

The Black–Scholes equation knows nothing about what kind of option we are valuing.

This is dealt with by the **final condition**. We must specify the option value V as a function of the underlying at the expiry date T. That is, we must prescribe $V(S,T)$, the payoff.

For example, if we have a call option then we know that

$$V(S,T) = \max(S-E,0).$$

Options on dividend-paying equities

Assume that the asset receives a continuous and constant dividend yield, D.

- Thus in a time dt each asset receives an amount $DS\,dt$.

This must be built into the derivation of the Black–Scholes equation.

Take up the Black–Scholes argument at the point where we are looking at the change in the value of the portfolio:

$$d\Pi = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}dt - \Delta\, dS - D\Delta S\, dt.$$

The last term on the right-hand side is the amount of the dividend per asset, $DS\,dt$, multiplied by the number of the asset held, $-\Delta$.

The Δ must still be the rate of change of the option value with respect to the underlying for the elimination of risk.

End result:

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D)S\frac{\partial V}{\partial S} - rV = 0.$$

Currency options

Options on currencies are handled in exactly the same way.

- In holding the foreign currency we receive interest at the foreign rate of interest r_f.

This is just like receiving a continuous dividend:

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - r_f) S \frac{\partial V}{\partial S} - rV = 0.$$

Commodity options

- The relevant feature of commodities requiring that we adjust the Black–Scholes equation is that they have a **cost of carry**. That is, the storage of commodities is not without cost.

Let us introduce q as the fraction of the value of a commodity that goes towards paying the cost of carry.

To be precise, for each unit of the commodity held an amount $qS\,dt$ will be required during short time dt to finance the holding:

$$\frac{\partial V}{\partial t}+\tfrac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+(r+q)S\frac{\partial V}{\partial S}-rV=0.$$

Solving the equation and the greeks

The Black–Scholes equation has simple solutions for calls, puts and some other contracts. Now we are going to go quickly through the derivation of these formulæ.

The 'delta,' the first derivative of the option value with respect to the underlying, occurs as an important quantity in the derivation of the Black–Scholes equation. In this lecture we see the importance of other derivatives of the option price, with respect to the variables and with respect to some of the parameters.

- These derivatives are important in the hedging of an option position, playing key roles in risk management.

Derivation of the formulæ for calls, puts and simple digitals

The Black–Scholes equation is

$$\frac{\partial V}{\partial t}+\tfrac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+rS\frac{\partial V}{\partial S}-rV=0. \tag{7}$$

This equation must be solved with final condition depending on the payoff: each contract will have a different functional form prescribed at expiry $t=T$, depending on whether it is a call, a put or something else.

The first step in the manipulation is to change from present value to future value terms.

Recalling that the payoff is received at time T but that we are valuing the option at time t this suggests that we write

- $$V(S,t) = e^{-r(T-t)}U(S,t).$$

$$\frac{\partial V}{\partial t} = re^{-r(T-t)}U + e^{-r(T-t)}\frac{\partial U}{\partial t}.$$

This takes our differential equation to

$$\frac{\partial U}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 U}{\partial S^2} + rS\frac{\partial U}{\partial S} = 0.$$

(Remember this result for later, present valuing means that one of the terms disappears.)

The second step is really trivial. Because we are solving a backward equation we'll write

- $$\tau = T - t.$$

This now takes our equation to

$$\frac{\partial U}{\partial \tau} = \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 U}{\partial S^2} + rS\frac{\partial U}{\partial S}.$$

When we first started modeling equity prices we used intuition about the asset price *return* to build up the stochastic differential equation model. Let's go back to examine the return and write

- $$\xi = \log S.$$

With this as the new variable, we find that

$$\frac{\partial}{\partial S} = e^{-\xi}\frac{\partial}{\partial \xi} \quad \text{and} \quad \frac{\partial^2}{\partial S^2} = e^{-2\xi}\frac{\partial^2}{\partial \xi^2} - e^{-2\xi}\frac{\partial}{\partial \xi}.$$

Now the Black–Scholes equation becomes

$$\frac{\partial U}{\partial \tau} = \frac{1}{2}\sigma^2\frac{\partial^2 U}{\partial \xi^2} + \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial U}{\partial \xi}.$$

The last step is simple, but the motivation is not so obvious. Write

- $$x = \xi + \left(r - \tfrac{1}{2}\sigma^2\right)\tau \quad \text{and} \quad U = W(x, \tau).$$

This is just a 'translation' of the co-ordinate system. It's a bit like using the forward price of the asset instead of the spot price as a variable.

After this change of variables the Black–Scholes becomes the simpler

$$\frac{\partial W}{\partial \tau} = \tfrac{1}{2}\sigma^2 \frac{\partial^2 W}{\partial x^2}. \tag{8}$$

And you've seen this equation before!

You've even solved it to find a special solution—out of the infinite number of possible solutions—and exactly the same solution will be needed here. (Lucky!)

To summarize,

$$V(S,t) = e^{-r(T-t)}U(S,t) = e^{-r\tau}U(S,T-\tau) = e^{-r\tau}U(e^{\xi},T-\tau)$$

$$= e^{-r\tau}U\left(e^{x-\left(r-\frac{1}{2}\sigma^2\right)\tau}, T-\tau\right) = e^{-r\tau}W(x,\tau).$$

We are going to derive an expression for the value of any option whose payoff is a known function of the asset price at expiry.

This includes calls, puts and digitals. This expression will be in the form of an integral.

For special cases, we'll see how to rewrite this integral in terms of the cumulative distribution function for the Normal distribution. This is particularly useful since the function can be found on spreadsheets, calculators and in the backs of books.

But there are two steps before we can write down this integral.

- The first step is to find a special solution of (8), called the fundamental solution. This solution has useful properties.

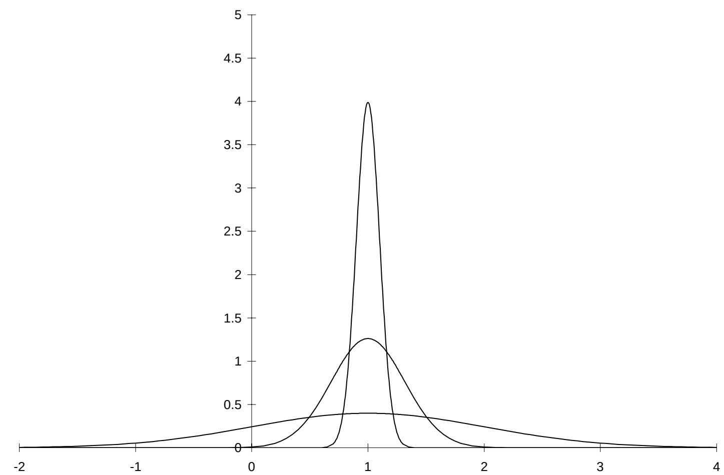
- The second step is to use the linearity of the equation and the useful properties of the special solution to find the *general solution* of the equation.

The first step is easy, just recall solving the equation from the earlier lecture. The solution we want is

- $$W_f(x,\tau;x') = \frac{1}{\sqrt{2\pi\tau}\,\sigma} e^{-\frac{(x-x')^2}{2\sigma^2\tau}}.$$

As you know, this is the probability density function for a Normal random variable x having mean of x' and standard deviation $\sigma\sqrt{\tau}$.

And this also strongly hints at a relationship between option values and probabilities. More anon!



Above is plotted W_f as a function of x' for several values of τ.

At $x' = x$ the function grows unboundedly, and away from this point the function decays to zero, as $\tau \to 0$.

Although the function is increasingly confined to a narrower and narrower region its area remains fixed at one.

- These properties of decay away from one point, unbounded growth at that point and constant area, result in a **Dirac delta function** $\delta(x' - x)$ as $\tau \to 0$.

The delta function has one important property, namely

$$\int \delta(x' - x) g(x') \, dx' = g(x)$$

where the integration is from any point below x to any point above x.

Thus the delta function 'picks out' the value of g at the point where the delta function is singular i.e. at $x' = x$.

In the limit as $\tau \to 0$ the function W becomes a delta function at $x = x'$. This means that

$$\lim_{\tau \to 0} \frac{1}{\sigma\sqrt{2\pi\tau}} \int_{-\infty}^{\infty} e^{-\frac{(x'-x)^2}{2\sigma^2\tau}} g(x')\, dx' = g(x).$$

Whoa! This is tricky!

I am going to 'cut to the chase' and quote the solution:

- $$V(S,t) = \frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}}$$

$$\int_0^\infty e^{-\left(\log(S/S')+\left(r-\frac{1}{2}\sigma^2\right)(T-t)\right)^2/2\sigma^2(T-t)}\,\text{Payoff}(S')\frac{dS'}{S'}. \qquad (9)$$

This 'formula' works for any European, non path-dependent, option on a single lognormal underlying asset, all you need to know is the payoff function.

Observations

- This is a general formula (see above conditions on type of option)
- It is of the form of a) a discounting term multiplied by b) the integral of the payoff multiplied by c) another function
- This other 'function' is known as a Green's function
- This function can be interpreted as a probability
- The whole expression can be interpreted as the present value of the expected payoff

Let's look at special cases.

Formula for a call

The call option has the payoff function

$$\text{Payoff}(S) = \max(S - E, 0).$$

Expression (9) can then be written as

$$\frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \int_E^\infty e^{-\left(\log(S/S') + \left(r - \frac{1}{2}\sigma^2\right)(T-t)\right)^2 / 2\sigma^2(T-t)} (S' - E) \frac{dS'}{S'}.$$

Return to the variable $x' = \log S'$, to write this as

$$\frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \int_{\log E}^{\infty} e^{-\left(-x'+\log S+\left(r-\frac{1}{2}\sigma^2\right)(T-t)\right)^2/2\sigma^2(T-t)} \left(e^{x'}-E\right) dx'$$

$$= \frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \int_{\log E}^{\infty} e^{-\left(-x'+\log S+\left(r-\frac{1}{2}\sigma^2\right)(T-t)\right)^2/2\sigma^2(T-t)} e^{x'}\, dx'$$

$$-E\frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \int_{\log E}^{\infty} e^{-\left(-x'+\log S+\left(r-\frac{1}{2}\sigma^2\right)(T-t)\right)^2/2\sigma^2(T-t)}\, dx'.$$

Both integrals in this expression can be written in the form

$$\int_d^\infty e^{-\frac{1}{2}x'^2}\,dx'$$

for some d (the second is just about in this form already, and the first just needs a completion of the square).

Thus the option price can be written as two separate terms involving the cumulative distribution function for a Normal distribution:

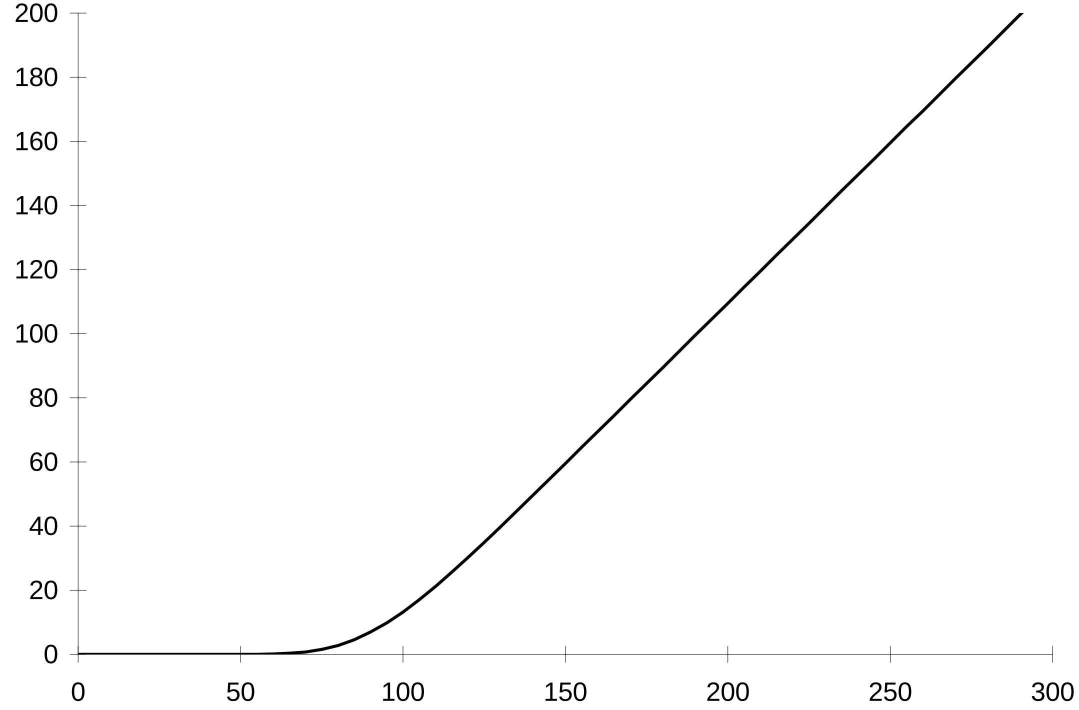
$$\text{Call option value } = SN(d_1) - Ee^{-r(T-t)}N(d_2)$$

where

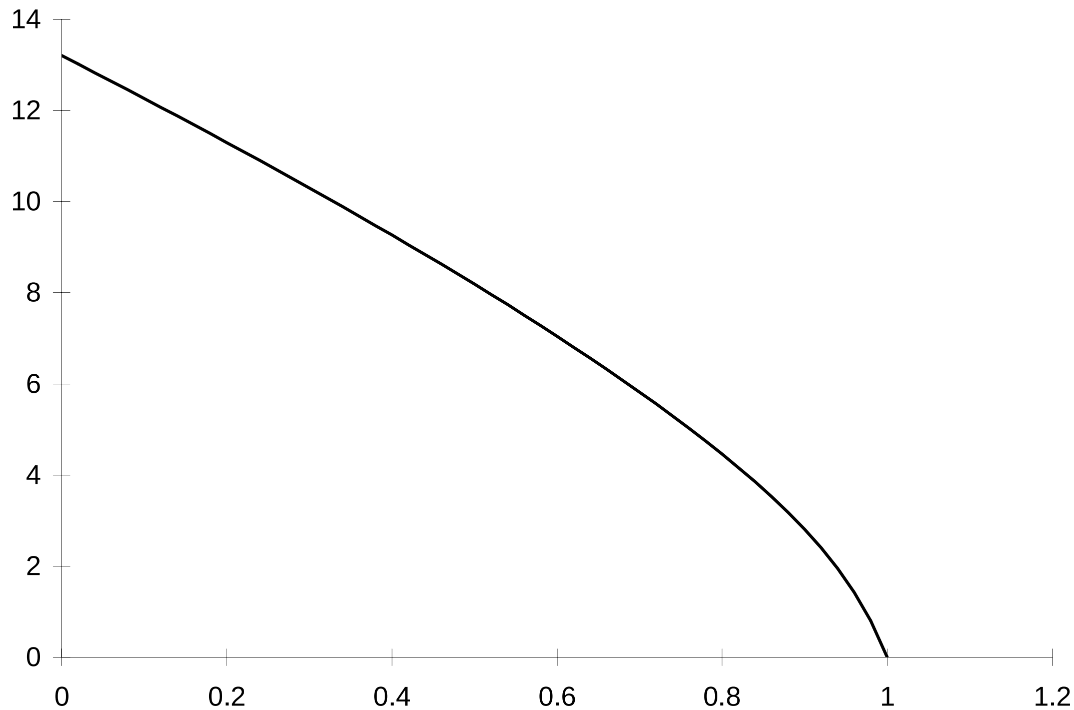
$$d_1 = \frac{\log(S/E) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \quad \text{and}$$

$$d_2 = \frac{\log(S/E) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}.$$

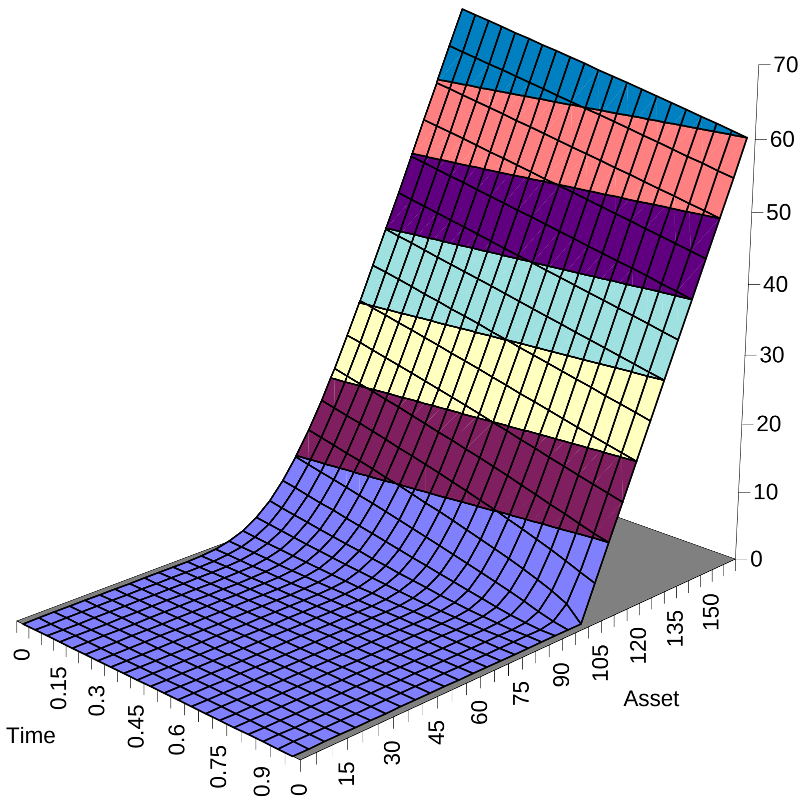
$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}\phi^2} d\phi.$$



The value of a call option as a function of the underlying at a fixed time before expiry.



The value of an at-the-money call option as a function of time.



The value of a call option as a function of asset and time.

When there is continuous dividend yield on the underlying, or it is a currency, then

Call option value

$$Se^{-D(T-t)}N(d_1) - Ee^{-r(T-t)}N(d_2)$$

$$d_1 = \frac{\log(S/E)+(r-D+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = \frac{\log(S/E)+(r-D-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t}$$

When the asset is 'at-the-money forward,' i.e. $S = E^{-(r-D)(T-t)}$, and the option is close to expiration then there is a simple approximation for the call value (Brenner & Subrahmanyam, 1994):

$$\text{Call } \approx 0.4\, Se^{-D(T-t)}\sigma\sqrt{T-t}.$$

Formula for a put

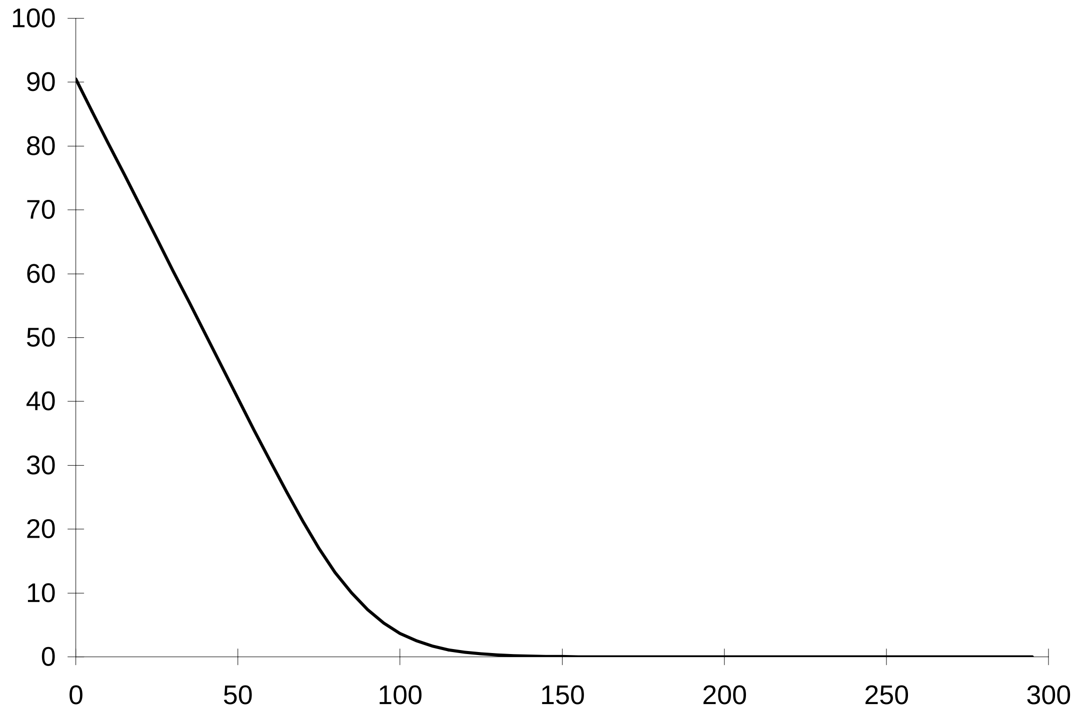
The put option has payoff

$$\text{Payoff}(S) = \max(E - S, 0).$$

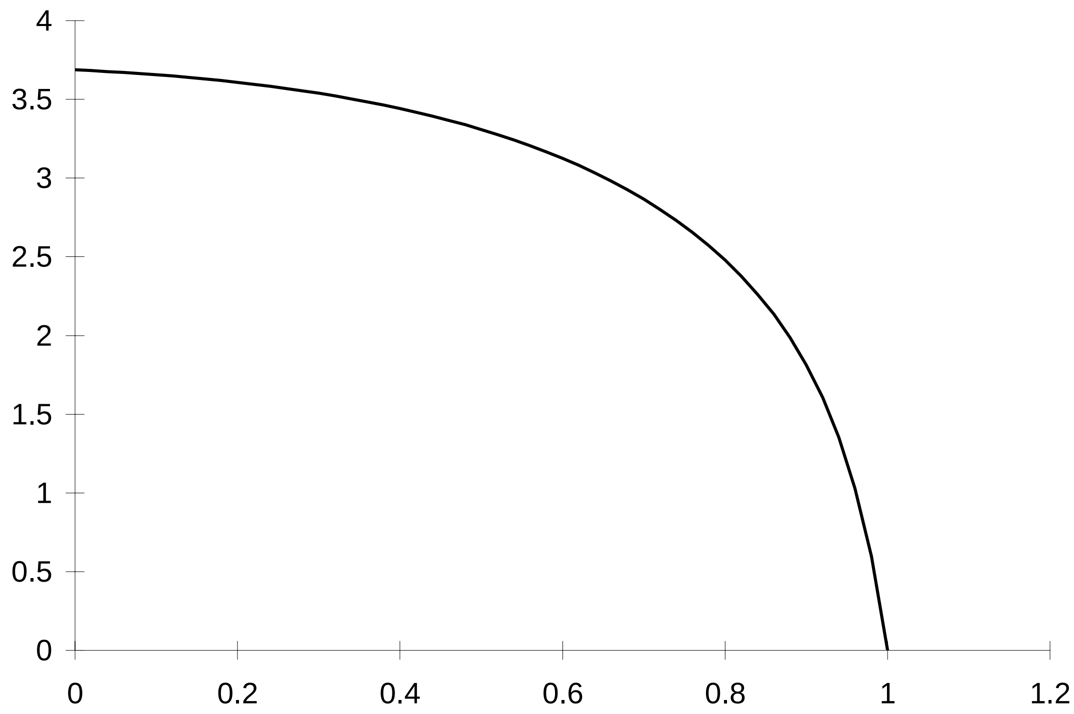
The value of a put option can be found in the same way as above, or using put-call parity

$$\text{Put option value } = -SN(-d_1) + Ee^{-r(T-t)}N(-d_2),$$

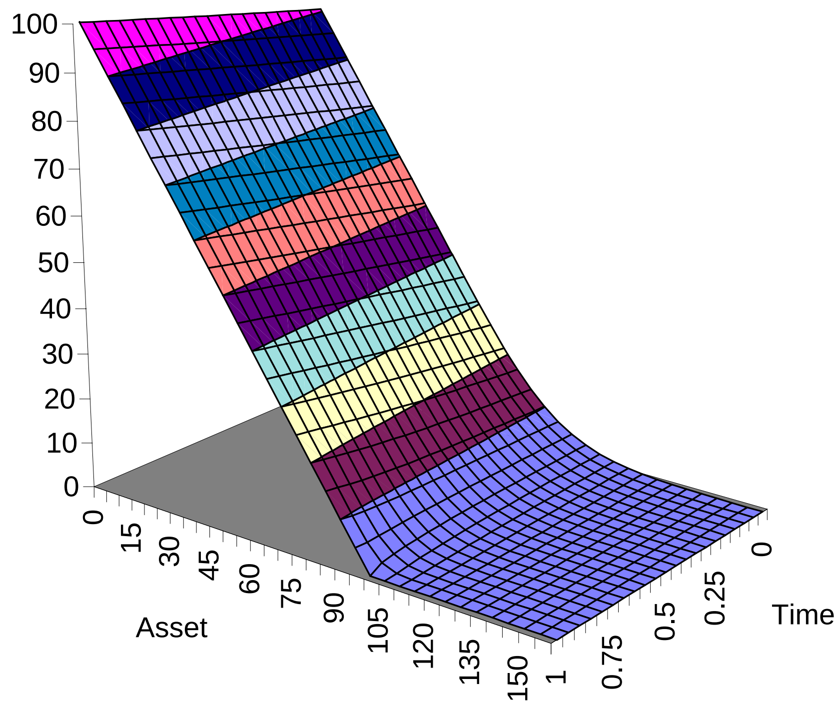
with the same d_1 and d_2.



The value of a put option as a function of the underlying at a fixed time to expiry.



The value of an at-the-money put option as a function of time to expiry.



The value of a put option as a function of the underlying and time to expiry.

When there is continuous dividend yield on the underlying, or it is a currency, then

Put option value

$$-Se^{-D(T-t)}N(-d_1) + Ee^{-r(T-t)}N(-d_2)$$

When the asset is at-the-money forward and the option is close to expiration the simple approximation for the put value (Brenner & Subrahmanyam, 1994) is

$$\text{Put } \approx 0.4\, Se^{-D(T-t)}\sigma\sqrt{T-t}.$$

Formula for a binary call

The binary call has payoff

$$\text{Payoff}(S) = \mathcal{H}(S - E),$$

where $\mathcal{H}$ is the Heaviside function taking the value one when its argument is positive and zero otherwise.

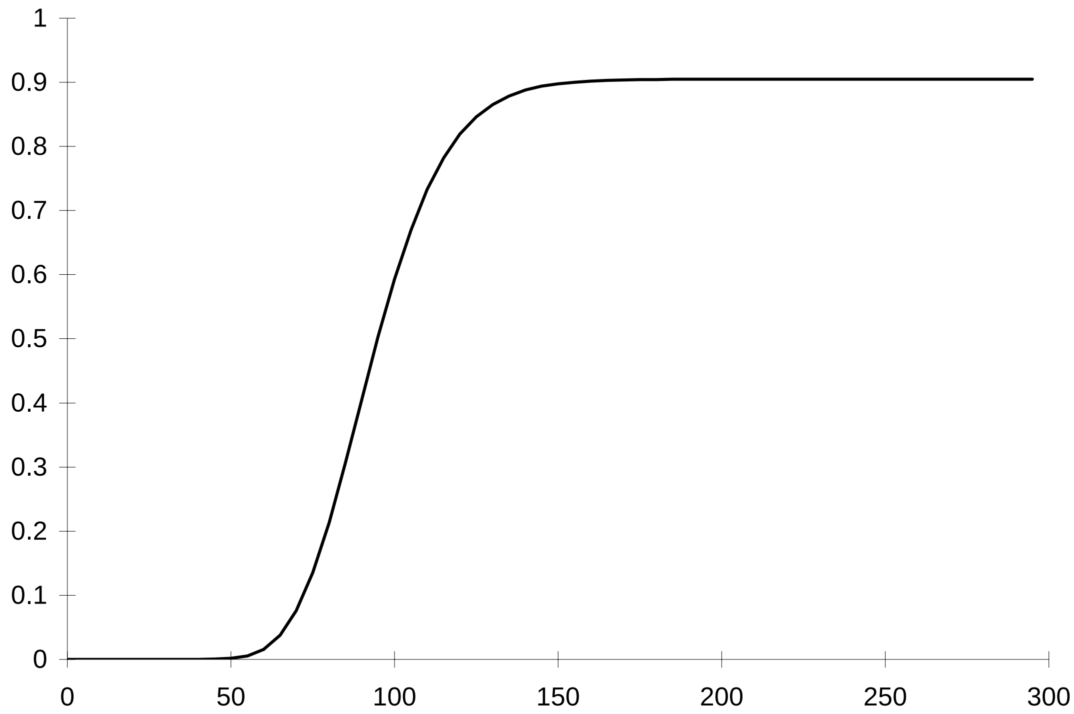
Incorporating a dividend yield, we can write the option value as

$$\frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}}\int_{\log E}^{\infty} e^{-\left(x'-\log S-\left(r-D-\frac{1}{2}\sigma^2\right)(T-t)\right)^2/2\sigma^2(T-t)}\,dx'.$$

This term is just like the second term in the call option equation and so...

Binary call option value

$$e^{-r(T-t)}N(d_2)$$



The value of a binary call option.

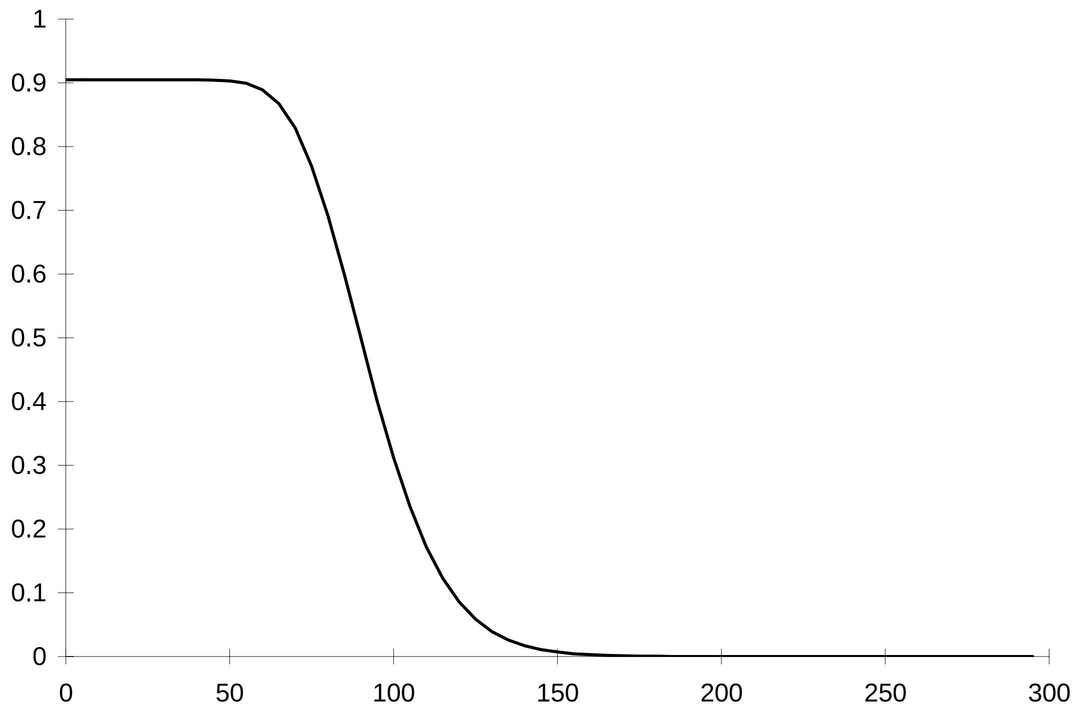
Formula for a binary put

The binary put has a payoff of one if $S < E$ at expiry. It has a value of

Binary put option value

$$e^{-r(T-t)}(1 - N(d_2))$$

A binary call and a binary put must add up to the present value of \$1 received at time T.



The value of a binary put option.

Delta

The **delta** of an option or a portfolio of options is the sensitivity of the option or portfolio to the underlying. It is the rate of change of value with respect to the asset:

$$\boxed{\Delta = \frac{\partial V}{\partial S}}$$

Here V can be the value of a single contract or of a whole portfolio of contracts. The delta of a portfolio of options is just the sum of the deltas of all the individual positions.

- The theoretical device of delta hedging for eliminating risk is far more than that, it is a very important practical technique.

Delta hedging means holding one of the option and short a quantity Δ of the underlying.

Delta can be expressed as a function of S and t.

This function varies as S and t vary.

- This means that the number of assets held must be continuously changed to maintain a **delta neutral** position, this procedure is called **dynamic hedging**.

Changing the number of assets held requires the continual purchase and/or sale of the stock. This is called **rehedging** or **rebalancing** the portfolio.

Here are some formulæ for the deltas of common contracts (all formulæ assume that the underlying pays dividends or is a currency):

Deltas of common contracts

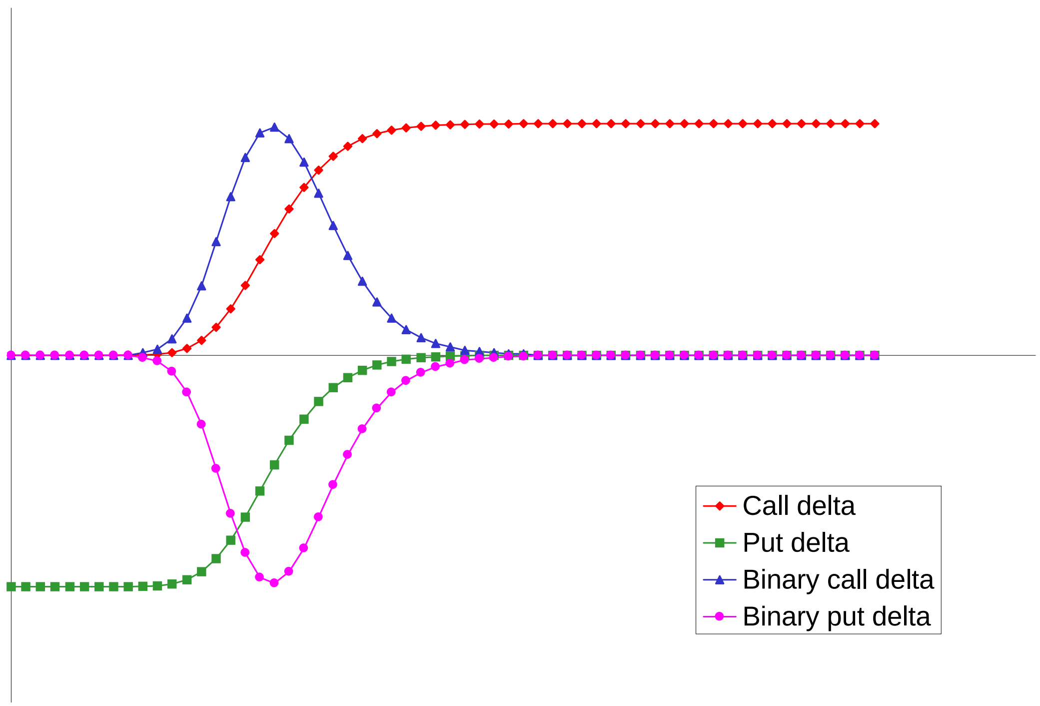
Call $e^{-D(T-t)}N(d_1)$

Put $e^{-D(T-t)}(N(d_1)-1)$

Binary call $\frac{e^{-r(T-t)}N'(d_2)}{\sigma S\sqrt{T-t}}$

Binary put $-\frac{e^{-r(T-t)}N'(d_2)}{\sigma S\sqrt{T-t}}$

$$N'(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}$$



The deltas of a call, put and binary options. (The deltas of the binaries have been scaled.)

Gamma

The **gamma**, Γ, of an option or a portfolio of options is the second derivative of the position with respect to the underlying:

$$\boxed{\Gamma = \frac{\partial^2 V}{\partial S^2}}$$

This is, of course, just

$$\frac{\partial\left(\frac{\partial V}{\partial S}\right)}{\partial S}.$$

Since gamma is the sensitivity of the delta to the underlying it is a measure of by how much or how often a position must be rehedged in order to maintain a delta-neutral position.

Because costs can be large and because one wants to reduce exposure to model error it is natural to try to minimize the need to rebalance the portfolio too frequently.

Since gamma is a measure of sensitivity of the hedge ratio Δ to the movement in the underlying, the hedging requirement can be decreased by a gamma-neutral strategy.

- This means buying or selling more *options*, not just the underlying.

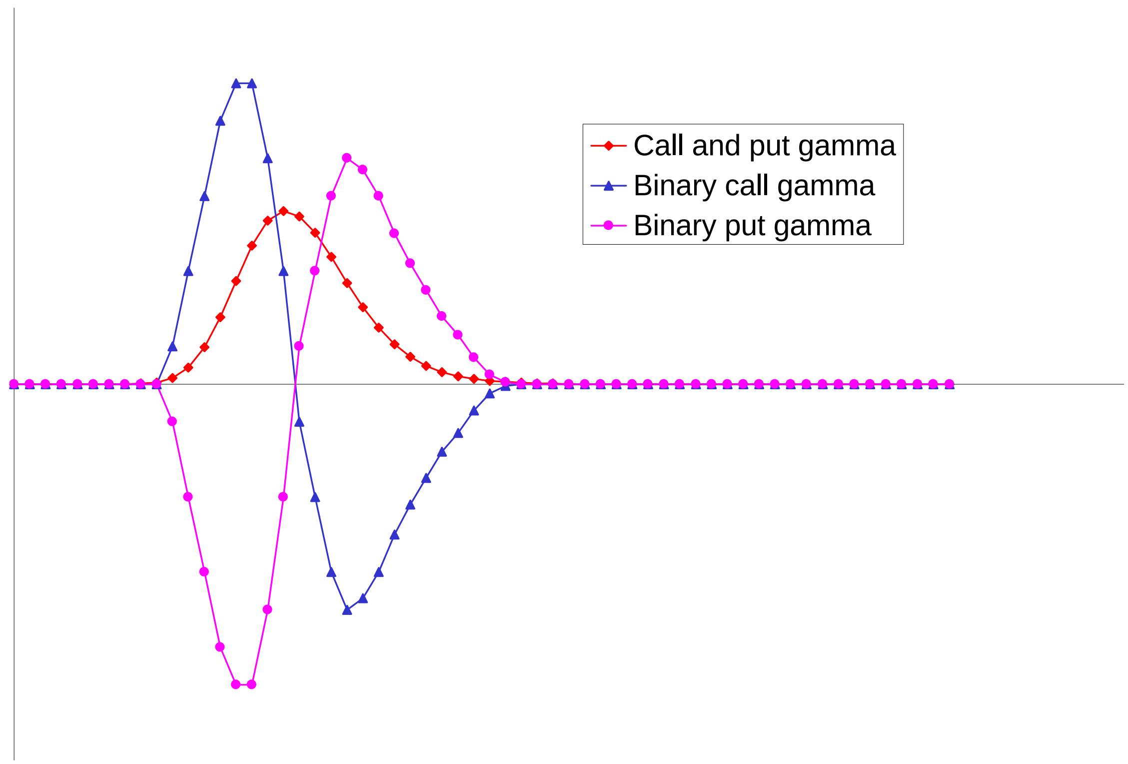
Because the gamma of the underlying (its second derivative) is zero, we cannot add gamma to our position just with the underlying.

- We can have as many options in our position as we want, we choose the quantities of each such that both delta and gamma are zero.

Here are some formulæ for the gammas of common contracts:

Gammas of common contracts

Call	$\frac{e^{-D(T-t)}N'(d_1)}{\sigma S\sqrt{T-t}}$
Put	$\frac{e^{-D(T-t)}N'(d_1)}{\sigma S\sqrt{T-t}}$
Binary call	$-\frac{e^{-r(T-t)}d_1 N'(d_2)}{\sigma^2 S^2(T-t)}$
Binary put	$\frac{e^{-r(T-t)}d_1 N'(d_2)}{\sigma^2 S^2(T-t)}$



The gammas of a call, put and binary options.

Theta

Theta, Θ, is the rate of change of the option price with time.

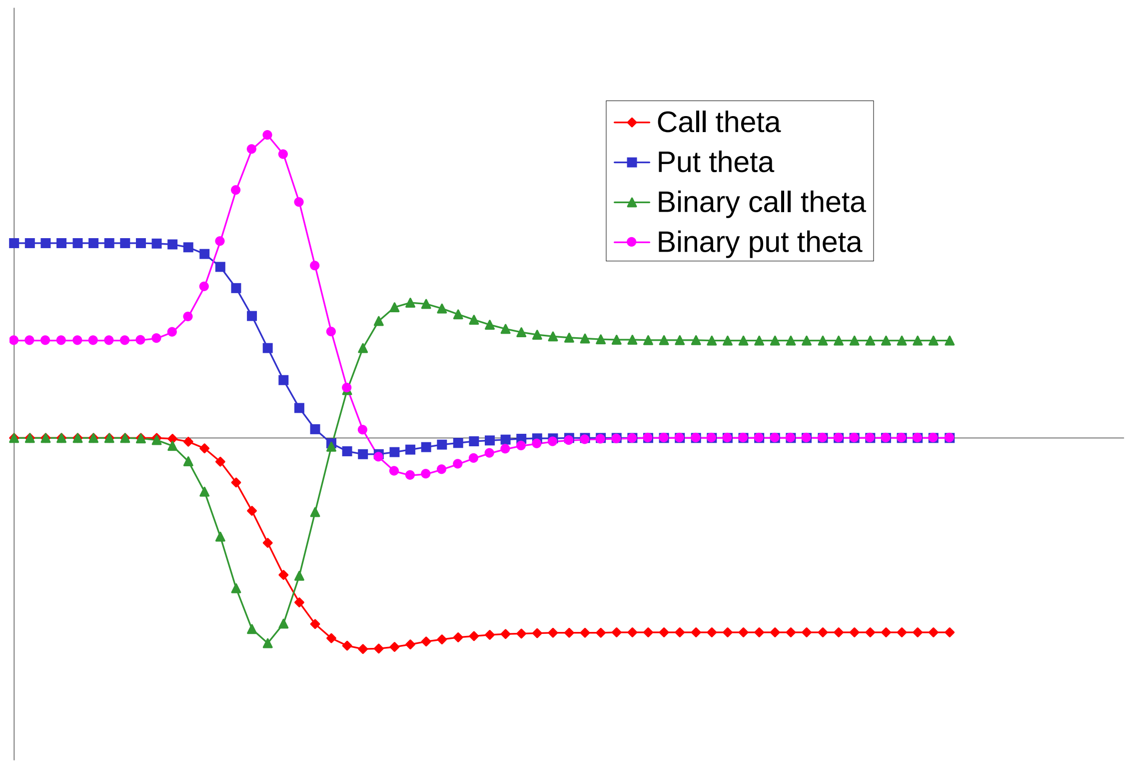
$$\boxed{\Theta = \frac{\partial V}{\partial t}}$$

The theta is related to the option value, the delta and the gamma by the Black–Scholes equation. In a delta-hedged portfolio the theta contributes to ensuring that the portfolio earns the risk-free rate.

Here are some formulæ for the thetas of common contracts:

Thetas of common contracts

Call	$-\frac{\sigma S e^{-D(T-t)} N'(d_1)}{2\sqrt{T-t}} + DSN(d_1)e^{-D(T-t)} - rEe^{-r(T-t)}N(d_2)$
Put	$-\frac{\sigma S e^{-D(T-t)} N'(-d_1)}{2\sqrt{T-t}} - DSN(-d_1)e^{-D(T-t)} + rEe^{-r(T-t)}N(-d_2)$
Binary call	$re^{-r(T-t)}N(d_2) + e^{-r(T-t)}N'(d_2)\left(\frac{d_1}{2(T-t)} - \frac{r-D}{\sigma\sqrt{T-t}}\right)$
Binary put	$re^{-r(T-t)}(1 - N(d_2)) - e^{-r(T-t)}N'(d_2)\left(\frac{d_1}{2(T-t)} - \frac{r-D}{\sigma\sqrt{T-t}}\right)$



The thetas of a call, put and binary options.

Vega

Vega (also zeta and kappa) is a very important but confusing quantity. It is the sensitivity of the option price to volatility.

$$\boxed{\text{Vega} = \frac{\partial V}{\partial \sigma}}$$

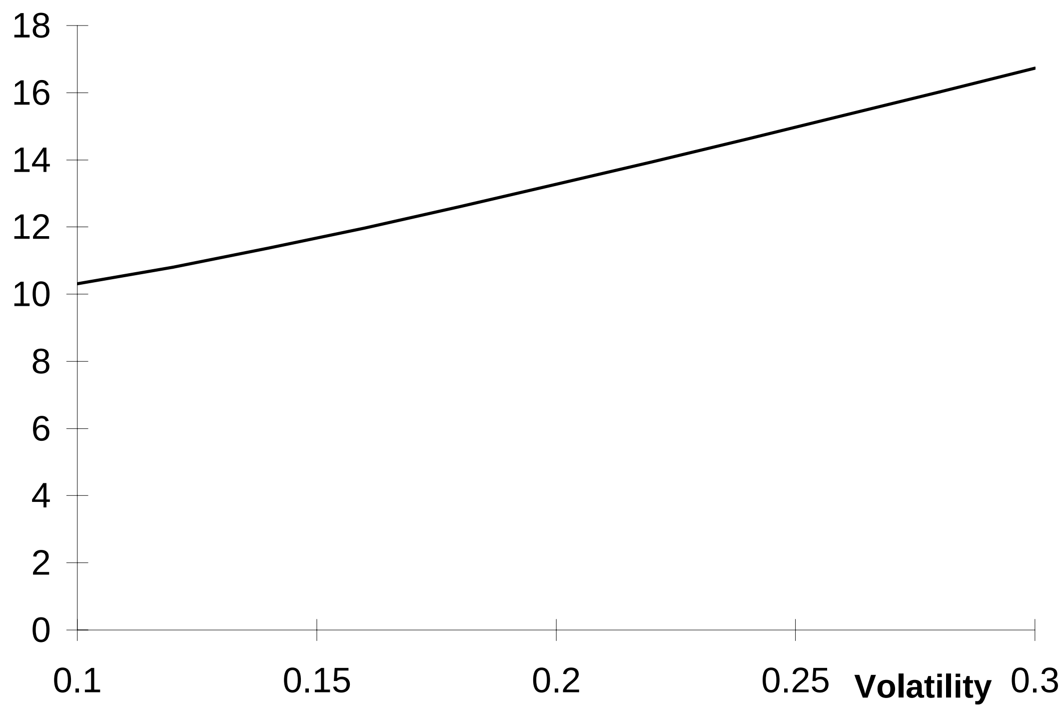
This is a completely different from the other Greeks since it is a derivative with respect to a parameter and not a variable.

- As with gamma hedging, one can vega hedge to reduce sensitivity to the volatility. This is a major step towards eliminating some model risk, since it reduces dependence on a quantity that is not known very accurately.

Here are formulæ for the vegas of common contracts:

Vegas of common contracts

Call	$S\sqrt{T-t}e^{-D(T-t)}N'(d_1)$
Put	$S\sqrt{T-t}e^{-D(T-t)}N'(d_1)$
Binary call	$-e^{-r(T-t)}N'(d_2)\left(\sqrt{T-t}+\frac{d_2}{\sigma}\right)$
Binary put	$e^{-r(T-t)}N'(d_2)\left(\sqrt{T-t}+\frac{d_2}{\sigma}\right)$



An at-the-money call option as a function of the volatility.

Rho

Rho, ρ, is the sensitivity of the option value to the interest rate used in the Black–Scholes formulæ:

$$\boxed{\rho = \frac{\partial V}{\partial r}}$$

Here are some formulæ for the rhos of common contracts:

Rhos of common contracts

Call $E(T-t)e^{-r(T-t)}N(d_2)$

Put $-E(T-t)e^{-r(T-t)}N(-d_2)$

Binary call $-(T-t)e^{-r(T-t)}N(d_2) + \frac{\sqrt{T-t}}{\sigma}e^{-r(T-t)}N'(d_2)$

Binary put $-(T-t)e^{-r(T-t)}(1-N(d_2)) - \frac{\sqrt{T-t}}{\sigma}e^{-r(T-t)}N'(d_2)$

The sensitivities of common contract to the dividend yield or foreign interest rate are given by the following formulæ:

Sensitivity to dividend for common contracts

Call $-(T-t)Se^{-D(T-t)}N(d_1)$

Put $(T-t)Se^{-D(T-t)}N(-d_1)$

Binary call $-\frac{\sqrt{T-t}}{\sigma}e^{-r(T-t)}N'(d_2)$

Binary put $\frac{\sqrt{T-t}}{\sigma}e^{-r(T-t)}N'(d_2)$

Early exercise and American options

American options are contracts that may be exercised early, *prior* to expiry.

For example, if the option is a call, we may hand over the exercise price and receive the asset whenever we wish.

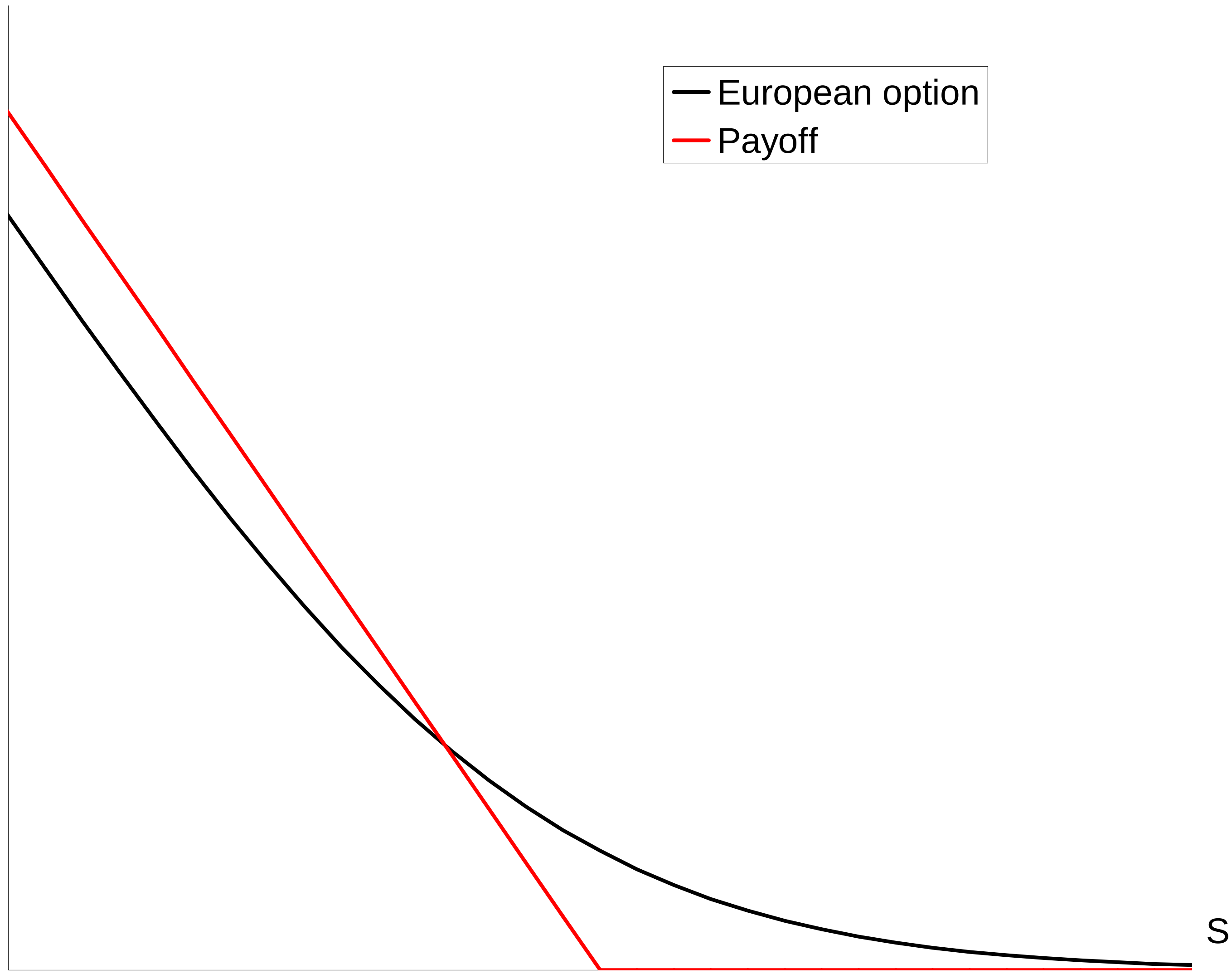
These options must be contrasted with European options for which exercise is only permitted *at* expiry.

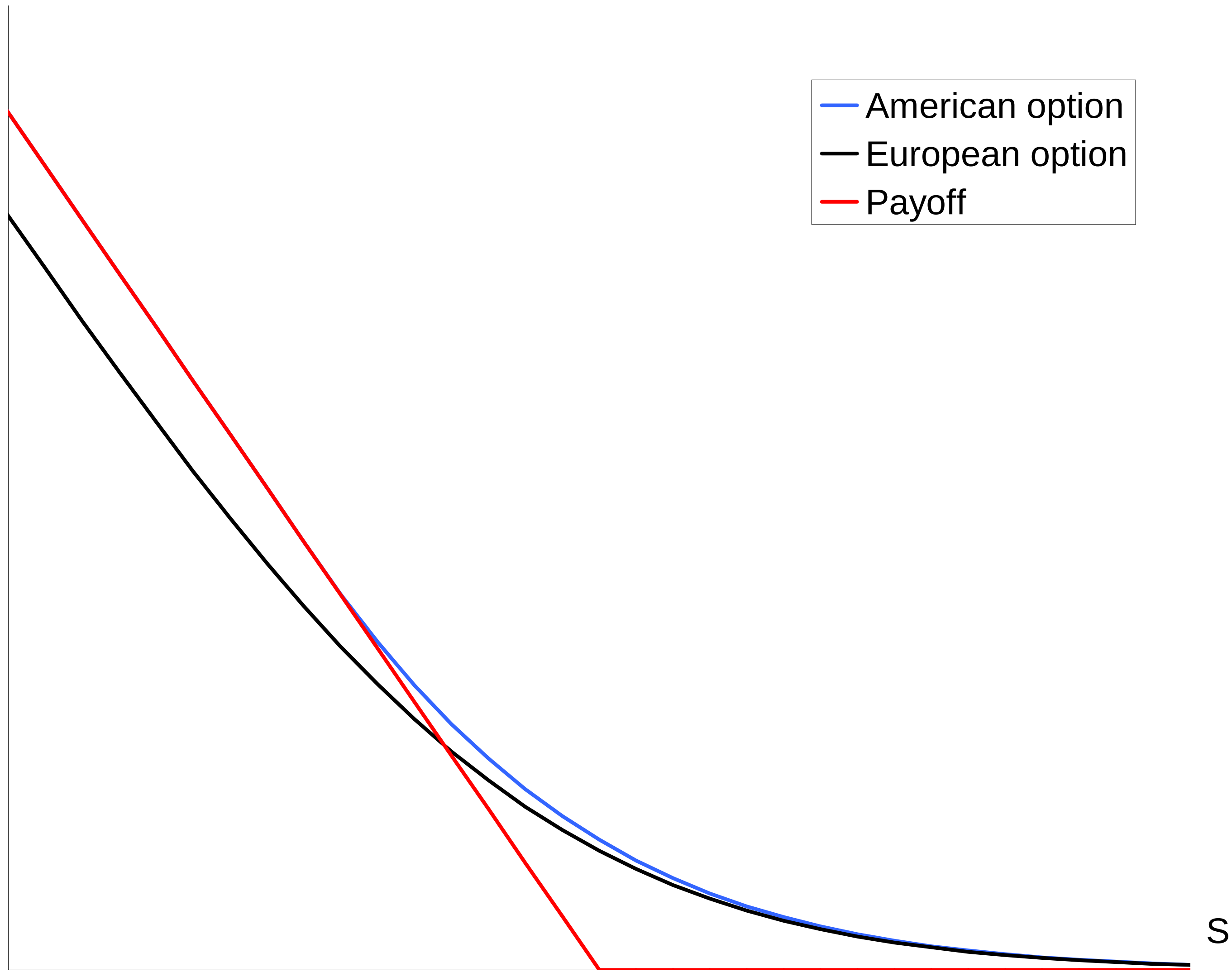
Most traded stock and futures options are American style, but most index options are European.

The right to exercise at any time at will is clearly valuable.

The value of an American option cannot be less than an equivalent European option.

Part of the valuation problem is deciding when is the best time to exercise. This is what makes American options much more interesting than their European cousins.


European option
Payoff
S


American option
European option
Payoff
S

$$\boxed{\text{The American option value is maximized by an exercise strategy that makes the option value and option delta continuous}}$$

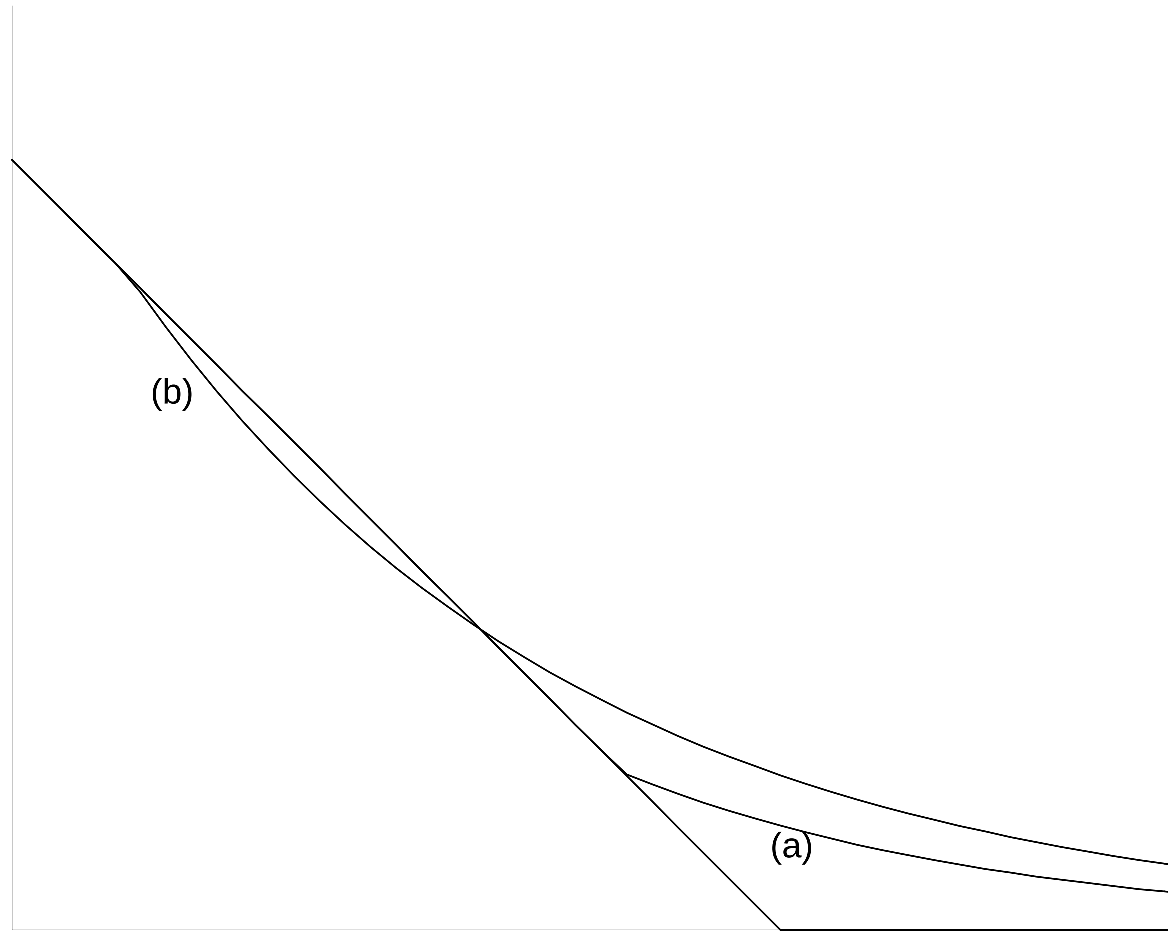
If we want to maximize our option's value by a careful choice of exercise strategy, then this is equivalent to solving the Black–Scholes equation with continuity of option *value* and option *delta*, the slope.

This is called the **high-contact** or **smooth-pasting condition**.

Another way of looking at the condition of continuity of delta is to consider what happens if the delta is not continuous at the exercise point. The two possibilities are shown in the next figure.

In this figure the curve (a) corresponds to exercise that is not optimal because it is premature, the option value is lower than it could be.

In case (b) there is clearly an arbitrage opportunity. If we take case (a) but progressively delay exercise by lowering the exercise point, we will maximize the option value everywhere when the delta is continuous.



Option price when exercise is (a) too soon or (b) too late.

Bermudan options

It is common for contracts that allow early exercise to permit the exercise only at certain specified times, and not at *all* times before expiry.

For example, exercise may only be allowed on Thursdays between certain times. An option with such intermittent exercise opportunities is called a **Bermudan option**.

All that this means mathematically is that the constraint (**??**) is only 'switched on' at these early exercise dates.

The relationship between option prices and expectations

Recall the binomial model.

That model also used the concept of delta hedging.

That model also resulted in the Black–Scholes partial differential equation (after going to the infinitesimal time step limit).

One of the most important insights from the binomial model was the idea that the option value can be interpreted as the present value of the risk-neutral expected payoff.

Can we get the same intuition from the Black–Scholes equation?

To get the same intuition from the Black–Scholes partial differential equation we need to remember one of the equations for the transition probability density function:

If

$$dS = \mu S\, dt + \sigma S\, dX$$

then the backward Kolmogorov equation for the transition probability density function $p(S, t)$ is

$$\frac{\partial p}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 p}{\partial S^2} + \mu S \frac{\partial p}{\partial S} = 0.$$

This is the equation that can be used for calculating the expected value of some quantity in the future.

Example 1: What is the expected value of the stock price at time $t = T$, given that it is S at time t? Let's call the answer $U(S, t)$.

We must solve

$$\frac{\partial U}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 U}{\partial S^2} + \mu S \frac{\partial U}{\partial S} = 0.$$

with

$$U(S, T) = S.$$

(The right-hand side of this is just what we are finding the expected value of.)

The answer is

$$U = e^{\mu(T-t)} S.$$

Example 2: What is the expected value of the square of the stock price at time $t = T$, given that it is S at time t? Again, let's call the answer $U(S,t)$.

We must solve

$$\frac{\partial U}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 U}{\partial S^2} + \mu S \frac{\partial U}{\partial S} = 0.$$

with

$$U(S,T) = S^2.$$

Only the final condition changes.

Example 3: What is the expected value of a call option at time $t = T$ (i.e. expiration), given that it is S at time t? Let's call the answer $U(S,t)$.

We must solve

$$\frac{\partial U}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 U}{\partial S^2} + \mu S \frac{\partial U}{\partial S} = 0.$$

with

$$U(S,T) = \max(S - E, 0).$$

That's how the backward Kolmogorov equation is used.

Now let's compare the backward Kolmogorov equation (for an expectation) and the Black–Scholes equation (for an option value):

$$\frac{\partial U}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 U}{\partial S^2} + \mu S \frac{\partial U}{\partial S} = 0$$

and

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0.$$

There are two differences:

1. The Black–Scholes equation has r instead of μ
2. The Black–Scholes equation has one extra term

The first difference is a simple change of parameter value.

The second change is the time value difference between now and expiration.

Options as expectations

There are three steps to getting from an expectation to an option value.

1. Replace μ with r

2. Calculate an expectation (via a partial differential equation or a simulation)

3. Take the present value of the expectation

This is the basis for the Monte Carlo method for valuing options.

Summary

Please take away the following important ideas

- Using tools from stochastic calculus we can build up an option pricing model from our lognormal asset price random walk model
- There are some 'simple' formulæ for the prices of simple contracts
- The greeks are important measures of the sensitivities of the option value to variables and parameters
- American options must always have a value greater than the payoff
- Option values can be interpreted in terms of expectations