Ludwig-Maximilian Universität München

WIRTSCHAFTSMATHEMATIK

Optimal Hedging Strategy for Minimum Guarantees in Life Insurance Policies

Diplomarbeit

von

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Hiermit erkläre ich, dass ich die Diplomarbeit selbstständig angefertigt und nur die angegebenen Quellen verwendet habe.

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Chapter 1

Introduction

Life insurance is a contract between the policy owner and the insurer, where the insurer agrees to pay a sum of money upon the occurrence of the insured event, it could be the policy owner's death, survival during a certain period or occurrence of some critical illnesses. In return the policy owner agrees to pay a stipulated amount called a premium at regular intervals.

There are many types of life insurance policies, but they tend to fall into two major categories, protection policies and investment policies. The first one is designed to provide a mortality benefit in the event the policy holder dies, the latter one is to facilitate the growth of capital. In Germany, the most common life insurance contract is a participating life insurance product. These policies are characterised by the fact that the insurers profits are shared with the policy owner. These contracts are commonly coupled with a minimum interest rate guarantee. The policy owner will pay a premium to the insurer, which can either be paid once at the beginning or regularly, such as monthly or annually. The insurer will invest the obtained premium less all costs in the financial markets. The profit from that investment will be shared between the two parties of the contract. The policy owner has claim on 90 percent of the profit, the other 10 percent will be remained by insurer as earnings. These policies always guarantee the policy owner a minimum interest rate. This means, if the investment doesn't have a good performance, the policy owner will still get the guaranteed interest rate on the entire sum of the premium he has paid. In Germany insurers currently offer the policy owners a minimum guarantee of at least 2.25 %.

In this thesis we will focus on such a life insurance policy, which lasts 12 years, with single premium at the beginning of the contract, in which the investment portfolio is annually adjusted according to the performance of the various asset classes in it. We consider a simple case: at time $t = 0$ the policy owner pays 125 Euro, 20% of this premium are pure costs, where the other 80% are going to be invested by the insurance company on the financial market, like in equities, bonds and some derivatives. At the end of the contract the insurer must pay the policy owner back all at one time. This contract is coupled with a minimum interest rate guarantee, say 2.25%. If the investment strategy from the insurer offers a better result than 2.25%, then the insurer may keep 10% of the entire profit from the investment. Hence the payoff at the maturity T is given by

$$\max(100(1+0.0225)^{12}, 90\% \text{ real performance}).$$

This means, the insurer will either get nothing or potentially even lose money, if the investment strategy didn't work well or earn 10% of the profit from the investment. This means, the aim of the insurer is to maximise the return of the portfolio under the constraints that the probability that the portfolio return is less than the minimum guarantee should be acceptable, i.e. less than a small number ϵ, usually ϵ is equal to a few basis points. The goal of this thesis is to find an optimal strategy for the insurer, which delivers maximal return under minimal risk.

Therefore the first part of this thesis will be focused on finding a tractable and realistic model for financial markets. At first we study the case of equity markets (or stock markets). Then we describe the interest rate markets. There are certain differences between the two models, because modelling interest rates is much more complicated than modelling stocks. The reason is that there are a variety of interest rates which are also correlated with each other in some way. So it just makes sense to model the behaviour of the entire interest rate curve. As a second step we try to combine the two models into a unique financial market model, we call it a hybrid model. The basic idea is to study the correlation structure between the equity market and the interest rate market. After deriving an adequate correlation structure we can use the interest rate market as basis and build a new equity market with the consideration of the correlation structure.

In the second part of this thesis we will simulate the models we constructed by using the Monte Carlo method. The stocks, interest rates are assumed to be log normal distributed. In this case Monte Carlo simulation can offer us a huge number of possible paths of the stock development and interest rate curves for various scenarios. For every possible path of the assets we can calculate the payoff of a derivative written on the underlying assets. Then if we take the mean of all possible payoffs of this derivative, we will get the expected value of this derivative. The price of this derivative is then the discounted expected value. So we can use the simulation results for pricing derivatives. We can also calculate the total portfolio value in every scenario. With all the possible values we can derive the distribution function for the portfolio value, which will allow us to do the analyses for an optimal strategy.

Now we have the base for doing scenario analyses. At first we must change our point of view from the risk neutral world to a real world concept, because in a risk neutral world the expected returns of all assets are equal. Under this assumption it doesn't make sense to vary the investment strategy. In the real world a more risky asset would also offer a higher return at the end. The difference between the return of a security and the return of the risk free asset is called the *excess return*. The excess return per one unit risk is defined as the *market price of risk*. Mathematically, to change to the real world viewpoint, we need to convert the dynamic for all assets or interest rates under the risk neutral measure to the real world measure. The density between the two measures can be expressed by a stochastic exponential using the market price of risk process, and Girsanov's change of measure theorem allows us to rewrite the dynamic easily. The single critical point in this procedure is the estimation of the market price of risk. Although the estimation of the market price of risk is a central component of every risk and return model in finance, it is surprisingly haphazard in practice. The standard approach remains to be the historical analysis. This method is easy to understand and to implement, but what we really want to know is tomorrow's market price of risk, and this method is sometimes not very reliable, when only have short periods of historical data available. Therefore we will introduce some methods from the academic studies, which don't require historical data. However the historical analysis is still the most common method with reasonable outputs, if we take a long enough period for the analysis, therefore in this thesis we will still use the historical analysis for the market price of risk to change the risk neutral measure to the real world measure. With the real world scenarios we can work with different strategies. At first we must define various strategy classes. Then we can run the simulations for every strategy to get the distributions of the portfolio values. The results of every strategy will be compared by using Value at Risk to quantify the associated risk. Value at Risk is a risk measure which expresses the possible loss of an investment under a certain probability level in a settled time period. If we have two different strategies, the distribution for each strategy can be calculated from a Monte Carlo simulation. With a predetermined probability level, saying 3 basis points and the same time period as the insurance contract $[0, 12]$, we can calculate Value at Risk for both strategies. Using this Value at Risk concept all various strategies can be compared with each other, so that we can derive an optimal investment strategy at the end.

Chapter 2

Stock Markets

On a financial market many different financial instruments are traded, such like stocks, bonds and derivatives. In this chapter we want to introduce a model that describes the development of stock prices. The price of a stock fluctuates fundamentally due to the theory of supply and demand. In the theory of financial mathematics the stock prices are always considered to be stochastic processes. The most common and popular stochastic process used in financial engineering is the Brownian motion. At first we will introduce the basics about pricing and hedging in a Brownian motion driven market.

In the first section of this chapter we will recall some mathematical principals for stochastic processes, stochastic integrals, stochastic differential equations as well as a few useful theorems in this branch.
In the second section we will use these concepts to construct an adapted mathematical model for stock price developments, which is used in a wide range both in the theory and practice.
In the third section we will explain what derivatives are and we will give a few examples about the most common derivatives.

2.1 Stochastic integral

As usual we consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

A Brownian motion is often used in mathematical model to describe random movements. It is among the simplest continuous time stochastic processes with normally distributed increments. The formal definition is as follows:

Definition 2.1 (One dimensional standard Brownian motion) *A one dimensional standard Brownian motion on* $[0, T]$ *is a stochastic process* $\{B(t),\ 0 \le t \le T\ \}$ *with the following properties:*

(i) $B(0) = 0$ *a.s.;*

(ii) $B(t)$ *has continuous paths, this means* $t \to B(t)$ *is continuous for every* ω*;*

(iii) $B(t)$ *has independent increments, this means* $B(t_i) - B(t_{i-1})$ *are independent for* $0 = t_0 \leq t_1 \leq \cdots \leq t_i \leq \cdots \leq t_n = T$*;*

(iv) $B(t) - B(s) \sim \mathcal{N}(0, t-s)$*.*

With this definition some properties of a standard Brownian motion can be derived:

(i) $B(t) = B(t) - B(0) \sim \mathcal{N}(0, t)$;

(ii) $\mathbb{E}[B(t)] = 0$ and $Var[B(t)] = t$;

(iii) $Cov(B(t), B(s)) = \min(s, t)$;

(iv) Brownian motion is among the family of the Markov processes. This means, only the present value of a variable is relevant to predict the future. The history of the variable is irrelevant;

(v) One can calculate the conditional expectation for the Brownian motion.

$$\mathbb{E}[B(t)|\mathcal{F}_s] = B(s) \quad for\ s < t$$

where $(\mathcal{F}_s)_{0 \leq s \leq t}$ is the from B(t) generated σ-algebra.
This means the Brownian motion $B(t)$ is a martingale.

The definition of Brownian motion can also be extended to multi dimensional case.

Definition 2.2 (Multi dimensional Brownian motion) *A d-dimensional Brownian motion* $B = (B_1, \cdots, B_d)$ *is a d-dimensional stochastic process, where* B_i *are independent one dimensional Brownian motions.*

The theory of stochastic integral generates the concept of integral with usual definition, because it uses the stochastic processes with infinite variations as integrator, particularly the Brownian motion. Therefore the stochastic integral represents the basic for stochastic analyses and the stochastic differential equations (SDE). With the following theorem the stochastic integral with Brownian motion as integrator is defined:

Theorem 2.3 (Stochastic Integral) *Let* $\phi = (\phi_1, \cdots, \phi_d)$ *be a predictable process and*

$$\int_0^\infty \|\phi(t)\|^2 dt < \infty \quad a.s.$$

Then we say $\phi \in \Lambda^2_{loc}$ *and one can define the stochastic integral*

$$(\phi \cdot B)_t = \int_0^t \phi(s) dB(s) = \sum_{j=0}^{d} \int_0^t \phi_j(s) dB_j(s)$$

If moreover $\mathbb{E}[\int_0^\infty \|\phi(t)\|^2 dt] < \infty$, *then we say* $\phi \in \Lambda^2$ *and we obtain that* $(\phi \cdot B)$ *is a martingale and the Itô isometry holds:*

$$\mathbb{E}\left[\left(\int_0^t \phi(s)dB(s)\right)^2\right] = \mathbb{E}\left[\int_0^t \|\phi(s)\|^2 ds\right]$$

With the extension of integral concept we can describe a further stochastic process:

Definition 2.4 (One dimensional diffusion Brownian motion) *For deterministic* $\mu(t)$ *and* $\sigma(t) > 0$ *we call a process* $\{X(t),\ 0 \leq t \leq T\}$ *a Brownian motion with drift* $\mu(t)$ *and diffusion coefficient* $\sigma(t)^2$, *if* $X(t)$ *satisfies the following SDE*

$$dX(t) = \mu(t)dt + \sigma(t)dB(t)$$

A further type of stochastic process is known as an Itô process. It is a generalised Brownian motion where the drift μ and diffusion coefficient σ^2 are functions of the underlying variable:

Definition 2.5 (Itô process) *We call a process* $\{X(t),\ 0 \leq t \leq T\}$ *an Itô process, if* $X(t)$ *satisfies the following SDE*

$$dX(t) = \mu(X,t)dt + \sigma(X,t)dB(t)$$

A very useful lemma in stochastic calculus is the Itô's formula. This lemma is used to find the differential of a function of an Itô process. This lemma is widely employed in mathematical finance, because a contingent claim can always be expressed as a function of an underlying asset, assumed that the dynamic of this asset is known, then one can easily calculate the dynamic of the contingent claim using Itô's lemma.

Theorem 2.6 (Itô's Formula) *Let* $f \in \mathcal{C}(\mathbb{R}^n)$ *and*

$$X(t) = X(0) + \int_0^t \alpha(s)ds + \int_0^t \phi(s)dB(s)$$

be a n-dimensional Itô process. Then $f(X)$ *is also an Itô process and*

$$f(X(t)) = f(X(0)) + \sum_{i=1}^n \int_0^t \frac{\partial f(X(s))}{\partial x_i} dX_i(s) + \frac{1}{2}\sum_{i,j=1}^n \int_0^t \frac{\partial^2 f(X(s))}{\partial x_i \partial x_j} d < X_i, X_j >_s$$

where

$$d < X_i, X_j >_s := \sum_{k=1}^d \phi_{ik}(s)\phi_{jk}(s)ds$$

In the financial mathematics there is a very important theorem, namely the Girsanov's theorem. It tells how stochastic processes change under changes in probability measure. Applied in finance it tells how to convert from the physical measure which describes the probability that an underlying instrument will take values to the risk neutral measure which is a very useful tool for evaluating the value of derivatives on the underlying. Before we state the theorem we must introduce a further type of processes at first.

Definition 2.7 (Stochastic exponential) *Processes Z_t of the form*

$$Z(t) = \exp\left(\int_0^t \phi(s)dB(s) - \frac{1}{2}\int_0^t \|\phi_s\|^2 ds\right)$$

are called stochastic exponentials and denoted $Z_t = \mathcal{E}(\phi B)_t$, where $\phi \in \Lambda^2_{loc,d}$.

The stochastic exponential has following properties:

Theorem 2.8 (Properties of the stochastic exponential) *Let $\phi \in \Lambda^2_{loc,d}$ and $Z_t = \mathcal{E}(\phi B)_t$, then:*

(i) Z_t is a local martingale and $Z_t = 1 + \int_0^t Z_s \phi_s^{tr} dB_s$;

(ii) Z_t is positive super martingale. $\forall t,\ Z_t \to Z_\infty$ a.s.;

(iii) $\mathbb{E}[Z_t] \leq 1,\ \forall t \in [0,\infty]$;

(iv) If $\exists T > 0$ such that $\mathbb{E}[Z_T] = 1$, then Z_t is a martingale for $t \in [0,T]$;

(v) If $\mathbb{E}[Z_\infty] = 1$, then $Z_t = \mathbb{E}[Z_\infty|\mathcal{F}_t]$ a.s.

Consider $Z_t = \mathcal{E}(\phi B)_t$ and assume $\mathbb{E}[Z_\infty] = 1$. Since Z_t is positive, we can introduce the probability measure $\mathbb{Q} \sim \mathbb{P}$ with density Z_∞. Furthermore we can find the associated Brownian motion with respect to $\mathbb{Q}$. This is the statement of the Girsanov's theorem.

Theorem 2.9 (Girsanov's change of measure Theorem) *Let γ be such that the stochastic exponential $\mathcal{E}(\gamma B)_\infty$ is a uniformly integrable strictly positive martingale. Then*

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \mathcal{E}(\gamma B)_\infty \quad \left(\Rightarrow \frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_t} = \mathcal{E}(\gamma B)_t\right)$$

defines an equivalent measure $\mathbb{Q} \sim \mathbb{P}$ and the process

$$\tilde{B}(t) = B(t) - \int_0^t \gamma(s)ds$$

is a $\mathbb{Q}$-Brownian motion.

Applications of this theorem can be seen in later chapters, for example when we derive the unique equivalent martingale measure in the Black Scholes model, or apply the change of numeraire.

2.2 Mathematical model

Now we describe the mathematical model for the stock market.

We assume that the underlying financial market satisfies the following hypothesis:
1.Trading takes place continuously in time;
2.Borrowing and lending money possible without amount restrictions and at the same rate;
3.There are no transaction costs;
4.Short sales are admitted;
5.The market is liquid;
6.There is no information asymmetry.

The stochastic basis for financial market is a filtered probability space (Ω, $\mathcal{F}$, $(\mathcal{F}_t)_{t\geq 0}$,$\mathbb{P}$) satisfying the usual conditions and carrying a d-dimensional $(\mathcal{F}_t)$-adapted Brownian motion $B = (B_1, ..., B_d)$.

We consider the financial market with a risk free asset S_0 and n risky assets S_i $(i = 1, ..., n)$. These $n + 1$ assets are following strictly positive Itô Processes:

$$(1) \quad dS_0(t) = r(t)S_0(t)dt, \quad S_0(0) = 1.$$

$S_0(t)$ is called the money market account and represents the possibility of borrowing/lending money from a bank to a short rate $r(t)$, where $r(t)$ is a progressive process such that $\mathbb{E}[S_0(t)] < \infty, \quad \forall\ t \geq 0$. From now on for the sake of simplicity we assume that $r(t)$ is bounded.

$$(2) \quad dS_i(t) = S_i(t)\mu_i(t)dt + \sum_{j=1}^{d} S_i(t)\sigma_{ij}(t)dB_j(t), \quad S_i(0) > 0, \quad i = 1, \cdots, n$$

Here $S_i(t)$ represents the risky assets. The n-dimensional drift μ and the $n \times d$-dimensional volatility σ are assumed to form predictable processes which meet the required integrability conditions so that all of the above integrals are well-defined.

Example 1 (The Black Scholes Model) *In the classical Black Scholes Model there are only the money market account S_0 and one risky asset S_1. The parameters r, μ and σ are assumed to be constant. So the dynamics of the assets are:*

$$dS_0(t) = rS_0(t)dt, \quad S_0(0) = 1 \Leftrightarrow S_0(t) = e^{rt}$$

$$dS_1(t) = S_1(t)(\mu dt + \sigma dB(t)), \quad S_1(0) = s_1 \Leftrightarrow S_1(t) = s_1 e^{(\mu - \frac{\sigma^2}{2})t + \sigma B(t)}$$

S_1 is called the geometric Brownian motion.

In this paper we mainly focus on the Black Scholes model, which means that the volatility is assumed to be constant. For further improvement one can also use other financial models with stochastic volatility like the Heston model.

Definition 2.10 (Strategy/Portfolio) *A trading strategy or portfolio*

$$\xi(t) = (\xi_0(t), ..., \xi_n(t))$$

is a $(n+1)$-dimensional predictable process, where $\xi_i(t)$ represents the unit of asset number i which an investor holds at time t.

Definition 2.11 (Self financing strategy) *$\xi(t)$ is a self financing strategy, if the portfolio value $V(t)$ associated with $\xi(t)$:*

$$V(t) = \xi(t)S(t) = \sum_{i=0}^{n} \xi_i(t)S_i(t)$$

satisfies

$$dV(t) = \xi(t)dS(t),$$

or in discrete case

$$\triangle V(t) = \xi(t)\triangle S(t).$$

Definition 2.12 (Admissible self financing strategy) *A self financing strategy $\xi(t) = (\xi_0(t), ..., \xi_n(t))$ is called admissible, if the corresponding portfolio value $V(t) = \xi(t)S(t)$ is (t, ω)-a.s. bounded from below, i.e. there exists $K > 0$ such that $V(t, w) > -K$ for almost every $(t, \omega) \in [0, T] \times \Omega$. We can interpret K as the maximal debt we are capable to assume.*

Definition 2.13 (Arbitrage) *An arbitrage portfolio is an admissible self-financing portfolio with value process satisfying:*

$$V(0) = 0, \quad V(T) \geq 0 \quad and \quad \mathbb{P}[V(T) > 0] > 0 \quad for\ some\ T > 0.$$

In other words, an arbitrage opportunity is an investment strategy that yields with positive probability a positive profit without any downside risk. A financial Market is called arbitrage free, if there exist no arbitrage portfolios. In the real world, arbitrage opportunities are rather hard to find. If such an opportunity would show up, it would generate a large demand, prices would adjust and the opportunity would disappear. Therefore we usually assume that the market is arbitrage free.

Definition 2.14 (Discounted assets) *We often consider the discounted asset processes. Usually we use the risk free asset S_0 as numeraire, so that the discounted process $X(t)$ is:*

$$X(t) = \frac{S(t)}{S_0(t)} = \left(1, \frac{S_1(t)}{S_0(t)}, ..., \frac{S_n(t)}{S_0(t)}\right),$$

and the discounted portfolio value is:

$$\bar{V}(t) = \sum_{i=0}^{n} \xi_i(t)X_i(t).$$

Now we are going to characterise those market models which do not admit any arbitrage opportunities. At first we introduce a very important definition:

Definition 2.15 (Equivalent martingale Measure) *A probability measure $\mathbb{Q}$ is called an equivalent (local) martingale measure, if it is equivalent to $\mathbb{P}$ and the discounted process $X(t)$ is a $\mathbb{Q}$-(local) martingale.*

When does such a martingale measure $\mathbb{Q}$ exist? According to the definition the discounted asset prices X should be a local martingale under $\mathbb{Q}$, so let's calculate at first the dynamics of X. The dynamics of the asset prices S are known, using Itô's Formula we can calculate the dynamics of the discounted asset prices $X(t)$:

$$dX_i(t) = X_i(t)\Big[(\mu_i(t) - r(t))dt + \sigma_i(t)dB(t)\Big].$$

Suppose there is an equivalent martingale measure $\mathbb{Q}$ with the density

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \mathcal{E}(\gamma B)_T$$

where B is the $\mathbb{P}$ Brownian motion. According to the Girsanov's theorem we know that

$$\tilde{B}(t) = B(t) - \int_0^t \gamma(s)ds$$

defines a $\mathbb{Q}$-Brownian motion, therefore the dynamics for $X(t)$ under $\mathbb{Q}$ can be rewritten as:

$$\begin{aligned}
dX_i(t) &= X_i(t)\Big[(\mu_i(t) - r(t))dt + \sigma_i(t)d(\tilde{B}(t) + \int_0^t \gamma(s)ds)\Big] \\
&= X_i(t)\Big[(\mu_i(t) - r(t))dt + \sigma_i(t)d(\tilde{B}(t)) + \gamma(t)dt\Big] \\
&= X_i(t)\Big[(\mu_i(t) - r(t) + \sigma_i\gamma(t))dt + \sigma_i(t)d(\tilde{B}(t))\Big]
\end{aligned}$$

If γ satisfies

$$\mu_i(t) - r(t) + \sigma_i(t)\gamma(t) = 0 \quad dt \otimes d\mathbb{Q} - a.s. \quad for\ all\ i = 1, \cdots, n,$$

then X_i is a martingale under $\mathbb{Q}$, so $\mathbb{Q}$ is then truly an equivalent martingale measure.

If σ is non-degenerate (in particular $d \leq n$ and $rank(\sigma) = d$), then γ is uniquely specified by

$$-\gamma = \sigma^{-1}(\mu - r\mathbb{I}),$$

where $\mathbb{I} = (1, \cdots, 1)^T$ has the same dimension as μ.
And $-\gamma$ is also called the *market price of risk* with respect to $\mathbb{Q}$.

Now we can characterise arbitrage free market models:

Theorem 2.16 (The fundamental Theorem of Asset Pricing) *Suppose there exists an equivalent (local) martingale measure (E(L)MM) $\mathbb{Q}$, then the model is arbitrage-free, in the sense that there exists no admissible arbitrage strategy.*

Proof. Assume $\bar{V}(t) = \sum_{i=0}^{n} \xi_i(t) X_i(t)$ is the discounted portfolio value associated with a self financing strategy ξ with $\bar{V}(0) = 0$ and $\bar{V}(T) \geq 0$. Then the stochastic differential equation for $\bar{V}$ is:

$$\begin{aligned} d\bar{V}(t) &= \xi(t) dX(t) \\ &= \sum_{k=1}^{d} \Big(\sum_{i=1}^{n} \frac{\xi_i(t)\sigma_{ik}(t)}{S_0(t)} \Big) d\tilde{B}_k(t) \\ &=: \psi(t) d\tilde{B}(t). \end{aligned}$$

$\psi(t)$ defined above belongs to Λ^2, hence $\bar{V}(t)$ is a martingale under $\mathbb{Q}$. A positive martingale is a super martingale, so we have

$$0 \leq \mathbb{E}[\bar{V}(T)] \leq \bar{V}(0) = 0 \quad \Rightarrow \quad \bar{V}(T) = 0.$$

This means we can't have an arbitrage opportunity if there exists an EMM $\mathbb{Q}$.

Example 2 (The Black Scholes Model is arbitrage free) *If we apply this result to the Black Scholes model introduced before, we can see that this model is arbitrage free. γ is uniquely defined by:*

$$\gamma = -\frac{\mu - r}{\sigma}$$

Now we introduce an another characteristic of the market models:

Definition 2.17 (Complete model) *A contingent claim X due at T is attainable if there exists a strategy ξ which replicates X: $V(T, \xi) = X$.*
A market is called complete, if every contingent claim is attainable.

In the case of complete model we can derive a pricing rule for all T-claims:

Theorem 2.18 (Pricing contingent claim in complete model) *In a complete model the fair price prevailing at t of a T-claim X is given by*

$$V(t, \xi) = S_0(t) \mathbb{E}_{\mathbb{Q}} \Big[\frac{X}{S_0(T)} | \mathcal{F}_t \Big].$$

Example 3 (The Black Scholes Model is complete) *One can prove that an arbitrage free market model is complete if and only if there exists exactly one martingale measure. We know that the martingale measure in Black Scholes model is unique. Therefore the Black Scholes model is complete, and one can use the pricing rule for the contingent claims in the BS model.*

2.3 Derivatives

In the financial market one may face many risks like interest rate risk, foreign exchange risk, credit risk and so on. Derivatives are effective tools for managing these many kinds of financial risks. Derivatives are financial securities written on an underlying asset's value. Underlying assets can be another kind of financial security like stocks or bonds, commodities or currencies or other kinds of indices (weather indices, catastrophe indices ...). The price of the derivatives is driven by the price movement of the underlying assets.

There are many types of derivatives, three of the most common are options, futures and forwards.

(1). **Futures**
A future contract is a standardised contract, traded on a futures exchange, to buy or sell a certain underlying instrument at a certain date in the future, the delivery date, at a specified price, the future price. A future contract gives the holder the obligation to buy or sell.

Consider a future contract with delivery date T to buy the underlying asset S, which has a value $S(t)$ at time t. The future price is predetermined as F. So the value of this future contract at delivery date T is :

$$P(T) = S(T) - F.$$

(2). **Forward**
A forward contract is an agreement between two parties to buy or sell an asset at a pre-agreed future point in time. It is very similar to a future contract, the difference between them is that futures are always traded on exchange, whereas forwards are always traded over the counter, so that the prices of them are always differently determined.
Furthermore futures are effectively settled everyday through the customers margin account and therefore have a constant duration of 1, while forward only settle at maturity and therefore have a discounted present value.

(3). **Option**
An option is a financial contract which gives the buyer the right not the obligation to buy an underlying asset (Call option) or sell an underlying asset (Put option) at a set price (strike) either at the pre determined exercise date (European option) or during a time interval (American option).

An European call option with strike K on an underlying asset S provides a payoff

$$C(T) = (S(T) - K)^+,$$

whereas a put option with same conditions offers

$$P(T) = (K - S(T))^+.$$

For valuing derivatives in a risk neutral world we can use the results from the previous section. The theorem about pricing contingent claim tells us that the valuation of a derivative is nothing else than taking the expectation of the payoff of this derivative under the risk neutral measure. But for complex payoffs this expectation can not be calculated analytically anymore, we need some numerical procedure for solving this problem. The widely used class of numerical methods in finance is the Monte Carlo simulation, which will be explained in later chapters.

Chapter 3

Interest Rate Models

In the previous chapter we have introduced a model for describing stocks. Other important financial instruments like bonds or derivatives that depend in some way on interest rate levels need a more complex model, because the movements of these products are dependent on the interest rates movements, which are somehow correlated with each other and therefore much more difficult to model. It doesn't make sense to model every single interest rate separately because of their correlation structure. For this reason we need a model that describes the possible future development of a variety of interest rates simultaneously.

There exist many different interest rate models. According to the type of interest rates described in the model they can be classified into the following classes:

(1) **Short rate models**
(2) **Instantaneous forward rate models**
(3) **Market models**

We will explain these models in later sections. In this article we will focus on the Market models because they are much more market oriented than other models.

In the first two sections of this chapter we will explain the different types of interest rates and how a bank account is defined as well as the relationships between them.
In the next section we will give a brief introduction to interest rate derivatives.
In the last section we will introduce the LIBOR market model which will be used in the next chapter for Monte Carlo simulations and the valuation of derivatives.

3.1 Interest rates

At first we will explain what are zero rates, forward rates, swap rates and the relationships between them.

A zero coupon bond is an investment that starts today and lasts n years which will return the buyer the interest and the principal at the end of n years. In a zero coupon bond there are no intermediate payments.

The rate of the interest earned from a zero coupon bond with maturity in n years is called the n-year *zero rate*.

The price of a zero coupon bond at time t which pays 1 dollar at time T is defined as *bond price* t with maturity T, denoted by $P(t,T)$.
In theory we always assume that

- $P(T,T) = 1$;
- $P(t,T)$ is continuously differentiable in T.

The bond price can also be explained as the time t value of 1 dollar at time T.

Given the bond price $P(t,T)$ many interest rates can be defined. Consider $t \leq T \leq S$:

- **Simple forward rate** for $[T,S]$ prevailing at time t
$$F(t,T,S) = \frac{1}{S-T}\Big(\frac{P(t,T)}{P(t,S)} - 1\Big)$$
- **Continuously compounded forward rate** for $[T,S]$ prevailing at time t
$$R(t,T,S) = -\frac{logP(t,S) - logP(t,T)}{S-T}$$
- **Simple spot rate** for $[t,T]$
$$F(t,T) := F(t,t,T) = \frac{1}{T-t}\Big(\frac{1}{P(t,T)} - 1\Big)$$
- **Continuously compounded spot rate** for $[t,T]$
$$R(t,T) := R(t,t,T) = -\frac{logP(t,T)}{T-t}$$
- **Instantaneous forward rate** with maturity T prevailing at time t
$$f(t,T) := \lim_{S\to T} R(t,T,S) = -\frac{\partial logP(t,T))}{\partial T}$$

  With this definition we can prove that under the assumption $P(T,T) = 1$ we have the relationship between $P(t,T)$ and $f(t,T)$ as follows:
$$P(t,T) = \exp(-\int_t^T f(t,u)du)$$

- **Short rate** at time t
$$r(t) := f(t,t) = \lim_{T \to t} R(t,T)$$

A very important interest rate is the LIBOR rate. The name LIBOR stays for London interbank offered rate. It is the most important rate at which deposits between banks are exchanged and at which swap transactions between banks occur. There is a series of LIBOR rates with various maturities. These rates are quoted on a simple compounding basis. For example the one-year forward LIBOR rate for the period $[T, T+1]$ at time t is given by $F(t, T, T+1)$. So we define the **spot LIBOR rate** as:

$$L(t,T) := \frac{1 - P(t,T)}{(T-t)P(t,T)},$$

and we define the **forward LIBOR rate** as:

$$F(t;T,S) := \frac{1}{S-T}\left(\frac{P(t,T)}{P(t,S)} - 1\right).$$

Now we have completed the definition of the basic interest rates. There is still another important interest rate, the swap rate. Swap rates can be observed directly in the market, so that it's very convenient to use them. A swap rate at time t is defined in such a way so that an interest rate swap (IRS) has value zero at time t for both parties of this contract.

An IRS is a contract, where a payment stream at a fixed rate of interest is exchanged for a payment stream at a floating rate, usually LIBOR rate. There are two types of IRS: The buyer of a *Payer IRS* pays fixed rate at certain future dates and receives floating rate, whereas in a *Receiver IRS* fixed payments will be received at future time points and floating rates are paid.

More precisely, an IRS is specified by:

- A number of future dates $T_0 < T_1 < \cdots < T_n$ with $T_i - T_{i-1} = \delta$;
- A fixed rate K;
- A nominal value N.

Suppose the valuation date is at time t. Cash flows take place at time $T_1, \cdots, T_n$:

- fixed $K\delta N$ is paid (Payer Swap) or received (Receiver Swap);
- floating $F(t, T_{i-1}, T_i)\delta N$ is received (Payer Swap) or paid (Receiver Swap).

The net cash flows at T_i is thus:

$$(F(t, T_{i-1}, T_i) - K)\delta N \quad (PayerSwap),$$

or

$$(K - F(t, T_{i-1}, T_i))\delta N \quad (ReceiverSwap).$$

The total value of the swap at time t is then:

$$\Pi_p = N\Big[P(t, T_0) - P(t, T_n) - K\delta \sum_{i=1}^{n} P(t, T_i)\Big],$$

$$\Pi_r = N\Big[K\delta \sum_{i=1}^{n} P(t, T_i) - P(t, T_0) - P(t, T_n)\Big].$$

The swap rate at time t $R_{swap}(t)$ is the fixed rate K which gives $\Pi_p = \Pi_r = 0$:

$$\Rightarrow \quad R_{swap}(t) = \frac{P(t, T_0) - P(t, T_n)}{\delta \sum_{i=1}^{n} P(t, T_i)}.$$

3.2 Bank account

With bank account or money market account we want to express the risk free asset insofar as its future value at time $t + \triangle t$ is known at time t. We define it as follows:

Assume we have 1 Euro today. We can invest it in a $\triangle t$-zero bond, then we will get $\frac{1}{P(0,\triangle t)}$ at time $\triangle t$;

At time $\triangle t$ we invest the obtained money again in a $\triangle t$-zero bond, so we will get $\frac{1}{P(0,\triangle t)P(\triangle t, 2\triangle t)}$ etc.

This strategy will lead to the discrete bank account $B_d(t)$, where:

$$B_d(t) = \frac{1}{P(0, \triangle t)} \cdots \frac{1}{P(t - \triangle t, t)}.$$

The strategy above is known as " rolling over " strategy. It leads in the limit to the continuous bank account $B(t)$ as follows:

$$\frac{1}{P(0, \triangle t)} = \exp\Big(\int_0^{\triangle t} f(0, u)du\Big) = 1 + r(0)t + o(\triangle t),$$

$$\frac{1}{P(0, \triangle t)} \frac{1}{P(\triangle t, 2\triangle t)} = (1 + r(0)\triangle t)(1 + r(\triangle t)\triangle t) + o(\triangle t).$$

So we get

$$B(t + \triangle t) = B(t)(1 + r(t)\triangle t) + o(\triangle t).$$

For $\triangle t \to 0$ this converges to

$$dB(t) = r(t)B(t)dt,$$

with $B(0) = 1$ we obtain

$$B(t) = \exp\Big(\int_0^t r(s)ds\Big).$$

The short rate expresses the interest rate at the current time for a infinitesimal small time interval, therefore it can be seen as the risk free rate.

In the case of interest rate market we define equivalent martingale measure (EMM) with respect to the numeraire $S(t)$ as the probability measure $\mathbb{Q}$, which is equivalent to the real world measure $\mathbb{P}$ and makes the process $\frac{P(t,T)}{S(t)}$ to a $\mathbb{Q}$- martingale. The processes $B(t)$ and $B_d(t)$ are usually used as numeraire. We denote $\mathbb{Q}^B$ and $\mathbb{Q}^{B_d}$ as the associated EMM.

So according to Theorem (2.16) the pricing rule for any contingent claim H is

$$V_t = \mathbb{E}_{\mathbb{Q}^B}\left[\frac{B(t)}{B(T)}H|\mathcal{F}_t\right].$$

Using this equation the relation between bond price and bank account can be calculated by setting $H_T = 1$:

$$\begin{aligned} P(t,T) &= \mathbb{E}_{\mathbb{Q}^B}\left[\frac{B(t)}{B(T)}1|\mathcal{F}_t\right] \\ &= \mathbb{E}_{\mathbb{Q}^B}\left[exp(-\int_t^T r_s ds)|\mathcal{F}_t\right] \\ &= \mathbb{E}_{\mathbb{Q}^B}\left[D(t,T)|\mathcal{F}_t\right] \end{aligned} \tag{3.1}$$

where

$$D(t,T) = exp(-\int_t^T r_s ds) = \frac{B(t)}{B(T)}$$

expresses the discount factor.

The discount factor is stochastic, hence it is not known with certainty at time t. But there is a close connection to the deterministic discount factor given by $P(t, T)$ according to the equation (3.1).

3.3 Interest rate derivatives

Interest rate derivatives are instruments whose payoffs are dependent in some way on the level of interest rates. A key challenge for derivatives traders is to find good robust procedures for pricing and hedging these products. The most important interest rate derivatives are caps, floors and swaptions. In this thesis our main focus will be swaptions and we will calibrate our model also according to the market swaption prices. Therefore we introduce at first the definition of swaption and explain how their price can be calculated.

A swaption is an option on an IRS. A payer (receiver) Swaption contract will give the holder the right to enter into a payer (receiver) IRS at a certain time in the future. This future time is called the swaption maturity. The length of the underlying swap is called tenor of this swaption.

Consider at time t a payer swaption contract with strike rate K. The reset dates of the underlying swap are $T_0, \cdots, T_n$ with $T_i - T_{i-1} = \delta$. Usually the swaption maturity coincides with the first reset date T_0. So at time T_i the swap has the payoff

$$\delta F(T_0, T_{i-1}, T_i) - K$$

with value at the maturity given by

$$\Pi_p = N \sum_{i=1}^{n} P(T_0, T_i)\delta(F(T_0, T_{i-1}, T_i) - K).$$

If this value is greater than zero, the buyer of the swaption will exercise this call option, otherwise it will disclaim the right to enter the swap contract. So the payoff of the swaption at the maturity is

$$N\Big(\sum_{i=1}^{n} P(T_0, T_i)\delta(F(T_0, T_{i-1}, T_i) - K)\Big)^+.$$

One can prove that it is equivalent to

$$N\delta\Big(R_{swap}(T_0) - K\Big)^+ \sum_{i=1}^{n} P(T_0, T_i).$$

Analogously for the receiver swaption

$$N\delta\Big(K - R_{swap}(T_0)\Big)^+ \sum_{i=1}^{n} P(T_0, T_i).$$

Accordingly payer (receiver) swaptions with strike K at time t are said to be at the money (ATM), in the money (ITM) or out of the money (OTM) if:

- $K = R_{swap}(t)$ (ATM);
- $K < (>) R_{swap}(t)$ (ITM);
- $K > (<) R_{swap}(t)$ (OTM).

For valuation swaptions there is a Black's formula applicable.

$$Swpt_p(t) = N\delta\Big(R_{swap}(t)\phi(d_1(t)) - K\phi(d_2(t))\Big)\sum_{i=1}^{n} P(t,T_i),$$

$$Swpt_r(t) = N\delta\Big(K\phi(-d_2(t)) - R_{swap}(t)\phi(-d_1(t))\Big)\sum_{i=1}^{n} P(t,T_i),$$

where

$$d_{1/2} := \frac{\log\left(R_{swap}(t) - K\right) \pm 1/2\sigma(t)^2(T_0 - t)}{\sigma(t)\sqrt{T_0 - t}}.$$

and $\sigma(t)$ is the swaption volatility.

3.4 LIBOR market model

For pricing complex interest rate derivatives like swaptions we need the dynamics of interest rates.

We consider 3 classes of models.

The first type is the *short rate model.* In these models the short rate is assumed to be the only source of uncertainty. Using this type of model there will be simple implementations of other rates and bond prices. But this model also has a few drawbacks. The short rate has a theoretical nature. Although there are available overnight rates in the market, but they tend to be somewhat unstable; or one can use for example three month rates for approximation of the short rate. Another point is that short rate models offer problem with calibration on the real market data. Moreover, using short rate as the only uncertainty source involves the fact that all interest rates are perfectly correlated, so that the interest rate structure can not be modeled appropriately.

An improvement of the short rate model is designed from Heath-Jarrow-Morton approach. In the HJM framework the development of the entire instantaneous forward rate curve is described instead of one single point of the curve. Therefore this model allows a more realistic description of the interest rate curve. But the drawback is still that the instantaneous forward rates are not directly observable in the market.

To overcome the weakness of the HJM model the concept of market model has been introduced. The market model is expressed in terms of the forward LIBOR rates that traders are used to working with. In this thesis we will focus on the market model, because it is more market oriented than the other models.

LIBOR market model

In LIBOR market model (LMM) we model directly the forward Libor rates.

$$F(t,T,S) = \frac{1}{S-T}\left(\frac{P(t,T)}{P(t,S)} - 1\right)$$

For sake of simplicity we will do it in yearly steps in a time interval of 20 years. This means we want to model the one year forward LIBOR rates

$$F_k(t) := F(t,k,k+1) = \left(\frac{P(t,k)}{P(t,k+1)} - 1\right)$$

for $t = 0, ..., 19$; $k = t, ..., 19$.
Of course the time horizon can be extended to 30 or 40 years and monthly steps can be used instead of yearly step.

In the frame of interest rate models, forward measures can make calculations more convenient:

Definition 3.1 (Forward measure) *The equivalent martingale measure $\mathbb{Q}_T \sim \mathbb{Q}$ on $\mathcal{F}_T$ by:*

$$\frac{d\mathbb{Q}_T}{d\mathbb{Q}} = \frac{1}{P(0,T)B(T)}$$

is called forward measure.

Under this measure $\frac{P(t,S)}{P(t,T)}$ is a $\mathbb{Q}_T$ martingale for every $S > 0$.

At first we can describe the forward rates as a drift-less stochastic process under the forward measures with the SDE

$$dF_i(t) = \sigma_i F_i(t) dB_i(t)$$

where B_i is the Brownian motion associated with $\mathbb{Q}_i$. This means that $F_i(t)$ is a $\mathbb{Q}_i$ martingale.

The second assumption is that F is log normal distributed.

At the next step we can describe the dynamics of the forward rates with maturity k under the forward measure $\mathbb{Q}_i$ with maturity i as:

$$dF_k(t) = \begin{cases} \mu_i^k(t, F(t))\sigma_k(t)F_k(t)dt + \sigma_k(t)F_k(t)dZ_k^i(t) & \text{if } i < k, \\ \sigma_k(t)F_k(t)dZ_k^k(t) & \text{if } i = k, \\ -\mu_k^i(t, F(t))\sigma_k(t)F_k(t)dt + \sigma_k(t)F_k(t)dZ_k^i(t) & \text{if } i > k, \end{cases}$$

where

$$\mu_l^m := \sum_{j=l+1}^{m} \tau_j \frac{\sigma_j(t)F_j(t)\rho_{m,j}}{1+\tau_j F_j(t)}$$

See [11].

It may happen that in simulating forward rates F_k under the forward measure $\mathbb{Q}_i$ with i much larger or smaller than k, the effect of the discretization procedure worsens the approximation with respect to cases where i is close to k. A remedy to this situations is to adopt the measure $\mathbb{Q}_{t+1}$. If we choose $P(t, t+1)$ instead of $P(t, i)$ as numeraire and apply change of numeraire technique starting from $dF_k(t) = \sigma_k F_k(t)dB_k(t)$, we can obtain the dynamic under $\mathbb{Q}_{t+1}$:

$$dF_k(t) = \sigma_k(t)F_k(t) \sum_{j=t+1}^{k} \frac{\rho_{k,j}\tau_j\sigma_j(t)F_j(t)}{1+\tau_j F_j(t)} dt + \sigma_k(t)F_k(t)dZ_k(t), \tag{3.2}$$

where $\tau_j = T_j - T_{j-1} = 1$ in our case, Z_k's are $\mathbb{Q}_{t+1}$ Brownian motions for every k, σ_k is the volatility of the k-th forward rate and $\rho_{k,j}$ is the correlation between the Brownian motions Z_k, Z_j.
The first summand in the equation is essentially the convexity adjustment for the forward rate curve. See [5].

Using Itô formula we get

$$d\ln(F_k(t)) = \sigma_k(t) \sum_{j=t+1}^{k} \frac{\rho_{k,j}\sigma_j(t)F_j(t)}{1+F_j(t)} dt - \frac{\sigma_k(t)^2}{2} dt + \sigma_k(t)dZ_k.$$

For small time interval $\triangle t$ the change of $\ln(F_k(t))$ can be expressed as

$$\ln(F_k(t+\triangle t)) = \ln(F_k(t)) + \sigma_k(t) \sum_{j=t+1}^{k} \frac{\rho_{k,j}\sigma_j(t)F_j(t)}{1+F_j(t)} \triangle t - \frac{\sigma_k(t)^2}{2} \triangle t + \sigma_k(t)\Big(Z_k(t+\triangle t) - Z_k(t)\Big).$$

This is equivalent to

$$F_k(t+\triangle t) = F_k(t) exp\left[\sigma_k(t) \sum_{j=t+1}^{k} \frac{\rho_{k,j}\sigma_j(t)F_j(t)}{1+F_j(t)} \triangle t - \frac{\sigma_k(t)^2}{2} \triangle t + \sigma_k(t)\sqrt{\triangle t}\epsilon\right], \tag{3.3}$$

where ϵ should be standard normally distributed.

We are using yearly steps so that $\triangle t = 1$ in our case, and $k = t, \cdots, 19$, therefore we have 19 Brownian motions Z_k with non trivial correlation matrix. To reduce the dimension of the underlying Brownian motions and make them independent, a principal component analyses (PCA) approach is needed. This procedure will be introduced in the next chapter.

Chapter 4

Monte Carlo Simulation

As explained in the previous chapter the pricing of a derivative either with stocks or with interest rates as underlying consists of taking the expectation of the future payoffs of the derivatives under an equivalent martingale measure, i.e. of calculating an integral which is sometimes analytically not solvable. Therefore we need to solve such problems numerically. The Monte Carlo (MC) simulation is one of the most common numerical method. Thereby the Law of Large Numbers is used as the mathematical justification.

In the first section we will give at first a brief introduction about the principle of MC simulation and how efficient this method is.
In the second section we will discuss how to apply the MC simulation for modelling the stock market and calculating the value of a derivative based on a stock. Thereby we will use at first the famous Black Scholes model for modelling one single risky asset and then extend this model for N with each other correlated assets. Using the simulated model the price of derivatives can be calculated.
In the third section a MC simulation for forward rates curve is composed based on the LIBOR market model introduced in Chapter 3. To reduce the dimension of the correlated randomness of forward rates and get independent randomness of them we established a PCA analyses for the historical forward rates. At the end we need to calibrate this model according to market swaption prices so that this model can be well used for valuation swaptions.

4.1 Principle of Monte Carlo simulation

The simplest example to use MC simulation is the following:
Consider we need to calculate an expected value of a function $f(X)$ where X has the distribution density $\psi(x)$.

$$\mu = \mathbb{E}[f(X)] = \int_x f(x)\psi(x)dx$$

We can do the following steps to get the solution:

(1)Establish a procedure of drawing variable x from the target distribution $\psi(x)$ and then draw a variate vector $(x_i)_{1\leq i\leq N}$, for example we can use a vector $(x_i)_i$ generated from computer programs like matlab;

(2)Evaluate $f_i = f(x_i)$;

(3)So we get N possible values for the random variable $f(X)$. We define the MC estimator:

$$\hat{\mu}_N := \frac{1}{N}\sum_{i=1}^{N} f(x_i).$$

The strong Law of Large Numbers from Kolmogorov ensures that if the number of draws increases to infinite, then the estimator $\hat{\mu}_N$ converges to the correct value μ almost for sure.

Theorem 4.1 (The strong Law of Large Numbers) *Consider a sequence of independent identically distributed (i.i.d) random variables X_i with $\mathbb{E}(X_i) = \mu$. Define*

$$S_N := \frac{1}{N}\sum_{i=1}^{N} X_i.$$

Then we have

$$S_N \overrightarrow{n \to \infty} \mu \quad a.s.$$

The central limit theorem provides information about the likely magnitude of the error in the estimate after a finite number of draws.

A fundamental implication of asset pricing theory is that under certain circumstances, the price of a derivative security can be usefully represented as an expected value. Valuing derivatives thus reduces to computing expectations. In many cases, if we were to write the relevant expectation as an integral, we would find that its dimension is large or even infinite. This is precisely the sort of setting in which MC method becomes attractive.

The standard errors of the MC method is $\frac{\sigma}{\sqrt{N}}$. The convergence of MC is usually slow. There exit many methods for increasing the efficiency of MC simulation by reducing the variance of simulation estimates, called variance reduction techniques. For further details we refer to [12].

4.2 Monte Carlo simulation for stock market

4.2.1 Monte Carlo simulation for one single asset

At first, we simplify the case by assuming that we only have one equity.
According to the stock market model we have one risky asset S which is an Itô process satisfying

$$dS(t) = \mu(t)S(t)dt + \sigma(t)S(t)dB(t).$$

Using Itô's Formula of $\ln(S)$ we get

$$d\ln(S(t)) = (\mu(t) - \frac{\sigma(t)^2}{2})dt + \sigma(t)dB(t).$$

If we take a very small time interval $\triangle t$, then we can write the change of $\ln(S(t))$ to $\ln(S(t+\triangle t))$ as:

$$\triangle \ln(S(t)) := \ln(S(t+\triangle t)) - \ln(S(t)) = (\mu(t) - \frac{\sigma(t)^2}{2})\triangle t + \sigma(t)\triangle B(t).$$

Because $\triangle B(t) = B(t+\triangle t) - B(t)$ is $\mathcal{N}(0, \triangle t)$ distributed, we have:

$$\triangle \ln(S(t)) := \ln(S(t+\triangle t)) - \ln(S(t)) = (\mu(t) - \frac{\sigma(t)^2}{2})\triangle t + \sigma(t)\epsilon\sqrt{\triangle t}$$

where ϵ is standard normally distributed. Hence:

$$S(t+\triangle t) = S(t)\exp[(\mu(t) - \frac{\sigma(t)^2}{2})\triangle t + \sigma(t)\epsilon\sqrt{\triangle t}] \tag{4.1}$$

Using the equation (4.1) we can perform the MC simulation. The idea is that for every time step we generate a value for the standard normally distributed ϵ. With predetermined $\mu(t)$, $\sigma(t)$, $S(t)$ and $\triangle t$ we can calculate a possible value for $S(t+\triangle t)$. We repeat this procedure N times and get N possible paths for S, whose mean value should converges to the real path of S due to the Law of Large Numbers.
Let's describe this procedure stepwise more clearly.

Procedure

(i) Consider the whole time interval is $[0, T]$. We divide it into n short time intervals with length $\triangle t$. ($\triangle t$ as small as possible). So we get $n+1$ different time points 0, $\triangle t$, $2\triangle t$, $\cdots$, $n\triangle t = T$.

(ii) Generate n random numbers $\epsilon_1, ...\epsilon_n$ which are standard normally distributed with a predetermined method. Here we are using the matlab generator.

(iii) Consider this asset starts with an initial value $S(0)$.
Using the formula (4.1) we get successively $S(t+\triangle t)$ for $t = \triangle t$, $2\triangle t$, ..., $(n-1)\triangle t$, T:

$$S(\triangle t) = S(0)\exp\Big((\mu(t) - \tfrac{\sigma(t)^2}{2})\triangle t + \sigma(t)\sqrt{\triangle t}\epsilon_1\Big),$$

$$\vdots$$

$$S(T) = S(n\triangle t) = S((n-1)\triangle t)\exp\Big((\mu(t) - \tfrac{\sigma(t)^2}{2})\triangle t + \sigma(t)\sqrt{\triangle t}\epsilon_n\Big).$$

So we get $S(0), S(\triangle t), ..., S(T)$ as one possible path of S.

(iv) Repeat step 2 and 3 N times to get N sample values.

Now we make a simple concrete example to make this procedure clearly.

Example $T = 10$; $\triangle t = 1$; $S(0) = 100$; $\mu = 0.2$ constant; $\sigma = 0.3$ constant:

$$\begin{aligned} S(t+\triangle t) &= S(t)\exp[(\mu - \sigma^2/2)\triangle t + \sigma\epsilon\sqrt{\triangle t}] \\ &= S(t)\exp[(0.2 - 0.3^2/2)\cdot 1 + 0.3\cdot 1\cdot \epsilon] \\ &= S(t)\exp(0.155 + 0.3\cdot\epsilon) \end{aligned}$$

Assume now a random sample for ϵ as in the second column in the following table, then the possible path for S can be calculated:

t	ϵ	$exp[(\mu - \frac{\sigma^2}{2})\triangle t + \sigma\epsilon\triangle t] =: A(t)$	S
0			100
1	-0.4326	$\exp(0.155 + 0.3\cdot\epsilon_1) = 1.0256$	$S(0)\cdot A(1) = 102.5552$
2	-1.6656	$\exp(0.155 + 0.3\cdot\epsilon_2) = 0.7085$	$S(1)\cdot A(2) = 72.6552$
3	0.1253	$\exp(0.155 + 0.3\cdot\epsilon_3) = 1.2124$	$S(2)\cdot A(3) = 88.0870$
4	0.2877	$\exp(0.155 + 0.3\cdot\epsilon_4) = 1.2729$	$S(3)\cdot A(4) = 112.1265$
5	-1.1465	$\exp(0.155 + 0.3\cdot\epsilon_5) = 0.8278$	$S(4)\cdot A(5) = 92.8223$
6	1.1909	$\exp(0.155 + 0.3\cdot\epsilon_6) = 1.6691$	$S(5)\cdot A(6) = 154.9281$
7	1.1892	$\exp(0.155 + 0.3\cdot\epsilon_7) = 1.6682$	$S(6)\cdot A(7) = 258.4521$
8	-0.0376	$\exp(0.155 + 0.3\cdot\epsilon_8) = 1.1545$	$S(7)\cdot A(8) = 298.3957$
9	0.3273	$\exp(0.155 + 0.3\cdot\epsilon_9) = 1.2881$	$S(8)\cdot A(9) = 384.3710$
10	0.1746	$\exp(0.155 + 0.3\cdot\epsilon_{10}) = 1.2305$	$S(9)\cdot A(10) = 472.9549$

Plot this possible path in a diagram one can see the following figure:

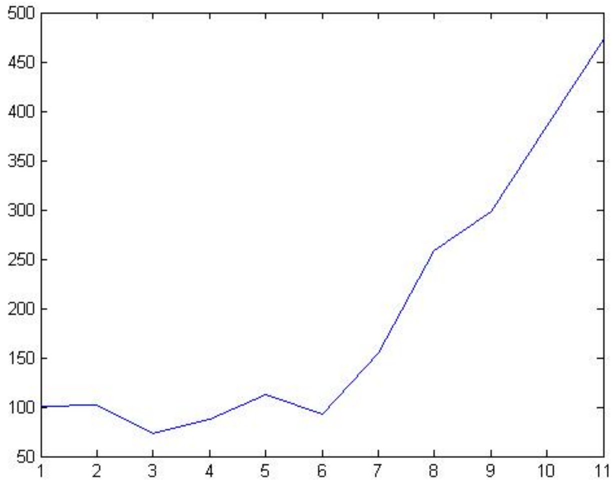

Figure 4.1: This graphic shows one possible path of this equity over the period $[0, 10]$ with $\triangle t = 1$

If we now take $\triangle t$ very small, for example $t = 0.01$ we will get a possible path for S as:

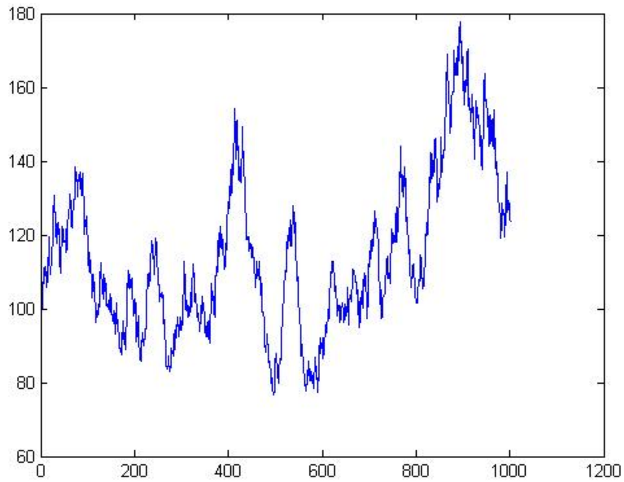

Figure 4.2: This graphic shows one possible path of this equity over the period $[0, 10]$ with $\triangle t = 0.01$

We can now sample repeatedly for ϵ from the standard normally distribution. And we can also plot the N paths in a diagram. For $N = 20$ we have the following picture:

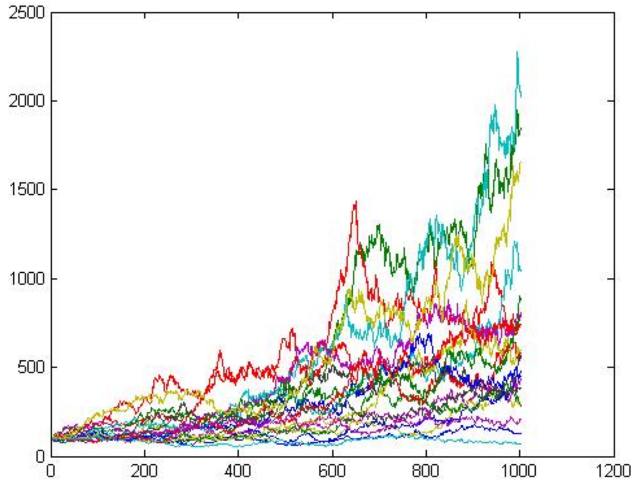

Figure 4.3: This graphic shows 20 possible path of this equity over the period $[0, 10]$ with $\triangle t = 0.01$

In this paper we do all simulation with the program matlab. The matlab files for doing this procedure can be found in the appendix.

4.2.2 Monte Carlo simulation for N correlated assets

Now we assume that we have N risky assets in our portfolio, which are dependent on each other.

For $i = 1, ...N$ we have:

$$dS_i(t) = \mu_i(t)S_i(t)dt + \sum_{j=1}^{d} S_i(t)\sigma_{ij}(t)dB_j(t),$$

where $B(t)$ is a d-dimensional Brownian motion.
For simplicity we can assume that each asset is only dependent on one single Brownian motion, so we consider $N = d$ and

$$(\sigma_{ij})_{i,j} = \begin{pmatrix} \sigma_1 & 0 & \dots & 0 \\ 0 & \sigma_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_N \end{pmatrix}$$

such that

$$dS_i(t) = \mu_i(t)S_i(t)dt + S_i(t)\sigma_i(t)dB_i(t) \quad i = 1, ..., N.$$

Consider $C = (\rho_{ij})_{i,j}$ as the correlation matrix of the n Brownian motion $B_1, ..., B_N$, and we want to generate possible paths for $S_1, ...S_N$ with this dependence structure.

We generate at first N independent random variables X_i $(1 \leq i \leq N)$, which are standard normally distributed. The required variables are ϵ_i $(1 \leq i \leq N)$ where $corr(\epsilon_i, \epsilon_j) = \rho_{ij}$. The idea is to get ϵ from X_i. Define:

$$\epsilon_i = \sum_{k=1}^{i} \alpha_{ik} X_k$$

where the $\alpha' s$ are defined such that:

(i) $\mathbb{E}(\epsilon_i) = 0$.
This means we must have:
$$\mathbb{E}(\sum_{k=1}^{i} \alpha_{ik} X_k) = 0$$
This is true because the expectation of X_i is zero for each i=1,..,N.

(ii) $Var(\epsilon_i) = 1$.
This means we must have:

$$
\begin{aligned}
Var(\sum_{k=1}^{i} \alpha_{ik} X_k) &= \sum_{k=1}^{i} Var(\alpha_{ik} X_k) \\
&= \sum_{k=1}^{i} \alpha_{ik}^2 Var(X_k) \quad | \, Var(X_k) = 1 \\
&= \sum_{k=1}^{i} \alpha_{ik}^2 \\
&= 1
\end{aligned}
$$

(iii) $corr(\epsilon_i, \epsilon_j) = \rho_{ij}$.
This means we must have:

$$
\begin{aligned}
corr(\epsilon_i, \epsilon_j) &= \frac{Cov(\sum_{k=1}^{i} \alpha_{ik} X_k, \sum_{l=1}^{j} \alpha_{jl} X_l)}{\sqrt{Var(\sum_{k=1}^{i} \alpha_{ik} X_k)}\sqrt{Var(\sum_{l=1}^{j} \alpha_{jl} X_l)}} \\
&= \frac{\sum_{k=1}^{i} \sum_{l=1}^{j} Cov(\alpha_{ik} X_k, \alpha_{jl} X_l)}{\sqrt{\sum_{k=1}^{i} Var(\alpha_{ik} X_k)}\sqrt{\sum_{l=1}^{j} Var(\alpha_{jl} X_l)}} \\
&= \frac{\sum_{k=1}^{i} \sum_{l=1}^{j} \alpha_{ik}\alpha_{jl} Cov(X_k, X_l)}{\sqrt{\sum_{k=1}^{i} \alpha_{ik}^2}\sqrt{\sum_{l=1}^{j} \alpha_{jl}^2}} \\
&= \sum_{k=1}^{j} \alpha_{ik}\alpha_{jk} Cov(X_k, X_k) \\
&= \sum_{k=1}^{j} \alpha_{ik}\alpha_{jk} \\
&= \rho_{ij}
\end{aligned}
$$

In this calculation we used the fact that:

$$
Cov(X_k, X_l) = \begin{cases} 0 & \text{if } k \neq l, \\ 1 & \text{if } k = l. \end{cases}
$$

So we must construct a lower triangular matrix A

$$
\begin{pmatrix}
\alpha_{11} & 0 & \dots & 0 \\
\alpha_{21} & \alpha_{22} & \dots & 0 \\
\alpha_{31} & \alpha_{32} & \dots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_{N1} & \alpha_{N2} & \dots & \alpha_{NN}
\end{pmatrix}
$$

whose elements α_{ij} fulfill the two conditions derived from before:

$$\sum_{k=1}^{i} \alpha_{ik}^2 = 1 \tag{4.2}$$

$$\sum_{k=1}^{j} \alpha_{ik}\alpha_{jk} = \rho_{ij} \tag{4.3}$$

This is exactly the Cholesky decomposition.

Procedure

(i) At first we consider N independent Brownian motions and we can generate random samples x_i $(1 \leq i \leq N)$ for them;

(ii) Perform the Cholesky decomposition on the correlation matrix $C = (\rho_{ij})_{i,j}$ and get $C = A \cdot A'$ where A is a lower triangle matrix with elements α_{ij} (4.2), (4.3).

(iii) Write the random samples x_i as a vector

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix}$$

Now define $E = AX$ to get a N-dimensional vector E, which is the required sample for N correlated Brownian motion with the given correlation matrix.

(iv) We also can generate m groups of samples x_i in a matrix form

$$X = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1m} \\ x_{21} & x_{22} & \cdots & x_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ x_{N1} & x_{N2} & \cdots & x_{Nm} \end{pmatrix}.$$

Define $E = AX$. So we get a $N \times m$-dimensional matrix E with each column a sample for the N Brownian motions.

Example 4 (Cholesky decomposition) *Consider we have three assets S_1, S_2, S_3, which are driven by the Brownian motions B_1, B_2, B_3. Between the B_i's exists certain dependence structure which is described by the correlation matrix $C = (\rho_{ij})_{i,j}$:*

$$\begin{pmatrix} 1 & 0.8 & 0.2 \\ 0.8 & 1 & 0.5 \\ 0.2 & 0.5 & 1 \end{pmatrix}$$

With the Cholesky decomposition we get the lower triangular matrix A as follows:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0.8 & 0.6 & 0 \\ 0.2 & 0.5667 & 0.7993 \end{pmatrix}$$

We can generate with matlab 1000 groups of samples for standard normally distributed random variables via the command $X = randn(3, 1000)$.
Now we calculate $E = AX$ and get a $3 \cdot 1000$-dimensional matrix E. Each column of E is a sample of realization of B_1, B_2, B_3.

Now we can use the Monte Carlo Simulation for the equity which was introduced in the previous section, but instead of using samples from independent Brownian motions we use now the samples from the three dependent Brownian motions, namely the 1000 samples in the matrix E.

4.2.3 Derivative pricing using simulation

According to Theorem (2.18) we can calculate the price at time 0 for a derivative with payoff X at time T:

$$V_0 = S_0(0)\mathbb{E}_{\mathbb{Q}}\left[\frac{X}{S_0(T)}|\mathcal{F}_0\right] = \mathbb{E}_{\mathbb{Q}}\left[\frac{X}{S_0(T)}\right] \tag{4.4}$$

We compute (4.4) by using the Monte Carlo method, i.e. we generate many independent samples of the random variable and take the empirical average.

Procedure

(i) Simulate the random paths for the underlying of this derivative;

(ii) Use each path to calculate the payoff X;

(iii) Take the mean of all payoffs to get the expected payoff;

(iv) Discount the expected payoff at the risk free rate.

Example 5 (Option valuation) *Consider a European call option with one single asset as underlying.*
The underlying asset is an equity with a start value $S(0) = 100$. The volatility of this equity is 20 %, and the expected return 10 %. The call option has a strike $X = 100$ and a maturity $T = 1$. The risk free rate r is 10 %.
As described above we can simulate the development of S with MC method, and then calculate the possible payoffs of this option ,and discount it to the current time point. The matlab file for this procedure can be found in the appendix. To show the result of the MC simulation we compare the price from MC simulation with the price from the calculation using Black Scholes formula in the following graphic:

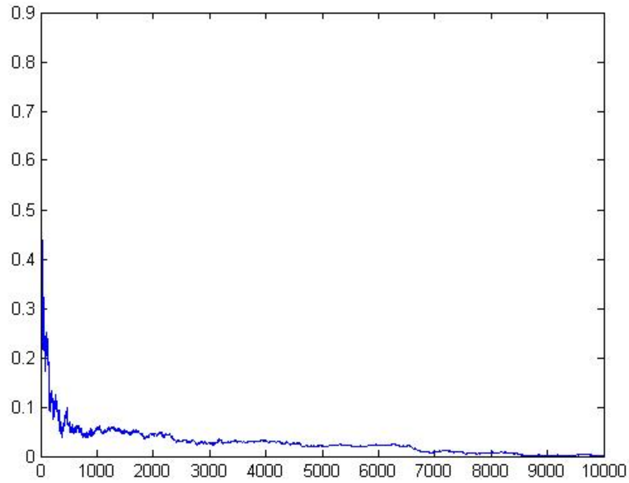

Figure 4.4: This graphic tells the difference between the option prices with the two different methods, on the x-axis we have the number of simulations. We can see that for a good approximation of the option price one need a huge number of simulations.

4.3 Monte Carlo simulation for forward rate curve

4.3.1 PCA method for forward rate curve

Consider now we want to model the forward LIBOR rates for the next 20 years with a yearly step. This means we want to model $F_k(t)$ for $t = 1, \cdots, 19$; $k = t, \cdots, 19$.

According to the LIBOR market model introduced in the last chapter the forward rates dynamics under the spot measure $\mathbb{Q}_{t+1}$ can be defined as follows:

$$dF_k(t) = \sigma_k(t)F_k(t)\sum_{j=t+1}^{k}\frac{\rho_{k,j}\tau_j\sigma_j(t)F_j(t)}{1+\tau_jF_j(t)}dt + \sigma_k(t)F_k(t)dZ_k$$

for $t = 0, \cdots, 19$ and $k = t, \cdots, 19$
From this formula we can see that the forward rate curve is dependent on 19 different Brownian motions $Z_1, \cdots, Z_{19}$, where the Brownian motions are correlated with each other. To reduce the dimension of Brownian motions that need to be simulated and to avoid the complicated structure between them we can apply a principal component analysis (PCA) on the forward rates data. A general introduction about the PCA can be found in appendix. Here we briefly explain why the requirement above can be satisfied with a PCA approach.

With a PCA we can convert the 19 correlated Z_i to 19 uncorrelated random variables $\tilde{Z}_i$, where Z_i are linear combinations of $\tilde{Z}_i$:

$$\begin{aligned} Z &= \begin{pmatrix} Z_1 \\ \vdots \\ Z_k \\ \vdots \\ Z_{19} \end{pmatrix} = P_1\tilde{Z}_1 + \cdots + P_{19}\tilde{Z}_{19} \\ &= \begin{pmatrix} P_{11}\tilde{Z}_1 + \cdots + P_{19,1}\tilde{Z}_{19} \\ \vdots \\ P_{1k}\tilde{Z}_1 + \cdots + P_{19,k}\tilde{Z}_{19} \\ \vdots \\ P_{1,19}\tilde{Z}_1 + \cdots + P_{19,19}\tilde{Z}_{19} \end{pmatrix} \end{aligned}$$

P_i are called the principal components (PC) and are ordered according to their importance in explaining the covariance structure of the Z_i. Particularly the first three PCs are most important, they can describe almost 95% of the interest rate movements, so that the other PCs can be ignored. Therefore we can simplify our case to

$$Z_k = P_{1k}\tilde{Z}_1 + \cdots + P_{19,k}\tilde{Z}_{19} \approx P_{1k}\tilde{Z}_1 + \cdots + P_{3,k}\tilde{Z}_3$$

So now we just have to simulate 3 independent random variables $\tilde{Z}_1, \tilde{Z}_2, \tilde{Z}_3$ instead of simulating 19 dependent $Z_1, \cdots, Z_{19}$.

Now we can introduce the PCA application for the forward rate curves.

Step 1

We have 19 randomness $Z_1, \cdots, Z_{19}$. At first we have to collect the data for the Z_k. From the formula we know that F_k are log normal distributed and the Z_k expresses the log changes of the F_k. Therefore we can at first collect the data for $F_k(t)$ from the history and then calculate the log changes of the forward rates. We choose a time horizon of 10 years and a time step of 1 month for the historical data, so we will get 120 samples for the 19 forward rates. And we define the log changes of the forward rates as:

$$z_{ij} := \ln \frac{F_{j+\frac{1}{12}}(i + \frac{1}{12})}{F_j(i)}.$$

We will have a 119×19- dimensional data matrix $A = (z_{i,j})$.

Remark In the PCA we are not looking at exactly the same forward rates. We always compare the rates for different time periods, i.e., we are comparing the forward rate $F_{j+\triangle t}(i+\triangle t)$ with $F_j(i)$, since the maturity date is always moving forward by one month. But in the MC simulation we have to simulate $F_j(i+\triangle t)$ from $F_j(i)$. This time shift makes the two procedures not entirely coincident and therefore weakens this method a little.

Step 2

Apply the PCA method using covariance matrix of the data matrix A.
Define $C := Cov(A, A)$:

$$C_{ij} := \frac{1}{119-1}\sum_{k=1}^{119}(z_{ik}-\bar{z}_i)(z_{jk}-\bar{z}_j),$$

where

$$\bar{z}_i := \sum_{k=1}^{119} z_{ik}.$$

Step 3

Decompose the covariance matrix C into $C = L \cdot \Lambda \cdot L^T$, where $\Lambda$is the diagonal matrix with eigen values of C on the diagonal and L is matrix with orthogonal eigenvector $P_1, \cdots, P_{19}$ of C as columns.

Step 4

Define $\tilde{Z} = L \cdot Z$, we get

$$Z = P_1\tilde{Z}_1 + \cdots + P_{19}\tilde{Z}_{19}.$$

With the properties of Λ and L one can easily prove that

$$\mathbb{E}[\tilde{Z}_i] = 0; \quad Cov(\tilde{Z}, \tilde{Z}) = \Lambda$$

This means

$$Cov(\tilde{Z}_i, \tilde{Z}_j) = \begin{cases} 0 & \text{if } i \neq j, \\ \lambda & \text{if } i = j. \end{cases}$$

We can see that $P_1, \cdots P_{19}$ defines the PC and $\tilde{Z}_1, \cdots, \tilde{Z}_{19}$ are the new independent sources of randomness.

Step 5

The explained variance by P_i is:

$$\frac{Var(\tilde{Z}_i)}{\sum_{k=1}^{19} Var(\tilde{Z}_k)} = \frac{\lambda_i}{\sum_{k=1}^{19}\lambda_k}$$

As we can see in the later examples the first three PC always explain the most of the variance, therefore we can ignore the other PC, so that

$$Z_k = P_{1k}\tilde{Z}_1 + P_{2k}\tilde{Z}_2 + P_{3k}\tilde{Z}_3 = P_{1k}\sqrt{\lambda_1}\epsilon_1 + P_{1k}\sqrt{\lambda_1}\epsilon_2 + P_{2k}\sqrt{\lambda_3}\epsilon_3$$

where $\epsilon_i \sim \mathcal{N}(0,1)$.

4.3.2 Monte Carlo simulation

PCA allows us to express dependent stochastic processes $\begin{pmatrix} Z_1 \\ \vdots \\ Z_{19} \end{pmatrix}$ through independent stochastic processes $\begin{pmatrix} \tilde{Z}_1 \\ \vdots \\ \tilde{Z}_{19} \end{pmatrix}$. And in our three factor model we approximate $\begin{pmatrix} Z_1 \\ \vdots \\ Z_{19} \end{pmatrix}$ by

$$P_1\sqrt{\lambda_1}\epsilon_1 + \cdots + P_3\sqrt{\lambda_3}\epsilon_3$$

Using this approximation and formula (3.3) and taking $\triangle t = 1$ we have :

$$F_k(t+1) = F_k(t)exp\left[\sigma_{k-t}\sum_{j=t+1}^{k}\frac{\rho_{k-t,j-t}\sigma_{j-t}F_j(t)}{1+F_j(t)} - \frac{\sigma_{k-t}^2}{2} + \sqrt{\lambda_1}P1^{k-t}\epsilon_1 + \sqrt{\lambda_2}P2^{k-t}\epsilon_2 + \sqrt{\lambda_3}P3^{k-t}\epsilon_3\right] \tag{4.5}$$

where ϵ_i are $\mathcal{N}$(0,1) distributed.

Remark
The principal components get shifted as t changes to $t+1$. This is based on the assumption that the changes in short-term resp. long-term forward rates can be better compared with each other than the changes in the forward rates with constant maturity. (The forward rates which are used to compute the change should ideally have the same maturity). Moreover, considering changes in the forward rates with constant maturity does not provide sufficient data for the PCA since there would be very limited amount of data for changes in the forward rates which are short-term.

Therefore we can derive the procedure for MC simulation of the forward rate curves as follows:

Procedure

(i) Generate 3 independent $\mathcal{N}(0,1)$ distributed random variable $\epsilon_1, \cdots, \epsilon_3$ for every time step t;

(ii) Calculate $F_k(t+1)$ from $F_k(t)$ using the formula (4.5) to get the entire forward rate curve;

(iii) Repeat these two steps N times and we will get N samples for the forward rate curve.

This procedure can be well done with matlab, the matlab file for MC simulation for forward rates curve can be found in the Appendix.

4.3.3 A concrete example

Let's see a concrete example for the MC simulation for the forward rate.

Input data

For $t = 0$ there is no stochastic in $F(0, k, k+1)$, we can calculate these rates from the given market swap rates. The method for calculation of forward rates will be explained in appendix.

For example we can get the forward rates $F(0, k, k+1)$ on 30. June 2007 as follows:

k	$FwdLIBORrates$	k	$FwdLIBORrates$
1	4.78	11	5.20
2	4.80	12	5.23
3	4.84	13	5.23
4	4.87	14	5.23
5	4.91	15	5.14
6	4.96	16	5.15
7	5.00	17	5.15
8	5.07	18	5.15
9	5.12	19	5.15
10	5.17		

If we consider now monthly return of the swap rates for ten years, we have 120 samples of the swap rates, and we will get 120 samples for forward rates. We write all the samples of the 19 forward rate variables into a matrix F, and calculate the logarithm changes of forward rate in monthly step to get matrix A. The concrete data of this example can be found in the Appendix.

PCA analyses

If we perform the PCA procedure of the matrix A, this means, calculate the covariance matrix C of A, do the decomposition of C to get eigenvalues and eigenvectors of C, define the eigenvectors as PC, calculate the new independent variables and calculate the explained variance of the PC, we will get the following result:

The first three PC are:

$P1$	$P2$	$P3$
1.0000	−1.0000	0.3208
0.9149	−0.6385	0.2518
0.8977	−0.3096	0.0064
0.7754	−0.1211	0.0351
0.7233	0.0033	−0.0952
0.6409	0.1192	−0.4433
0.6277	0.2845	−0.3189
0.5654	0.1684	−0.4872
0.5267	0.1953	−0.3920
0.4630	0.2740	−0.7136
0.4614	0.2789	−0.7229
0.4601	0.2836	−0.7316
0.4590	0.2880	−0.7399
0.4582	0.2922	−0.7479
0.4470	0.3702	0.9153
0.4461	0.3756	0.9374
0.4453	0.3810	0.9587
0.4446	0.3864	0.9795
0.4440	0.3918	1.0000

The explained variance for P1, P2 and P3 are 63.57%, 19.14% and 7.22%, i.e. the first three PCs can explain 89.92% of the total variance. The following picture presents the first three PCs:

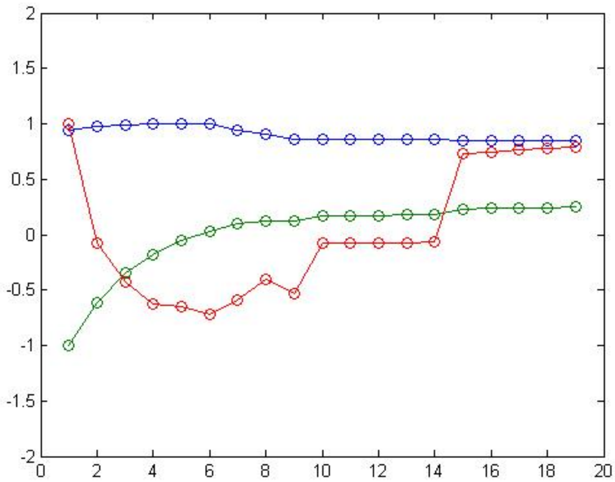

Figure 4.5: This graphic shows the first three principal components

We can see that

- The first PC is roughly flat, it can be expressed as the parallel shift of the forward rate curve
- The second PC is upward sloping
- The third PC is hump-shaped

MC simulation
As start point we have the forward rates at time t=0, $F_k(0)$ for $k = 0, \cdots, 19$ are already known.

Generating three independent standard normally distributed random numbers and using formula (3.3) we get successively $F_k(1), \cdots, F_k(19)$. At the end we will get an upper triangle matrix for the forward rates:

$$R = \begin{pmatrix} F_0(0) & F_1(0) & \cdots & \cdots & F_{19}(0) \\ 0 & F_1(1) & \cdots & \cdots & F_{19}(1) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & F_t(t) & \cdots & F_{19}(t) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \cdots & F_{19}(19) \end{pmatrix}$$

For example a possible path for the forward rates can be as follows:

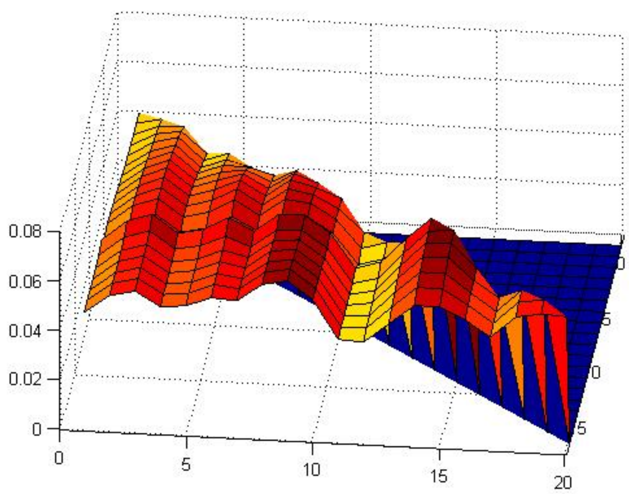

Figure 4.6: This graphic shows one possible path of the simulated interest rates

Repeat this procedure for N times, we can get N such matrices, every matrix is then one possible interest rate curve in the future.

4.3.4 Interest rate derivative pricing using simulation

The payoff of an interest rate derivative is dependent on the level of interest rates, from the possible interest rates in the future simulated with Monte Carlo we can calculate the possible payoffs of the derivative and the mean value of all possible payoffs should be the expected payoff of this derivative. Discount the expected payoff with risk free rate we will get the price of this instrument.

Example 6 (Valuation of swaption) *Consider a swaption contract with maturity T and Tenor n, this means, the holder of the swaption has in T year the right to decide whether he want to enter into a n-year IRS or not. The strike of the IRS is K, the principal N. The reset dates are in yearly distance.*

As explained in the previous chapter the payoff of the swaption at maturity T is

$$N \cdot \Big(R_{swap}(T) - K\Big)^+ \sum_{i=1}^{n} P(T, T+i)$$

Therefore for valuating a swaption we need to calculate the discount factors $P(T, T+i)$, the forward swap rate $R_{swap}(T)$ and the risk free rate for discounting.

(i) Discount factors
We already simulated the forward rate curve. For each path we have a 20×20 matrix R. The matrix of discount factors can be calculated easily:

$$Df := \begin{pmatrix} P(0,1) & P(0,2) & \cdots & \cdots & P(0,20) \\ 0 & P(1,2) & \cdots & \cdots & P(1,20) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & P(t,t+1) & \cdots & P(t,20) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \cdots & P(19,20) \end{pmatrix}$$

$$= \begin{pmatrix} (\frac{1}{1+F_0(0)}) & (\frac{1}{1+F_0(0)}\frac{1}{1+F_1(0)}) & \cdots & \cdots & (\frac{1}{1+F_0(0)} \cdots \frac{1}{1+F_{19}(0)}) \\ 0 & (\frac{1}{1+F_1(1)}) & \cdots & \cdots & (\frac{1}{1+F_1(1)} \cdots \frac{1}{1+F_{19}(1)}) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & (\frac{1}{1+F_t(t)}) & \cdots & (\frac{1}{1+F_t(t)} \cdots \frac{1}{1+F_{19}(t)}) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \cdots & (\frac{1}{1+F_{19}(19)}) \end{pmatrix}$$

(ii) *Forward swap rate*
Using the formula for calculating R we can get the forward swap rate from the matrix Df.

$$R_{swap}(T) = \frac{P(T,T) - P(T,Tenor)}{\sum_{i=1}^{n} P(T,T+i)}$$

(iii) *Risk free rate for discounting*
To discount the expected payoff one needs the bank account $B(t)$:

$$B(t) = (1 + F_0(0)) \cdot (1 + F_1(1)) \cdots (1 + F_{t-1}(t-1))$$

because the $F_t(t)$ on the diagonal of the matrix R are the spot rates seen at time t, therefore can be seen as the risk free rates.

With one path of the forward rate we can calculate one possible value for this swaption.

Execute this procedure for the N paths, we will get N possible values for the swaption price, the average value of them is then the swaption price.

4.3.5 Calibration of the model

Calibration is a method of implying volatility parameters from the prices of actively traded options.

The volatility parameters are inferred from market data on actively traded options. These will be referred to the calibrating instruments. In this paper we choose swaption prices for calibration of the model, because we have to use this model for valuating swaptions. We choose the most important swaptions for us, for example swaptions with maturity in 2 or 6 years and tenor as 4 or 6 years, and find market prices for these swaptions. We can change our input parameters for the model so that the prices from our model can fit the market prices well. The input parameters are for example the PCs and their weights.

The PCs coming from PCA are not always smooth, see Figure (4.5). That is because they reflect certain specific market effects concerning swaptions. Since these market effects do not influence the evaluation of our financial instruments, we smooth the PCs in order to eliminate these market effects. Furthermore the PCs and their weights derived from historical data may not fit the future situation anymore. So we can keep the shape of the PCs and try to smooth the curves of the PCs, so that they can deliver us the wished volatility structure of the forward rates.

Smoothed PCs give better behaved and more continuous volatility surfaces, which are more reasonable for ex-ante valuation.

For example a pair of smoothed PC can look like as follows:

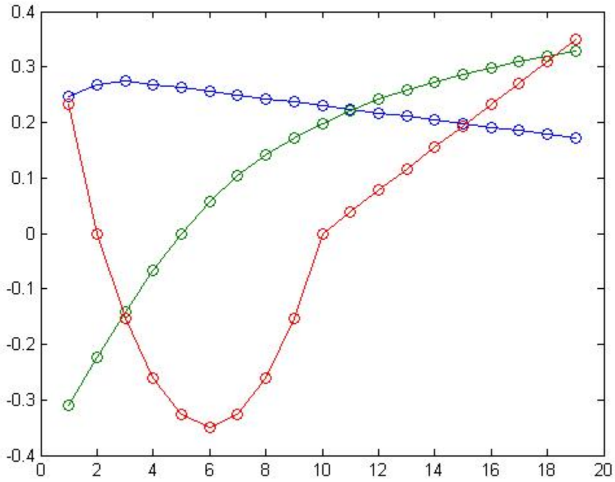

Figure 4.7: This graphic shows the new created principal components, which are more smooth now

The volatility and correlation of the forward rates using the smoothed PCs are quite reasonable in comparison with historical data. For the volatility we can use the historical data to get the level of the forward rates volatility. Furthermore we can get the current implied volatility for swaptions. It is not exactly the volatility of the forward rates, but the magnitude should be the same. Our volatility with the smoothed PCs, which is displayed in figure (4.8), satisfies these points very well. For the correlation we also can use historical estimation to get the correlation matrix of the forward rates. In [5] such an analysis was made, where the historical correlation between the forward rate in 1 year and the other forward rates in 1,...,19 years is from 100% to about 25%, which are quite similar to our result with the smoothed PCs.

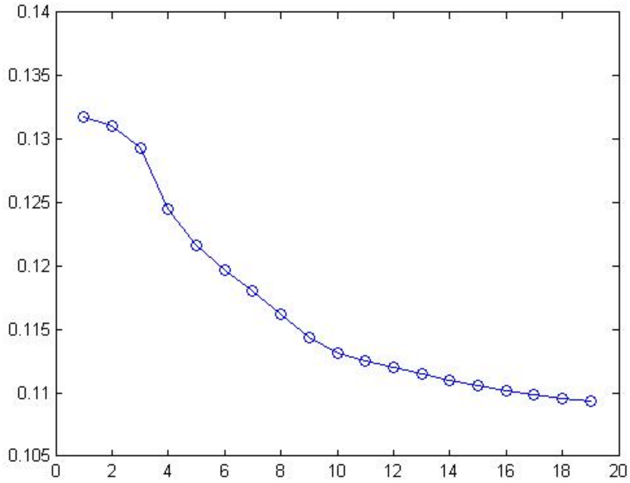

Figure 4.8: This graphic shows the volatility of the forward rates in 1,...,19 years. We can see they are between 10.5% and 13.4%.

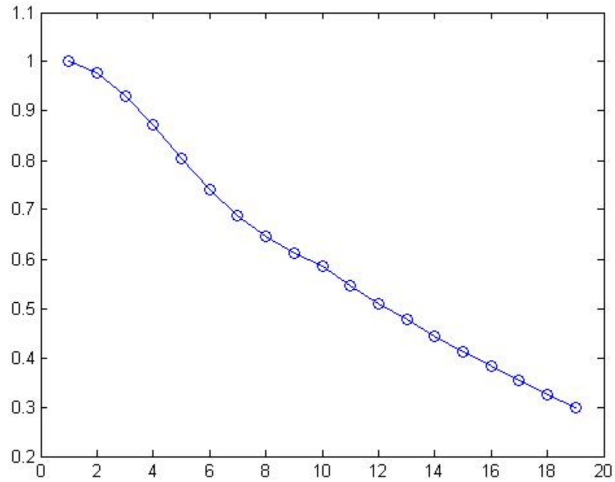

Figure 4.9: This graphic shows the correlation between the forward rate in 1 year and the forward rates in 1,...,19 years. We can see it is from 100% to about 30%.

Chapter 5

Hybrid model

In the recent years the financial industry has witnessed the growth of a new class of derivative products, so called *hybrid*, because they combine risks from different markets such as interest rates, equity and credit. For handling with these financial products a model for dynamics of different markets in a tractable and realistic form is of extreme importance.

In this chapter we will introduce a hybrid model that combines the interest rate market and the equity market taking into account of the correlation structure between the two markets. Using this model we can price hybrid derivatives.

The basic idea is to use the three factor LIBOR market model and build an equity model based on it. In our interest rate model we decomposed the randomness of the forward rates into three independent Brownian motions. With a historical analyses we may get the correlation coefficient between these three Brownian motions and the Brownian motion which the equity movement is driven by. So we just need to generate a Brownian motion deriving the equity movements.

5.1 Monte Carlo simulation for the hybrid model

In the previous chapter we have built a stock market and an interest rate model separately. Now we have to combine them. Our interest rate model is using three factors, all interest rate randomness are assumed to depend only on these three factors, which are somehow correlated with the randomness in the stock market. If we can find out the correlation structure between them, then we can generate them simultaneously and then use the correlated random numbers to simulate the stocks and interest rates with the MC method.

So the problem goes down to an adequate measure for the correlation structures. There are many methods for the description of correlations. For example one can describe them by using Garch process or as described in the following. To simplify the case we will derive the correlations from an analysis of historical data.

Define correlation

(i) we have calculated the PC from historical data. Define P=(PC1 PC2 PC3) and $B := (B1\ B2\ B3) := A \cdot P$. So we get a 119×3 matrix B. And because P is an orthogonal matrix we have $B \cdot P^{-1} = B \cdot P^T = A$ and:

$$
\begin{aligned}
B \cdot P^T &= \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ \vdots & \vdots & \vdots \\ B_{119,1} & B_{119,2} & B_{119,3} \end{pmatrix} \cdot \begin{pmatrix} PC_{11} & PC_{12} & \cdots & PC_{1,19} \\ PC_{21} & PC_{22} & \cdots & PC_{2,19} \\ PC_{31} & PC_{32} & \cdots & PC_{3,19} \end{pmatrix} \\
&= B1 \cdot PC1^T + B2 \cdot PC2^T + B3 \cdot PC3^T.
\end{aligned}
$$

Therefore $B1, B2, B3$ are the vectors with the historical data of the three randomness that influence the logarithm changes of the forward rates we use in the model.

(ii) For a certain equity, for example the MSCI Europe, we can find the historical prices for this equity in the same period, namely 10 years from today in monthly step as well. Then we also can calculate the logarithm changes in the equity prices and define this data as a vector E.

(iii) Now we have the three randomness factors for interest rate changes $B1, B2, B3$, they are uncorrelated with each other, and we have the factor for equity changes E. Define:

$$Corr(B1, E) = \rho_1 \quad Corr(B2, E) = \rho_2 \quad Corr(B3, E) = \rho_3,$$

where ρ_1, ρ_2 and ρ_3 describe the correlation between the three factors and equity.

The drawback on this method is that we only define the correlation as constants.

MC simulation

(i) Now we have to simulate 4 Brownian motions, one for equity and the other for interest rate model. Their correlation structure can be explained with a correlation matrix:

$$Corr = \begin{pmatrix} 1 & 0 & 0 & \rho_1 \\ 0 & 1 & 0 & \rho_2 \\ 0 & 0 & 1 & \rho_3 \\ \rho_1 & \rho_2 & \rho_3 & 1 \end{pmatrix}$$

(ii) As explained in the section about simulation of N correlated assets we have to do the Cholesky decomposition of the matrix Corr: $Corr = A \cdot A^T$ and we can calculate A in this case very easily:

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \rho_1 & \rho_2 & \rho_3 & \sqrt{1-\rho_1^2-\rho_2^2-\rho_3^3} \end{pmatrix}$$

(iii) The whole MC procedure can be split in the following steps:

(a) Generate 4 independent standard normally distributed random numbers $B1,\ B2,\ B3,\ B4$.

(b) Define

$$X = A \cdot \begin{pmatrix} B1 \\ B2 \\ B3 \\ B4 \end{pmatrix} = \begin{pmatrix} B1 \\ B2 \\ B3 \\ \rho_1 \cdot B1 + \rho_2 \cdot B2 + \rho_3 \cdot B3 + \sqrt{1-\rho_1^2-\rho_2^2-\rho_3^3} \cdot B4 \end{pmatrix}$$

(c) Use $X1, X2, X3$ to simulate the interest rate model and use $X4$ to simulate equity. We should have the hybrid model with the wished correlation structure.

5.2 Derivative pricing using simulation in a hybrid model

With the hybrid model described in the last section, we can calculate now fair price for a hybrid option. A hybrid option is an option which combines two or more different securities such as equities and bonds. An example for a hybrid option could be a convertible bond, i.e. a bond, which can be exchanged with equity shares.

We consider now an example for a hybrid option and try to price it.

Example 7 (Hybrid option valuation) *At first consider a classical ("Plain vanilla") put option, which uses an equity S as underlying. The payoff of the put option at maturity T is:*

$$\max(X - S(T), 0)$$

where X expresses the strike, $S(T)$ is the value of the asset S at time T. As explained in Example 5 in Chapter 4 we can calculate the price of this option in the hybrid model. At first simulate the forward interest rates. Then assume $X = 100$, $S(0) = 100$, $T = 12$, *the volatility of S is 20%, and set the expected return of the equity in every time period equal to the risk free rate for that period, that is, the forward spot rate. Then simulate with these assumption 10,000 paths of the equity price process. Calculate for every path the payoff of the put option and take the discounted mean value of all payoffs. Then we will get the option price, for example, we get with 10,000 simulations a price of 4.8054.*

Now consider a hybrid put option with all the same inputs. The only difference is, that the payoff is now multiplied with a nominal value N, which is dependent on the interest rate level. The payoff is now:

$$N \times \max(X - S(T), 0)$$

where

$$N = \min((0.08 - \frac{\sum_{t=0}^{11} F(t, t+1)}{12}) \cdot 25, 1);$$

$F(t, t+1)$ is the spot rate at time point t for the period $[t, t+1]$, which we obtain from the MC simulation. With the same method as above we calculate the price for this hybrid option, given by 4.2166.

We can see that the hybrid option is less expensive than the classical one. The reason is, that the hybrid option only give us the put protection if the interest rates are going down, whereas the classical option will always deliver us the put protection.

Chapter 6

Scenario Analyses

In the previous chapters we discussed models for equities, interest rates and a hybrid model combining the first two models. We considered these models from a risk neutral viewpoint, because for valuation derivative instruments one need to calculate the expected payoff of the derivatives under a risk neutral probability measure. But in such a world it doesn't make sense to vary investment strategies because all assets should offer the same rate of return, namely the risk free rate. Therefore in order to test various investment strategies, we will change our attention from a risk neutral economy to a real world economy. This means in a real world an asset containing more risk should also offer more expected return, otherwise it is not worth investing in this asset. The excess return divided by the risk of this asset is called market price of risk. It measures the relationship between risk and return.
We will have various assets in our portfolio, like bonds, equities. In general equities are more volatile than bonds and also provide higher returns. The question is how to allocate our money in the assets, so that the relationship between return and risk becomes optimal. To compare different strategies we can use the Value at Risk (VaR) approach. It enables us to describe different types of risk with one comparable characteristic number.

Therefore in this chapter we will at first explain how to change the point of view from a risk neutral economy to real world for both stock model and interest rate model. Then we will introduce the VaR concept and how to use it to compare different strategies. Finally we will simulate the future world in the sense of real world scenarios and use VaR to evaluate some strategies to choose the optimal strategy.

6.1 From risk neutral to real world

In a risk neutral economy all expected returns are equal to the risk free rate. When we change our attention from a risk neutral economy to the real world economy, expected returns then involve a risk factor determined by the preferences of the investors concerning future price uncertainties and their relationship with a market portfolios.
After making assumption about risk preferences, it is possible to derive transformations from a risk neutral density to a real world density.

In this chapter we will discuss how to change the risk neutral dynamic in the stock market and in the interest rate market to a real world dynamic. After changing the dynamics one can use the same MC method to generate possible portfolio values and get the distribution of them.

6.1.1 Market price of risk and changing numeraire

At first let's discuss the general idea about market price of risk and changing the point of view to the real world one.

Given $(\Omega, \mathcal{F}, \mathbb{P})$ as underlying probability space. Consider the risky asset S_t with dynamics

$$\frac{dS(t)}{S(t)} = \mu(t)dt + \sigma(t)dB(t)$$

where $B(t)$ is a $\mathbb{P}$ Brownian motion.

Consider an equivalent martingale measure (EMM) $\mathbb{Q}$.

The density between $\mathbb{P}$ and $\mathbb{Q}$ can be expressed as a stochastic exponential defined in Chapter 2:

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \mathcal{E}(\gamma B)_T$$

By Girsanov's Theorem we can also change between the Brownian motions under $\mathbb{P}$ and $\mathbb{Q}$:

$$\tilde{B}(t) = B(t) - \int_0^t \gamma(s)ds$$

where $\tilde{B}(t)$ expresses a Brownian motion under $\mathbb{Q}$.

Then we can calculate with Itô's formula the dynamics of X under the measure $\mathbb{Q}$ associated with S_0.

$$\begin{aligned}
dX(t) &= dS(t)\frac{1}{S_0} + d(\frac{1}{S_0(t)})S + d < \frac{1}{S_0}, S >_t \\
&= (\mu(t)dt + \sigma(t)dB(t))X(t) - (r(t)dt)X(t) \\
&= (\mu(t) - r(t))X(t)dt + \sigma(t)X(t)(d\tilde{B}(t) + \gamma(t)dt) \\
&= X(\mu(t) - r(t) + \sigma(t) \cdot \gamma(t))dt + X\sigma(t)d\tilde{B}(t)
\end{aligned}$$

To make X a martingale under $\mathbb{Q}$ we need

$$\mu(t) - r(t) = \sigma(t)\gamma(t) \quad \Leftrightarrow \quad -\gamma(t) = \frac{\mu(t) - r(t)}{\sigma(t)}$$

$-\gamma$ is called *market price of risk* of this asset S. It measures the excess return of one unit risk.

Let's consider now two derivatives whose payoffs are dependent on this asset S, such as:

$$\frac{df_1(t)}{f_1(t)} = \mu_1(t)dt + \sigma_1(t)dB(t)$$
$$\frac{df_2(t)}{f_2(t)} = \mu_2(t)dt + \sigma_2(t)dB(t)$$

where $\mu_{1/2}(t), \sigma_{1/2}(t)$ are functions of S and t.

We consider now a portfolio consisting of $\sigma_2 f_2$ units of the first derivative and $(-\sigma_1 f_1)$ units of the second derivative, i.e.

$$V = (\sigma_2 f_2) f_1 - (\sigma_1 f_1) f_2.$$

One can prove that this portfolio is risk less, since the two random terms are counterbalanced in this portfolio.

Proof. For the portfolio value V it holds:

$$\begin{aligned} dV(t) &= \sigma_2(t)f_2(t)df_1(t) - \sigma_1(t)f_1(t)df_2(t) \\ &= \sigma_2(t)f_2(t)f_1(t)(\mu_1(t)dt + \sigma_1(t)dB(t)) - \sigma_1(t)f_1(t)f_2(t)(\mu_2(t)dt + \sigma_2(t)dB(t)) \\ &= (\mu_1(t)\sigma_2(t) - \mu_2(t)\sigma_1(t))f_1(t)f_2(t)dt \end{aligned}$$

Then we will have for a very short time period $\triangle t$:

$$\triangle V := V(t+) - V(t) = (\mu_1(t)\sigma_2(t)f_1(t)f_2(t) - \mu_2(t)\sigma_1(t)f_1(t)f_2(t))\triangle t$$

On the other hand the portfolio is instantaneous risk free. Therefore it must earn the risk free rate r:

$$\triangle V = r(t)V\triangle t = r(t)((\sigma_2(t)f_2(t))f_1(t) - (\sigma_1(t)f_1(t))f_2(t))\triangle t$$

Compare the two equations above we get:

$$\frac{\mu_1(t) - r(t)}{\sigma_1(t)} = \frac{\mu_2(t) - r(t)}{\sigma_2(t)} =: \lambda(t)$$

We see that this is not dependent on the nature of the derivative f_1 or f_2, it just depends on the asset S. At any given time the relationship between excess return and risk must be the same for all derivatives that are dependent only on S and t.

Furthermore if S is an investment asset, one can prove that $-\gamma = \lambda$. See [15].

We have already explained the concept of the market price of risk. Why is this important in the sense of changing the viewpoint to another perspective? And how can we take advantage of this by using this procedure? Now let's discuss the relationship between them.

The first point here is that changing your point of view from one world to another is nothing else than changing the probability measure in the first world to the probability measure in the other world. For example, in our case, what we have to do is only to change the dynamics of all assets from under the risk neutral probability measure $\mathbb{Q}$ to the dynamics under the real world measure $\mathbb{P}$.

The second point here is that changing probability measure can be expressed in another way, namely changing market price of risk from under the old measure to under the new measure.
Suppose we still have this asset S. We see that

$$\frac{dS(t)}{S(t)} = \mu(t)dt - \sigma(t)dB(t) \quad under \quad \mathbb{P},$$
$$\frac{dS(t)}{S(t)} = r(t)dt - \sigma(t)dB(t) \quad under \quad \mathbb{Q}.$$

The market price of risk of S is zero under $\mathbb{Q}$ and is $-\gamma(t) = \lambda(t) := \frac{\mu(t)-r(t)}{\sigma(t)}$ under $\mathbb{P}$. The change from the dynamic under $\mathbb{Q}$ to under $\mathbb{P}$ is nothing else than add the term $\lambda(t) \cdot \sigma(t)dt$ in the drift.

If we want to change to a world where the market price of risk is $\bar{\lambda}(t)$ from the world where the market price of risk is $\tilde{\lambda}(t)$, we have to change:

$$\frac{dS(t)}{S(t)} = (r(t) + \sigma(t)\tilde{\lambda}(t))dt + \sigma(t)dB(t)$$

to

$$\frac{dS(t)}{S(t)} = (r(t) + \sigma(t)\bar{\lambda}(t))dt + \sigma(t)dB(t)$$

This means we just have to add a drift term

$$\sigma(t)(\bar{\lambda}(t) - \tilde{\lambda}(t))dt$$

in the SDE for S.

The third point here is that every martingale measure is associated with a numeraire. Therefore changing the probability measure is equivalent to changing numeraire. There is also a relationship between the numeraire and the market price of risk. Suppose we are under the measure $\mathbb{Q}$. If we want to have that the discounted price is a martingale under $\mathbb{Q}$, we must have that the volatility of the numeraire is equal to the market price of risk under this measure. The proof of this point can be found in [15].

6.1.2 Estimate the market price of risk

Market price of risk is a very important concept by changing of probability measures, and it measures the relationship between the excess return and the risk, so it describes the rate of risk aversion of the investors, therefore the estimation of it is a central component of every risk and return model. In this section we will introduce some methods for the estimation of the market price of risk.

Market price of risk of an asset is the excess return of this asset divided by the risk contained in this asset. The risk of an asset is usually defined as the historical standard deviation of the realized return of this asset around its expected return. The remaining problem is the estimation of the excess return. The most common approach for that is to use the historical analysis. One can measure the difference between the return of this asset and the risk free return. But in this method, the risk premium estimates vary across users because of differences in time period used, the choice of securities as the risk free rate. Furthermore, a market price of risk from a historical analysis may not fit the future market price of risk. Therefore other methods, which require no historical data, would be useful.

Srdjan D. Stojanovic has introduced a HARA solution for this problem in [22], where the market price of risk is determined via a solution of a certain nonlinear PDE. He used a stochastic volatility model. Consider a security Y, which is described as an Itô process. Consider also an option using Y as underlying. Define $R \in (0, \infty)$ as the rate of risk aversion. An utility function ϕ from the HARA class should measure the utility of the wealth. The statement is, that each R implies a selection of the market price of risk. More precisely, for a risk aversion R, the corresponding fair option price for the option is a solution of a generalised Black-Scholes PDE, which implies, that the market price of risk can be expressed as a function of the risk aversion R. For more details see [22].

Specially for equity excess return, there are two different methods. The first one is a dividend based method, the second one is a earning based method. The dividend based approach says that returns are function of dividends and their future growth. Consider an example with a single stock that today is priced at 100, pays a constant 3% dividend yield, but we also expect the dividend to grow at 5% per year. Then calculate the dividend for each year with the growth rate, and then you can calculate the stock price via dividing the dividend pay out by the yield. Then we can see, if the dividend grows 5% each year, the stock price must go up 5% per year too. So the total return of this stock is then 5% plus 3% dividend yield. The key assumption in this method is that the dividend yield is fixed. Subtracting the risk free rate from the total return we can get the excess return for that asset. In the earnings based approach the idea is that the markets expected long-run real return is equal to the current earnings yield. For example, a stock has a P/E (Price and Earning) ratio as 25. Then the expected return of this stock should be equal to its earning yield, which is $1/25 = 4\%$.

These two methods are only applicable for equities, but not for bonds or interest rates. And they also require many assumptions, which are usually not realistic. Furthermore a historical analysis with a long time period has a quite small standard error of the estimation. Therefore we will prefer a historical analysis here to define our market price of risk. The result from the historical data is that the market price of risk for equity is about 30% and for bonds about 25%.

6.1.3 Real world stock model

At first let's discuss the case for the stock market.

It's quite simple in this case. It's the same method as explained for general cases. We have the dynamic for the risky asset S under the EMM $\mathbb{Q}$ as follows:

$$dS(t) = S(t)[r(t)dt + \sigma(t)d\tilde{B}(t)]$$

And we have to change it to

$$dS(t) = S(t)[r(t)dt + \sigma(t)dB(t)] + (-\gamma(t)\sigma(t))dt) = S(t)[\mu(t)dt + \sigma(t)dB(t)]$$

for the real world scenario.

Usually the market price of risk of an equity asset is assumed to be about 30%, this ratio can be derived from an analysis about historical data. The σ is the volatility of the equity, it is assumed to be 15% per annum. This is a simple assumption, in the reality the volatility is not always constant, but it follows also a stochastic process. However for the sake of simplicity we assume it here to be constant. One can later still introduce a stochastic for σ to improve this model.

6.1.4 Real world interest rate model

For the interest rate market we have to find the dynamic under the real world measure $\mathbb{P}$ giving the dynamic under the forward measures $\mathbb{Q}_{t+1}$ explained in Chapter 3.

Let's discuss how to do this. In this section we will introduce two alternative methods for the change to real world scenarios. The first one is simple to understand and avoid complex calculations, but the drawback of this method is that we don't have the exact definition of the market price of risk for forward rates. We can just make an assumption that it is again constant about 20%. This method causes that we can just add an extra drift for forward rates and we don't know how this drift will effect our bond prices. The second one can define the market price of risk of forward rates separately for different maturities and can add the drift exactly for bond prices, what we ultimately want, but it requires more complex calculations, the assumption about the market price of risk of bond prices and the definition of the volatilities of the bond prices.

The first method is exactly the idea about changing numeraire. As explained in the LIBOR market model, we define the dynamic of forward rates at first under the forward measure $\mathbb{Q}_k$. The rates are drift less under $\mathbb{Q}_k$:

$$\frac{dF_k(t)}{F_k(t)} = \sigma_k(t)dB_k(t).$$

Then we have defined the dynamics for the rates under the forward measure $\mathbb{Q}_{t+1}$, actually using the method of changing market price of risk. Here we mention the fact that the forward measure $\mathbb{Q}_k$ is using the bond prices $P(t,k)$ as numeraire for the forward rate $F_k(t)$, and the spot measure $\mathbb{Q}_{t+1}$ is using the bond prices $P(t,t+1)$ as numeraire for $F_k(t)$. Therefore

$$\frac{dF_k(t)}{F_k(t)} = \sigma_k(t)dB(t) + \sigma_k(t)(v(t,t+1) - v(t,k))dt,$$

where $\sigma_k(t)$ is the volatility of the forward rate $F_k(t)$ and $v(t,k)$ is the volatility of the bond price $P(t,k)$, and as mentioned in the last section, $v(t,k)$ is also the market price of risk of $F_k(t)$ using $P(t,k)$ as numeraire, because $\mathbb{Q}_k$ and $\mathbb{Q}_{t+1}$ are both EMM.

If we use now the relationship between forward rates and bond prices, we can calculate $v(t,t+1) - v(t,k)$ and we will get exactly the same formula (3.2) for forward rates as in the Chapter 3 about LIBOR market model. More details can be found in the chapter about Libor market model in [15].

The next step we have to do is to change the spot measure $\mathbb{Q}_{t+1}$ to the real world measure $\mathbb{P}$. It's exactly the same process as for equity, we need to add an extra drift term:

$$\sigma_k(t)(\lambda_k(t) - v(t,t+1))dt$$

where λ expresses the market price of risk of $F_k(t)$ under $\mathbb{P}$.
With the relationship $P(t,t+1) = 1/(1+F_k(t))$ and using Itô formula we will get

$$v(t,t+1) = -\sigma_t(t)\frac{F_t(t)}{1+F_t(t)}$$

It's not easy to define the market price of the forward rates, because there is an entire curve of the rates, and they are also correlated somehow. But we can perform a historical analysis, which tells us, that the market price of risk of bond prices, that is the excess return divided by the volatility containing in the bond, is always around 20%. So for simplicity we just assume in this model, that for every forward rate $\lambda_k(t)$ is constant by 20%.

The second method tries to define the dynamics of forward rates in such a way, that the dynamic of bond prices is exactly that what we want. To explain this method we introduce at first an alternative method for calculation the dynamics of the forward rates in risk neutral world , and then we variegate this method to get the dynamics of forward rates in real world.

We assume that the forward rates follows a stochastic process under the risk neutral measure,

$$\frac{dF_k(t)}{F_k(t)} = m_k(t)dt + \sigma_k(t)dB(t)$$

The drift term $m_k(t)$ is necessary, because the discounted bond prices must be a martingale: since the function between forward rates and bond prices is a convex function, the forward rates cannot be a drift less process, otherwise the discounted bond prices will not be a martingale. More precisely, the discounted bond prices should fulfill:

$$\mathbb{E}_{\mathbb{Q}}\Big[\frac{P(t,k+1)}{B(t)}|\mathcal{F}_{t-1}\Big] = \frac{P(t-1,k+1)}{B(t-1)} \tag{6.1}$$

Therefore the term $m_k(t)$ is also called *convexity adjustment*.

Using Itô's formula we get:

$$d\ln(F_k(t)) = (m_k(t) - \frac{1}{2}\sigma_k(t)^2)dt + \sigma_k(t)dB(t).$$

Then

$$F_k(t) = F_k(t-1)\cdot\exp((m_k(t) - \frac{1}{2}\sigma_k(t)^2) + \sigma_k(t)\epsilon),$$

where $\epsilon \sim \mathcal{N}(0,1)$.

We have now to find the exact definition of m, such that the martingale property for discounted bond prices above martingale should be fulfilled. Define

$$A_k(t) := (m_k(t) - \frac{1}{2}\sigma_k(t)^2) + \sigma_k(t)\epsilon$$

Condition (6.1) can be rewritten as:

$$\begin{aligned}
& \mathbb{E}_{\mathbb{Q}}\Big[\frac{P(t,k+1)}{B(t)}|\mathcal{F}_{t-1}\Big] \\
= \;& \mathbb{E}_{\mathbb{Q}}\Big[\frac{1}{1+F_t(t)}\cdots\frac{1}{1+F_k(t)}\cdot\frac{1}{B(t)}|\mathcal{F}_{t-1}\Big] \\
= \;& \frac{1}{1+F_t(t-1)}\cdots\frac{1}{1+F_k(t-1)}\cdot\frac{1}{B(t-1)}
\end{aligned}$$

$B(t)$ is predictable, this means, it is known at time $t-1$, hence we can take it to the outside of the conditional expectation. Multiply the both side with $B(t)$

$$B(t) := [1+F_0(0)]\cdots[1+F_{t-1}(t-1)]$$

we get

$$\mathbb{E}_{\mathbb{Q}}\Big[\frac{1}{1+F_t(t)}\cdots\frac{1}{1+F_k(t)}|\mathcal{F}_{t-1}\Big] = \frac{1}{1+F_t(t-1)}\cdots\frac{1}{1+F_k(t-1)} \tag{6.2}$$

We calculate the $A_k(t)$ inductive by k using the equation (6.2).

(i) **For** $k=1$
 t can only be 1.

$$\begin{aligned}
& \mathbb{E}_{\mathbb{Q}}\Big[\frac{1}{1+F_t(t)}\cdots\frac{1}{1+F_k(t)}|\mathcal{F}_{t-1}\Big] \\
= \;& \mathbb{E}_{\mathbb{Q}}\Big[\frac{1}{1+F_1(1)}|\mathcal{F}_0\Big] \\
= \;& \mathbb{E}_{\mathbb{Q}}\Big[\frac{1}{1+F_1(0)\exp A_1(1)}\Big] \\
\approx \;& \mathbb{E}_{\mathbb{Q}}\Big[\frac{1}{1+F_1(0)} - A_1(1)\frac{F_1(0)}{(1+F_1(0))^2} + \frac{1}{2}A_1(1)^2(\frac{2F_1(0)^2}{(1+F_1(0))^3} - \frac{F_1(0)}{(1+F_1(0))^2})\Big] \\
= \;& \frac{1}{1+F_1(0)} + \mathbb{E}_{\mathbb{Q}}\Big[-A_1(1)\frac{F_1(0)}{(1+F_1(0))^2} + \frac{1}{2}A_1(1)^2(\frac{2F_1(0)^2}{(1+F_1(0))^3} - \frac{F_1(0)}{(1+F_1(0))^2})\Big]
\end{aligned}$$

Compare this with the equation (6.2) it must hold:

$$-\mathbb{E}_{\mathbb{Q}}[A_1(1)] + \mathbb{E}_{\mathbb{Q}}[A_1(1)^2](\frac{F_1(0)}{1+F_1(0)} - \frac{1}{2}) = 0$$

$$\mathbb{E}_{\mathbb{Q}}[A_1(1)^2] = \mathbb{E}_{\mathbb{Q}}[A_1(1)]^2 + \mathbb{V}(A_1(1)) \approx \sigma_1(1)^2$$

$$\mathbb{E}_{\mathbb{Q}}[A_1(1)] \approx \sigma_1(1)^2(\frac{F_1(0)}{1+F_1(0)} - \frac{1}{2})$$

(ii) **For** $k = 2$
t can be 1 or 2. If $t = 2$, there is the same calculation as above for $k = 1$, $t = 0$. We now see the case $t = 1$.

$$\begin{aligned}
& \mathbb{E}_{\mathbb{Q}}\Big[\frac{1}{1+F_t(t)} \cdots \frac{1}{1+F_k(t)} |\mathcal{F}_{t-1}\Big] \\
= \; & \mathbb{E}_{\mathbb{Q}}\Big[\frac{1}{1+F_1(1)} \frac{1}{1+F_2(1)} |\mathcal{F}_0\Big] \\
= \; & \mathbb{E}_{\mathbb{Q}}\Big[\frac{1}{1+F_1(0)\exp(A_1(1))} \frac{1}{1+F_2(0)\exp(A_2(1))}\Big] \\
\approx \; & \mathbb{E}_{\mathbb{Q}}\Big[\frac{1}{1+F_1(0)} \frac{1}{1+F_2(0)} - A_1(1)\frac{F_1(0)}{(1+F_1(0))^2}\frac{1}{F_2(0)} \\
& -A_2(1)\frac{F_2(0)}{(1+F_2(0))^2}\frac{1}{F_1(0)} \\
& +\frac{1}{2}A_1(1)^2(\frac{2F_1(0)^2}{(1+F_1(0))^3} - \frac{F_1(0)}{(1+F_1(0))^2})\frac{1}{1+F_2(0)} \\
& +\frac{1}{2}A_2(1)^2(\frac{2F_2(0)^2}{(1+F_2(0))^3} - \frac{F_2(0)}{(1+F_2(0))^2})\frac{1}{1+F_1(0)} \\
& +A_1(1)A_2(1)(\frac{F_1(0)}{(1+F_1(0))^2}\frac{F_2(0)}{(1+F_2(0))^2})\Big]
\end{aligned}$$

According to (6.2) this should be equal to $\frac{1}{1+F_1(0)}\frac{1}{1+F_2(0)}$. Then:

$$
\begin{aligned}
& \mathbb{E}_{\mathbb{Q}}\Big[-A_1(1)\frac{F_1(0)}{(1+F_1(0))^2)}\frac{1}{1+F_2(0)} \\
& +\frac{1}{2}A_1(1)^2(\frac{2F_1(0)^2}{(1+F_1(0))^3)}-\frac{F_1(0)}{(1+F_1(0))^2)})\frac{1}{1+F_2(0)}\Big] \\
& +\mathbb{E}_{\mathbb{Q}}\Big[-A_2(1)\frac{F_2(0)}{(1+F_2(0))^2)}\frac{1}{1+F_1(0)} \\
& +\frac{1}{2}A_2(1)^2(\frac{2F_2(0)^2}{(1+F_2(0))^3)}-\frac{F_2(0)}{(1+F_2(0))^2)})\frac{1}{1+F_1(0)}\Big] \\
& +\mathbb{E}_{\mathbb{Q}}\Big[A_1(1)A_2(1)\frac{F_1(0)}{(1+F_1(0))^2}\frac{F_2(0)}{(1+F_2(0))^2}\Big] \\
= \;& 0
\end{aligned}
$$

The first expectation should be zero according to the first calculation for $k = 1$, $t = 0$. Therefore it must hold:

$$-\mathbb{E}_{\mathbb{Q}}[A_2(1)] + \mathbb{E}_{\mathbb{Q}}[A_2(1)^2](\frac{F_2(0)}{1+F_2(0)}-\frac{1}{2}) + \mathbb{E}_{\mathbb{Q}}[A_1(1)A_2(1)]\frac{F_1(0)}{1+F_1(0)} = 0$$

Using the following approximation:

$$\mathbb{E}_{\mathbb{Q}}[A_2(1)^2] = \mathbb{E}_{\mathbb{Q}}[A_2(1)]^2 + \mathbb{V}(A_2(1)) \approx \sigma_2(1)^2$$

$$\mathbb{E}_{\mathbb{Q}}[A_1(1)A_2(1)] = \mathbb{E}_{\mathbb{Q}}[A_1(1)]\mathbb{E}_{\mathbb{Q}}[A_2(1)]+Cov(A_1(1), A_2(1) \approx Cov(A_1(1), A_2(1)) = \rho_{12}\sigma_1\sigma_2$$

we get:

$$\mathbb{E}_{\mathbb{Q}}[A_2(1)] \approx \sigma_2(1)^2(\frac{F_2(0)}{1+F_2(0)}-\frac{1}{2}) + \rho_{12}\sigma_1\sigma_2\frac{F_1(0)}{1+F_1(0)}$$

(iii) **For** $k > 2$
With the relationship $F_j(t) = F_j(t-1)\cdot\exp(A_j(t))$ and approximating to the second order, we obtain

$$
\begin{aligned}
& \mathbb{E}_{\mathbb{Q}}\Big[\frac{1}{1+F_t(t)}\cdots\frac{1}{1+F_k(t)}|\mathcal{F}_{t-1}\Big] \\
\approx\ & \mathbb{E}_{\mathbb{Q}}\Big[\frac{1}{1+F_t(t-1)}\cdots\frac{1}{1+F_k(t-1)} \\
& -A_t(t)\frac{F_t(t-1)}{(1+F_t(t-1))^2}\prod_{j\neq t}\frac{1}{1+F_j(t-1)} \\
& \cdots \\
& -A_k(t)\frac{F_k(t-1)}{(1+F_k(t-1))^2}\prod_{j\neq k}\frac{1}{1+F_j(t-1)} \\
& +\frac{1}{2}A_t(t)^2\Big(\frac{2F_t(t-1)^2}{(1+F_t(t-1))^3}-\frac{F_t(t-1)}{(1+F_t(t-1))^2}\Big)\prod_{j\neq t}\frac{1}{1+F_j(t-1)} \\
& \cdots \\
& +\frac{1}{2}A_k(t)^2\Big(\frac{2F_k(t-1)^2}{(1+F_k(t-1))^3}-\frac{F_k(t-1)}{(1+F_k(t-1))^2}\Big)\prod_{j\neq k}\frac{1}{1+F_j(t-1)} \\
& +\sum_{i=t}^{k}\sum_{j>i}A_i(t)A_j(t)\Big(\frac{F_i(t-1)}{(1+F_i(t-1))^2}\frac{F_j(t-1)}{(1+F_j(t-1))^2}\Big)\prod_{k\neq i,j}\frac{1}{1+F_k(t-1)}|\mathcal{F}_{t-1}\Big]
\end{aligned}
$$

According to (6.2) this should be equal to $\frac{1}{1+F_t(t-1)}\cdots\frac{1}{1+F_k(t-1)}$. Then:

$$\mathbb{E}_{\mathbb{Q}}[A_k(t)|\mathcal{F}_{t-1}] = \sigma_k(t)^2\frac{F_k(t-1)}{1+F_k(t-1)} + \sigma_k(t)\sum_{j=t}^{k-1}\frac{\rho_{kj}\sigma_jF_j(t-1)}{1+F_j(t-1)} - \frac{1}{2}\sigma_k(t)^2 \tag{6.3}$$

Compare (6.3) with the dynamics of forward rates we defined under the spot measure, they are exactly the same.

Now we have tried an another way to get the dynamics of forward rates, which make the discounted bond prices as a martingale. If now we turn our view point from risk neutral to real world, this would mean, that the discounted bond prices are not martingales anymore, they should have now a drift term in their dynamic, which should explain the market price of risk for the bond prices. To get this point, we only have to change the equation (6.1) to the following equation:

$$\mathbb{E}_{\mathbb{P}}\Big[\frac{P(t,k+1)}{B(t)}|\mathcal{F}_{t-1}\Big] = \frac{P(t-1,k+1)}{B(t-1)} + \lambda_k(t) \tag{6.4}$$

We can now do the same calculation for forward rate using this basic condition. We get

$$
\begin{aligned}
\mathbb{E}_{\mathbb{P}}[A_k(t)|\mathcal{F}_{t-1}] &= \sigma_k(t)^2 \frac{F_k(t-1)}{1+F_k(t-1)} + \sigma_k(t) \sum_{j=t}^{k-1} \frac{\rho_{kj}\sigma_j F_j(t-1)}{1+F_j(t-1)} \\
&- \frac{1}{2}\sigma_k(t)^2 - \lambda_k(t) B(t) \frac{(1+F_k(t-1))^2}{F_k(t-1)} \Pi_{j=t}^{k-1}(1+F_j(t-1))
\end{aligned} \tag{6.5}
$$

The remaining problem is to determine $\lambda_k(t)$. We have that $\lambda_k(t)$ is certainly connected with the market price of risk of bond prices in some way. Denote the market price of risk of bond price as $-\gamma$. Then we should have:

$$
\mathbb{E}_{\mathbb{P}}\Big[\frac{P(t,k+1)}{B(t)}|\mathcal{F}_{t-1}\Big] = \frac{P(t-1,k+1)}{B(t-1)} \exp(-\gamma v(t,k+1))
$$

where $v(t,k+1)$ expresses the volatility of the bond price $P(t,k+1)$.

Then we can express $\lambda_k(t)$ with $-\gamma$ as follows:

$$
\lambda_k(t) = \frac{(\exp[-\gamma v(t,k+1)] - 1)P(t-1,k+1)}{B(t-1)}
$$

Set this $\lambda_k(t)$ into the equation (6.5) we get:

$$
\begin{aligned}
\mathbb{E}_{\mathbb{P}}[A_k(t)|\mathcal{F}_{t-1}] &= \sigma_k(t)^2 \frac{F_k(t-1)}{1+F_k(t-1)} + \sigma_k(t) \sum_{j=t}^{k-1} \frac{\rho_{kj}\sigma_j F_j(t-1)}{1+F_j(t-1)} - \frac{1}{2}\sigma_k(t)^2 \\
&\quad -(\exp[-\gamma v(t,k+1)] - 1) \frac{1+F_k(t-1)}{F_k(t-1)}
\end{aligned}
$$

Now we have for both stock model and interest rate model the dynamic in a real world economy. We can use the new dynamics to simulate the equity prices and bond prices.

6.2 VaR

In economics and finance, Value at Risk (VaR) is a measure of how the market value of an asset or a portfolio of assets is likely to decrease over a certain period. Let's at first see the precise definition of VaR.

Definition 6.1 (Risk measure) *Consider X is a financial position: $\Omega \to \mathbb{R}$ and $\mathcal{X}$ is the set of bounded maps X and also contains constants. A map ρ: $\mathcal{X} \to \mathbb{R}$ is called a risk measure, if it satisfies :*

(a) cash invariance: $\rho(X+m) = \rho(X) - m$

(b) monotonicity: if $X \geq Y$, then $\rho(X) \leq \rho(Y)$

The risk measure $\rho(X)$ can be explained as the amount of money that one has to add to X to make this financial position "acceptable", the concept of being acceptable can be defined in several ways, for example it should always be positive, or the maximal loss of it should not exceed some level or the conditional expectation should reach some given level and so on.

Definition 6.2 (Convex risk measures) *A risk measure ρ is called convex, if*

$$\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y) \quad \forall\, \lambda \in [0,1].$$

Definition 6.3 (Positive homogeneous risk measures) *A risk measure ρ is called positive homogeneous, if*

$$\rho(\lambda X) = \lambda\rho(X) \quad \forall\, \lambda > 0$$

Definition 6.4 (Value at risk) *VaR is a risk measure defined as follows:*

$$VaR_\lambda(X) := \inf\Big\{m|\mathbb{P}(X+m<0) \leq (1-\lambda)\Big\}$$

This means, $VaR_\lambda(X)$ is the minimal amount that has to be added to X to make X acceptable, acceptable means in this case that the probability that $X < 0$ is smaller than $1-\lambda$. It can also be explained in the other way, it is the maximal loss that the financial position X will not exceed with probability λ during a certain period.

We can prove that:

$$VaR_\lambda(X) = -q_X^+(1-\lambda).$$

The right-hand side is the opposite value of the the right $(1-\lambda)$-quantile of X.

If we have then the distribution of X , we can easily calculate the λ-VaR of X.

Let's see a few simple examples to clear the VaR concept.

Example 8 (VaR for one asset) *We consider a financial position X over a 10 day period. X can for example be the change of a certain equity, for the sake of simplicity we assume this equity has a value of 1. The daily volatility of X is assumed to be 2 %, then the volatility of X in 10 days is $2\% \cdot \sqrt{10} = 6.32\%$. Assume X is normally distributed. Set the confidence level λ as 99 %. With all the assumptions we can calculate the 10-day VaR for X:*

$$\begin{aligned}
\mathbb{P}(X + m < 0) &= \mathbb{P}(X < -m) \\
&= \mathbb{P}(\frac{X-0}{6.32\%} < \frac{-m-0}{6.32\%}) \\
&= \mathbb{P}(Y < \frac{-m-0}{6.32\%}) \\
&= \Phi(\frac{-m-0}{6.32\%}) \\
&= 1 - 0.99 = 0.01
\end{aligned}$$

where Y is a random variable with standard normal distribution and Φ is the distribution function of a standard normally distributed random variable.

$$m = -\Phi^{-1}(0.01) \cdot 6.32\% = 2.33 \cdot 6.32\% = 0.1474$$

m is according to the definition the VaR of X.
Or with the another definition of VaR we just need to calculate the opposite number of $(1-\lambda)$-quantile of $X \sim \mathcal{N}(0, 6.32\%)$. We get the same result.

Example 9 (VaR for a normal distributed portfolio) *We consider a portfolio with now two financial positions X and Y with a correlation of 0.7 over a 10 day period. The standard deviation σ_X of X is still $2\% \cdot \sqrt{10} = 6.32\%$, and σ_Y for Y is $0.5\% \cdot \sqrt{10} = 1.58\%$. Both X and Y are normal distributed with mean equals to zero. We assume that the joint distribution of X and Y is still normal distribution with mean zero, the standard deviation is then equal to:*

$$\sigma_{X+Y} = \sqrt{\sigma_X^2 + \sigma_Y^2 + 2 \cdot 0.7 \sigma_X \sigma_Y} = 7.52\%$$

Then with the same method we can calculate the VaR for the financial position $X + Y$:

$$VaR_{0.99}(X + Y) = -\Phi^{-1}(0.01) \cdot 7.52\% = 0.1752$$

Moreover

$$VaR_{0.99}(X) = -\Phi^{-1}(0.01) \cdot 6.32\% = 0.1474,$$

$$VaR_{0.99}(Y) = -\Phi^{-1}(0.01) \cdot 1.58\% = 0.0368.$$

We can see

$$VaR_{0.99}(X) + VaR_{0.99}(Y) = 0.1842 > 0.1752 = VaR_{0.99}(X+Y)$$

This represents the benefits of diversification.

Remark The relationship

$$VaR_\lambda(X) + VaR_\lambda(Y) \geq VaR_\lambda(X+Y)$$

is not always true. VaR is NOT a convex risk measure.

Example 10 (VaR using MC simulation) *We explained how to calculate VaR, if the distribution of the financial position is known, but for the cases that the joint distribution of a portfolio cannot be calculated easily, we can use the MC method for calculating VaR. The procedure is as follows:*

(a) Calculate the current portfolio value

(b) Execute MC simulation for all assets in the portfolio to get N samples of possible future values for each of them

(c) Calculate with the outputs in the previous step to get N samples for the future portfolio value

(d) Calculate the N samples of the change of the portfolio values over the certain time period, we call this vector $\triangle V$

(e) Calculate the opposite value of the $(1-\lambda)$-quantile of $\triangle V$.

For example the initial portfolio value is 100. We get with MC simulation 10 possible values of the portfolio value in 10 days:

$$V_T = \begin{pmatrix} 108.36 & 119.65 & 109.88 & 111.68 & 104.47 & 116.79 & 110.51 & 116.99 & 119.91 & 119.27 \end{pmatrix}$$

Now calculate the opposite value of the 1%-quantile of $V_T - 100$, that gives the VaR of this portfolio.

6.3 The optimal strategy

Now we come back to our primary problem. We have a life insurance contract which lasts 12 years. The initial premium less costs is 100 Euro that should be invested in the financial market. There exist a variety of alternatives of the investment, therefore in this section we will run the real world model and test different investment

strategies. We will compare them according to the return they generate and the risk they contain, in order to choose the best strategy.
At first we should fix the set of strategies, which should be investigated. We will define five families of strategies to be tested. The first family of strategies is just using equity and bonds. The second type of strategies is containing equity, bonds and put options, where the put options are used for hedging against the risk of the equity. The third type of strategies is composed of bonds and swaptions, where the swaptions are used for hedging against the reinvestment risk of the coupons from bonds. In the fourth family we will test strategies containing equity, bond, put options and swaptions. And finally we will test strategies with equity, bond and a hybrid option.
For the equity we simply assume that the start value $S(0)$ is equal to 100. The market price of risk is 20% and the volatility is 15%. With these assumptions the real world model for equity introduced in the first section will deliver us simulated future values for this equity.
For the bond we have chosen 4 bonds with different maturities. The first one is a 12 year coupon bond with principal 100. The coupons, which are established in the way, that the initial value of the bond is equal to the principal, are paid at the end of each year. The second one is a 6 year coupon bond with the same description as the 12 year bond. We buy one of this bond at the begin of the insurance contract, and in 6 year, when this bond is due, we will buy an another 6 year bond, which will be mature at the end of the contract. The third strategy is to buy 4 times coupon bonds with maturity in 3 years, and the last way of investing is to buy a one year-bond every year.
For the put option we just choose a 12 year at the money put option and for the swaptions we choose 6y 6y, 4y 8y, 8y 4y, 2y 10y and 10y 2y swaptions. And we use a ATM hybrid put option with 12 year maturity. We can calculate the prices of the put option and the swaptions simply with the MC method explained in Chapter 4, and of course, under the risk neutral viewpoint.

Then we have everything except the development of the values of the different bonds. We take the 12 year bond as example for how to calculate the bond prices.

Example 11 (Bond price) *The bond has maturity equal to 12 year, the principal is 100. Coupon payment at the end of each year, so the payoff of this bond is C at $t = 1, \cdots, 11$ and $100 + C$ at $t = 12$. Denote Df as the vector of discount factors, this means, $Df(i)$ is the discount factor from $t = i$ to $t = 0$, then it must hold:*

$$100 = C \cdot Df(1) + C \cdot Df(2) + \cdots + C \cdot Df(11) + (100 + C) \cdot Df(12)$$

The vector Df is known from the MC simulation and it is path dependent, solve this equation for C, it is then the coupon rate for this bond.

According to the equation above, the value of the bond at time $t = 0$ is exactly 100. At time t, we already received some coupons, they will be invested in the risk free asset. Therefore the value of the bond at time t is the sum of the present value *of the expected future cash flows at time t and the value of the bank account at time t. Denote R as the vector of risk free rates, this means, $R(i)$ is the risk free rate during the period $[i, i+1]$, then the value of the bond at time t can be expressed as:*

$$C\prod_{i=1}^{t-1}(1+R(i))+C\prod_{i=2}^{t-1}(1+R(i))+\cdots+C\cdot(1+R(t-1))+C+C\cdot Df(t)+\cdots+(100+C)\cdot Df(12)$$

C and Df are known, so does the vector R, it is also simulated with the MC, and this vector is also path dependent. So the value of this bond can be calculated for every $t = 0, \cdots, 12$ for every path of simulated interest rates.

With this method we can calculate the bond prices of all bonds for every path of the interest rates. The matlab file for calculating the bond prices is attached in appendix.

Now we have all the inputs of values we need and begin to compare different strategies.

The first family of strategies is equity plus bonds. Our initial portfolio value $V(0)$ is 100. We will divide this 100 Euro in the equity S and in the 4 different bonds. Since we already simulated the possible future values of the equity and the bonds, we can calculate for every path the portfolio value $V(T)$ at the maturity. The return of this portfolio is then:

$$\left(\frac{V(T)}{V(0)}\right)^{\frac{1}{12}} - 1$$

And we also have the distribution of the $V(T)$ so that we can calculate the distribution of

$$L := V(T) - V(0) \cdot (1 + minimumGuarantee)^{12}.$$

Then we can calculate the VaR for this variable L. For example we want to have the 97%-VaR, this is the 3%-quantile of L. And we also can calculate how great is the probability that the return by $V(T)$ is less than the minimum guarantee, while we take the number of $V(T)$, if $V(T) < V(0) \cdot (1 + minimumGarantee)^{12}$, divided by the number of simulations. It should hold, that if this probability is exactly 3%, then the VaR should be exactly zero. (The test results showed, that this relationship is fulfilled in our model). For all the tests below we have chosen a minimum guarantee as 4%. This procedure can be easily expressed with a matlab file, the matlab files for calculating portfolio values, returns and VaR will be passed over.

The following table show the result with a set of strategies, which are containing equities and the bonds:

		Bond				Expected	Value	Probability
Strategy	Equity	1 * 12 Y	2 * 6Y	4 * 3Y	12 * 1 Y	Return	at Risk	(return<mG)
1	0%	100%	0%	0%	0%	4.76%	-6.85	0.00%
2	0%	0%	100%	0%	0%	4.59%	8.61	20.33%
3	0%	0%	0%	100%	0%	4.47%	15.04	32.86%
4	0%	0%	0%	0%	100%	4.37%	19.34	39.69%
5	0%	25%	25%	25%	25%	4.55%	8.76	22.43%
6	5%	95%	0%	0%	0%	4.93%	-4.42	0.03%
7	5%	80%	5%	5%	5%	4.89%	-1.55	1.26%
8	5%	50%	15%	15%	15%	4.81%	4.20	9.53%
9	5%	20%	25%	25%	25%	4.72%	10.02	19.74%
10	5%	0%	95%	0%	0%	4.77%	8.77	16.98%
11	5%	5%	80%	5%	5%	4.76%	8.82	17.14%
12	5%	15%	50%	15%	15%	4.74%	9.01	17.72%
13	5%	25%	20%	25%	25%	4.73%	9.34	18.66%
14	5%	0%	0%	95%	0%	4.65%	15.03	26.97%
15	5%	5%	5%	80%	5%	4.67%	13.83	25.31%
16	5%	15%	15%	50%	15%	4.70%	11.38	21.79%
17	5%	25%	25%	20%	25%	4.73%	9.01	17.90%
18	5%	0%	0%	0%	95%	4.56%	19.18	32.84%
19	5%	5%	5%	5%	80%	4.59%	17.01	30.19%
20	5%	15%	15%	15%	50%	4.67%	12.87	24.64%
21	5%	25%	25%	25%	20%	4.74%	8.80	17.48%
22	10%	90%	0%	0%	0%	5.08%	-0.55	2.47%
23	10%	60%	10%	10%	10%	5.00%	4.60	9.44%
24	10%	30%	20%	20%	20%	4.92%	10.31	17.05%
25	10%	0%	90%	0%	0%	4.93%	10.99	17.44%
26	10%	10%	60%	10%	10%	4.92%	11.29	18.01%
27	10%	20%	30%	20%	20%	4.90%	11.61	18.49%
28	10%	0%	0%	90%	0%	4.82%	16.78	24.96%
29	10%	10%	10%	60%	10%	4.85%	14.50	22.19%
30	10%	20%	20%	30%	20%	4.89%	12.26	19.71%
31	10%	0%	0%	0%	90%	4.73%	20.87	29.63%
32	10%	10%	10%	10%	60%	4.81%	16.75	24.97%
33	10%	20%	20%	20%	30%	4.88%	12.72	20.21%
34	15%	85%	0%	0%	0%	5.22%	3.71	7.21%
35	15%	40%	15%	15%	15%	5.10%	11.15	16.56%
36	15%	25%	20%	20%	20%	5.06%	13.84	19.35%
37	15%	0%	85%	0%	0%	5.08%	13.82	18.77%
38	15%	15%	40%	15%	15%	5.06%	14.33	19.57%
39	15%	20%	25%	20%	20%	5.05%	14.43	19.94%
40	15%	0%	0%	85%	0%	4.98%	19.05	24.50%
41	15%	15%	15%	40%	15%	5.03%	15.78	21.44%
42	15%	20%	20%	25%	20%	5.05%	14.74	20.27%
43	15%	0%	0%	0%	85%	4.89%	22.91	28.73%
44	15%	15%	15%	15%	40%	5.00%	16.84	22.64%
45	15%	20%	20%	20%	25%	5.04%	14.97	20.54%
46	20%	80%	0%	0%	0%	5.36%	8.19	12.18%
47	20%	50%	10%	10%	10%	5.28%	12.89	17.09%
48	20%	20%	20%	20%	20%	5.19%	17.67	21.42%
49	20%	0%	80%	0%	0%	5.23%	17.13	20.26%
50	20%	10%	50%	10%	10%	5.21%	17.35	20.90%
51	20%	20%	20%	20%	20%	5.19%	17.67	21.42%
52	20%	0%	0%	80%	0%	5.13%	21.66	24.87%
53	20%	10%	10%	50%	10%	5.16%	19.70	23.13%
54	20%	20%	20%	20%	20%	5.19%	17.67	21.42%
55	20%	0%	0%	0%	80%	5.05%	25.08	28.00%
56	20%	10%	10%	10%	50%	5.12%	21.26	24.95%
57	20%	20%	20%	20%	20%	5.19%	17.67	21.42%
58	25%	75%	0%	0%	0%	5.48%	12.86	15.97%
59	25%	30%	15%	15%	15%	5.36%	19.64	21.52%
60	25%	19%	19%	19%	19%	5.33%	21.39	22.54%
61	25%	0%	75%	0%	0%	5.36%	20.69	21.61%
62	25%	15%	30%	15%	15%	5.33%	21.20	22.43%
63	25%	0%	0%	75%	0%	5.27%	24.81	25.13%
64	25%	15%	15%	30%	15%	5.31%	22.11	23.08%
65	25%	0%	0%	0%	75%	5.19%	27.90	28.07%
66	25%	15%	15%	15%	30%	5.30%	22.71	23.73%
67	30%	70%	0%	0%	0%	5.60%	17.45	18.94%
68	30%	40%	10%	10%	10%	5.52%	21.59	21.82%
69	30%	18%	18%	18%	18%	5.45%	25.08	23.65%
70	30%	0%	70%	0%	0%	5.48%	24.36	23.03%
71	30%	10%	40%	10%	10%	5.46%	24.79	23.38%
72	30%	0%	0%	70%	0%	5.40%	28.28	25.71%
73	30%	10%	10%	40%	10%	5.43%	26.39	24.57%
74	30%	0%	0%	0%	70%	5.33%	31.19	28.13%
75	30%	10%	10%	10%	40%	5.40%	27.88	25.75%

Figure 6.1: This graphic shows the result of portfolios containing equity and bonds

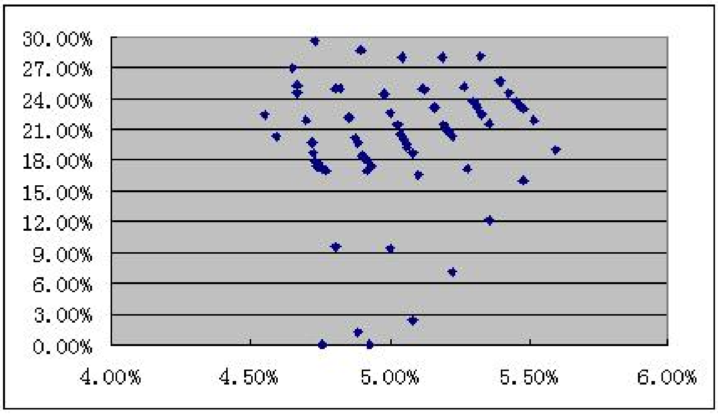

Figure 6.2: This graphic shows in the x-axis the return and in the y-axis the probability that the return of the portfolio is less than the minimum Guarantee for every portfolio in the table above

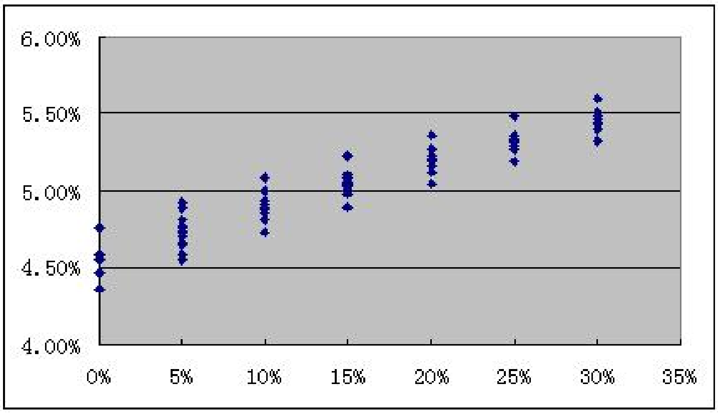

Figure 6.3: This graphic tells the relationship between the equity quote in the x-axis and the return in the y-axis

Conclusion (Investment Model 1)

(a) From the figure 6.3 we can see that the returns of the portfolios increase with the equity shares. But in the table we also see that the risk also increase with the equity shares. So we should choose the equity shares as high as possible with the constraint that the risk is under certain wished level.

(b) The strategies with the 12-year bond dominates the strategies with the other bonds, in terms of that the returns are higher and the risks are lower, more precisely, the 97%-VaR is lower, which means, the maximal loss with the strategies under 3% probability is smaller.

(c) We define the acceptable strategies as the strategies, whose 97% VaR is less or equal to zero, this means, we are 97% sure that we won't make loss with the strategies, if we promise the insurance owner to pay them at least the minimum guarantee. Or in other words, the probability, that the return from the investment strategy is less than the minimum guarantee should be held under 3%. The best acceptable strategy is an acceptable strategy with the highest return. In this sense, if we look at the table 6.1, the best acceptable strategy should be the strategy number 22, where we should invest 10% in equity and 90% in a 12-year bond.

The second family of strategies is containing equity, bonds and put option. The number of the put option is equal to the number of the equity shares, so that the risk from the equity price fluctuation can be totally hedged. The put option has a price at time $t = 0$, that will be the cost of this put option, the rest of the 100 Euro subtracted the cost of the option should be divided into equity and bonds. The portfolio value V(T) is composed of the equity value at time T, the value of the bond and the payoff from the put option at time T. The following table shows the result from using these strategies:

		Bond				Put	Expected	Value	Probability
Strategy	Equity	1 * 12 Y	2 * 6Y	4 * 3Y	12 * 1 Y	12 Y	Return	at Risk	(return<mG)
1	5%	94%	0%	0%	0%	0.05	4.88%	-6.22	0.00%
2	5%	79%	5%	5%	5%	0.05	4.84%	-3.26	0.13%
3	5%	49%	15%	15%	15%	0.05	4.76%	2.73	8.01%
4	5%	20%	25%	25%	25%	0.05	4.68%	8.90	20.27%
5	5%	0%	94%	0%	0%	0.05	4.72%	7.74	17.05%
6	5%	5%	79%	5%	5%	0.05	4.72%	7.66	17.31%
7	5%	15%	49%	15%	15%	0.05	4.70%	7.92	18.02%
8	5%	25%	20%	25%	25%	0.05	4.68%	8.16	19.18%
9	5%	0%	0%	94%	0%	0.05	4.61%	13.86	27.88%
10	5%	5%	5%	79%	5%	0.05	4.62%	12.65	26.18%
11	5%	15%	15%	49%	15%	0.05	4.66%	10.20	22.37%
12	5%	25%	25%	20%	25%	0.05	4.69%	7.85	18.34%
13	5%	0%	0%	0%	94%	0.05	4.51%	18.16	34.36%
14	5%	5%	5%	5%	79%	0.05	4.55%	16.07	31.69%
15	5%	15%	15%	15%	49%	0.05	4.62%	11.81	25.12%
16	5%	25%	25%	25%	20%	0.05	4.69%	7.63	17.98%
17	10%	88%	0%	0%	0%	0.10	4.99%	-5.29	0.00%
18	10%	59%	10%	10%	10%	0.10	4.91%	0.55	3.97%
19	10%	29%	20%	20%	20%	0.10	4.83%	6.57	15.89%
20	10%	0%	88%	0%	0%	0.10	4.84%	7.55	16.64%
21	10%	10%	59%	10%	10%	0.10	4.83%	7.70	17.38%
22	10%	20%	29%	20%	20%	0.10	4.81%	8.01	18.19%
23	10%	0%	0%	88%	0%	0.10	4.74%	13.34	25.92%
24	10%	10%	10%	59%	10%	0.10	4.77%	10.99	22.80%
25	10%	20%	20%	29%	20%	0.10	4.80%	8.65	19.50%
26	10%	0%	0%	0%	88%	0.10	4.65%	17.32	31.70%
27	10%	10%	10%	10%	59%	0.10	4.72%	13.15	26.71%
28	10%	20%	20%	20%	29%	0.10	4.79%	9.05	20.24%
29	14%	82%	0%	0%	0%	0.14	5.09%	-4.29	0.00%
30	14%	39%	14%	14%	14%	0.14	4.97%	4.37	12.39%
31	14%	24%	19%	19%	19%	0.14	4.93%	7.29	17.99%
32	14%	0%	82%	0%	0%	0.14	4.95%	7.63	17.22%
33	14%	14%	39%	14%	14%	0.14	4.93%	7.94	18.37%
34	14%	19%	24%	19%	19%	0.14	4.92%	8.01	18.94%
35	14%	0%	0%	82%	0%	0.14	4.85%	12.94	25.68%
36	14%	14%	14%	39%	14%	0.14	4.90%	9.52	21.09%
37	14%	19%	19%	24%	19%	0.14	4.92%	8.33	19.51%
38	14%	0%	0%	0%	82%	0.14	4.77%	16.63	30.78%
39	14%	14%	14%	14%	39%	0.14	4.88%	10.58	22.97%
40	14%	19%	19%	19%	24%	0.14	4.91%	8.55	19.82%
41	19%	76%	0%	0%	0%	0.19	5.18%	-3.30	0.00%
42	19%	48%	10%	10%	10%	0.19	5.10%	2.36	8.54%
43	19%	19%	19%	19%	19%	0.19	5.02%	8.14	20.02%
44	19%	0%	76%	0%	0%	0.19	5.05%	7.71	18.21%
45	19%	10%	48%	10%	10%	0.19	5.04%	7.86	19.04%
46	19%	19%	19%	19%	19%	0.19	5.02%	8.14	20.02%
47	19%	0%	0%	76%	0%	0.19	4.96%	12.67	26.08%
48	19%	10%	10%	48%	10%	0.19	4.99%	10.34	23.02%
49	19%	19%	19%	19%	19%	0.19	5.02%	8.14	20.02%
50	19%	0%	0%	0%	76%	0.19	4.88%	16.09	30.56%
51	19%	10%	10%	10%	48%	0.19	4.95%	12.07	25.91%
52	19%	19%	19%	19%	19%	0.19	5.02%	8.14	20.02%
53	24%	71%	0%	0%	0%	0.24	5.26%	-2.32	0.03%
54	24%	28%	14%	14%	14%	0.24	5.15%	6.10	18.01%
55	24%	18%	18%	18%	18%	0.24	5.12%	8.25	21.36%
56	24%	0%	71%	0%	0%	0.24	5.15%	7.84	19.37%
57	24%	14%	28%	14%	14%	0.24	5.12%	8.10	20.95%
58	24%	0%	0%	71%	0%	0.24	5.06%	12.41	26.58%
59	24%	14%	14%	28%	14%	0.24	5.11%	9.08	22.57%
60	24%	0%	0%	0%	71%	0.24	4.99%	15.58	30.70%
61	24%	14%	14%	14%	28%	0.24	5.09%	9.70	23.76%
62	28%	65%	0%	0%	0%	0.28	5.34%	-1.37	0.32%
63	28%	37%	9%	9%	9%	0.28	5.27%	4.15	15.82%
64	28%	16%	16%	16%	16%	0.28	5.21%	8.36	23.01%
65	28%	0%	65%	0%	0%	0.28	5.23%	7.97	20.93%
66	28%	9%	37%	9%	9%	0.28	5.22%	8.20	22.19%
67	28%	0%	0%	65%	0%	0.28	5.15%	12.22	27.43%
68	28%	9%	9%	37%	9%	0.28	5.18%	9.99	25.17%
69	28%	0%	0%	0%	65%	0.28	5.09%	15.13	31.07%
70	28%	9%	9%	9%	37%	0.28	5.16%	11.28	27.01%

Figure 6.4: This graphic shows the result of portfolios containing equity, bonds and put options

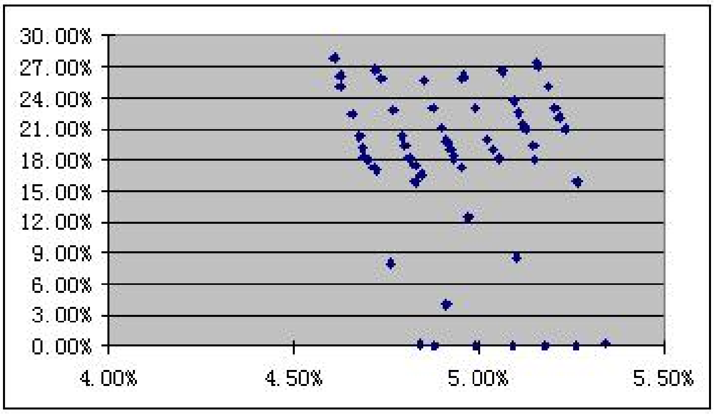

Figure 6.5: This graphic shows in the x-axis the return and in the y-axis the probability that the return of the portfolio is less than the minimum Guarantee for every portfolio in the table above

Conclusion (Investment Model 2)

(a) Actually we have chosen the same strategies as in the first family with equity share is greater then zero, and add a put option to the strategies, so that we can compare them.

(b) We can see that with the same strategies but an additional put option, the risk contained in the strategies are clearly reduced, and that the returns are also higher than before. So we really benefit a lot from adding put options.

(c) The best acceptable strategy is now the strategy number 62. We should buy 0.03 put option and invest 30% of the rest in equity and 70% in a 12-year bond. The equity share is now much higher, and we also gain about 1% return more than before, although the VaR is still below zero.

The third family of strategies is using only bonds and swaptions, to see, how effective do the swaptions hedge against the interest rate risk containing in the bonds. The swaptions has a principal of 100, with the maturities 6y 6y, 4y 8y, 8y 4y, 2y 10y and 10y 2y. The results from these strategies can be seen in the following table:

		Bond				Swaption					Expected	Value	Probability
Strategy	Equity	1 * 12 Y	2 * 6Y	4 * 3Y	12 * 1 Y	6 Y 6 Y	4 Y 8 Y	8 Y 4 Y	2 Y 10 Y	10 Y 2 Y	Return	at Risk	(return<mG)
1	0%	99%	0%	0%	0%	1	0	0	0	0	4.79%	-10.10	0.00%
2	0%	0%	99%	0%	0%	1	0	0	0	0	4.64%	-1.26	1.01%
3	0%	0%	0%	99%	0%	1	0	0	0	0	4.52%	6.11	21.70%
4	0%	0%	0%	0%	99%	1	0	0	0	0	4.42%	10.72	33.70%
5	0%	25%	25%	25%	25%	1	0	0	0	0	4.60%	-0.20	2.54%
6	0%	99%	0%	0%	0%	0	1	0	0	0	4.80%	-8.04	0.00%
7	0%	0%	99%	0%	0%	0	1	0	0	0	4.65%	1.23	5.27%
8	0%	0%	0%	99%	0%	0	1	0	0	0	4.53%	6.26	21.49%
9	0%	0%	0%	0%	99%	0	1	0	0	0	4.43%	10.48	33.01%
10	0%	25%	25%	25%	25%	0	1	0	0	0	4.61%	1.07	5.18%
11	0%	99%	0%	0%	0%	0	0	1	0	0	4.78%	-11.43	0.00%
12	0%	0%	99%	0%	0%	0	0	1	0	0	4.62%	2.87	9.62%
13	0%	0%	0%	99%	0%	0	0	1	0	0	4.50%	8.97	26.21%
14	0%	0%	0%	0%	99%	0	0	1	0	0	4.41%	13.21	35.70%
15	0%	25%	25%	25%	25%	0	0	1	0	0	4.58%	2.76	10.12%
16	0%	99%	0%	0%	0%	0	0	0	1	0	4.80%	-6.45	0.00%
17	0%	0%	99%	0%	0%	0	0	0	1	0	4.64%	5.26	11.67%
18	0%	0%	0%	99%	0%	0	0	0	1	0	4.52%	9.56	25.69%
19	0%	0%	0%	0%	99%	0	0	0	1	0	4.42%	13.30	35.00%
20	0%	25%	25%	25%	25%	0	0	0	1	0	4.60%	4.56	11.95%
21	0%	100%	0%	0%	0%	0	0	0	0	1	4.85%	-11.36	0.00%
22	0%	0%	100%	0%	0%	0	0	0	0	1	4.69%	4.49	11.62%
23	0%	0%	0%	100%	0%	0	0	0	0	1	4.56%	10.69	26.05%
24	0%	0%	0%	0%	100%	0	0	0	0	1	4.46%	15.08	34.04%
25	0%	25%	25%	25%	25%	0	0	0	0	1	4.65%	4.41	12.59%

Figure 6.6: This graphic shows the result of portfolios containing bonds and swaptions

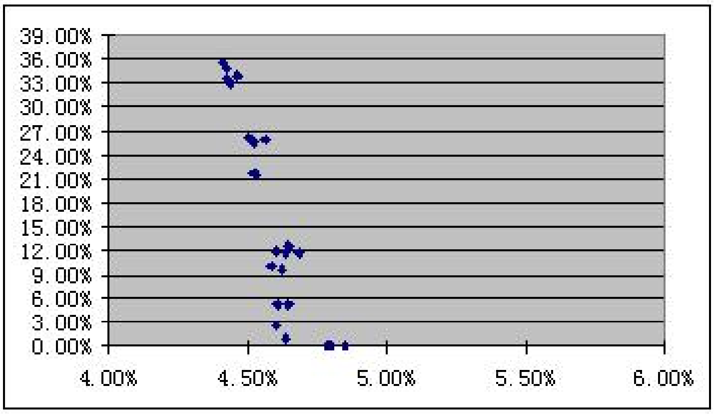

Figure 6.7: This graphic shows in the x-axis the return and in the y-axis the probability that the return of the portfolio is less than the minimum Guarantee for every portfolio in the table above

Conclusion (Investment Model 3)

(a) We want to test how the swaption effect the portfolios, therefore we choose strategies composed only of the different bonds and different swaptions. So we can compare these strategies with the first 5 strategies in the first family, which are only containing the different bonds.

(b) We can see with a swaption the risk contained in the bonds are really reduced, especially for the strategies with 2 time 6 year bonds. With a 6y 6y swaption this strategy also becomes acceptable.

(c) The returns from the strategies are now all higher than before.

(d) But the result still tells us that the strategies with the 12-year bond always perform better than the other bonds, i.e. with higher return and lower risk, therefore we will consider from now only the strategies with the 12-year bond. And if we consider a swaption in the strategy we will use the 6y 6y swaption.

Now we want to test strategies with all the financial instruments mentioned above. Below is the table of the results:

Strategy	Equity	Bond 1 * 12 Y	Put 12Y	Swaption 6 Y 6 Y	Expected Return	Value at Risk	Probability (return<mG)
1	0%	100%	0%	0	4.76%	-6.85	0.00%
2	0%	98%	0%	1	4.79%	-10.10	0.00%
3	5%	95%	0%	0	4.93%	-4.42	0.03%
4	5%	94%	5%	0	4.88%	-6.22	0.00%
5	5%	93%	0%	1	4.96%	-7.96	0.00%
6	5%	92%	5%	1	4.91%	-9.35	0.00%
7	10%	90%	0%	0	5.08%	-0.55	2.47%
8	10%	88%	10%	0	4.99%	-5.29	0.00%
9	10%	88%	0%	1	5.11%	-3.99	0.33%
10	10%	86%	10%	1	5.02%	-8.05	0.00%
11	15%	85%	0%	0	5.22%	3.71	7.21%
12	14%	82%	14%	0	5.09%	-4.29	0.00%
13	15%	83%	0%	1	5.25%	0.36	3.42%
14	14%	80%	14%	1	5.12%	-6.75	0.00%
15	20%	80%	0%	0	5.36%	8.19	12.18%
16	19%	76%	19%	0	5.18%	-3.30	0.00%
17	20%	78%	0%	1	5.38%	4.88	8.31%
18	19%	75%	19%	1	5.21%	-5.46	0.00%
19	25%	75%	0%	0	5.48%	12.86	15.97%
20	24%	71%	24%	0	5.26%	-2.32	0.03%
21	24%	73%	0%	1	5.50%	9.49	12.64%
22	23%	69%	23%	1	5.29%	-4.18	0.00%
23	30%	70%	0%	0	5.60%	17.45	18.94%
24	28%	65%	28%	0	5.34%	-1.37	0.32%
25	29%	68%	0%	1	5.62%	14.01	16.08%
26	27%	64%	27%	1	5.36%	-2.91	0.01%

Figure 6.8: This graphic shows the result of portfolios containing equity, bonds,put option and swaptions

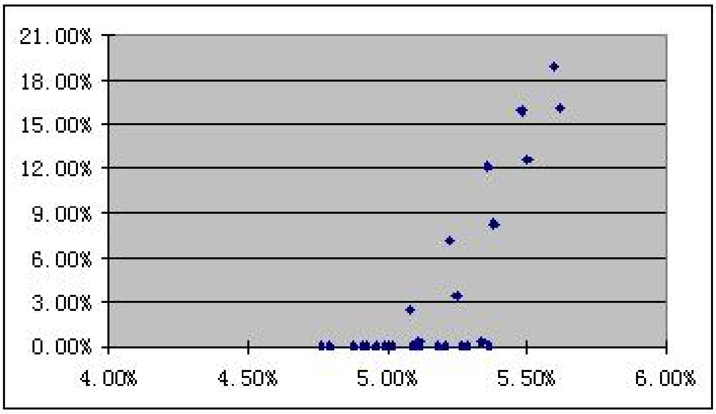

Figure 6.9: This graphic shows the return and the probability that the return of the portfolio is less than the minimum Guarantee for every portfolio in the table above

Conclusion (Investment Model 4)

From all theses strategies investigated above the best strategy is to buy 0.03 put option, 1 swaption and invest 30% of the rest in equity, 70% in the 12-year bond. At last we try to use a hybrid option to hedge against the risk contained in equity

and bonds. This put option has the payoff at maturity $T = 12$:

$$Payoff = \min((0.08 - \frac{\sum_{t=0}^{11} F(t, t+1)}{12}) \cdot 25, 1) \times \max(X - S(T), 0).$$

N is 1 if the average value of the spot rates is between 0% and 4%, if the average interest rate increases above 4%, the nominal N will decrease. If the average interest rate reaches 8%, N will be 0. This means, if the interest rates are going up, then we don't have the reinvestment risk of the coupons from the bonds, so maybe we don't need to hedge the entire risk in the equity, we can reduce the put option volume; if the interest rates are going down, then we will have bad result from the bond securities, so maybe we need more protection from the put option. We replace the classical put option with the hybrid option and test this family of strategies. The result can be seen in the following table:

		Bond				Hybrid Put	Expected	Value	Probability
Strategy	Equity	1 * 12 Y	2 * 6Y	4 * 3Y	12 * 1 Y	12 Y	Return	at Risk	(return<mG)
1	5%	94%	0%	0%	0%	0.05	4.87%	-6.32	0.00%
2	5%	79%	5%	5%	5%	0.05	4.83%	-3.34	0.10%
3	5%	49%	15%	15%	15%	0.05	4.75%	2.79	8.14%
4	5%	20%	25%	25%	25%	0.05	4.66%	8.96	20.33%
5	5%	0%	94%	0%	0%	0.05	4.71%	7.57	17.07%
6	5%	5%	79%	5%	5%	0.05	4.71%	7.63	17.39%
7	5%	15%	49%	15%	15%	0.05	4.69%	7.92	17.83%
8	5%	25%	20%	25%	25%	0.05	4.67%	8.22	18.98%
9	5%	0%	0%	94%	0%	0.05	4.60%	14.09	28.08%
10	5%	5%	5%	79%	5%	0.05	4.61%	12.83	26.23%
11	5%	15%	15%	49%	15%	0.05	4.65%	10.34	22.56%
12	5%	25%	25%	20%	25%	0.05	4.68%	7.86	18.28%
13	5%	0%	0%	0%	94%	0.05	4.50%	18.02	34.96%
14	5%	5%	5%	5%	79%	0.05	4.54%	15.95	31.90%
15	5%	15%	15%	15%	49%	0.05	4.61%	11.82	25.74%
16	5%	25%	25%	25%	20%	0.05	4.68%	7.66	17.72%
17	10%	88%	0%	0%	0%	0.10	4.97%	-5.33	0.00%
18	10%	59%	10%	10%	10%	0.10	4.89%	0.56	4.10%
19	10%	29%	20%	20%	20%	0.10	4.81%	6.54	16.15%
20	10%	0%	88%	0%	0%	0.10	4.82%	7.58	17.12%
21	10%	10%	59%	10%	10%	0.10	4.81%	7.73	17.51%
22	10%	20%	29%	20%	20%	0.10	4.79%	8.01	18.71%
23	10%	0%	0%	88%	0%	0.10	4.72%	13.44	26.95%
24	10%	10%	10%	59%	10%	0.10	4.75%	11.13	23.86%
25	10%	20%	20%	29%	20%	0.10	4.78%	8.67	20.00%
26	10%	0%	0%	0%	88%	0.10	4.63%	17.19	33.60%
27	10%	10%	10%	10%	59%	0.10	4.70%	13.12	27.73%
28	10%	20%	20%	20%	29%	0.10	4.77%	9.04	20.84%
29	14%	82%	0%	0%	0%	0.14	5.06%	-4.33	0.00%
30	14%	39%	14%	14%	14%	0.14	4.94%	4.38	12.37%
31	14%	24%	19%	19%	19%	0.14	4.90%	7.33	19.03%
32	14%	0%	82%	0%	0%	0.14	4.93%	7.63	18.32%
33	14%	14%	39%	14%	14%	0.14	4.90%	7.92	19.58%
34	14%	19%	24%	19%	19%	0.14	4.89%	8.06	20.16%
35	14%	0%	0%	82%	0%	0.14	4.82%	13.10	27.54%
36	14%	14%	14%	39%	14%	0.14	4.87%	9.62	22.87%
37	14%	19%	19%	24%	19%	0.14	4.89%	8.38	20.91%
38	14%	0%	0%	0%	82%	0.14	4.74%	16.54	33.38%
39	14%	14%	14%	14%	39%	0.14	4.85%	10.52	24.92%
40	14%	19%	19%	19%	24%	0.14	4.88%	8.57	21.36%
41	19%	76%	0%	0%	0%	0.19	5.15%	-3.05	0.05%
42	19%	48%	10%	10%	10%	0.19	5.07%	2.40	8.55%
43	19%	19%	19%	19%	19%	0.19	4.99%	8.16	23.08%
44	19%	0%	76%	0%	0%	0.19	5.02%	7.73	20.24%
45	19%	10%	48%	10%	10%	0.19	5.00%	7.94	21.46%
46	19%	19%	19%	19%	19%	0.19	4.99%	8.16	23.08%
47	19%	0%	0%	76%	0%	0.19	4.92%	12.76	28.91%
48	19%	10%	10%	48%	10%	0.19	4.96%	10.48	26.37%
49	19%	19%	19%	19%	19%	0.19	4.99%	8.16	23.08%
50	19%	0%	0%	0%	76%	0.19	4.85%	16.02	34.20%
51	19%	10%	10%	10%	48%	0.19	4.92%	11.99	29.53%
52	19%	19%	19%	19%	19%	0.19	4.99%	8.16	23.08%
53	24%	71%	0%	0%	0%	0.24	5.22%	-1.51	0.92%
54	24%	28%	14%	14%	14%	0.24	5.10%	6.12	23.15%
55	24%	18%	18%	18%	18%	0.24	5.07%	8.26	26.70%
56	24%	0%	71%	0%	0%	0.24	5.10%	7.89	23.21%
57	24%	14%	28%	14%	14%	0.24	5.08%	8.18	26.13%
58	24%	0%	0%	71%	0%	0.24	5.02%	12.50	31.25%
59	24%	14%	14%	28%	14%	0.24	5.06%	9.15	27.83%
60	24%	0%	0%	0%	71%	0.24	4.95%	15.54	35.56%
61	24%	14%	14%	14%	28%	0.24	5.05%	9.67	29.18%
62	28%	65%	0%	0%	0%	0.28	5.29%	0.47	3.87%
63	28%	37%	9%	9%	9%	0.28	5.21%	4.31	24.36%
64	28%	16%	16%	16%	16%	0.28	5.15%	8.38	31.00%
65	28%	0%	65%	0%	0%	0.28	5.18%	8.14	27.21%
66	28%	9%	37%	9%	9%	0.28	5.17%	8.22	29.40%
67	28%	0%	0%	65%	0%	0.28	5.10%	12.28	33.97%
68	28%	9%	9%	37%	9%	0.28	5.13%	10.07	32.37%
69	28%	0%	0%	0%	65%	0.28	5.04%	15.06	36.96%
70	28%	9%	9%	9%	37%	0.28	5.10%	11.16	34.10%

Figure 6.10: This graphic shows the result of portfolios containing equity, bonds and hybrid put option

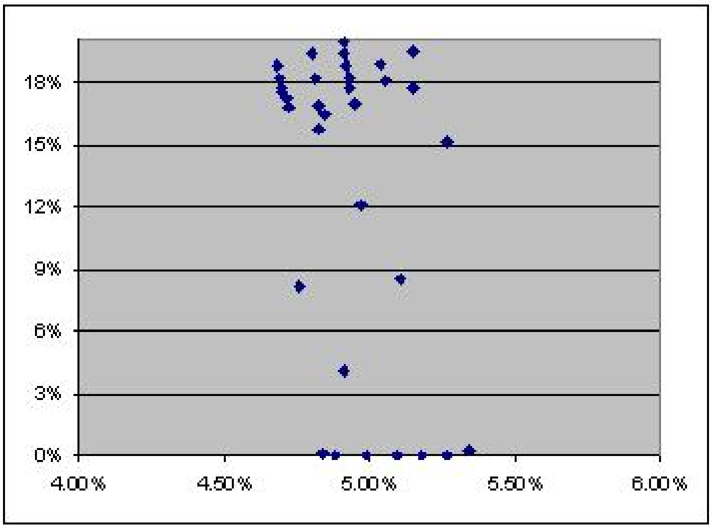

Figure 6.11: This graphic shows in the x-axis the return and in the y-axis the probability that the return of the portfolio is less than the minimum Guarantee for every portfolio in the table above

Conclusion (Investment Model 5)

This hybrid option doesn't deliver us the best hedging strategy in comparison with the classical put option. The reason could be that in this product the equity risk is dominant. Therefore we will choose a strategy with a classical put option, so that the risk of the equity can be totally hedged.
As the final conclusion we will choose the strategy from the fourth family with 30% equity, 70% 12-year bond, 0.3 put option and 1 swaption, which offers the highest return because of the high equity share, and small risk because of the put option and swaption.

Chapter 7

Conclusion

In this thesis we tried to solve the problem of finding the optimal hedging strategy for a life insurance contract with a minimum guarantee. More precisely, the policy owner paid the insurance company an amount of premium, which should be invested in the financial market by the insurer. There are many ways of investing, one can choose different securities in the market. We considered in this thesis mainly the equity, bonds, option, swaption and hybrid option. The insurer tries to get the maximal return from the investment with a small risk. Therefore we defined different families with strategies and tested them according to their return and risk. We mean by risk the possibility, that the investment return is less than the minimum guarantee at the end of the contract. In that case the insurance company will still have to pay the policy owner the guaranteed return, so it will suffer a loss.

In order to test different strategies we have to build an adequate model first. In the first step we build separately an equity model and an interest rate model. For the equity model we used the Black Scholes model, which has the assumption of a log normal distribution of the stock prices and a constant volatility. After describing the stock process with a SDE, we used the Monte Carlo method to simulate them. MC simulation is one of the most common numerical method, which uses the Law of Large Numbers as its mathematical justification. For the interest rate model we used the LIBOR market model approach, which described the forward LIBOR rates directly. Then we simulated the forward rates with the MC method. Thereby a principal component analysis is used to reduce the dimension of the correlated randomness and get independent randomness, we call them principal components, and the result shows that 3 PCs can already explain the most interest rate movements. Therefore we only used these most important 3 PCs to simulate the entire interest rate curve. At the second step we tried to build a hybrid model. We analysed the correlation structure between the equity and the forward rates by means of the PCA, since we just needed to calculate the correlation between the equity and the 3 principal components from the PCA. This can be done with the historical data. With this correlation structure we built an equity model on the base of the interest rate model. Thereby one just need to do the Cholesky decomposition to get the correlated randomness for equity and interest rates.

With the described model we could price derivatives. The fair price of a derivative at time t is the conditional expectation of the payoff of this derivative under the risk neutral probability measure with respect to $\mathcal{F}_t$. Calculating an expectation can be well done with the MC method. We can generate the process of the underlying, for example an equity, or the forward rates, or both of them. Then calculate the payoff of this derivative for every path. Taking the average of the payoffs and discounting this with the risk free rate should offer the fair price of the derivative.

Before we tested the strategies, we had to change our point of view from the risk neutral world to a real world view. For option pricing one needs to use the risk neutral probability measure, but in this risk neutral world all assets have the same rate of expected return, namely the risk free rate. It doesn't make sense to have different investment strategies. We stated in Chapter 6, that changing viewpoint from risk neutral world is equivalent to changing the risk neutral probability measure to the real world measure, which is again equivalent to changing the market price of risk or numeraire under the different probability measures. Market price of risk is defined as the excess return of an asset divided by the risk contained in this asset. This concept is very important, because if we can define the process of the market price of risk, we can change the probability measures using the Girsanov's theorem. But the estimation of the market price of risk still remains difficult, we used the most common method with the historical data to estimate it.

After converting to the real world model we can test our investment strategies. We run the real world scenarios, and calculate for every strategy its expected return and the probability that its return is less than the guaranteed rate. If the probability is greater than 3%, we consider the strategy as unacceptable. We tested strategies with equity and bonds, strategies with equity, bonds and a put option, strategies with bonds and a swaption, strategies with equity, bonds, options and swaptions and finally strategies with equity, bonds and an hybrid option. The result shows that higher equity shares bring more return, which sounds very reasonable, since the equity is riskier than bonds and should therefore offer higher return than bonds. But without hedging, the strategies with high equity share are too risky, therefore put options are needed to cover the equity risk. Swaptions should hedge against the interest rate risks. In our case, if we only use the 12-year coupon bond, this is the re-investment risk of the coupons. But in this product, the equity risk is dominant, this can explain, why an hybrid option, which only offers contingent put protection, doesn't offer as good results as classical option. The final conclusion is that we should invest 30% in equity, 70% in bonds, and that we should buy 0.3 put option and a swaption to hedge against the risk.

Appendix A

Principal component analysis

Principal components analysis (PCA) is a technique used to reduce multidimensional data sets to lower dimensions for analysis. The applications include exploratory data analysis data and for generating predictive models. PCA involves the computation of the eigenvalue decomposition or singular value decomposition of a data set, usually after mean centring the data for each attribute.

Let X be a p-dimensional random vector. If we draw for every X_i n observations, we will get a $n \times p$-matrix A.

$$A = \begin{pmatrix} x_{11} & \cdots & x_{1p} \\ \vdots & \ldots & \vdots \\ x_{n1} & \cdots & x_{np} \end{pmatrix}$$

These p variables are typically correlated in some way. Hence the matrix A contains redundant information. The basic idea of PCA is to construct uncorrelated variables, namely the Principal components (PC), so that the correlated variables are linear combinations of the PCs. The PCs should be ordered according to their explaining weights for the variance. The first PC takes the maximal parts on the sample variance, the second PC which is not correlated with the first PC takes the maximal part on the rest of the variance and so on.

There are two algorithms for using PCA. The first one is using covariance matrix of the data, the second one is using correlation matrix of the data. We are focusing on the first method.

The starting point of this method is the covariance matrix of the data matrix A. Define C=Cov(A,A) as the covariance matrix of A:

$$C_{ij} = \frac{1}{n-1}\sum_{k=1}^{n}(x_{ki} - \bar{x_i})(x_{kj} - \bar{x_j})$$

where

$$\bar{x_i} := \frac{1}{n}\sum_{k=1}^{n} x_{ki}$$

C is positive definite and assumed to be invertible.

C can be decomposed in the form $C = L \cdot \Lambda \cdot L^T$, where

(a) Λ is a diagonal matrix with the eigenvalues $\lambda_1, \cdots, \lambda_p$ of C on the diagonal.

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & 0 \\ \vdots & \cdots & \cdots & \vdots \\ 0 & \cdots & 0 & \lambda_p \end{pmatrix}$$

(b) $L = (P_1, \cdots, P_p)$ is a orthogonal matrix with $P_1, \cdots, P_p$ the eigenvectors associated with $\lambda_1, \cdots, \lambda_p$. This means

$$A \cdot P_i = \lambda_i \cdot P_i \quad i = 1, \cdots, p$$

and

$$P_i^T \cdot P_j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

The p eigenvalues λ_i of the positive definite matrix C should be sorted in decreasing order.

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p > 0$$

The principal components of the random vector X are $p \times 1$- dimensional vectors P_1,...,P_p.

Define $Z := L^T \cdot A$. Then we get

$$A = L^T \cdot Z = P_1 \cdot Z_1 + \cdots + P_p \cdot Z_p$$

and

$$Cov(Z, Z) = \mathbb{E}[Z \cdot Z^T] = \mathbb{E}[L \cdot A \cdot A^T \cdot L] = \Lambda$$

This means, Z_i and Z_j are not correlated for $i \neq j$ and $\mathbb{V}(Z_i) = \lambda_i$.

We call Z the factor exposure.

The importance of component P_i is determined by the size of the corresponding eigenvalue λ_i, which indicates the amount of variance explained by P_i

$$Explained\,Variance\ by\ P_i = \frac{\lambda_i}{\sum_{j=1}^{n} \lambda_j}.$$

Appendix B

Calculation forward rates

In this part we explain how to calculate forward LIBOR rates and forward swap rates with the current market swap rates.

In the Chapter 3 we have already calculated the formula for forward LIBOR rates and forward swap rates:

$$F(t,T,S) = \frac{1}{S-T}\Big(\frac{P(t,T)}{P(t,S)} - 1\Big) \tag{B.1}$$

$$R_{swap}(t) = \frac{P(t,T_0) - P(t,T_n)}{\delta \sum_{i=1}^{n} P(t,T_i)} \tag{B.2}$$

We can see that we need to calculate the discount factors at time t.

In Chapter 3 we also explained the relationship between discount factors $P(t,T)$ and some other interest rates, like zero coupon rates and instantaneous forward rates. $P(t,T)$ is actually the rate earned by a zero coupon bond between time t and T, and it can be calculated with the formula

$$P(t,T) = \exp(-\int_t^T f(t,u)du)$$

The problem is that zero coupon rates and instantaneous forward rates are not directly observable at the market. In practice the trader always use the current swap rates. Consider for example the swap rates on 30. June 2007 for various maturities:

$Maturity$	$Swap\ rates$	$Maturity$	$Swap\ rates$
1	4.59	11	4.90
2	4.68	12	4.91
3	4.72	13	4.93
4	4.75	14	4.95
5	4.77	15	4.96
6	4.79	16	4.97
7	4.81	17	4.97
8	4.83	18	4.98
9	4.85	19	4.99
10	4.87	20	4.99

The method for calculating the discount factors from swap rates is known as the "Bootstrap" method. We can calculate the discount factors inductively. Let's use the swap rates above to explain this method.

(a) P(0,1)
Consider the one year swap with principal 100. The contract begins with 100. At the end of one year one party of the contract will pay the flex rate and the other party will pay 4.59 coupon, therefore the contract ends with 104.59. Discount this end value to today's time point, it must be equal to 100 because of the pricing rule in arbitrage free world. The discount factor between 0 and 1 therefore is simply:

$$P(0,1) = \frac{100}{100 + 4.59}.$$

(b) P(0,2)
Consider the two year swap with principal 100. The contract begins with 100. At the end of the first year one party of the contract will pay the flexible rate and the other party will pay coupon 4.68; At the end of the second year one party will pay the flexible rate and the other party will pay again the coupon 4.68. Therefore we have the relationship

$$100 = 4.68 \cdot P(0,1) + (100 + 4.68) \cdot P(0,2).$$

Therefore the discount factor between 0 and 2 is:

$$P(0,2) = \frac{100 - 4.68P(0,1)}{100 + 4.68}$$

(c) P(0,t) for $t > 2$
Suppose $P(0, t-1)$ are all calculated. Then consider a t year swap with principal 100. The coupon payment is at the end of every year the t year swap rate, which can be found in the table above, denote $Swap_t$, and at the end of the swap the payment is $100 + Swap_t$. Therefore we get:

$$100 = Swap_t \cdot P(0,1) + \cdots Swap_t \cdot P(0,t-1) + (100 + Swap_t) \cdot P(0,t)$$

Because all discount factors up to t-1 are known, we can calculate the discount factor P(0,t) as:

$$P(0,t) = \frac{100 - Swap_t \sum_{j=1}^{t-1} P(0,j)}{100 + Swap_t}$$

Using this Bootstrapping method one can get the discount factors for 1,..,19 years using the swap rates above like:

$Maturity$	$discountfactors$	$Maturity$	$discountfactors$
1	0.9561	11	0.5897
2	0.9125	12	0.5605
3	0.8707	13	0.5326
4	0.8305	14	0.5062
5	0.7919	15	0.4810
6	0.7549	16	0.4574
7	0.7192	17	0.4351
8	0.6849	18	0.4138
9	0.6519	19	0.3935
10	0.6201	20	0.3743

With the discount factors we can use the formula (B.1) and (B.2) to calculate the one year forward LIBOR rates in 1,...,19 years and the forward swap rates $R_{swap}(0)$ for various maturities and tenors of an IRS.

$Maturity$	$FwdLIBORrates$	$Maturity$	$FwdLIBORrates$
1	4.78	11	5.20
2	4.80	12	5.23
3	4.84	13	5.23
4	4.87	14	5.23
5	4.91	15	5.14
6	4.96	16	5.15
7	5.00	17	5.15
8	5.07	18	5.15
9	5.12	19	5.15
10	5.17		

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
1	4,78%	4,79%	4,81%	4,82%	4,84%	4,86%	4,87%	4,89%	4,91%	4,93%	4,95%	4,97%	4,99%	5,00%	5,00%	5,01%	5,02%	5,02%	5,02%
2	4,80%	4,82%	4,84%	4,85%	4,87%	4,89%	4,91%	4,93%	4,96%	4,98%	4,99%	5,01%	5,02%	5,03%	5,03%	5,04%	5,04%	5,05%	
3	4,84%	4,85%	4,87%	4,89%	4,91%	4,94%	4,96%	4,98%	5,00%	5,02%	5,03%	5,05%	5,05%	5,06%	5,06%	5,06%	5,07%		
4	4,87%	4,89%	4,91%	4,93%	4,96%	4,98%	5,00%	5,02%	5,04%	5,06%	5,07%	5,07%	5,08%	5,08%	5,09%	5,09%			
5	4,91%	4,93%	4,95%	4,98%	5,01%	5,03%	5,05%	5,07%	5,08%	5,10%	5,10%	5,10%	5,10%	5,11%	5,11%				
6	4,96%	4,98%	5,01%	5,03%	5,06%	5,08%	5,10%	5,11%	5,12%	5,12%	5,13%	5,13%	5,13%	5,13%					
7	5,00%	5,04%	5,06%	5,09%	5,11%	5,13%	5,14%	5,15%	5,15%	5,15%	5,15%	5,15%	5,15%						
8	5,07%	5,09%	5,12%	5,14%	5,15%	5,17%	5,17%	5,17%	5,17%	5,17%	5,17%	5,16%							
9	5,12%	5,14%	5,16%	5,18%	5,19%	5,19%	5,19%	5,18%	5,18%	5,18%	5,18%								
10	5,17%	5,19%	5,20%	5,21%	5,21%	5,20%	5,19%	5,19%	5,19%	5,18%									
11	5,20%	5,22%	5,22%	5,22%	5,21%	5,20%	5,19%	5,19%	5,18%										
12	5,23%	5,23%	5,23%	5,21%	5,20%	5,19%	5,19%	5,18%											
13	5,23%	5,23%	5,20%	5,19%	5,18%	5,18%	5,17%												
14	5,23%	5,19%	5,18%	5,17%	5,17%	5,16%													
15	5,14%	5,15%	5,15%	5,15%	5,15%														
16	5,15%	5,15%	5,15%	5,15%															
17	5,15%	5,15%	5,15%																
18	5,15%	5,15%																	
19	5,15%																		

Figure B.1: This graphic shows the matrix of the forward LIBOR rates

Appendix C

Matlab files

The matlab files used in this paper

C.1 Equity simulation

At first the m file for simulation of one single asset:

```
%Monte Carlo Simulation for Stock prices
%S_start = initial stock price
%mu = expected return
%s = Volatility
%T = Maturity
%h = length of the small time intervals
%delta_S= change in the stock price S in the small interval of time
%delta_S= (mu*delta_t*S+sigma*sqrt(delta_t)*S*epsilon)— model of stock
%price behavior

T = 10;
S_start = 100;
r = 0.2;
mu = r;
s = 0.3;
n = 1000;
h = T/n;
nSimulation = 1;

dW=randn(n,nSimulation);

S = S_start * ones(n + 1, nSimulation);
for i = 1 : n
S(i + 1, :) = S(i, :). * exp((r - s^2/2) * h + s * sqrt(h) * dW(i, :));
```

```
end
```

As next we have simulated N assets which are correlated with each other with a given correlation matrix.

```
%Monte Carlo Simulation for N correlated assets
%S0(i) = initial stock price of the i-th asset
%mu(i) = expected return of the i-th asset
%s = Volatility
%T = Maturity
%h = length of the small time intervals
%delta_S= change in the stock price S in the small interval of time
%delta_S= (mu*delta_t*S+sigma*sqrt(delta_t)*S*epsilon)— model of stock price behavior

T = 12;
S_start = 100;
r = 0.0436;
s = 0.2;
n = 50 * 12;
h = T/n;
nSimulation = 1000;

N = 3; % number of Assets
Rho = [10.80.2; 0.810.5; 0.20.51]; % The correlation matrix of the N assets

X=chol(Rho)'; % The Cholesky decomposition of the Correlation matrix

for k = 1 : nSimulation
dW = randn(n, N);
dW2 = (X * dW')';
S = S_start * ones(n + 1, N);
fori = 1 : n
S(i + 1, :) = S(i, :). * exp((r - s^2/2) * h + s * sqrt(h) * dW2(i, :));
end
S_3d(:, :, k) = S;
end
```

C.2 Option prices

We can calculate a plain vanilla option with the Black Scholes formula. The matlab for define this formula is as follows:

The matlab file Call.m defines the function of pricing a call option:

```
function result = call(S,t,X,r,sigma,T)
d1 = (log(S./X) + (r + 0.5 * sigma^2) * (T - t))/(sigma * sqrt(T - t));
d2 = d1 - sigma * sqrt(T - t);
n1 = 0.5 * (1 + erf(d1/sqrt(2))); % This equation calculate the value n1 equals N(d1)
n2 = 0.5 * (1 + erf(d2/sqrt(2))); % Equivalent calculation for N(d2)
result = S. * n1 - X * exp(-r * (T - t)) * n2;
```

The matlab file Put.m defines the function of pricing a put option:

```
function result = put(S,t,X,r,sigma,T)
d1 = (log(S/X) + (r + 0.5 * sigma^2) * (T - t))/(sigma * sqrt(T - t));
d2 = d1 - sigma * sqrt(T - t);
n1 = 0.5 * (1 + erf(-d1/sqrt(2)));
n2 = 0.5 * (1 + erf(-d2/sqrt(2)));
result = X * exp(-r * (T - t)) * n2 - S. * n1;
```

The matlab file OptionBS.m calculate price for an option with given inputs:

```
type =' c';
S = 100; % S is the start price of the underlying equity
X = 100; % X is the strike of the option
T = 1; % T is the maturity of the option
r = 0.1; % r is the risk free rate
sigma = 0.2; % sigma is the volatility of the option


hold on
for t = 0 : 0.2 : 1
if type=='c'
C=call(S,t,X,r,sigma,T);
else
P=put(S,t,X,r,sigma,T);
end
end
```

Another way to calculate an option price is to use the Monte Carlo simulation explained in chapter 4.

```
randn('state',3)
X = 100;
```

```
r = 0.1;
sigma = 0.2;
T = 1;
S_start = 100;
n = 50;
h = T/n;
nSimulation = 10000;


dW = sqrt(h) * randn(n, nSimulation);
S = S_start * ones(n + 1, nSimulation);

for i = 1 : n
S(i + 1, :) = S(i, :). * exp((r - sigma^2/2) * h + sigma * dW(i, :));
end

payoff = max(S(n + 1, :) - X, 0);
V = exp(-r * T) * (cumsum(payoff)./(1 : nSimulation));
ePayoff = mean(payoff);
eV = V(nSimulation);

Vexakt = call(S_start, 0, X, r, sigma, T);
plot(abs(V - Vexakt * ones(1, nSimulation))/Vexakt);
```

C.3 Principal component analysis

```
load data.dat % in data.dat ist the matrix saved on which the PCA analyses should be applied
n = 120; % number of the observations
Cov = cov(data); % calculate the covariance matrix for a data matrix
E = eig(Cov); % E is the vector of the eigenvalues of Cov
[V, D] = eig(Cov); % This command decomposes the Cov in two matrices
% V is the orthogonal matrix with eigenvectors
% D is the diagonal matrix with eigenvalues on the diagonal
EV = E/sum(E); % This command will calculate the explained variance for each factor
P = zeros(19, 3); % initialization of the Principle component matrix
P(:, 1) = V(:, 19); % The first component is the eigenvector associated with the largest eigenvalue
P(:, 2) = V(:, 18); % The second component is the eigenvector associated with the second largest eigenvalue
P(:, 3) = V(:, 17); % The third component is the eigenvector associated with the third largest eigenvalue

Q = [P1P2P3];
F = (Q' * data')'; % Factor exposure
Z1=F(:,1)';
Z2=F(:,2)';
```

```
Z3=F(:,3)';
plot([P1 P2 P3],'o-')
% data = P1*Z1+ P2*Z2+ P3*Z3 approximately
```

C.4 Interest rate simulation - Risk neutral

This file is for simulation the interest rate curves in risk neutral world.

```
% Monte Carlo Simulation for the forward rates curve
EV=xlsread('Input.xls','E');
PC=xlsread('Input.xls','PC');
S=xlsread('Input.xls','Swap');
F=xlsread('Input.xls','Fwd');
s = sqrt(diag(EV,0));
clear EV
n = size(S,1);
nSimulation = size(epsilon,3);
R_Matrix = zeros(n,n);
R_3d = zeros(n,n,nSimulation);
delta_R = zeros(n-1,n);
epsilon = randn(3,n-1,nSimulation);

sigma = zeros(1,19);
for i = 1:19
sigma(i) = sqrt(s(1)^2*PC(i,1)^2 + s(2)^2*PC(i,2)^2 + s(3)^2*PC(i,3)^2);
end

rho = zeros(19,19);
for i = 1:19
for j = 1:19
rho(i,j) = (s(1)^2*PC(i,1)*PC(j,1) + s(2)^2*PC(i,2)*PC(j,2) + s(3)^2*PC(i,3)*PC(j,3))/(sigma(i)*
sigma(j));
end
end

for i = 1:nSimulation
P = [zeros(1,3);PC(:,:)];
R_Matrix(1,:) = [S(1);F]';
for j = 1:n-1
delta_R(j,:) = (P(:,1)*s(1)*epsilon(1,j,i) + P(:,2)*s(2)*epsilon(2,j,i) + P(:,3)*s(3)*epsilon(3,j,i))';
R_Matrix(j+1,:) = R_Matrix(j,:).*exp(delta_R(j,:));
for k = j+1:n
R_Matrix(j+1,k) = R_Matrix(j+1,k)*exp(-sigma(k-j)^2/2 + sigma(k-j)*sum(((R_Matrix(j,j+1:
```

$k). * rho(k-j, 1:k-j)). * sigma(1:k-j)). * (1./(1+R\_Matrix(j, j+1:k)))));$

end

$P(n,:) = [];$

$P = [zeros(1,3); P];$

end

$R\_3d(:,:,i) = R\_Matrix;$

end

$R\_mean = zeros(20, 20);$

$for\ i = 1:20$

$for\ j = 1:20$

$R\_mean(i,j) = mean(R\_3d(i,j,:));$

end

end

$save('R\_3d', 'R\_3d', 'R\_mean', 'nSimulation', 'n');$

C.5 Discount factors

This matlab file tries to calculate the discount factors ,or in other way expressed the bond prices with the simulated forward rates. The calculation is of course path dependent.

$load\ R\_3d.mat$

$mdf = zeros(n, n, nSimulation);$

$df0 = zeros(n, nSimulation);$

$dumvar = ones(n, n, nSimulation);$

$df = dumvar./(dumvar + R\_3d);$

$for\ i = 1 : nSimulation \quad df(:,:,i) = df(:,:,i) - tril(df(:,:,i), -1) + tril(dumvar(:,:,i), -1);$

$df(:,1,i) = df(:,1,i);$

$df0(:,i) = 1./(1 + diag(R\_3d(:,:,i)));$

end

$mdf = cumprod(df, 2);$

$df0 = cumprod(df0);$

$save('Mdf', 'mdf', 'df0', 'nSimulation', 'n');$

C.6 Swaption prices

This matlab file calculates the swaption prices using the monte carlo simulation.

$T = 1;$

$Tenor = 3;$

$K = 0.04393;$
$type =' P';$
$L = 10000000;$
$V = zeros(nSimulation, 1);$
$R = zeros(nSimulation, 1);$

$for\ i = 1 : nSimulation$
$R(i, :) = (1 - mdf(T + 1, T + Tenor, i))/sum(mdf(T + 1, T + 1 : T + Tenor, i));$
$D1 = sum(mdf(T + 1, T + 1 : T + Tenor, i)) * df0(T, i);$
$iftype ==' P'$
$V(i, :) = max(R(i, :) - K, 0) * D1;$
$else$
$V(i, :) = max(K - R(i, :), 0) * D1;$
end
end

$P = L * mean(V);$
$M = mean(R);$
$B = std(log(R));$

C.7 Equity model hybrid

load R_3d.mat;
$T = 12;$
$S\_start = 100;$
$sigma = 0.15;$
$n = 20;$
$h = T/n;$

$rho1 = 0.15;$
$rho2 = -0.3;$
$rho3 = 0.15;$

$S = S\_start * ones(n, nSimulation);$
$epsilon4 = randn(1, 19, nSimulation);$
$dW = zeros(1, 19, nSimulation);$

$for\ i = 1 : nSimulation$
$dW(:, :, i) = rho1 * epsilon(1, :, i) + rho2 * epsilon(2, :, i) + rho3 * epsilon(3, :, i) + sqrt(1 - rho1^2 - rho2^2 - rho3^2) * epsilon4(:, :, i);$
$for\ j = 1 : (n - 1)$
$S(j + 1, i) = S(j, i) * exp((R\_3d(j, j, i) - sigma^2/2) * h + sigma * sqrt(h) * dW(1, j, i));$
end

```
end
```

C.8 Interest rate simulation - Real world

Here the method for changing from the risk neutral viewpoint to the real world one is executed in the m files. The first one explains this method with the first alternative , the second file with the another one.

```
EV = xlsread('Input.xls','E');
PC = xlsread('Input.xls','PC');
S = xlsread('Input.xls','Swap','A1:A20');
Discf = xlsread('Input.xls','Swap','B1:B19');
F = xlsread('Input.xls','Fwd');
epsilon = randn(3,19,1000);
s = sqrt(diag(EV,0));
clearEV
n = size(S,1);
nSimulation = size(epsilon,3);
R_Matrix = zeros(n,n);
R_3d = zeros(n,n,nSimulation);
delta_R = zeros(n-1,n);
sigma = zeros(1,19);
for i = 1:19
sigma(i) = sqrt(s(1)^2*PC(i,1)^2 + s(2)^2*PC(i,2)^2 + s(3)^2*PC(i,3)^2);
end

rho = zeros(19,19);
for i = 1:19
for j = 1:19
rho(i,j) = (s(1)^2*PC(i,1)*PC(j,1) + s(2)^2*PC(i,2)*PC(j,2) + s(3)^2*PC(i,3)*PC(j,3))/(sigma(i)*
sigma(j));
end
end

gamma = -0.2;

for i = 1:nSimulation
P = [zeros(1,3);PC(:,:)];
R_Matrix(1,:) = [S(1);F]';
for j = 1:n-1
delta_R(j,:) = (P(:,1)*s(1)*epsilon(1,j,i) + P(:,2)*s(2)*epsilon(2,j,i) + P(:,3)*s(3)*epsilon(3,j,i))';
R_Matrix(j+1,:) = R_Matrix(j,:).*exp(delta_R(j,:));
for k = j+1:n
```

```
%Lognormal distribution adjustment
R_Matrix(j+1,k) = R_Matrix(j+1,k)*exp(-sigma(k-j)^2/2);
%Convexity adjustment
R_Matrix(j+1,k) = R_Matrix(j+1,k)*exp(sigma(k-j)*sum(((R_Matrix(j,j+1:k).*rho(k-j,1:k-j)).*sigma(1:k-j)).*(1./(1+R_Matrix(j,j+1:k)))));
%Real World adjustment
R_Matrix(j+1,k) = R_Matrix(j+1,k)*exp(sigma(k-j)*gamma+sigma(k-j)*sigma(1)*R_Matrix(j+1,j+1)/(1+R_Matrix(j+1,j+1)));
end
P(n,:) = [];
P = [zeros(1,3);P];
end
R_3d(:,:,i) = R_Matrix;
end

R_mean = zeros(20,20);
for i = 1:20
for j = 1:20
R_mean(i,j) = mean(R_3d(i,j,:));
end
end

save('R_3d','R_3d','R_mean','nSimulation','n');



EV = xlsread('Input.xls','E');
PC = xlsread('Input.xls','PC');
S = xlsread('Input.xls','Swap','A1:A20');
Discf = xlsread('Input.xls','Swap','B1:B19');
F = xlsread('Input.xls','Fwd');
epsilon = randn(3,19,1000);
s = sqrt(diag(EV,0));
clearEV
n = size(S,1);
nSimulation = size(epsilon,3);
R_Matrix = zeros(n,n);
R_3d = zeros(n,n,nSimulation);
delta_R = zeros(n-1,n);
sigma = zeros(1,19);
for i = 1:19
sigma(i) = sqrt(s(1)^2*PC(i,1)^2+s(2)^2*PC(i,2)^2+s(3)^2*PC(i,3)^2);
end

rho = zeros(19,19);
```

```
for i = 1 : 19
for j = 1 : 19
rho(i, j) = (s(1)^2 * PC(i, 1) * PC(j, 1) + s(2)^2 * PC(i, 2) * PC(j, 2) + s(3)^2 * PC(i, 3) * PC(j, 3))/(sigma(i) *
sigma(j));
end
end

gamma = -0.2;
vol = 0.1 * 0.04;
hhh = exp(gamma * vol) - 1;

for i = 1 : nSimulation
P = [zeros(1, 3); PC(:, :)];
R_Matrix(1, :) = [S(1); F]';
for j = 1 : n - 1
delta_R(j, :) = (P(:, 1) * s(1) * epsilon(1, j, i) + P(:, 2) * s(2) * epsilon(2, j, i) + P(:, 3) * s(3) * epsilon(3, j, i))';
R_Matrix(j + 1, :) = R_Matrix(j, :). * exp(delta_R(j, :));
for k = j + 1 : n
%Lognormal distribution adjustment
R_Matrix(j + 1, k) = R_Matrix(j + 1, k) * exp(-sigma(k - j)^2/2);
%Convexity adjustment
R_Matrix(j + 1, k) = R_Matrix(j + 1, k) * exp(sigma(k - j) * sum(((R_Matrix(j, j + 1 : k). * rho(k - j, 1 :
k - j)). * sigma(1 : k - j)). * (1./(1 + R_Matrix(j, j + 1 : k)))));
%Real World adjustment
R_Matrix(j + 1, k) = R_Matrix(j + 1, k) * exp(-hhh * (1 + R_Matrix(j, k))/R_Matrix(j, k));
end
P(n, :) = [];
P = [zeros(1, 3); P];
end
R_3d(:, :, i) = R_Matrix;
end

R_mean = zeros(20, 20);
for i = 1 : 20
for j = 1 : 20
R_mean(i, j) = mean(R_3d(i, j, :));
end
end

save('R_3d', 'R_3d', 'R_mean', 'nSimulation', 'n');
```

C.9 Bond price calculation

Here the method for calculating bond prices. We just give an example for the 12-year bond, the method for other bonds is the same.

```
BondT = 12;
N = 100;
BPrice = zeros(4, BondT + 1);
% Calculate the first bond with maturity 12

Bond1 = zeros(nSimulation, BondT + 1);
D1 = zeros(BondT + 1, 1);

for i = 1 : nSimulation
C1 = (N - N * mdf_RW(1, BondT, i))/(sum(mdf_RW(1, 1 : BondT, i)));
for j = 1 : BondT
Bond1(i, j) = D1(j) + C1 * (sum(mdf_RW(j, j : BondT, i))) + N * mdf_RW(j, BondT, i);
D1(j + 1) = D1(j)/mdf_RW(j, j, i) + C1;
end
Bond1(i, BondT + 1) = D1(BondT + 1) + N;
end

BPrice(1, :) = mean(Bond1);
```

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