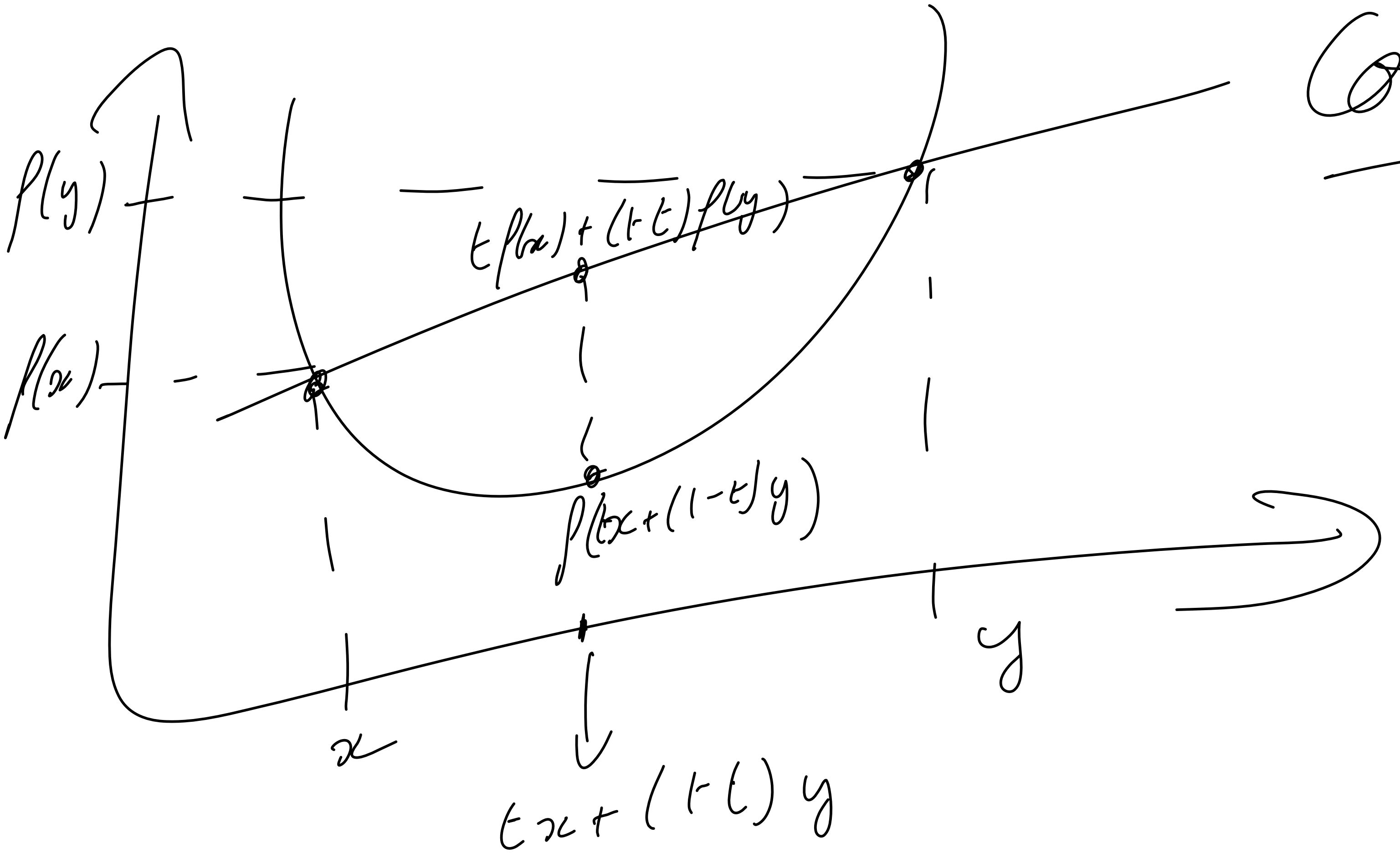
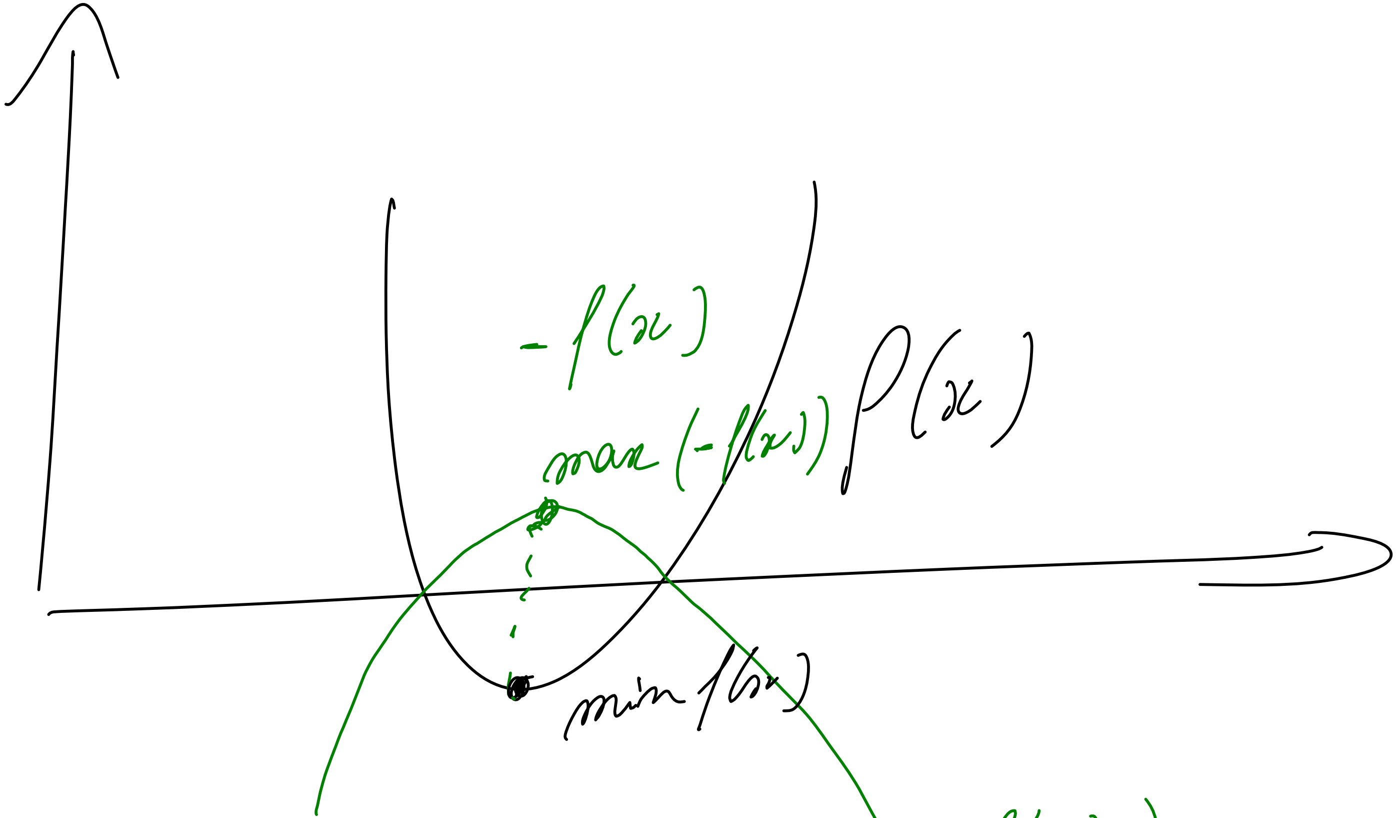
Convexity



$t \in (0,1)$



$$\min f(x) = -\max(-f(x))$$

Calculus: a function $\rightarrow$ stationary point.

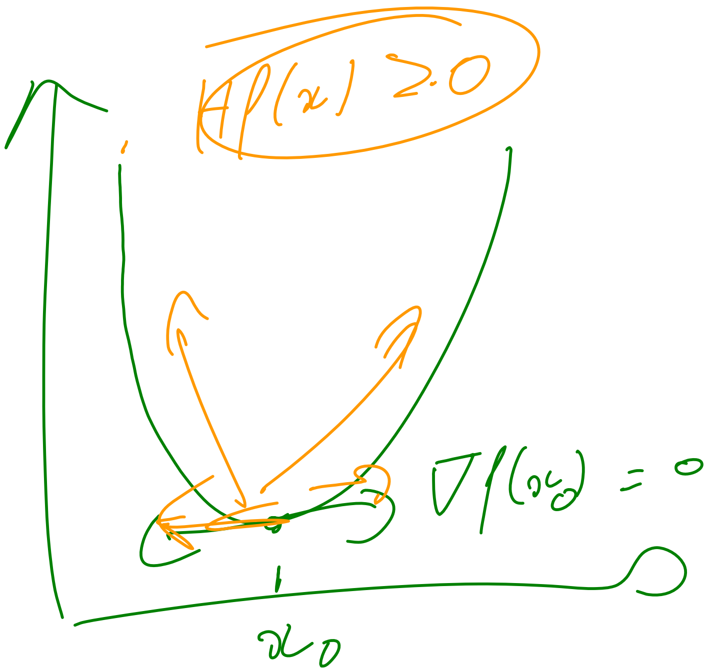
a stationary point is any point where

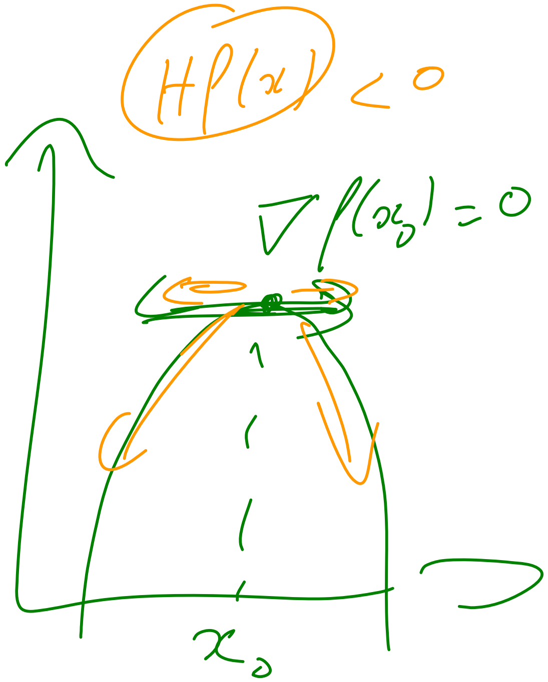
1st order derivative $\nabla f(x) = 0$

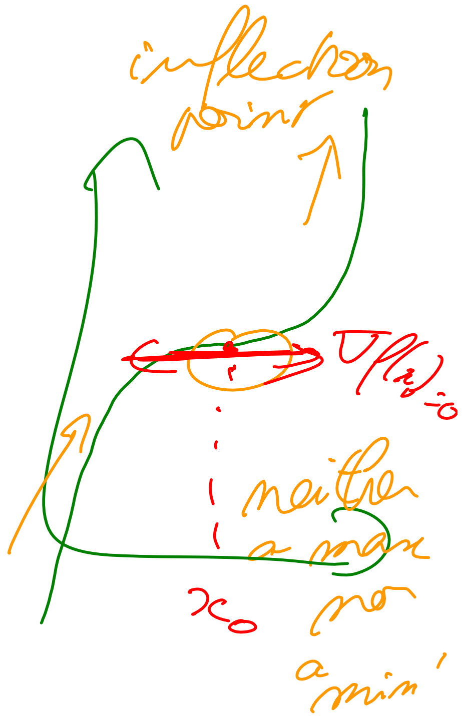
2nd order: $Hf(x)$

$\rightarrow$ 3 types of stationary points:

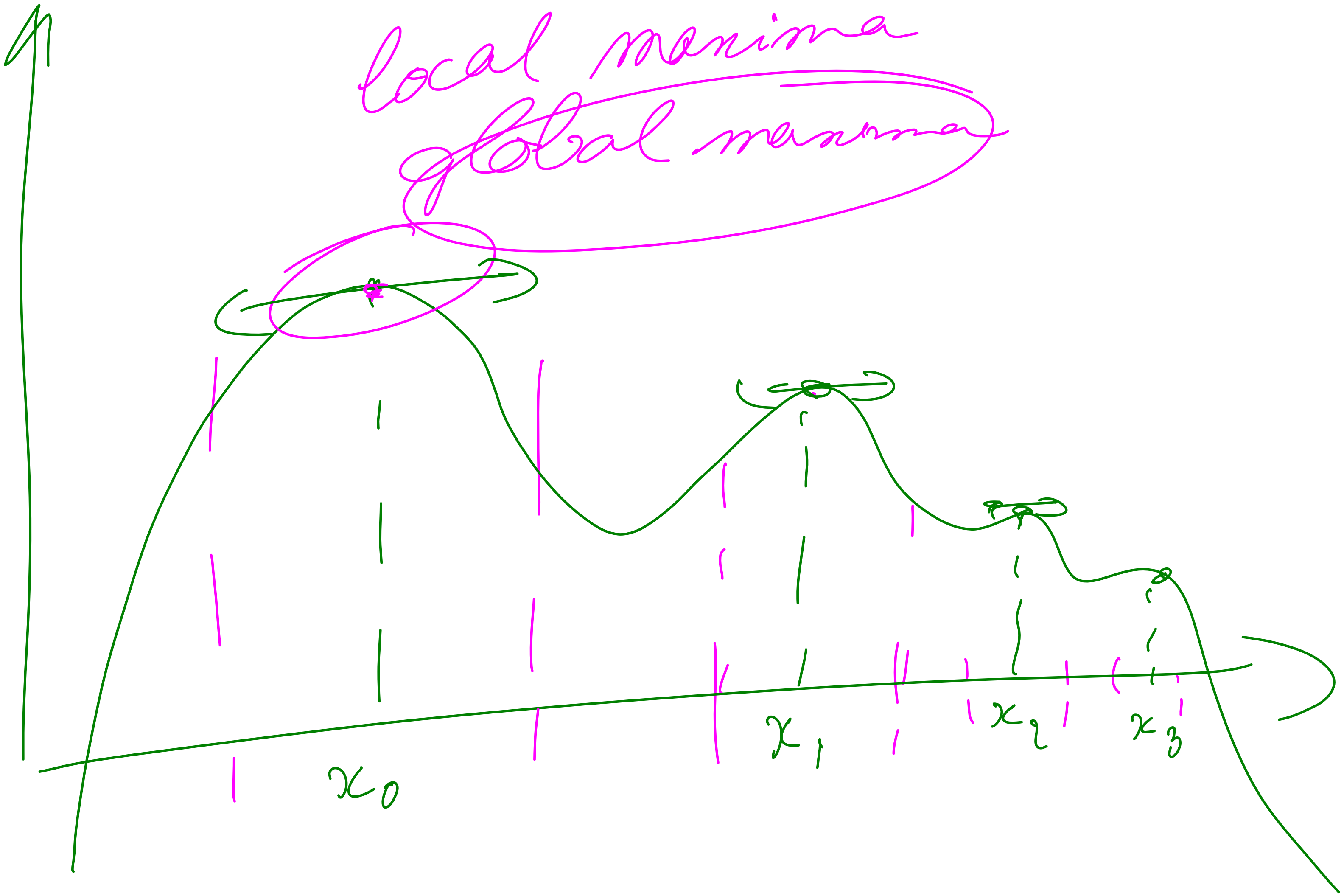
(higher: saddle points)







Among maxima and minima ;



To recap: $\min_{x} \ (\max_{x}) \ f(x)$

① Necessary condition

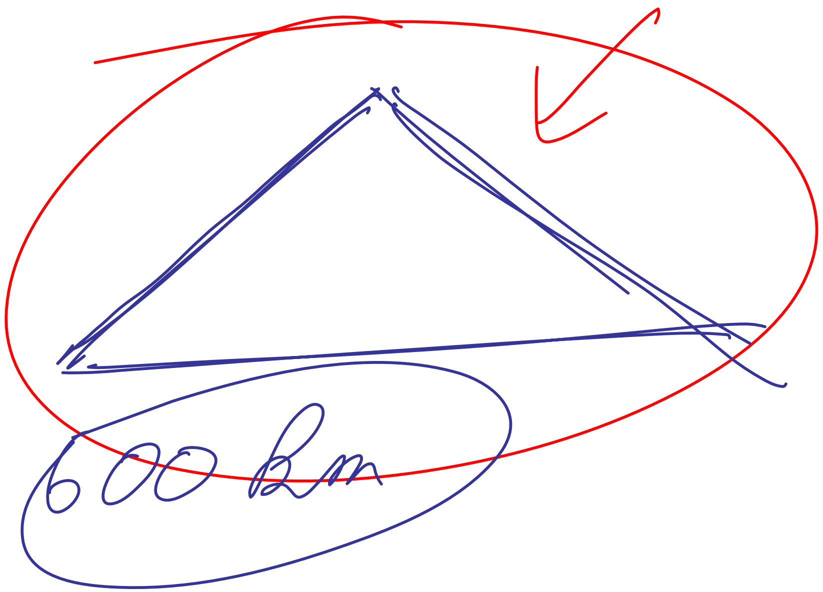
$$\nabla f(x_0) = 0$$

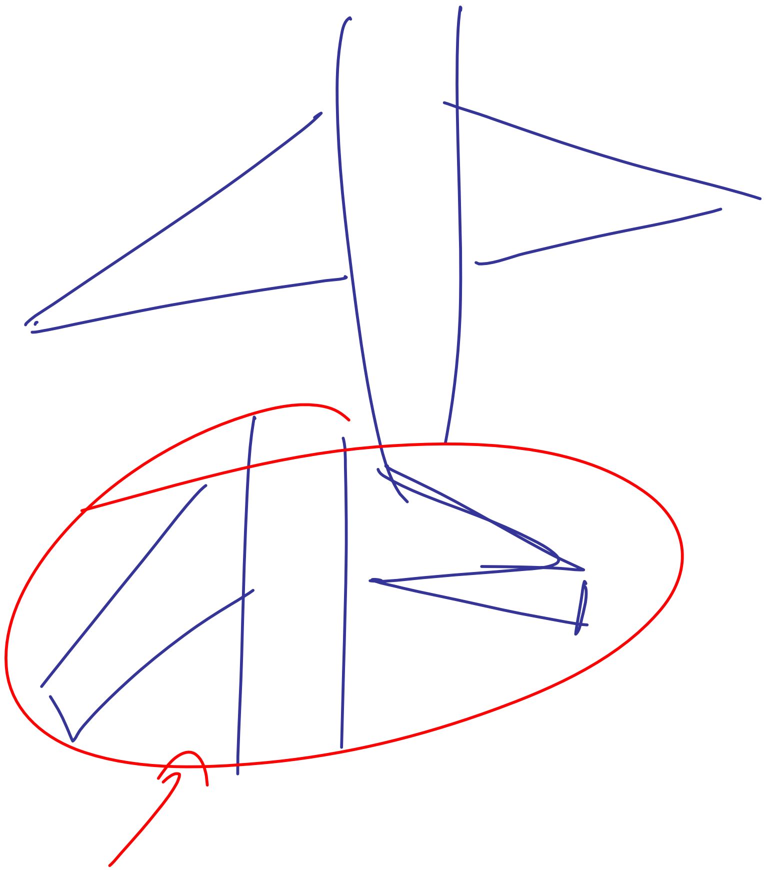
$$\nabla f(x_0) = 0$$

② Sufficient condition

$$Hf(x_0) > 0$$

$$< 0$$


600 km



$$\begin{pmatrix} z_1 & \cdots & z_n \end{pmatrix} H \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} =$$

$$1 \times n \quad n \times n \quad n \times 1 \quad = \underline{1 \times 1}$$

scala.

- $H > 0$ iff $z' H z > 0 \quad \forall z \in \mathbb{R}^n$
- $H < 0$ iff $z' H z < 0 \quad \forall z \in \mathbb{R}^n$.

Slide 15 Pb in standard form

$$\max_{x,y} \; A(x,y) = xy$$

subject

$$P(x,y) = 2(x+y) = p$$

1 constraint

Slide 18

~~1st we form the Lagrange function~~

$$L(x, y, \lambda) = A(x,y) - \lambda\left(P(x,y) - p\right)$$

decision variable

$$= xy - \lambda\left(2(x+y) - p\right)$$

$$\max_{\cancel{x,y,\lambda}} L(x, y, \lambda)$$

1st order Condition :

$$0 = \frac{\partial L}{\partial x} = y - 2\lambda \Rightarrow y = 2\lambda \Rightarrow y = \frac{p}{4}$$

$$0 = \frac{\partial L}{\partial y} = x - 2\lambda \Rightarrow x = 2\lambda \Rightarrow x = \frac{p}{4}$$

$$0 = \frac{\partial L}{\partial \lambda} = 2(x+y) - p \Rightarrow \lambda = \frac{p}{8}$$

Candidate solution $\left(\frac{p}{4}, \frac{p}{4}, \cancel{\frac{p}{8}}\right)$

2nd order condition

$$H = \begin{pmatrix} \frac{\partial^2 A}{\partial x^2} & \frac{\partial^2 A}{\partial x \partial y} \\ \frac{\partial^2 A}{\partial y \partial x} & \frac{\partial^2 A}{\partial y^2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

neither !

$$(\alpha \ \beta) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\left\{\begin{array}{l} x = \frac{p}{4} - \varepsilon \\ y = \frac{p}{4} + \varepsilon \end{array}\right\} \longrightarrow \quad \varepsilon \text{ small}$$

$$P(x, y) = p. \checkmark$$

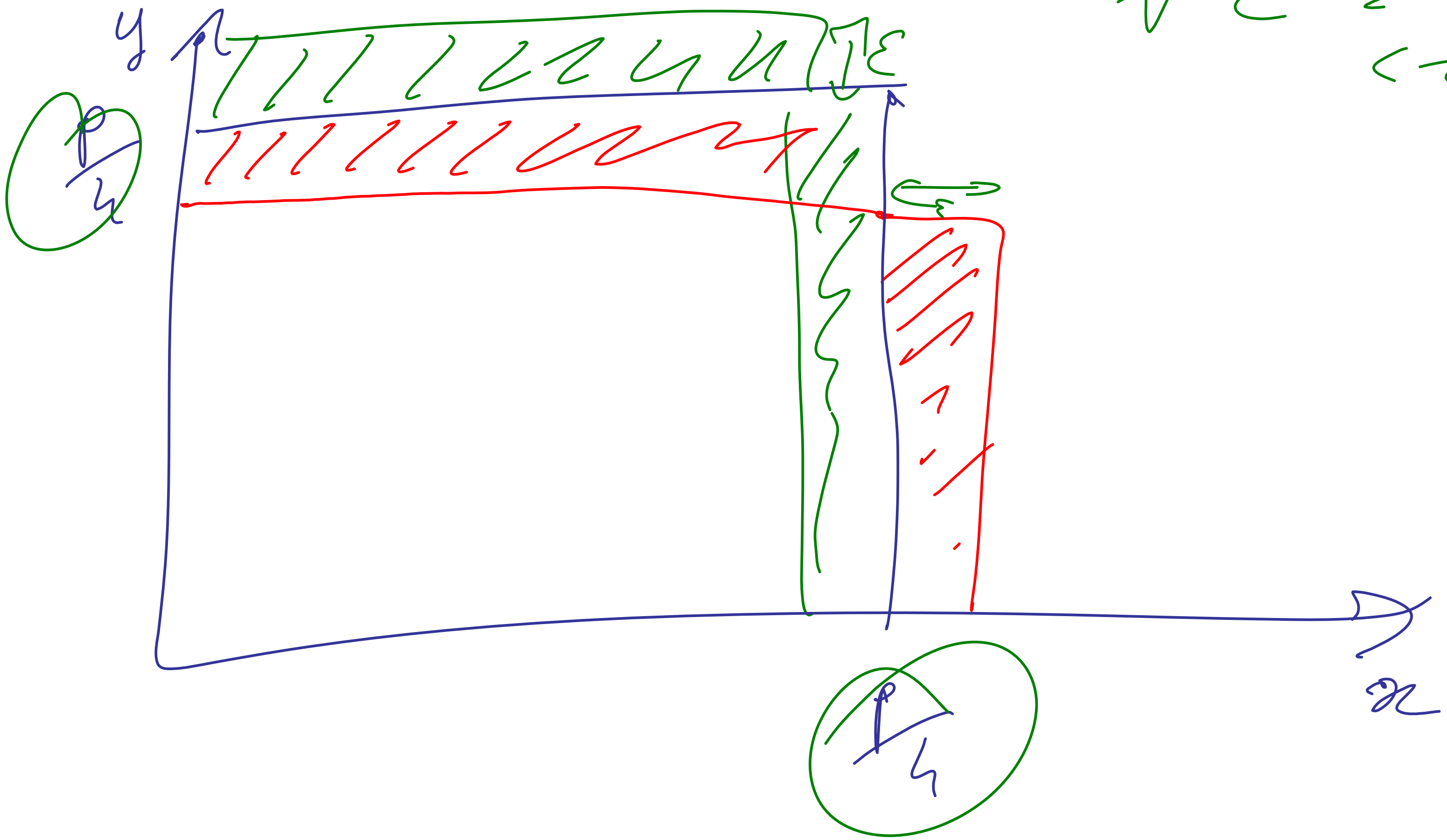
$$A(x,y) = \left(\frac{p}{4} - \varepsilon\right)\left(\frac{p}{4} + \varepsilon\right) = \frac{p^2}{16} - \varepsilon^2$$

$$\varepsilon^2 \geq 0$$

$$\leq \frac{p^2}{16}$$

$$A\left(\frac{p}{4}, \frac{p}{4}\right) = \frac{p^2}{16}$$

$\rightarrow$ max reached at $\left(\frac{p}{4}, \frac{p}{4}\right)$



$\forall \varepsilon > 0$

< 0

Portfolio return

Portfolio standard deviation.

Portfolio return $\mu_\pi = \mu^T w$

$$(\mu_1 \cdots \mu_i \cdots \mu_n) \begin{pmatrix} w_1 \\ \vdots \\ w_i \\ \vdots \\ w_n \end{pmatrix} = \sum_{i=1}^{n} w_i \mu_i$$

scalar.

$$\sigma_\pi = \sqrt{\sigma_\pi^2}$$

$n = 2$ or $n = 3$

$$= \sqrt{w^T \Sigma w}$$

$$= \sqrt{(w_1 \; \ldots \; w_n) \begin{pmatrix} \sigma_1^2 & \rho_{12}\sigma_1\sigma_2 & \cdots \\ \vdots & \ddots & \vdots \\ \cdots & & \sigma_n \end{pmatrix} \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}}$$

$$= \sqrt{\sum_{i=1}^{n} w_i^2 \sigma_i^2 + 2 \sum_{\substack{i=1 \\ j>i}} w_i w_j \rho_{ij} \sigma_i \sigma_j}$$

1st order condition:

$$0 = \frac{\partial L}{\partial w} = w^T \Sigma - \lambda \mu^T - \gamma \mathbb{1}^T$$

$$0 = \frac{\partial L}{\partial \lambda} = m - \mu^T w$$ ← return constraint

$$0 = \frac{\partial L}{\partial \gamma} = 1 - \mathbb{1}^T w$$ ← budget constraint

$$L(\omega, \lambda, \gamma) = \frac{1}{2}\,\omega^T \Sigma\, \cancel{\omega} + \lambda(m - \mu^T \cancel{\omega}) + \gamma(1 - \cancel{1^T\omega})$$

$$\underbrace{x\,a\,x}\qquad m - b\,x \qquad 1 - 1\,x$$

$$\approx x^2 a$$

$$\frac{d(\,)}{dx} = 2xa \qquad \frac{d(.)}{dx} = -b \qquad \frac{d(.)}{dx} = -1$$

$$\frac{\partial L}{\partial \omega} = \cancel{\frac{1}{2}} \times \cancel{2}\,[\omega^T \Sigma] - \lambda \mu^T - \gamma\, 1^T$$

$1 \times \cancel{m} \times \cancel{m} \times m$

$1 \times m$ $\quad$ $1 \times m$ $\quad$ $1 \times m$

$$w^T \Sigma - \lambda \mu^T - \gamma \mathbb{1}^T = 0$$

$$\Sigma w - \lambda \mu - \gamma \mathbb{1} = 0$$

$$\Sigma w = \lambda \mu + \gamma \mathbb{1}$$

$$\cancel{\Sigma^{-1} \Sigma} w = \Sigma^{-1} (\lambda \mu + \gamma \mathbb{1})$$

$$\underbrace{}_{I}$$

$$w = \Sigma^{-1} (\lambda \mu + \gamma \mathbb{1})$$

candidate point

Checking the 2^{nd} order condition

$$L(w, \lambda, \gamma) = \frac{1}{2} w^T \Sigma w + \lambda (m - \mu^T w) + \gamma (1 - \mathbb{1}^T w)$$

$$\frac{\partial L}{\partial w} = w^T \Sigma - \lambda \mu^T - \gamma \mathbb{1}^T$$

$$\frac{\partial^2 L}{\partial w^2} = \Sigma > 0$$

covariance matrix

problem has a unique minimum

$$\frac{\partial L}{\partial \lambda} = 0 \iff \mu^T \omega = m$$

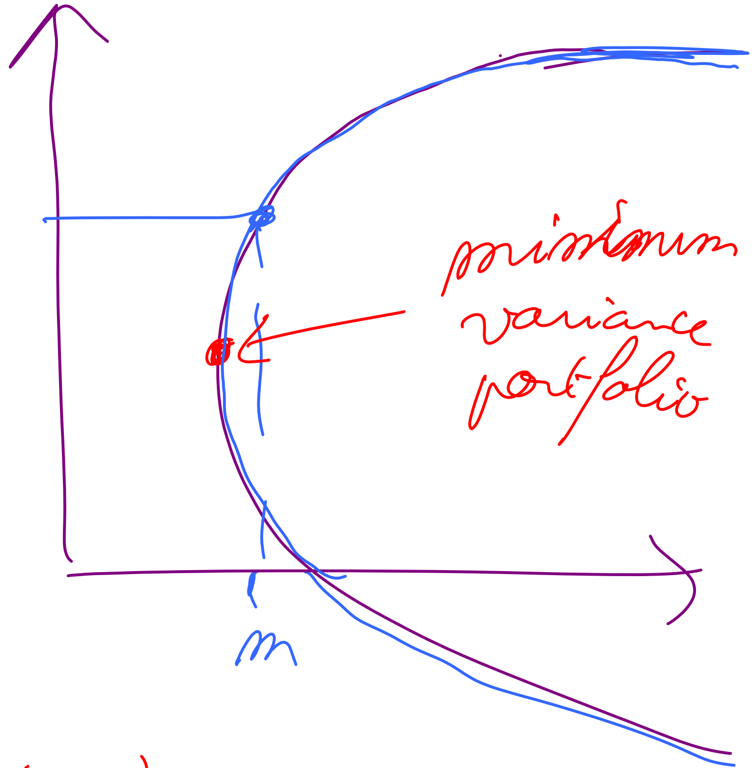
$$\frac{\partial L}{\partial \gamma} = 0 \iff \mathbb{1}^T \omega = 1$$

. Find the ω which minimizes the variance of the portfolio return.

$$\min_{\omega} \sigma_{\Pi}^2(\omega)$$

$$\omega = \Sigma^{-1}(\lambda\mu + \gamma\mathbb{1})$$

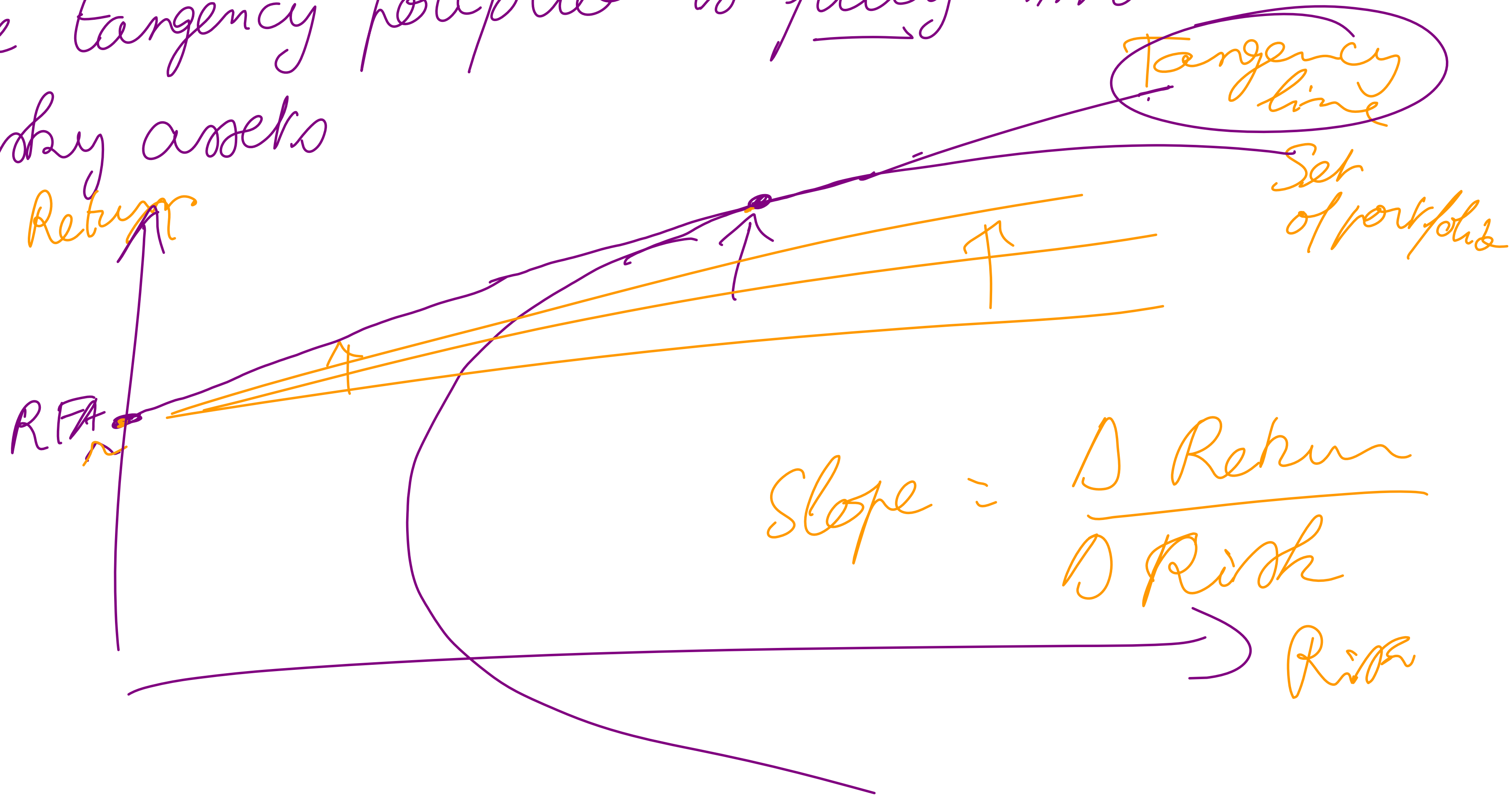
on the boundary of the opportunity set.



$\rightarrow \omega(m) \rightarrow \sigma_{\Pi}^2(m)$ return

Tangency Portfolio:

The tangency portfolio is fully invested in risky assets



Slope = $\dfrac{\Delta \text{ Return}}{\Delta \text{ Risk}}$

$$m_t = w_t^T \mu = \mu^T w$$

substitute this into the asset allocation on slide 72

$$w_t^* = \frac{(m_t - r)\, \Sigma^{-1} (\mu - r \mathbb{1})}{(\mu - r \mathbb{1})^T \Sigma^{-1} (\mu - r \mathbb{1})}$$

$$= \frac{(w_t^T \mu - r)\, \Sigma^{-1} (\mu - r \mathbb{1})}{(\mu - r \mathbb{1})^T \Sigma^{-1} (\mu - r \mathbb{1})}$$

$$w_t (\mu - r \mathbb{1})^T \Sigma^{-1} (\mu - r \mathbb{1}) = w_t \mu^T \Sigma^{-1} (\mu - r \mathbb{1}) - r \Sigma^{-1} (\mu - r \mathbb{1})$$

Developping and cancelling identical terms,

$$-w_t \; r \mathbb{1}^T \Sigma^{-1}(\mu - r\mathbb{1}) = -\Sigma^{-1}(\mu - r\mathbb{1})$$

Rearrange and note $\mathbb{1}^T \Sigma^{-1}(\mu - r\mathbb{1}) = B - Ar$

We finally get

$$\boxed{w_t = \frac{\Sigma^{-1}(\mu - r\mathbb{1})}{B - Ar}}$$

$$\sigma_{\pi}^{2}(m) = \frac{Am^2 - 2Bm + C}{AC - B^2}$$

$$\underbrace{\mathbb{1}^T \Sigma^{-1} \mathbb{1}}_{\geq 0} \geq 0$$

$$\boxed{\min_{m} \ \sigma_{\pi}^{2}(m)}$$

← unconstrained Problem.

$$\frac{\partial \sigma_{\pi}^{2}(m)}{\partial m} = \frac{2Am - 2B}{AC - B^2} = 0$$

$$\boxed{m^* = \frac{B}{A}}$$

$$\frac{\partial^2 \sigma_{\pi}^{2}(m)}{\partial m^2} = 2A > 0 \checkmark$$