The Feynman-Kac Formula

18.177 Final Project

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December 14, 2007

1 Introduction

In quantum mechanics a nonrelativistic particle of mass m moving in a conservative field of potential $V(x)$ is described by a *wave function* $\psi(t,x) : \mathbb{R}^{1+3} \to \mathbb{C}$, which satisfies the Schrödinger equation:

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + V(x)\psi, \quad \psi(0,x) = \psi_0(x). \tag{1}$$

Here $\hbar$ denotes the Planck constant and $\psi_0(x)$ is the state of the particle at time $t = 0$. In a 1948 paper [1], Richard Feynman made the profound observation that the state $\psi(t,x)$ of the particle at time t, can be obtained by "averaging" over all possible paths $y(t)$ starting at x, the "amplitude" $e^{i\frac{S[y]}{\hbar}}$, where $S[y]$ is the action associated to the path y up to time t:

$$S[y] = \int_0^t \Big(\frac{m\dot{y}^2(s)}{2} - V(y(s))\Big)\, ds.$$

That is, the solution to (1) can be expressed as the "path integral":

$$\psi(t,x) = \mathcal{N}\int_{\mathcal{C}_x} e^{i\frac{S}{\hbar}}\ \psi_0(y(t))\ \mathcal{D}_t^\infty[y]. \tag{2}$$

Here $\mathcal{C}_x$ denotes the space of continuous functions $y : [0,\infty) \to \mathbb{R}^3$ such that $y(0) = x$; $\mathcal{N}$ is a symbolic renormalization constant and $\mathcal{D}_t^\infty[y]$ is a symbolic expression for the "infinite dimensional Lebesgue measure."

Although formula (2) is mathematically nonsensical, it does represent a very powerful idea, which Feynman incorporated in his celebrated reformulation of quantum mechanics. As is the case with all great ideas, this one too found a life outside its original realm. At the time Feynman and Mark Kac were both faculty members at Cornell University. Upon learning about his colleague's innovative approach to the Schrödinger equation, Kac realized that a similar scheme can be applied rigorously to the heat equation with an external cooling term:

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u - V(x)u, \quad u(0,x) = u_0(x). \tag{3}$$

In his 1949 paper [2] Kac proved the first version of what is now known as the Feynman-Kac formula: under appropriate assumptions (3) admits a solution given by:

$$u(t,x) = E^x\Big[u_0(B_t)e^{-\int_0^t V(B_s)\,ds}\Big], \tag{4}$$

where E^x denotes the expectation with respect to the translated Brownian motion $B_t + x$.

This formula is a manifestation of the remarkable connection between linear (elliptic and parabolic) PDE and diffusion processes. We shall explore the basics of this connection in the coming sections. We follow the exposition of the subject as in [3] and [4].

2 Itô diffusions

Let us set notation. Denote the probability space by $(\Omega, \mathcal{F}, P^0)$, and let $\mathcal{F}_t$ be a filtration of the σ-algebra $\mathcal{F}$, with respect to which the standard n-dimensional Brownian motion $B_t = (B_t^1, \ldots, B_t^n)$ is $\mathcal{F}_t$-adapted.

Let X_t be a (time-homogeneous) Itô diffusion in $\mathbb{R}^d$, i.e. a *strong* solution to the stochastic differential equation:

$$dX_t^i = \mu_i(X_t)dt + \sum_{j=1}^n \sigma_{ij}(X_t)dB_t^j \qquad i = 1, \ldots, d \tag{5}$$

where $\mu_i : \mathbb{R}^d \to \mathbb{R}$, $\sigma_{ij} : \mathbb{R}^d \to \mathbb{R}$ are Lipschitz continuous and of sublinear growth. Denote by X_t^x the solution determined by the initial condition $X_0 = x \in \mathbb{R}^d$. Let P^x denote the natural probability law associated to X_t, i.e.

$$P^x(X_{t_1} \in S_1, \ldots, X_{t_k} \in S_k) = P^0(X_{t_1}^x \in S_1, \ldots, X_{t_k}^x \in S_k)$$

for all sets $S_i \in \mathcal{B}(\mathbb{R}^d)$, the Borel σ-algebra of $\mathbb{R}^d$. Let E^x denote the expectation taken with respect to the probability measure P^x.

The notion that bridges Itô diffusions and PDE is that of an *infinitesimal generator.*

Definition 1 *The (infinitesimal) generator A of the Itô diffusion X_t is defined by:*

$$Af(x) = \lim_{t \downarrow 0} \frac{E^x[f(X_t)] - f(x)}{t} \qquad x \in \mathbb{R}^d, \tag{6}$$

We denote by $\mathcal{D}_A$ the set of functions $f : \mathbb{R}^d \to \mathbb{R}$, for which the limit exists for all $x \in \mathbb{R}^d$.

We will show that, for example, $\mathcal{D}_A$ contains the set of compactly supported C^2-functions $C_c^2(\mathbb{R}^d)$, and that the action of A on $C_c^2(\mathbb{R}^d)$ is given by the second order differential operator:

$$L = \sum_{i=1}^d \mu_i(x)\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i,j}(\sigma\sigma^T)_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j}. \tag{7}$$

This is a corollary of the next proposition. But before we state it, let us define the class $C_c^{1,2}([0,T]\times\mathbb{R}^d)$ to be the set of $C^{1,2}([0,T]\times\mathbb{R}^d)$-functions $f(t,x)$ such that the slice $f_t(x)$ is uniformly supported in a compact set $K(f)\subset\mathbb{R}^d$ $\forall t\in[0,T]$.

Proposition 1 *Let $Y_t^x : [0,T]_t\times\Omega\to\mathbb{R}^d$ be an d-dimensional Itô process of the form:*

$$(Y_t^x)^i(\omega) = x_i + \int_0^t u_i(s,\omega)\,ds + \sum_{j=1}^n \int_0^t v_{ij}(s,\omega)\,dB_s^j(\omega) \qquad i=1,\dots,d,$$

where $u_i\in L^1_{ad}(\Omega\times[0,T])$, $v_{ij}\in L^2_{ad}(\Omega\times[0,T])$.
Let $f\in C_c^{1,2}([0,T]\times\mathbb{R}^d)$ and let $\tau\le T$ be a bounded $\mathcal{F}_t$-stopping time. Then

$$E^x[f(\tau,Y_\tau)] = f(0,x) + E^x\Bigg[\int_0^\tau \Big(\frac{\partial f}{\partial t}(s,Y_s)+ + \sum_{i=1}^d u_i(s,\omega)\frac{\partial f}{\partial x_i}(s,Y_s) + \frac{1}{2}\sum_{i,j}(vv^T)_{ij}(s,\omega)\frac{\partial^2 f}{\partial x_i\partial x_j}(s,Y_s)\Big)ds\Bigg]. \tag{8}$$

Proof Apply Itô's formula to $f(t,Y_t)$. Denote $f_{,0}=\frac{\partial f}{\partial t}$, $f_{,i}=\frac{\partial f}{\partial x_i}$.

$$df(t,Y_t) = f_{,0}dt + \sum_i f_{,i}dY_t^i + \frac{1}{2}\sum_{i,j} f_{,ij}dY_t^i dY_t^j = $$
$$= f_{,0}dt + \sum_i u_i f_{,i}dt + \sum_{i,k} f_{,i}v_{ik}dB_t^k + \frac{1}{2}\sum_{i,j} f_{,ij}dY_t^i dY_t^j.$$

We have

$$dY_t^i dY_t^j = (u_i dt + \sum_k v_{ik}dB_t^j)(u_i dt + \sum_l v_{jl}dB_t^l) = (vv^T)_{ij}dt.$$

Thus,

$$f(t,Y_t)-f(0,Y_0) = \int_0^t \Big(f_{,0}+\sum_i u_i f_{,i} + \frac{1}{2}\sum_{i,j}(vv^T)_{ij}f_{,ij}\Big)ds + \sum_{i,k}\int_0^t f_{,i}v_{ik}\,dB_t^k.$$

Formula (8) will follow once we show that for all $i=1,\dots d$, $k=1\dots n$.

$$E^x\int_0^\tau v_{ik}f_{,i}(t,Y_t)\,dB_t^k = 0.$$

Denote $g(t,\omega)=v_{ik}(t,\omega)f_{,i}(t,Y_t)$. Since $f_{,i}(t,x)$ is uniformly bounded for all $t\in[0,T]$, $g(t,\omega)\in L^2_{ad}$. Also, the characteristic function $\chi_{t<\tau}$ is $\mathcal{F}_t$-measurable, so we can write

$$\int_0^\tau g(t,\omega)\,dB_t^k = \int_0^T g(t,\omega)\chi_{t<\tau}(\omega)\,dB_t^k(\omega).$$

as an Itô integral of an L^2_{ad}-process. Thus, its expectation is indeed 0. $\boxtimes$

Corollary 1 *The generator of the time-homogeneous Itô diffusion* (5) *acts on* $C_c^2(\mathbb{R}^d)$ *as the differential operator* (7).

Proof Apply Proposition 1 with $\hat{f}(t,x) = f(x)$, where $f \in C_c^2(\mathbb{R}^d)$ and

$$u_i(s,\omega) = \mu_i(X_s^x(\omega)), \quad v_{ij}(s,\omega) = \sigma_{ij}(X_s^x(\omega)), \quad \tau = t.$$

We get

$$g(t) := E^x[f(X_t)] = f(x) + E^x\Big[\int_0^t Lf(X_s^x)\,ds\Big] = f(x) + \int_0^t E^x[Lf(X_s)]\,ds,$$

where the interchange of E^x and the integral sign was justified by the fact that Lf is bounded. Also note that whenever $s_i \to s$, $Lf(X_{s_i}) \to Lf(X_s)$ a.s., since X_s is a continuous process. Then the Dominated convergence theorem yields $E^x[Lf(X_{s_i})] \to E^x[Lf(X_s)]$. Thus $E^x[Lf(X_s)]$ is continuous in s, which implies that $g(t)$ is differentiable $\forall t \in (0,T)$ and

$$g'(t) = E^x[Lf(X_t)].$$

Obviously, the same argument also shows that

$$\lim_{t\downarrow 0} \frac{g(t)-g(0)}{t} = E^x[Lf(X_0)] = Lf(x).$$

That is,

$$Af(x) = \lim_{t\downarrow 0} \frac{E^x[f(X_t)] - f(x)}{t} = Lf(x).$$

$\boxtimes$

Let us also state another immediate corollary of Proposition 1 in combination with Corollary 1.

Corollary 2 *Let* $f \in C_c^{1,2}([0,T]\times\mathbb{R}^d)$, X_t *be a time-homogeneous Itô diffusion* (5) *and* $\tau \le T$ *be a bounded* $\mathcal{F}_t$*-stopping time. Then*

$$E^x[f(\tau, X_\tau)] = f(0,x) + E^x\Big[\int_0^\tau \frac{\partial f}{\partial t}(s,X_s) + Af(s,X_s)\,ds\Big],$$

where A *acts on* $C_c^{1,2}$ *functions* $f(s,x)$ *by acting on its restriction* $f_t(x) \in C_c^2$ *for each fixed* t.

Now we can finally commence to the statement of the Feynman-Kac formula.

3 The Feynman-Kac formula

Theorem 1 *Let* X_t *be a time-homogeneous Itô diffusion* (5), *and let* A *be its generator. Assume that* $f \in C_c^2(\mathbb{R}^d)$ *and that* $V \in C(\mathbb{R}^d)$ *is bounded. Then*

$$u(t,x) = E^x\Big[f(X_t)e^{-\int_0^t V(X_s)\,ds}\Big], \qquad 0 \le t \le T \tag{9}$$

is a solution to

$$\frac{\partial u}{\partial t} = Au - Vu \tag{10}$$
$$u(0,x) = f(x).$$

Moreover, if $v \in C^{1,2}([0,T] \times \mathbb{R}^d)$ *is a bounded solution to* (10)*, then* $v(t,x) = u(t,x)$*, given by* (9)*.*

Proof Denote $Y_t = f(X_t)$, $Z_t = e^{-\int_0^t V(X_s)\,ds}$. Since f and V are bounded, it follows that Y_t and Z_t are uniformly bounded $\forall t \in [0,T]$ and thus $u(t,x) = E^x[Y_t Z_t]$ is bounded, as well. Fix $t \in (0,T)$ and consider

$$\frac{E^x[u(t,X_r)] - u(t,x)}{r} = \frac{1}{r} E^x[E^{X_r}[Y_t Z_t] - E^x[Y_t Z_t]].$$

for small enough $r > 0$. The Markov property of Itô diffusions implies

$$\begin{aligned}
E^{X_r}[Y_t Z_t] &= E^x\left[f(X_{t+r}) e^{-\int_0^t V(X_{s+r})\,ds} \middle| \mathcal{F}_r\right] = \\
&= E^x[Y_{t+r} e^{-\int_r^{t+r} V(X_s)\,ds} | \mathcal{F}_r] = E^x[Y_{t+r} Z_{t+r} e^{\int_0^r V(X_s)\,ds} | \mathcal{F}_r] = \\
&= E^x[Y_{t+r} Z_{t+r} | \mathcal{F}_r] e^{\int_0^r V(X_s)\,ds}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\frac{E^x[u(t,X_r)] - u(t,x)}{r} &= E^x[e^{\int_0^r V(X_s)\,ds} Y_{t+r} Z_{t+r} - Y_t Z_t] = \\
= \frac{u(t+r,x) - u(t,x)}{r} &+ \frac{1}{r} E^x\left[Y_{t+r} Z_{t+r} (e^{\int_0^r V(X_s)\,ds} - 1)\right] \qquad (11)
\end{aligned}$$

Let's first deal with the second term in (11). The fact that V is bounded implies that $\frac{1}{r}(e^{\int_0^r V(X_s)\,ds} - 1)$ is uniformly bounded for all $r \in [0,1]$. Moreover, since f and V are continuous, and X_t is a continuous process:

$$\begin{aligned}
Y_{t+r} = f(X_{t+r}) &\overset{r \to 0}{\to} f(X_t) = Y_t \quad \text{a.s.} \\
Z_{t+r} = e^{-\int_0^{t+r} V(X_s)\,ds} &\overset{r \to 0}{\to} e^{-\int_0^t V(X_s)\,ds} = Z_t \quad \text{a.s.} \\
\frac{1}{r}(e^{\int_0^r V(X_s)\,ds} - 1) &\overset{r \to 0}{\to} V(x) \quad \text{a.s.}
\end{aligned}$$

Hence, the Dominated convergence theorem implies

$$\frac{1}{r} E^x\left[Y_{t+r} Z_{t+r} (e^{\int_0^r V(X_s)\,ds} - 1)\right] \overset{r \to 0}{\to} V(x) E^x[Y_t Z_t] = V(x) u(t,x). \tag{12}$$

Let's now show that $u(t,x)$ is differentiable in t. We shall use the following trick:

$$e^{\int_0^t g(s)\,ds} = 1 + \int_0^t g(s) e^{\int_s^t g(r)\,dr}\,ds \qquad g \in C[0,T],$$

which follows from integration by parts. Plug in $g(s) = -V(X_s)$ to obtain

$$Z_t = 1 - \int_0^t V(X_s) e^{-\int_s^t V(X_r)\,dr}\,ds \qquad \text{a.s.}$$

Multiplying both sides by Y_t and taking expectation we obtain

$$u(t,x) = E^x[Y_t] - \int_0^t E^x\Big[Y_t V(X_s) e^{-\int_s^t V(X_r)\,dr}\Big]\,ds \tag{13}$$

The first term is differentiable in t by the proof of Corollary 1. To deal with the second, note that $Y_t V(X_s) e^{-\int_s^t V(X_r)\,dr}$ is bounded and continuous almost surely; so the dominated convergence theorem implies that the integrand $E^x\Big[Y_t V(X_s) e^{-\int_s^t V(X_r)\,dr}\Big]$ is bounded and continuous in s. Thus, the second term in (13) is differentiable in t, as well.

So, (11), (12) and (13) imply $u(t,x) \in \mathcal{D}_A$, and more precisely,

$$Au(t,x) = \frac{\partial u}{\partial t}(t,x) + V(x)u(t,x).$$

As $u(0,x) = E^x[Y_0 Z_0] = f(x)$, we verify that $u(t,x)$ solves (10).

Now let us prove the uniqueness statement of the theorem.

Let $v(t,x) \in C^{1,2}([0,T] \times \mathbb{R}^d)$ be a bounded solution to (10). Define for $(x,z) \in \mathbb{R}^{d+1}$,

$$Z_s = z + \int_0^s V(X_r)\,dr, \qquad H_s = (X_s^x, Z_s).$$

It is easy to see that H_s is a time-homogeneous Itô diffusion, whose generator

$$A_H \phi(x,z) = A\phi + V\frac{\partial \phi}{\partial z}, \qquad \phi \in C_c^2(\mathbb{R}^{d+1}).$$

Now fix $t \in [0,T]$ and let $\psi \in C_c^\infty(\mathbb{R}^{d+1})$ be a smooth bump function with $\psi \equiv 1$ on $|(x,z)| \le 2R$. Define

$$\phi(s,x,z) = \psi(x,z)e^{-z}v(t-s,x) \qquad 0 \le s \le t.$$

Therefore, $\phi \in C_c^{1,2}([0,t] \times \mathbb{R}^{d+1})$. Apply Corollary 2 to ϕ and the bounded stopping time $t \wedge \tau_R$, where $\tau_R = \inf\{t \ge 0 : |H_t| \ge R\}$:

$$E^{x,z}[\phi(t\wedge\tau_R, H_{t\wedge\tau_R})] = \phi(0,x,z) + E^{x,z}\Big[\int_0^{t\wedge\tau_R} (A_H + \frac{\partial}{\partial r})\phi(r, H_r)\,dr\Big]$$

We have

$$\Big(\frac{\partial}{\partial r} + A_H\Big)\phi(r,x,z) = \Big(\frac{\partial}{\partial r} + A + V\frac{\partial}{\partial z}\Big)(\psi(x,z)e^{-z}v(t-r,x))$$

and note that for the time interval of interest (up to $t \wedge \tau_R$), $\psi(H_r) \equiv 1$. Thus,

$$\Big(\frac{\partial}{\partial r} + A_H\Big)\phi(r, H_r) = e^{-z}\Big(-\frac{\partial v}{\partial r} + Av - Vv\Big)\Big|_{(x,z)=H_r} = 0.$$

So we can deduce that for all $R > 0$ large enough

$$\begin{aligned} v(t,x) = \phi(0,x,0) = E^{x,0}[\phi(t \wedge \tau_R, H_{t\wedge\tau_R})] = \\ = E^x\Big[e^{-\int_0^{t\wedge\tau_R} V(X_r)\,dr} v(t - t\wedge\tau_R, X_{t\wedge\tau_R})\Big] \\ \stackrel{R\to\infty}{\to} E^x\Big[e^{-\int_0^t V(X_r)\,dr} v(0, X_t)\Big], \end{aligned}$$

where the limiting argument is justified by the Dominated convergence theorem, bearing in mind that both factors in the expectation are bounded by assumption. Since $v(0, X_t) = f(X_t)$,

$$w(t,x) = E^x\Big[e^{-\int_0^t V(X_r)\,dr} f(X_t)\Big] = u(t,x).$$

This concludes the proof of the theorem. ⊠

Remark 1 *Note that the argument in the proof of uniqueness only uses the action of the generator A on $C_c^{1,2}$. Since $A = L$ on this class of functions, we get a proof of the following statement for free:*
If $v(t,x)$ is a bounded $C^{1,2}([0,T] \times \mathbb{R}^d)$ function which solves

$$\begin{aligned} \frac{\partial u}{\partial t} &= Lu - Vu \\ u(0,x) &= f(x). \end{aligned} \tag{14}$$

then $v(t,x)$ is given by (9)*. That is, any bounded $C^{1,2}$-solution of* (14) *is, in fact, a solution of* (10) *and is given by* (9)*.*

4 Concluding remarks

It is somewhat unfortunate that although the differential equations (10) and (14) are very closely related, they are still distinct. It is therefore natural to ask under what conditions the two equations coincide. In this spirit it is important to know when the $u(t,x)$, given by (9), is actually $C^{1,2}$. Then one can actually show that u solves (14), as well. This is a corollary of the following fact: closely related to the notion of an (infinitesimal) generator A of an Itô diffusion X_t is the notion of its *characteristic operator*

$$\mathcal{A}f(x) = \lim_{U \downarrow x} \frac{E^x[f(X_{\tau_U}] - f(x)}{E^x[\tau_U]}$$

where $U \downarrow x$ is any decreasing sequence of open sets containing x, and τ_U is the first exit time of X_t^x from U. If $\mathcal{D}_\mathcal{A}$ denotes the domain of functions for which the limit exists for all x, it is actually the case that $\mathcal{D}_A \subseteq \mathcal{D}_{\mathcal{A}}$ and that $Af = \mathcal{A}f \quad \forall f \in \mathcal{D}_A$. The advantage of the operator $\mathcal{A}$ is that it acts on C^2-functions just like L (cf. [5]). Obviously, this implies that whenever the u, given by (9), is in $C^{1,2}$, then $Au = Lu$.

In the case of $X_t = B_t$ and $A \leftrightarrow \frac{\Delta}{2}$, one can show that a sufficient condition for u to be $C^{1,2}$ is V being locally Hölder continuous. (cf. [4])

References

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