CQF Exercises 4.3 Calibration

1. Very briefly outline the difference between (one factor) *equilibrium* and *no-arbitrage* models for the spot rate.

   Sol: Equilibrium models start with assumptions about economic variables and derive the behaviour of interest rates. Thus these models have the property that initial term structure is an output from the model, whereas in a no-arbitrage model this is an input.

   The behaviour of interest rates in the latter are designed to ensure consistency with the initial term structure.

   So in an equilibrium model choose the drift and diffusion in

   $$dr = u(r,t)\,dt + w(r,t)\,dX \ ,$$

   to have a few parameters to get "best fit". Whereas in a no-arbitrage model we have an arbitrary function (of time) which allows us to fit the yield curve exactly.

2. Substitute the fitted function for $A(t\ ;T)$, using the Ho & Lee model, back into the solution of the bond pricing equation for a zero-coupon bond,

   $$Z(r,\ t\ ;T) = \exp\left(A(t\ ;T) - r(T-t)\right).$$

   The form for $A(t\ ;T)$ can be found on page 11 of the lecture notes. What do you notice when $t = t^*$?

   Sol: With a Ho & Lee model, the form of the fitted function for $A(t\ T)$ is

   $$A(t\ ;T) = \log\left(\frac{Z_M(t^*;\ T)}{Z_M(t^*;\ t)}\right) - (T-t)\frac{\partial}{\partial t}\log\left(Z_M(t^*;\ t)\right) - \frac{1}{2}c^2(t-t^*)(T-t)^2.$$

   Then

   $$\begin{aligned} Z(t\ ;T) &= e^{\log\left(\frac{Z_M(t^*;\ T)}{Z_M(t^*;\ t)}\right) - (T-t)\frac{\partial}{\partial t}\log(Z_M(t^*;\ t)) - \frac{1}{2}c^2(t-t^*)(T-t)^2 - r(T-t)} \\ &= \frac{Z_M(t^*;\ T)}{Z_M(t^*;\ t)} e^{-(T-t)\left(\frac{\partial}{\partial t}\log(Z_M(t^*;\ t)) + \frac{1}{2}c^2(t-t^*)(T-t) + r\right)}. \end{aligned}$$

   We note that that when $t = t^*$

   $$Z(t^*\ ;T) = \frac{Z_M(t^*;\ T)}{Z_M(t^*;\ t)} e^{-(T-t^*)\left(\frac{\partial}{\partial t}\log(Z_M(t^*;\ t)) + \frac{1}{2}c^2(t^*-t^*)(T-t^*) + r\right)} = Z_M(t^*;\ T).$$

3. Differentiate Equation (2) on page 16 of the lecture notes, twice to solve for the value of $\eta^*(t)$. What is the value of a zero-coupon bond with a fitted Vasicek model for the interest rate?

Solution: We have

$$\begin{aligned} & -\int_{t^*}^{T} \eta^*(s)\, B(s\,;T)\ ds + \frac{c^2}{2\gamma^2}\left((T-t^*) + \frac{2}{\gamma}e^{-\gamma(T-t^*)} - \frac{1}{2\gamma}e^{-2\gamma(T-t^*)} - \frac{3}{2\gamma}\right) \\ = \ & \log(Z_M(t^*\,;T)) + r^* B(t^*\,;T). \end{aligned}$$

Differentiating with respect to T,

$$\begin{aligned} & -\int_{t^*}^{T} \eta^*(s)\, \frac{\partial}{\partial T}B(s\,;T)\ ds - \eta^*(T)\, B(T\,;T) + \frac{c^2}{2\gamma^2}\left(1 - 2e^{-\gamma(T-t^*)} + e^{-2\gamma(T-t^*)}\right) \\ = \ & \frac{\partial}{\partial T}\log(Z_M(t^*\,;T)) + r^* \frac{\partial}{\partial T}B(t^*\,;T). \end{aligned}$$

Now

$$B(t\,;T) = \frac{1}{\gamma}\left(1 - e^{-\gamma(T-t)}\right) \quad \text{so} \quad B(T\,;T) = 0,$$

and

$$\frac{\partial}{\partial T}B(t\,;T) = e^{-\gamma(T-t)}.$$

Substituting back into the PDE

$$\begin{aligned} & -\int_{t^*}^{T} \eta^*(s)\, e^{-\gamma(T-s)}\ ds + \frac{c^2}{2\gamma^2}\left(1 - 2e^{-\gamma(T-t^*)} + e^{-2\gamma(T-t^*)}\right) \\ = \ & \frac{\partial}{\partial T}\log(Z_M(t^*\,;T)) + r^* e^{-\gamma(T-t^*)}. \end{aligned}$$

Differentiating again with respect to T,

$$\begin{aligned} & -\eta^*(T) + \gamma\int_{t^*}^{T} \eta^*(s)\, e^{-\gamma(T-s)}\ ds + \frac{c^2}{2\gamma^2}\left(2\gamma e^{-\gamma(T-t^*)} - 2\gamma e^{-2\gamma(T-t^*)}\right) \\ = \ & \frac{\partial^2}{\partial T^2}\log(Z_M(t^*\,;T)) - \gamma r^* e^{-\gamma(T-t^*)}. \end{aligned}$$

Substituting for the integral from the previous equation, we find

$$\begin{aligned} & -\eta^*(T) + \gamma\left(\frac{c^2}{2\gamma^2}\left(1 - 2e^{-\gamma(T-t^*)} + e^{-2\gamma(T-t^*)}\right) - \frac{\partial}{\partial T}\log(Z_M(t^*\,;T)) - r^* e^{-\gamma(T-t^*)}\right) \\ & + \frac{c^2}{2\gamma^2}\left(2\gamma e^{-\gamma(T-t^*)} - 2\gamma e^{-2\gamma(T-t^*)}\right) \\ = \ & \frac{\partial^2}{\partial T^2}\log(Z_M(t^*\,;T)) - \gamma r^* e^{-\gamma(T-t^*)}. \end{aligned}$$

This simplifies to

$$\eta^*(T) = -\frac{\partial^2}{\partial T^2}\log\left(Z_M\left(t^*\ ;T\right)\right) + \frac{c^2}{2\gamma} - \gamma\frac{\partial}{\partial T}\log\left(Z_M\left(t^*\ ;T\right)\right) - \frac{c^2}{2\gamma}e^{-2\gamma(T-t^*)},$$

and

$$\eta^*(t) = -\frac{\partial^2}{\partial t^2}\log\left(Z_M\left(t^*\ ;t\right)\right) - \gamma\frac{\partial}{\partial t}\log\left(Z_M\left(t^*\ ;t\right)\right) + \frac{c^2}{2\gamma}\left(1 - e^{-2\gamma(t-t^*)}\right).$$

We then have

$$A(t\ ;T) = -\int_t^T \eta^*(s)\,B(s\ ;T)\ ds + \frac{c^2}{2\gamma^2}\left((T-t) + \frac{2}{\gamma}e^{-\gamma(T-t)} - \frac{1}{2\gamma}e^{-2\gamma(T-t)} - \frac{3}{2\gamma}\right)$$

and substituting for η^* and integrating, we find

$$= \log\left(\frac{Z_M(t^*;\ T)}{Z_M(t^*;\ t)}\right) - B(t\ ;T)\frac{\partial}{\partial t}\log\left(Z_M(t^*;\ t)\right) - \frac{c^2}{4\gamma^3}\left(e^{-\gamma(T-t^*)} - e^{-\gamma(t-t^*)}\right)^2\left(e^{2\gamma(t-t^*)} - 1\right).$$

We know the value of a zero-coupon bond is

$$Z(r,\ t\ ;T) = \exp\left(A(t\ ;T) - rB(t\ ;T)\right),$$

with $A(t\ ;T)$ given by the above, and

$$B(t\ ;T) = \frac{1}{\gamma}\left(1 - e^{-\gamma(T-t)}\right).$$

4. Use spot rate data to find ν and β if we assume that interest rate movements are of the form

$$dr = u(r)\,dt + \nu r^{\beta}dX.$$

Does your estimated value of β lie close to that of any of the standard models? (Use any finance based website to down load interest rate data for this question). *To follow in separate document*

5. In problem sheet 4.2 we derived a BPE which gave zero coupon bonds of the form

$$V(r,t) = \exp\left(A(t) + rB(t)\right)$$

where

$$B = t - T$$

$$A(t) = -\int_t^T \left[a(s)\,(s-T)\right]ds - \frac{(t-T)^3}{6}$$

Suppose at time t^* bond prices are given for a continuous range of maturities, T, so that

$$V\left(r,t^*;T\right)$$

is known as a function of T. Determine $a\left(T\right)$ in terms of

$$\frac{\partial^2}{\partial T^2}\left(\log V\left(r,t^*;T\right)\right).$$

Sol: Now

$$\begin{aligned}
V\left(r,t^*;T\right) &= \exp\left(A\left(t^*\right)+rB\left(t^*\right)\right) \\
\log V\left(r,t^*;T\right) &= \left(A\left(t^*\right)+rB\left(t^*\right)\right) \\
\log V\left(r,t^*;T\right) &= -\int_{t^*}^{T}\left[a\left(s\right)\left(s-T\right)\right]ds-\frac{\left(t^*-T\right)^3}{6}+r\left(t^*-T\right)
\end{aligned}$$

Now differentiate both sides wrt T :

$$\frac{\partial}{\partial T}\log V\left(r,t^*;T\right)=\int_{t^*}^{T}\left[a\left(s\right)\right]ds+\frac{\left(t^*-T\right)^2}{2}-r$$

and again

$$\frac{\partial^2}{\partial T^2}\log V\left(r,t^*;T\right)=a\left(T\right)-\left(t^*-T\right)$$

hence

$$a\left(T\right)=\frac{\partial^2}{\partial T^2}\log V\left(r,t^*;T\right)+\left(t^*-T\right)$$