CQF Module 5, Session 6: Valuing CDOs Using Copulas

CQF

In this lecture...

- we will learn about pricing credit instruments;
- formulate a pricing model for static CDOs;
- define copulas;
- apply copulas to price and manage the risk of credit derivatives;

I - The Pricing Problem

If you were asked to price a static CDO, how would you proceed?

First, you would need a general pricing framework. Also, you would need to ask yourself what make CDOs special or different from other derivatives. This is what we will develop in the second section. Next, you would formulate a general pricing model for the CDO. This is section 3.

But, as we will find out, we do not yet have all the tools we need to solve the pricing problem and come up with a price. Therefore, the next step is to find a set of techniques to help finish solving the problem: a statistical method based on the idea of copula functions, which will be introduced in section 4 before we see how this idea can be applied to credit derivatives in section 5.

Now that all the pieces are set, we can finalize our CDO pricing model be designing a pricing algorithm (Section 6).

II - Pricing Credit Instruments

1 - Modelling Default Probabilities

Let $p(t) := p(\tau > t)$ be the probability that a security S defaults **after** time t. $p(t)$ therefore defines the **survival probability** of S over $[0, t]$.

The default probability over $[0, t]$, $d(t)$ can be defined as

$$d(t) := 1 - p(t)$$

These can be called "spot" probabilities since they are defined from today (by analogy with interest rates).

By Bayes' rule we can find the probability $p(s|t)$ of surviving up to time s conditional on surviving to time t (for $t \leq s$):

$$p(s|t) = \frac{p(s)}{p(t)}$$

Hence, the probability $f(t, s)$ of defaulting between times t and s is equal to

$$f(t, s) = 1 - p(s|t)$$

We will define $f(t, s)$ as the **forward default probability** since its starting point is a future time t.

We will now do a bit of calculus.

We will assume that the survival probability $p(t)$ is strictly positive and is continuous.

As s tends to t, we see that the instantaneous forward default rate is equal to

$$f(t) = -\frac{\frac{dp}{dt}(t)}{p(t)} \tag{1}$$

This is a separable ODE which we can solve directly. Integrating over $[0, t]$ and rearraging we get:

$$p(t) = e^{-\int_0^t f(u)du}$$

and hence

$$p(s|t) = e^{-\int_t^s f(u)du} \tag{2}$$

Aside: *Observations and Extensions of the Basic Framework*
Two observations can be made at this stage.

First, now that we have spot default rates and forward default rates, we can build a *term structure of default risk*. Since the mathematics of default and the mathematics of discounting are the same in continuous time, this is all the more useful when pricing bonds. Indeed, assuming no recovery, we can simply take the credit spread to be an appropriately calculated default rate.

Second, if f is constant then the probability of default up to time t

$$\begin{aligned} d(t) &= 1 - e^{-ft} \\ &= G_f(t) \end{aligned}$$

where G_f is the cumulative distribution function of the **exponential distribution** with parameter f. Recall that the probability distribution function g_λ of an exponentially distributed random variable is given by

$$g_\lambda(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

Hence, defaults are actually exponentially distributed random variables. This makes sense if we recall the definition and use of exponential distributions.

If f is not constant, we still have a more general form of the exponential distribution called an *inhomogeneous exponential distribution* (since f can vary).

(end of aside.)

2 - Fundamentals of Credit Instrument Valuation

The value of a credit instrument, be it a CDS, CDO or a defaultable bond, is driven by

- the default probability (also called hazard rate) λ;
- the loss given default. The loss given default is a function of the notional amount and of the recovery rate R which is the share of the notional recovered in the event a default takes place:

$$\text{Loss Given Default} = \text{Notional} \times (1 - R)$$

In this section, we will present a more general framework to enable us to price credit risky securities.

In the Black-Scholes world with no credit risk, the time t value $V(t, S_t)$ of a derivative on an underlying security S can be expressed as the following expectation under the so-called "risk-neutral" measure Q:

$$V(t, S_t) = \mathbf{E}^Q \left[e^{-\int_t^T r(s)ds} H(S_T) \right]$$

where H is the derivative's payoff function.

A similar probabilistic framework can be developed for defaultable securities.

The traditional way of looking at it is to start by modelling the default time, τ, as a random variable. Indeed, to know if our security defaults, all we need to do is compare a realization of τ with the maturity or expiration time T of the security we are pricing.

- if $\tau > T$ then by the time the issuer defaults, the security has already matured or expired. We may as well consider that there is no default on this security;
- if $\tau \leq T$, then the issuer default before the security matures. The investor only gets the recovery value.

This method is simple, works well with both discrete and continuous time models and is rather elegant.

To make things simple at this point, and focus more readily on default risk, we are going to restrict the type of securities we will consider to zero coupon bonds, since they have a very simple payoff function.

A defaultable zero-coupon bond has a the terminal payoff of 1 if the issuer does not default and R if the issuer defaults. R can therefore be interpreted as the recovery amount.

In the Black-Scholes Valuation framework, the value $D(t,T)$ of a defaultable bond can therefore be written as:

$$\begin{aligned} D(t,T) &= \mathbf{E}^Q\left[e^{-\int_t^T r(s)ds}\mathbf{1}_{\{\tau>T\}}\right] \\ &\quad + \mathbf{E}^Q\left[e^{-\int_t^\tau r(s)ds}\mathbf{1}_{\{\tau\leq T\}}R\right] \end{aligned}$$

where $\mathbf{1}$ is the indicator function, i.e. $\mathbf{1}_{\{x>a\}} = 1$ if $x > a$ and 0 otherwise.

In plain English, the time t price of a defaultable bond is either the present value of 1 if the issuer does not default before the bond matures, or the present value of R otherwise.

Unfortunately, this does not seem particularly insightful.

Can we do a bit better?

Yes we can!

First, we know from equation (2) that

$$P(\tau > T) = e^{-\int_t^T \lambda(u)du}$$

where λ is the **risk-neutral** instantaneous forward default probability.

Also, we will make one more assumption: there is no recovery, so $R = 0$.

Aside: *Actual and Risk Neutral Default Probabilities*
Real default probabilities are readily observable. Rating agencies such as S&P or Moody's estimate them based on historical default and survival statistics.

Unfortunately, for pricing purpose we need "risk neutral" rather than actual probabilities.

The usual way to get the "risk neutral" probabilities is to back them out through calibration. In short, we select a very liquid and easy to price defaultable security (such as a CDS). The risk neutral probability is then determined as the probability for which the theoretical price given by our model matches the market price of the asset. By reference to the Black-Scholes model, we could call this approach an "implied default probability" approach.

(end of aside.)

When $R = 0$ and default risk and interest rate are **independent**, then we can express $D(t, T)$ as

$$\begin{aligned} D(t,T) &= \mathbf{E}^Q\left[e^{-\int_t^T r(s)ds}\mathbf{1}_{\{\tau>T\}}\right] \\ &= \mathbf{E}^Q\left[e^{-\int_t^T r(s)ds}\right]\mathbf{E}^Q\left[\mathbf{1}_{\{\tau>T\}}\right] \\ &= B(t,T)P\left(\tau > T\right) \end{aligned}$$

where $B(t, T)$ is the time t price of a default-free zero coupon bond.

Not only does this relationship make intuitive sense, it is also **model independent** (i.e. particulary interesting for model testing and validation).

This relationship can alternatively be expressed as:

$$\begin{aligned} D(t,T) &= \mathbf{E}^{Q}\left[e^{-\int_t^T r(s)ds}\mathbf{1}_{\{\tau>T\}}\right] \\ &= \mathbf{E}^{Q}\left[e^{-\int_t^T r(s)+\lambda(s)ds}\right] \end{aligned}$$

In plain English, what this tells us is that the rules of the Time Value of Money have slightly changed. To price a defaultable security, we should not discount future cash flows at the risk-free rate. Instead, we should discount them at a default risk-adjusted rate.

Conversely, default risk can be priced in through a spread added to the (default-free) discount rate.

Now, still assuming zero recovery, we can generalize our pricing approach to apply to a defaultable security V with arbitrary terminal payoff function $G(\cdot)$ (which may or may not dependent on other variables such as asset prices, interest rates... etc).

$$\begin{aligned} V(t) &= \mathbf{E}^{Q}\left[e^{-\int_t^T r(s)ds}\mathbf{1}_{\{\tau>T\}}G(\cdot)\right] \\ &= \mathbf{E}^{Q}\left[e^{-\int_t^T (r(s)+\lambda)ds}G(\cdot)\right] \end{aligned}$$

Aside: *Filtration*
As usual, we have placed ourselves in a probability space $(\Omega, \mathcal{F}_t, \mathbf{P})$ where Ω is the sample space, $\mathcal{F}_t$ is the natural filtration generated by the market and $\mathbf{P}$ is the probability measure.

If we wanted to be "probabilistically correct", we would therefore need to write

$$V(t) \quad = \quad \mathbf{E}^{Q}\left[e^{-\int_t^T r(s)ds}\mathbf{1}_{\{\tau > T\}} G|\mathcal{F}_t\right]$$

Since the default time τ is a random variable (actually it is a special type of random variable called a stopping time), it can generate its own filtration $\mathcal{H}_t$, called the default filtration. $\mathcal{H}_t$ represents all the information available at time t related to the possibility of defaulting.

If we "remove" the default filtration from the market filtration, we get a third filtration that we will call the default-free filtration, $\mathcal{G}_t$. It is called default-free because we have removed from the market filtration all information related to the default event.

This last filtration is important since it is the filtration under which the second part of our pricing equation holds:

$$V(t) \quad = \quad \mathbf{E}^{Q}\left[e^{-\int_t^T (r(s)+\lambda)ds} G|\mathcal{G}_t\right]$$

In plain English, what this formula tells us is that by adding a credit spread to the discount rate, we no longer need to worry about the actual default. We have priced a defaultable security from a default-free perspective.

(end of aside.)

4 - Valuing Basket Credit Instruments

The methodology we have just developed works well enough when our instruments are driven by the default of one entity. But what would happen if we were to value a credit instrument depending on the the default of not one but several entities, as happens when you consider a basket credit derivatives established on a basket of underlying entities?

Let's take for example a CDS established on a basket of two issuers and which pays a given amount if both issuers default. [1]

Clearly, the value of such contract will depend not only on the (marginal) default probabilities for each of the two entities but also on the default "correlation" between the two entities.

[1] Such a contract is a simplified version of a kth to default CDS. For those interested in this, a copula-based pricing method is presented in the Appendix.

Let's illustrate this point with a simple experiment.

Take two companies A and B, each with a 10% chance of defaulting by the end of the year.

If the defaults are **independent**, then the probability of A and B defaulting is equal to

$$
\begin{aligned}
& \Pr\left[\text{“A defaults” and “B defaults”}\right] \\
= \; & \Pr\left[\text{“A defaults”}\right] \times \Pr\left[\text{“B defaults”}\right] \\
= \; & 1\%
\end{aligned}
$$

Now, what would happen if the defaults are not independent?

Say that, for example, if one of the companies defaults, the other has 60% chance of defaulting. Then, applying Bayes' formula, we find that

$$\begin{aligned} & \mathsf{Pr}\left[\text{“A defaults” and “B defaults”}\right] \\ = \quad & \mathsf{Pr}\left[\text{“A defaults”}\right] \times \mathsf{Pr}\left[\text{“B defaults given that A defaulted”}\right] \\ = \quad & 6\% \end{aligned}$$

What this experiment shows that clearly, the dependence of defaults matters enormously!

The value of the basket CDS will be much higher if the two entities have highly "correlated" default probabilities (i.e. two car makers) than if the two entities have "uncorrelated" or negatively "correlated" default probabilities (think about a European supermarket chain and a US mining company).

Hence, even if you are able to estimate the individual default probabilities accurately (big IF!), forgetting the interaction between defaults will result in huge pricing errors.

So, how are we going to model defaults?

Since looking at the individual (marginal) default probabilities is not enough, we will have to look at the joint distribution of defaults.

Section III: Pricing a CDO

1 - Introduction

In this section, we will develop a copula-based model for a passively-managed CDO. We will not get into the pricing of active CDOs, which would require some utility theory and game theory as well as many more assumptions.

While single tranche passive CDOs can be priced relatively easily and accurately, models for multi-tranche CDOs tend to become inaccurate for higher quality tranches. Understanding and solving this problem is actually an active research area.

The approach we will develop here is a slightly simplified version of the Laurent and Gregory (2003) methodology, which is one of the most widely accepted CDO pricing models.

2 - Assumptions

To make the pricing a bit easier and focus more readily on the dependence structure of defaults, we will assume that interest rates and default times are independent. This assumption is not fully satisfactory as there is a documented positive correlation between defaults and interest rates.

Moreover, we will assume that the recoveries are independent of both default time and interest rates.

Finally, we assume that the coupon paid by each tranche is LIBOR plus a premium paid as a spread over LIBOR. This assumption is consistent with synthetic CDOs and also with some (but not all) cash CDOs.

3 - Setting

We want to price the D to U tranche of a CDO maturing at time T, where D is the lower attachment point and U is the upper attachment point of the tranche.

In its collateral, the CDO has securities related to n different issuers. The nominal amount related to investor i is A_i and the expected recovery rate is denoted by R_i. Thus, the loss associated with issuer i is computed as

$$L_i := (1 - R_i)A_i$$

We denote by τ_i the default time of issuer i and introduce a counting process $N_i(t)$ defined as

$$N_i(t) := \mathbf{1}_{\{\tau_i < t\}}$$

So, N_i indicates whether issuer i has defaulted or not by time t: $N_i(t)$ returns 1 if the issuer has defaulted and 0 otherwise.

The total loss of the collateral portfolio at time t, $L(t)$, can be expressed as:

$$L(t) := \sum_{i=1}^{n} L_i N_i(t)$$

Notice that with this definition, $L(t)$ can be interpreted as a pure jump process. In addition notice that $L(t)$ is an increasing process.

We define the cumulative loss on our U to D tranche at time t, $M(t)$, as:

$$M(t) := \begin{cases} 0 & \text{if } L(t) \leq D \\ L(t) - D & \text{if } D < L(t) < U \\ U - D & \text{if } L(t) \geq U \end{cases}$$

Note that $M(t)$ is also an increasing pure jump process.

Finally, we introduce the premium.

We will denote the LIBOR rate for maturity time t by $f(0,t)$ and the instantaneous time t LIBOR forward rate by $f(t)$. The exact day count between time t and time τ in the day count convention associated with the LIBOR rate is denoted by $\delta(t,\tau)$

The instantaneous premium is denoted by X.

We are now ready to price!

4 - Pricing

Pricing a CDO tranche is not much different from pricing a swap or a CDS.

Just like swaps and CDS, Pricing a CDO tranche is actually two different problems:

1. at inception, the question is to determine the fair coupon to be paid to investors so that they are exactly compensated for the risk they take;
2. during the life of the CDO, the problem is to determine how the value of a tranche has evolved.

We will consider each of these problems in turn.

The **main trick** in this approach is to realize that since the premium can be decomposed in a base interest rate (here LIBOR) and a credit risk premium, the CDO can itself be decomposed (for pricing purpose) into two separate securities:

- a non-defaultable fixed income security (in our case an amortizing floating rate note);
- an “exotic” credit default swap in which the investor is the protection seller and the CDO manager is the protection buyer.

Since the fixed income security is non-defaultable, all the investor can expect is a base interest rate (LIBOR for floating rate notes and an appropriate government bond or AAA swap rate for fixed rate notes). That's for the non-defaultable part.

Now for the defaultable part. we will use the same fair pricing logic as for CDS pricing. Namely, for the "exotic" credit default swap to be fairly priced, the present value of the premia paid (the so-called **premium leg**) must be equal to the present value of losses due to default (the so-called **default leg**).

The Default Leg

The value of the default leg is given by the following risk-neutral expectation

$$\mathbf{E}^{Q}\left[\int_0^T B(0,t)dM(t)\right]$$

where $B(0,t) := e^{-\int_0^t r(s)ds}$ is the discount factor.

What is this equation trying to tell us?

Let's consider it forward in time. At time t, we check whether $M(t)$ has jumped or not. If it has, then $dM(t) > 0$ and we need to account for a new default. Multiplying by $B(0,t)$ present values the loss. Now we move to $t \leftarrow t + dt$ and repeat the operation up until we get to maturity time T.

Aside:

If we wanted to push the stochastic analysis a bit further, we could even write (using Fubini's Theorem and stochastic integration by parts):

$$\begin{aligned} &\mathbf{E}^Q\left[\int_0^T B(0,t)dM(T)\right] \\ = \quad & B(0,T)\mathbf{E}^Q\left[M(T)\right] + \int_0^T f(0,t)B(0,t)\mathbf{E}^Q\left[M(t)\right]dt \end{aligned}$$

where

$$f(0,t)B(0,t) = -\frac{dB(0,t)}{dt}$$

(end of aside.)

The Premium Leg

The premimum leg is not overly difficult to model, but there is a slight trick: the premium received at time t depends on the total cumulative loss up to time t.

If there have been no loss on the tranche (i.e $M(t) = 0$), then the premium $\Pi(t, t + dt)$ paid on the tranche over a short time period dt is equal to $\Pi(t, t + dt) = X(U - D)dt$.

On the other hand, if losses have accumulated beyond the upper attachment point (i.e $M(t) = U - D$), then a premium is no longer paid on the tranche.

Finally, if losses are somewhere in between (i.e $0 < M(t) = L(t) - D < U$), then the premium paid on the tranche over a short time period dt is equal to $\Pi(t, t + dt) = X(U - L_t)dt$.

A more concise, but more mathematical, way of expressing this idea is as

$$\Pi(t, t + dt) = X \min\left[\max\left[U - L_t, 0\right], U - D\right] dt$$

The value of the premium leg can now be expressed as the expected present value of premia paid:

$$X\mathbf{E}^Q\left[\int_0^T B(0, t) \min\left[\max\left[U - L_t, 0\right] U - D\right] dt\right]$$

The Fair Premium

The fair premium X is determined so that the premium leg compensates exactly the credit risk incurred by the investor, i.e.

$$\begin{aligned} & \mathbf{E}^{Q}\left[\int_0^T B(0,t)dM(T)\right] \\ = \quad & X\mathbf{E}^{Q}\left[\int_0^T B(0,t)\min\left[\max\left[U-L_t,0\right]U-D\right]dt\right] \end{aligned}$$

and thus

$$X = \frac{\mathbf{E}^{Q}\left[\int_0^T B(0,t)dM(T)\right]}{\mathbf{E}^{Q}\left[\int_0^T B(0,t)\min\left[\max\left[U-L_t,0\right]U-D\right]dt\right]}$$

Evaluating the Integral Terms

The great difficulty that we face to get a working pricing algorithm out of this mathematical model is in the evaluation of the integrals

$$\int_0^T B(0,t)dM(T) \tag{3}$$

and

$$\int_0^T B(0,t)\min\left[\max\left[U - L_t, 0\right]U - D\right]dt \tag{4}$$

First, both

- the cumulative loss on the CDO collateral $L(t)$, and;
- the cumulative loss on the tranche $M(t)$.

depend on the **joint probability of defaults**, which, as we have seen earlier is generally a difficult function to deal with.

In addition (and as a consequence of the previous point), we cannot expect to compute these integrals analytically. We will have to compute them **numerically**. And since the variables $L(t)$ and $M(t)$ are stochastic processes, the numerical method we will tend to use is **simulations**.

Hence, to evaluate these integrals, what we would like to find is

1. a convenient representation of the joint probability of defaults;
2. a method to simulate joint defaults.

The answer to our first request comes in the form of a statistical technique called **copulas**. As for our second question, it will need to be addressed later, once we have learned a bit about copulas.

Section IV: Copulas

1 - What is a copula and what does it do?

A copula is a functional relationship which encodes a dependence structure into a set of marginal cumulative density functions (CDF) in order to generate a joint CDF.

Concretely, a copula is a special type of multivariate **functions** which is used to “aggregate” a number of marginal CDFs into **one joint CDF**. To simplify somewhat:

- the marginal CDFs represent the probability associated with each random variable;
- the copula represents the relationship (a.k.a.) dependence structure between the various random variables.

In this section, we base our discussion on bivariate (2-dimensional) copulas for clarity.

However, this discussion can be extended very easily to n-dimensional copulas. This is actually one of the great strength and major appeals of copulas: the key concepts and results do not get much more complicated as the dimension of the problem increases. Copulas are therefore particularly well-suited for very high dimensionality problems.

2 - What a copula is NOT...

- *a technique to actually elicit the dependence structure*: you need to either know or assume a dependence structure in order to use Copulas; copula is a statistical tool acting directly at the probability distribution level. It is therefore incompatible with stochastic analysis and dynamic hedging
- *a dynamic tool*: copulas are a static tool.

3 - What a copula is useful for...

- *high dimensionality problems* in which the underlying asset is a basket with $n > 2$ securities (i.e. basket options, CDOs, basket default swaps)...
- ... *and with no dynamic decision structure* which means static CDOs, basket default swaps and European basket options.

Copulas have been quickly adopted as a useful toolbox to value and assess the risk of basket credit derivatives such as CDOs and kth to default credit default swaps. The premise in this case is that developing a model accurately representing the dependence structure of defaults in the basket is more important than modelling accurately individual default risks.

4 - Preliminary Definitions

Definition (Grounded Function)

Let $A, B \subset \mathbb{R}$ bet two non empty subsets and G be a function such that $G : A \times B \rightarrow \mathbb{R}$. Let a be the least element of A and b be the least element of B.

A function is said to be **grounded** if $\forall (v, z) \in A \times B$,

$$G(a, z) = 0 = G(v, b)$$

Definition (2-increasing function)

A function $G : A \times B \to \mathbb{R}$ is called **2 increasing** if for every "rectangle" $[v_1, v_2] \times [z_1 \times z_2]$ such that $v_1, v_2 \in A$ with $v_1 \leq v_2$ and $z_1, z_2 \in B$ with $z_1 \leq z_2$,

$$G(v_2, z_2) - G(v_2, z_1) - (G(v_1, z2) - G(v_1, z1)) \geq 0$$

Note: the LHS of the equation can be intrepreted as the "total mass" in the rectangle $[v_1, v_2] \times [z_1 \times z_2]$. 2-increasing functions therefore have positive mass in all rectangles.

5 - Subcopulas and Copulas

Definition (2-dimensional subcopula)

A **2-dimensional subcopula** is a bivariate real-valued function defined on $A \times B$ where $A, B \subset I = [0,1]$, A, B, nonempty and $\{0,1\} \in A, B$ if:

$$\mathcal{C} : A \times B \longrightarrow \mathbb{R}$$

is

- grounded (i.e. $C(v,0) = 0 = C(0,z) \quad \forall (v,z) \in A \times B$);
- such that $C(v,1) = v$, $C(1,z) = z \quad \forall (v,z) \in A \times B$;
- 2-increasing.

Definition (2-dimensional copula)

A **2-dimensional copula** is a 2-dimensional subcopula with $A = B = I$

6 - Examples

The functions:

- $\mathcal{C}(v, z) = \max(v + z - 1, 0)$;
- $\mathcal{C}(v, z) = \min(v, z)$;
- $\mathcal{C}(v, z) = vz$.

are copulas and therefore, they are also subcopulas.

7 - How does all this relates to probabilities?

First, we observe that copulas can be interpreted as joint density functions of standard uniform random variables:

$$\mathcal{C}(v,z) = P[U_1 \leq v, U_2 \leq z]$$

Also, recall that if the random variables X and Y have a marginal CDF given respectively by $F_1(X)$ and $F_2(Y)$ then

$$
\begin{aligned}
\mathcal{C}(F_1(x), F_2(y)) &= P[U_1 \leq F_1(x), U_2 \leq F_2(y)] \\
&= P[F_1^{-1}(U_1) \leq x, F_2^{-1}(U_2) \leq y] \\
&= P[X \leq x, Y \leq y] \\
&= F(x, y)
\end{aligned}
$$

where F is the joint CDF of X and Y.

This is the fundamental relationship that we use all the time when we apply copulas to pricing and risk management.

8 - How to express probabilities in terms of copulas?

Now that we have seen the fundamental relationship between probability and copulas, we can try to express various joint probabilities we may want to compute in terms of copulas. For example,

$$\begin{aligned}
P[U_1 \leq v, U_2 > z] &= v - \mathcal{C}(v, z) \\
P[U_1 > v, U_2 \leq z] &= z - \mathcal{C}(v, z) \\
P[U_1 \leq v | U_2 \leq z] &= \frac{\mathcal{C}(v, z)}{z} \\
P[U_1 \leq v | U_2 > z] &= \frac{v - \mathcal{C}(v, z)}{1 - z}
\end{aligned}$$

9 - THE Key Result: Sklar's Theorem

Sklar's theorem is important because:

1. it shows that every joint CDF can be split into its marginal CDF and a copula function;
2. it shows that if you have some marginal CDF and a copula function, you can create a joint CDF.

And now for a statement of Sklar's Theorem...

Theorem (Sklar's Theorem)

Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Let X *and* Y *be two* $\mathbb{R}$*-valued (measurable) random variables on* $(\Omega, \mathcal{F}, \mathbb{P})$. *Denote by* F_1, F_2 *and* F *the two marginal and the joint distribution.*

Let F_1 *and* F_2 *be two marginal density functions, then* $\forall (x, y) \in \mathbb{R}^2$:

- *if* $\mathcal{C}$ *is any subcopula whose domain includes the cross product of the ranges of* F_1 *and* F_2 *it follows that*

  $$\mathcal{C}(F_1(x), F_2(y))$$

  is a joint density function with marginal $F_1(x), F_2(y)$;

Theorem (Sklar's Theorem (continued...))

- *conversely, if $F(x, y)$ is a joint density function with margins $F_1(x), F_2(y)$, then there exists a unique subcopula $\mathcal{C}$ whose domain is equal to the cross product of the ranges of F_1 and F_2, such that*

$$F(x, y) = \mathcal{C}(F_1(x), F_2(y))$$

If moreover $F_1(x)$ and F_2 are continuous, then the subcopula is a copula. Otherwise, there exists a copula C such that

$$C(v, z) = \mathcal{C}(v, z)$$

for every (v, z) in the cross product of the ranges of F_1 and F_2.

we also have an equally important corollary which gives us the exact correspondence between the marginals, the joint CDF and the copula:

Corollary

Under the hypotheses of the converse in Sklar's theorem (second part), the unique subcopula $\mathcal{C}$ such that

$$F(x, y) = \mathcal{C}(F_1(x), F_2(y))$$

is

$$\mathcal{C}(v, z) = F(F_1^{-1}(x), F_2^{-1}(y))$$

Note that if the ranges of F_1 and of F_2 are equal to I then the subcopula is a copula.

To sum up, the sole role of the copula function is to provide a dependence structure.

This split between marginal and dependence is in fact the true power of the copula approach. It gives the modeller much more freedom than the traditional joint distribution approach, since now the modeller can use the two dimensions of dependence and marginal behaviour to fit the observed data.

10 - Survival Copulas

Another important concept is that of survival copulas. This concept is particularly useful when dealing with the pricing of credit derivatives, since we are equally thinking in terms of default probabilities as we are thinking in terms of survival probabilities.

The intuition is as follows. The joint survival probability, $\bar{F}$ is defined as

$$\bar{F}(v, z) = P[U_1 > v, U_2 > z]$$

Now, we would like to rewrite this definition in terms of copulas

$$
\begin{aligned}
& P[U_1 > v, U_2 > z] \\
= \; & P[U_1 > v] + P[U_2 > z] - 1 + P[U_1 \leq v, U_2 \leq z] \\
= \; & 1 - P[U_1 \leq v] + 1 - P[U_2 \leq z] - 1 + P[U_1 \leq v, U_2 \leq z] \\
= \; & 1 - v + 1 - z - 1 + C(v, z) \\
= \; & 1 - v - z + C(v, z)
\end{aligned}
$$

Definition (Survival copula)

The survival copula, $\bar{\mathcal{C}}$, associated with the copula $\mathcal{C}$ is defined as

$$\bar{\mathcal{C}}(v,z) = v + z - 1 + \mathcal{C}(1-v, 1-z)$$

This definition is a direct consequence of the derivation on the previous slide but computed at $(1-v, 1-z)$. Hence, the relation between joint survival function and survival copula is given by

$$\bar{F}(v,z) = \bar{\mathcal{C}}(1-v, 1-z)$$

Note that Sklar's theorem can also be expressed in terms of survival copulas rather than copulas.

Section V: Modelling Correlated Defaults

Now comes the time to apply what we have just seen about copulas to modelling multiple defaults.

To keep the discussion general enough, let's say that we want to price a credit derivative dependent on the survival or default of m different issuers. We define:

- by τ_i the default time of issuer i, with $i = 1, \ldots, m$;
- by $h_i(t)$ the default intensity (i.e. the risk-neutral instantaneous forward default rate) of issuer i at time t;
- by θ_i the default threshold of issuer i. It is a measure of how easy it is for issuer i to default.

The default time can be thought of as the first possible time in which the cumulative default probability exceeds the default threshold:

$$\tau_i = \inf \left\{ t : \int_0^t h_i(s) ds \geq \theta_i \right\} \tag{5}$$

From here, we can take three different avenues to model multiple defaults:

1. correlate default intensity;
2. model the joint survival function;
3. correlate the default thresholds.

Correlate Default Intensity:

Taking the default intensities $h_i, i = 1, \ldots, m$ to be stochastic processes, we can relatively easily correlated their driving Brownian motions or Poisson processes.

However, this approach is not fully satisfactory. Somewhere the link between intensity and default time breaks: the correlation structure we impose on default intensity does not translate into a strong dependence structure between default times.

Model the Joint Survival Function:

The joint survival function of all m issuers, U, is defined as

$$U(t_1, \ldots, t_m) := P\left(\tau_1 > t_1, \ldots, \tau_m > t_m\right)$$

By **Sklar's Theorem**, the survival function admits a copula representation as

$$U(t_1, \ldots, t_m) = \bar{\mathcal{C}}_{\tau_1,\ldots,\tau_m}\left(U_1(t_1), \ldots, U_m(t_m)\right)$$

where $\bar{\mathcal{C}}$ is the survival copula and U_i is the marginal survival function for issuer i.

This approach was first proposed by Li (2000) who used a Gaussian copula. Li's approach is also at the heart of the CreditMetrics methodology.

Correlate the Default Thresholds:

In this approach proposed by Giesecke and by Schönbucher and Shubert, both in 2001, a copula is assumed for the survival function K of the default thresholds θ_i. Specifically,

$$\begin{aligned} K(k_1, \ldots, k_m) &= P\left(\theta_1 > k_1, \ldots, \theta_m > k_m\right) \\ &= \bar{\mathcal{C}}_{\theta_1,\ldots,\theta_m}\left(K_1(k_1), \ldots, K_m(k_m)\right) \end{aligned}$$

The default time survival copula can then be expressed as

$$\begin{aligned} & \bar{\mathcal{C}}_{\tau_1,\ldots,\tau_m}\left(S_1(t_1), \ldots, S_m(t_m)\right) \\ = \quad & \mathbf{E}\left[\bar{\mathcal{C}}_{\theta_1,\ldots,\theta_m}\left(e^{-\int_0^{t_1} h_1(s)ds}, \ldots, e^{-\int_0^{t_m} h_m(s)ds}\right)\right] \end{aligned}$$

When default intensities **are assumed to be deterministic** the last two approaches coincide:

$$\bar{\mathcal{C}}_{\tau_1,\ldots,\tau_m} = \bar{\mathcal{C}}_{\theta_1,\ldots,\theta_m}$$

This is actually the assumption that we will make from now on. This assumption is quite acceptable from a practical pricing perspective as it helps us focus on modelling what matters the most: the dependence structure of defaults across issuers.

Let's now spend some time looking at the popular Li model.

A Brief Description of the Li Model

Denote by F_i the marginal distribution function of τ_i.

Li assumes a Gaussian copula with default time correlation matrix Σ. Under his approach, the joint distribution of default times is given by

$$F(t_1, \ldots, t_m) = \Phi_m\left(\Phi^{-1}(F_1(t_1)), \ldots, \Phi^{-1}(F_m(t_m))\right) \tag{6}$$

where Φ is the Normal univariate CDF and Φ_m is the m-dimensional Normal CDF.

This now leads us to the issue of simulating correlated default times.

Notice that each of the arguments taken by the copula function Φ_n is a random variable. Define the the ith argument in the copula function as a random variable Y_i, i.e.

$$Y_i = \Phi^{-1}(F_i(t_i))$$

Inverting the relationship (which can be done because there is a one-to-one correspondence between t and Y), we get:

$$\tau_i = F_i^{-1}(\Phi(Y_i))$$

Therefore, we can get τ_i from the transformation of a Normally distributed random variable Y_i!

Now, how do we correlate default times?

Simple. Instead of simulating each Y_i independently in order to independently get the τ_i's, we generate m correlated normally distributed random variables $Y_i, i = 1, \ldots, m$ using the correlation matrix Σ. This correlation structure will in turn translate into correlated default times τ_i.

With this information we can (almost) price any credit derivative we can think of.

To recap, the simulation process is as follows:

1. using a random number generator, simulate m normally distributed random variables $Y_i, i = 1, \ldots, m$ with correlation matrix Σ;
2. use the mapping $\tau_i = F_i^{-1}(\Phi(Y_i))$ to translate each random variable into a default time τ_i;
3. If required, feed into the Copula relationship (6);
4. Use this information alongside the payoff function and interest rates to compute the value of the derivative to price;
5. Repeat many, many times to obtain many many possible values for the derivative.
6. Take the expectation across all values of the derivative to find the price.
7. Done!

Important Remark on the Li Model

As presented above, the Li model could as well be used for pricing as for risk management. The structure of the Li model is independent from the probability measure you use or from the application we intend to develop.

For pricing applications, we would use risk-neutral default probability functions and take a risk-neutral expectation in Step 6.

For risk management applications, we would use actual probability functions, use loss functions rather than payoff functions in Step 4, and take a "regular" expectation under the physical measure in Step 6.

Alternatively, Li suggested the use of equity price correlation. After all, equity prices are readily available, so estimating correlations should be relatively easy. This solution presents as many problems as it solves. The main hurdle is that equity and credit markets are distinct: what is good for shareholders is not necessarily good for bondholders and vice versa. As a result, equity correlation may not be a god data to price credit derivatives.

Another possibility is to estimate the correlation between CDS spreads, since CDS are now a fairly liquid asset class. Once again, this is just a proxy for the real statistic, but at least it is one step closer since there is now a genuine credit connection.

The search is still on for a better estimation method!

Estimating the Correlation Matrix in the Li Model

One of the main problems with the Li model is how to estimate the correlation matrix Σ.

Ideally, what we really want is a the **correlation between default times**. However, accurately estimating default time correlation is virtually impossible:

- To start with, default times can obviously not be estimated until an issuer defaults, which is a very rare event.
- Moreover, generally, an issuer will only default once. Therefore, after an issuer defaults, any statistics on its default becomes essentially useless from a pricing perspective;
- And finally, good estimates of correlations require a large number of data points.

Section VI: Concluding on CDO Pricing

The integral terms (3) and (4) are rather difficult to evaluate analytically. For concrete applications, we will need to evaluate them numerically. Here is one simple (but computationally intensive) way to do this.

We are going to modify the Li algorithm (but we could as well use any of our favourite copula-based algorithm) to price the CDO.

1. using a random number generator, simulate m normally distributed random variables $Y_i, i = 1, \ldots, m$ with correlation matrix Σ;
2. use the mapping $\tau_i = F_i^{-1}(\Phi(Y_i))$ to translate each random variable into a default time τ_i;
3. Under this scenario, compute the number of default as a parameter d. Order all d default times τ_i so that $\tau_1 \leq \tau_2 \leq \ldots \leq \tau_d \leq \tau_{d+1} = T$. Now, evaluate the discretized version of the default leg integral, i.e.

$$\sum_{i=1}^{d+1} B(0, \tau_i) \left(M(\tau_i) - M(\tau_i + 1) \right)$$

4. Still under this scenario, evaluate the discretized version of the premium leg integral, i.e.

$$\sum_{i=1}^{m} \delta(t_{i-1}, t_i) B(0, t_i) \min\left[\max\left[D - L_{t_i}, 0\right] D - C\right]$$

   where m is the number of premium paid and t_i is the payment time of the ith premium.

5. Repeat many, many times to obtain many many values for each leg.

6. Compute the expected value of the default leg and of the premium leg.

7. Compute the fair premium X.

8. Done!

Section VII: Pricing During the Life of the CDO

This problem is seldom tackled in the CDO-oriented literature, so we will have to adapt the previous approach to answer this question.

Based on Laurent and Gregory, the price of the CDO tranche at time $0 \leq t \leq T$ can be decomposed in two: the price of the fixed income security $F(t)$ and the price of the credit component $C(t)$

$$V(t) = F(t) + C(t)$$

Let's assume that by the time s we are interested in selling our share in the D to U tranche,

- k issuers have defaulted,
- resulting in total losses of $L(s)$, and;
- resulting in losses for the tranche amounting to $M(s)$.

The value of the fixed income component can be readily computed.

Logically, the value of the credit component should be equal to:

$$\begin{aligned} C(s) &= X\mathbf{E}^{Q}\left[\int_{s}^{T} B(0,t)\min\left[\max\left[U-L_t,0\right]U-D\right]dt\right] \\ &\quad -\mathbf{E}^{Q}\left[\int_{s}^{T} B(0,t)dM(T)\right] \end{aligned}$$

Now, before rushing to rerun our pricing algorithm, we need to make few adjustments reflecting the fact that:

- we know X;
- since k issuers have defaulted we now only have a pool of $n - k$ issuers left;
- the total losses start at $L(s)$ rather than 0, and;
- the losses for the tranche start at $M(s)$ rather than 0.

Conclusion: What to Think of This Type of Model?

On the upside, this type of model is simple and take advantage of some interesting properties of copulas, such as

- the ability to de-couple the choice of marginal probability functions and of a dependence structure;
- the fact that the multivariate ($n \geq 3$) theory is a straight extension of bivariate theory, making high dimensionality problems not significantly more difficult to crack than low dimensionality problems;
- the very wide choice of dependency structures.

On the downside, this type of models is far from satisfactory:

- *we have assumed that interest rates and default risks are independent* while we know that they are not;
- because it is based on copulas, *this model is limited to a one-period time frame and cannot accommodate embedded decision*. Actively managed CDOs cannot be priced;
- *how do we choose the copula function?* So far we have limited ourselves to a Normal copula, which is just a multivariate normal distribution. However, if we wanted to choose another function for our copula, how would we proceed? The copula literature does not provide a clear answers, and the conventional wisdom is to try several copula functions and see which one presents the best to your data.
- *data!* Calibration and parameter estimation are difficult since the closest that can be used to approximate actual default probabilities are CDS rates. We would not only need default probabilities for all the securities held in the collateral... but

However, it is still very much considered as a “standard” model because

- no other model has proved that it can really do better!
- *the lack of data* is the major limitation when it comes to CDO pricing and they will affect every model. So beware!;
- *empirical analysis* have shown that most models gives prices that are quite close to market prices for the equity tranche but that the pricing accuracy drops rapidly as we get into mezzanine and senior tranches.

The search for a good CDO pricing model is still on!

In this lecture, we have seen...

- a valuation framework for credit derivatives;
- a pricing model for static CDOs;
- the meaning and basic properties of copulas;
- Sklar's theorem;
- how to model joint default with copulas;
- the Li model;
- the upside and downside of this type of model.