The Performance of Multi-Factor Term Structure Models for Pricing and Hedging Caps and Swaptions

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Abstract

In this paper we analyze the pricing and hedging of caps and swaptions using term structure models. Cap prices mainly depend on variances of forward interest rates, whereas swaption prices also depend on the correlations between these forward rates. We therefore compare one-factor models, that imply perfectly correlated interest rates, with multi-factor models, using US data on cap and swaption prices for 1995 until 1999. The caps and swaptions data contain wide ranges of both option maturities and maturities of the underlying interest rates and swaps. The models are estimated using either historical interest rate data or derivative prices. We compare the different models by analyzing the accuracy of predicting caps and swaption prices and the accuracy of hedging caps and swaptions. We find that models with two or three factors imply more accurate predictions for cap and swaption prices and more accurate hedging of caps and swaptions than one-factor models. However, the difference between the results for one-factor models and multi-factor models is economically small. For all models, estimation on the basis of derivative prices leads to more accurate out-of-sample prediction of cap and swaption prices than estimation on the basis of interest rate data.

JEL Codes: G12, G13, E43.

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1 Introduction

Two important applications of models for the term structure of interest rates are the pricing and hedging of interest rate derivatives. Many articles have examined the empirical performance of such models. A large part of this empirical literature has focused on estimating and testing term structure models using data on bond prices or interest rates (for example, Babbs and Nowman (1999), Dai and Singleton (1999), and Pearson and Sun (1994)). In general, the conclusion is that models that have one factor that drives interest rates of all maturities are rejected in favour of two- or three-factor models.

However, there exists little evidence concerning the performance of multi-factor models in terms of the pricing and hedging of interest rate derivatives. Longstaff, Santa-Clara, and Schwartz (1999) and Rebonato (1999) argue on a purely theoretical basis that, although a one-factor model might suffice for the pricing of caps, it is likely to be inappropriate for the pricing of swaptions, because swaption prices directly depend on the correlation between interest rates of different maturities. In one-factor models, these (instantaneous) correlations are equal to one, contradicting empirical observations. In this paper we perform an empirical analysis, to analyze the performance of both one-factor and multi-factor models for the pricing and hedging of caps and swaptions, using weekly data on cap and swaption prices from 1995 until 1999. The primary goal of this paper is to identify the impact of the choice of the *volatility structure* of forward interest rates and the *pattern of correlations* between forward interest rates on the accuracy of pricing and hedging caps and swaptions.

This paper is related to Amin and Morton (1994) and Buhler et al. (1999). Amin and Morton (1994) use Eurodollar futures options data and test several one-factor HJM models by analyzing the prediction of futures option prices, parameter stability and profits from model-based trading strategies. They conclude that the simplest one-factor model, the Ho-Lee (1986) model, is the most preferable one, although there is weak evidence for a hump-shaped volatility structure. They estimate the parameters of the model using the daily cross-section of option prices. In contrast, Buhler et al. (1999) estimate the parameters of one- and two-factor models using data on interest rate changes, and, subsequently, analyze how well these models price options on German government bonds. Our paper extends the two abovementioned articles in several ways.

First, in this paper we both apply the *option-based* estimation method of Amin and Morton (1994), as well as the *interest-rate-based* estimation method that is applied by Buhler et al. (1999)[1]. This allows us to analyze which data should be used for estimation, to obtain the best out-of-sample prediction of derivative prices. We also assess the sensitivity of the conclusions of Amin and Morton (1994) and Buhler

[1] Buhler et al. (1999) name their methodology a *global* approach, as opposed to the *local* approach of Amin and Morton (1994). Global or interest-rate-based estimation is also applied by Moraleda and Vorst (1996), whereas the local or option-based estimation method is also used by Flesaker (1993), Moraleda and Vorst (1997) and Moraleda and Pelsser (1998).

et al. (1999) to the specific estimation method chosen.

Second, we use different data compared to Buhler et al. (1999) and Amin and Morton (1994), namely panel data on prices of caps and swaptions. These derivative prices contain much information, because they contain both short- and long-maturity options, ranging from 1 month to 10 years, and these options are written on both single interest rates (caps) and combinations of interest rates of different maturities (swaptions). This variety in instruments enables us to analyze in detail both the entire volatility structure of forward interest rates and the correlations between these forward interest rates. In particular, we can distinguish between one-factor, two-factor and three-factor models. In Amin and Morton (1994), the Eurodollar futures options have maturities up to one year and the underlying interest rate has a maturity of three months. They note that 'with options on short-maturity instruments, we cannot distinguish between multiple additive factors'. In Buhler et al. (1999), the options have maturities up to three years, and have as underlying instrument only medium-term and long-term government bonds. They analyze both one- and two-factor models, and find in some cases that the two-factor models have larger pricing errors on the bond options than one-factor models. One explanation for this result might be a lack of variety in the derivative instruments in their data to identify multiple factors. Also, because their two-factor models do not nest the one-factor models, and estimation strategies differ amongst these models, it is difficult to determine what exactly causes the difference between their one- and two-factor models. In particular, the effect of non-perfect correlations between interest rates of different maturities for derivative pricing remains unclear.

Third, we analyze both the pricing of interest rate derivatives, and the size of the cap and swaption hedging errors for model-based delta-hedging strategies. Although pricing accuracy is often investigated, this does not apply to hedging accuracy. For equity options, several articles examine the effectiveness of delta-hedging (for example Dumas, Fleming and Whaley (1997)). For interest rate options, the empirical evidence seems to be scarce. In a simulated two-factor economy, Canabarro (1995) shows that one-factor models might yield accurate derivative price predictions, but these models poorly hedge interest rate options. We *empirically* analyze hedging accuracy, and focus especially on the differences in hedging errors between one-factor and multi-factor models. We also analyze hedge strategies based on different sets of hedge instruments.

The models that we analyze are all specified according to the Heath, Jarrow and Morton (HJM, 1992) approach. Many well-known term structure models, such as the Ho-Lee (1986) model and the Hull-White (1990) model, fit into this framework. In contrast to the equilibrium term structure models, such as the Vasicek (1977) model and Cox, Ingersoll and Ross (1985) model, HJM-models fit the current term structure of (forward) interest rates by construction. Especially for the pricing and hedging of interest rate derivative portfolios, it is important to price the underlying swaps and bonds without error. Also, the HJM models can price and hedge interest rate derivatives without assumptions on the market price of interest rate risk. One only needs to specify the volatilities and correlations of forward interest rates for all forward maturities.

Our empirical analysis consists of the following steps. First, we estimate the parameters of the models, using either the daily cross-section of caps and swaptions (option-based estimation) or time-series of interest rate changes (interest-rate-based estimation). Second, we analyze for each model and for both estimation strategies the accuracy of predicting the prices of caps and swaptions out-of-sample. Third, for each model, we assess the hedging accuracy for caps and swaptions, i.e., we analyze how much of the variability of cap and swaption prices is removed by delta-hedging strategies based on the model.

The empirical results can be summarized as follows. First, we find that models with a hump-shaped forward rate volatility structure predict cap prices best. However, for swaptions, models with a slowly decaying forward rate volatility structure yield the best prediction of prices. Taking everything together, a three-factor model, combined with option-based estimation, results in the best predictions for cap and swaption prices, but the differences in predictions with the one-factor models are economically not very large, and not always statistically significant. In particular, we find that the absolute pricing errors for swaptions decrease on average from 12.5% of the price, in case of perfectly correlated interest rates, to 8.7% of the price, if we allow for non-perfect correlation between forward interest rates.

Second, in all cases, option-based estimation leads to better predictions than interest-rate-based estimation, and the differences are both statistically significant and large in economic terms. Option-based estimation on average leads to estimates for forward rate volatilities that are of the same size or a little lower than for interest-rate-based estimation. For the multi-factor models, the correlations between forward interest rates are lowest in case of option-based estimation. Hence, the interest-rate correlations implicit in swaption prices are lower than the historically estimated interest rate correlations. If one compares different models on the basis of interest-rate-based estimation only, the multi-factor models on average lead to worse predictions of cap and swaption prices than a one-factor model. This result corresponds to the results of Buhler et al. (1999), who use interest-rate-based estimation only, and implies that conclusions on the basis of interest-rate-based estimation only might be premature.

Third, in almost all cases, the one-factor models on average underprice caps, and overprice swaptions. This could be the result of the one-factor assumption, because lower interest rate correlations imply lower swaption prices, whereas cap prices are insensitive to these correlations. We find, however, that the multi-factor models also persistently underprice caps and overprice swaptions over almost 5 years of derivative data.

Finally, we construct for each model strategies for delta-hedging caps and swaptions, assuming the underlying model is valid, using zero-coupon bonds as hedge instruments. We calculate how much of the variability of caps and swaptions decreases if one rebalances the hedge portfolio every 10 days. If we use as many hedge instruments as the number of factors in the model, which are the same for all caps and swaptions, we find large differences between the one-factor and multi-factor models; the reduction in derivative price variability due to delta-hedging with the three-factor model is almost twice as large as for the one-factor models. However, if we use for every cap and swaption a set of hedge instruments that covers all maturities at which the cap or swaption pays out, the differences between the one-factor and multi-factor models disappear. Hence, the choice of the number of hedge instruments and the maturities of these hedge instruments seems to be more important than the particular model choice.

The remainder of this paper is organized as follows. In section 2 we briefly review the literature on HJM models and the pricing of caps and swaptions. Section 3 describes the data. In section 4 we discuss the specification of the different models and the estimation methods that are used. Section 5 contains the estimation results. In section 6, we analyze the predictions of caps and swaption prices for both one-factor and multi-factor models, and we exactly determine the effect of non-perfectly correlated interest rates on the pricing of caps and swaptions. In section 7, we assess the hedging accuracy for caps and swaptions. Section 8 contains concluding remarks.

2 Pricing Caps and Swaptions with HJM Models

In this section we briefly review the HJM approach to modeling the term structure of (forward) interest rates. Let $f(t,T)$ denote the forward interest rate at time t for riskless and instantaneous borrowing or lending at date T. The key to the HJM approach is to start with modeling the processes of these instantaneous forward interest rates, given a current forward rate curve $f(0,T)$, as follows

$$df(t,T) = \mu(t,T)\,dt + \sum_{i=1}^{K} \sigma_i(t,T,f(t,T))\,dW_i(t), \tag{1}$$

where $W_i(t)$, $i=1,..,K$ represent independent Brownian Motions. The drift function $\mu(t,T)$ of the forward rates in principle depends on the evolution of forward rates of all different maturities from the date *0*, at which the forward rate curve is initiated, until date t. The function $\sigma_i(t,T,f(t,T))$ is called the volatility function of factor i. HJM (1992) show that in an arbitrage-free economy, the drift of forward rates under the equivalent martingale measure, with the money-market account as numeraire, is completely determined by the volatility functions,

$$\mu(t,T) = \sum_{i=1}^{K} \sigma_i(t,T,f(t,T)) \int_t^T \sigma_i(t,u,f(t,u))\,du. \tag{2}$$

This implies that for the pricing and hedging of interest rate derivatives, only the volatility functions need to be specified and estimated.

In this paper, we only analyze models with time-homogeneous volatility functions, that is, volatility functions that only depend on the dates t and T through their difference T-t, since estimation of models with time-inhomogeneous volatility functions from historical interest rate data is at the least very difficult. Also, we analyze only models that imply a Gaussian distribution for interest rates. In forward rate models, the probability distribution of forward rates is determined by the dependence of the volatility function on the forward interest rate. If the volatility function is independent of the forward interest rates, so that the volatility function only depends on the time-to-maturity *T-t*, all forward interest rates are normally distributed. As our analysis is based on prices of at-the-money options, we will not be able to obtain very precise results on the probability distribution of interest rates. Only if one observes a set of options with a wide range of strike prices, one will be able to make clear statements concerning the probability distribution of interest rates. For example, Ait-Sahalia and Lo (1998) nonparametrically estimate the risk-neutral density of equity prices using equity option prices with different strikes. Also, in the studies of Amin and Morton (1994) and Buhler et al. (1999) the differences between models that only differ through their dependence of the volatility function on forward rates are not large. We choose to analyze Gaussian models because of their analytical and numerical tractability and because these models are linked to the Duffie-Kan (DK, 1996) class of interest rate models[2]. The DK-class, that encompasses, for example, the Vasicek (1977) and Cox, Ingersoll and Ross (1985) models, can be modified to fit the current interest rate curve and, thus, fits into the HJM framework (see Frachot and Lesne (1993)).

In the Gaussian HJM models, volatilities of forward rates do not depend on the level of the forward rates. For US interest rates, there is some evidence that the volatility of interest rates depends on the level of interest rates (Chan et al. (1992)). However, as argued by Babbs and Nowman (1999), these results may (partly) be caused by the high and volatile interest rates in the period 1979-1982. More recent studies (Nowman (1997), Bliss and Smith (1998)) provide at the most very weak evidence for a relation between volatility and the interest rate level.

Thus, the models that we analyze have the following form

$$df(t,T) = \mu(t,T)\,dt + \sum_{i=1}^{K} \sigma_i(T-t)\,dW_i(t), \tag{3}$$

The models differ through the number of factors K and the specification of each volatility function $\sigma_i(T-t)$. Given the specification of such a Gaussian HJM model, pricing formulas for caps and swaptions are readily available. The price of a caplet at time t, that pays off $\delta(L_\delta(T,T)-k)^+$ at time $T+\delta$, where $L_\delta(t,T) = \frac{1}{\delta}(\frac{P(t,T)}{P(t,T+\delta)} - 1)$ is the forward Libor rate, has been shown to be equal to (see Brace and

[2]Besides Gaussian models, the DK class also contains models that have square-root processes for the underlying factors. These square-root models have the advantage that interest rates are always positive. However, for realistic parameter values, the probability of negative interest rates is small for Gaussian models (see Rogers (1997)).

Musiela (1994))

$$
\begin{aligned}
& P(t,T)N(-h) \; - \; (1+k\delta)P(t,T+\delta)N(-h-\xi),\\
\xi^2 \; &= \; Var(\log P(T,T+\delta)\,|\; I_t) \; = \; \int_0^{T-t} [\sum_{i=1}^{K} (\int_s^{s+\delta} \sigma_i(u)\,du)^2]\,ds \\
h \; &= \; \frac{1}{\xi}(\log \frac{(1+k\delta)P(t,T+\delta)}{P(t,T)} - \frac{1}{2}\xi^2).
\end{aligned}
\tag{4}
$$

Here $N(.)$ denotes the cumulative standard normal distribution function and I_t denotes the information set of time t. Inspection of these formulas reveals that, because the variance of the log-bond price is the relevant input for the price of the caplet, only the sum of the squared volatility functions of all factors are present in the pricing formula. The price of a caplet, and the price of a cap, which is a sum of caplets of different maturities, therefore only depends on the variance of interest rates, and not on the covariances of bond prices or interest rates of different maturities.

For the price of a payer swaption, that gives the right to enter a swap at time T with fixed rate k, where the swap has payment dates $T_1, T_2,.., T_n$, the following expression is derived by Brace and Musiela (1994)

$$
\int_{\mathbb{R}^K} [P(t,T)\,\phi_K(x) \; - \; \sum_{i=1}^{n} k\,P(t,T_i)\,\phi_K(x+\gamma_i)]^+ \,dx, \tag{5}
$$

where $\phi_K(x)$ is the density function of the K-dimensional standard normal distribution and $\gamma_1,..,\gamma_n$ are K-dimensional vectors such that

$$
\begin{aligned}
& \gamma_i^{/}\gamma_j \; = \; Cov(\log P(T,T_i), \log P(T,T_j)\,|\,I_t) \; = \\
& \int_0^{T-t} [\sum_{k=1}^{K} (\int_s^{T_i-T+s} \sigma_k(u)\,du)(\int_s^{T_j-T+s} \sigma_k(u)\,du)]\,ds, \quad i,j \; = \; 1,..,n
\end{aligned}
\tag{6}
$$

As opposed to caplets, the price of a swaption also depends on the covariances between bond prices or interest rates of different maturities. As a very important difference between one-factor models and multi-factor models lies in the implications for covariances and correlations of interest rates (one-factor models imply perfect instantaneous correlations between interest rates), swaption prices potentially contain information on the number of factors that determine interest rate movements.

Formula (5) is a special case of the pricing formula for the price of a put-option on a coupon-bond derived by Jamshidian (1989), as a payer swaption is equivalent to a put-option on a coupon-bond with coupon rate k and exercise price 1. For one-factor models, equation (5) leads to a closed-form expression for the swaption price. For multi-factor models, it is, in general, not possible to obtain closed-form solutions, and simulation is necessary to determine prices. We use the simulation methodology of Clewlow, Pang and Strickland (1996), who make use of control variates, to obtain prices for swaptions.

For our hedging analysis we also need partial derivatives of the prices of caps and swaptions with respect to zero-coupon prices, which are derived by Brace and Musiela (1994). For the sake of completeness, the appendix contains these hedge ratios.

3 Caps and Swaptions Data

We use two US datasets[3] for our analysis, data on money-market rates and swap rates on the one hand and implied Black (1976) volatilities of caps and swaptions on the other hand.

From January 1994 until June 1999 we have weekly data on US money-market rates with maturities of 1, 3, 6, 9 and 12 months, and data on US swap rates with maturities ranging from 2 years until 15 years. These data are used to construct the (forward) interest rate curve at each day in the dataset. We need these forward interest rates for two reasons. First, when pricing derivatives with HJM models, the initial forward interest rate curve is an input to the HJM model. Second, one way to estimate the parameters of the HJM volatility functions is based on the variances and covariances of historical forward rate changes of different maturities. As noted by Buhler et al. (1999), there is a trade off between these two requirements. In principle, for the pricing of derivatives at one day, one would like to perfectly fit the price of the underlying instrument. On the other hand, because estimates for forward interest rates are very sensitive to small differences between money market or swap rates of nearly the same maturity, a perfect fit of all underlying money market and swap rates leads to unreasonable high estimates for the volatilities of historical forward rate changes. Therefore we impose some smoothness conditions on the shape of the forward interest rate curve. Following Vasicek and Fong (1982) and many others, we parametrize the price of a zero-coupon bond as follows

$$P(t,T) = \exp(\beta_1 (T-t) + \ldots + \beta_d (T-t)^d + \sum_{j=1}^{s} \beta_{d+j} \max(0, T-t-k_j)^d) \tag{7}$$

This specification yields a continuous and differentiable forward interest curve $f(t,T)$. The parameters β in equation (7) are estimated each day by minimizing the sum of squared relative differences between

[3]The data are provided by ABN-Amro Bank, Amsterdam, the Netherlands.

observed money-market and swap rates and the money-market and swap rates that are implied by (7)[4]. In table 1, we give some summary statistics on the fit. It follows that the daily average absolute error is 0.46% for money-market rates and 0.25% for swap rates, which is equivalent to respectively 2.3 and 1.2 basis points, which seems satisfactory.

The derivatives data that we use are weekly quotes for the implied Black (1976) volatilities of at-the-money-forward, flat-yield US caps and swaptions, from January 2, 1995 until June 11, 1999. In total, this renders 232 weekly time-series observations on 63 instruments each week. The caps have maturities ranging from 1 year to 10 years, and their payoffs are defined on 3-month interest rates. The 1-year cap consists of 3 caplets with maturities of 3, 6, and 9 months, and the 10-year cap consists of 39 caplets, with maturities ranging from 3 months to 9 years and 9 months. The other caps are defined similarly. The strike of each cap is equal to the corresponding forward swap rate with quarterly compounding. Caps are quoted in the market by Black implied volatilities. Given the underlying forward interest rate curve, there is a one-to-one correspondence between the cap implied volatility and the price of a cap.

In table 2 we provide some summary statistics on the implied volatilities of the caps. Although these implied volatilities cannot be interpreted directly as volatilities of single interest rates, because a cap consists of several caplets, we can still conclude that there is some evidence for a humped volatility structure, which is in line with Amin and Morton (1994) and Moraleda and Vorst (1997). More formal evidence for humped volatility structures will be given later in this paper.

A swaption is characterized both by the option maturity and the swap maturity. In our data, the option maturities range from 1 month to 5 years, whereas the swap maturities range from 1 year to 10 years. We do not include prices of swaptions with total maturities, defined as the sum of the option maturity and the swap maturity, longer than 11 years, because the implied volatilities of these swaptions are not always updated in our data. The strike of an at-the-money swaption is equal to the corresponding forward par swap rate[5]. Hence, given the underlying forward interest rate curve, there is a one-to-one correspondence between swaption implied volatilities and swaption prices. In tables 3 and 4, we provide summary statistics for the swaption implied Black volatilities. Again, there is some informal evidence for a humped volatility structure. We also see that the variability over time in the swaption implied volatilities is somewhat lower than for cap implied volatilities.

[4]We choose d equal to 3, and the knot points k equal to 2 years and 4 years, which implies that s is equal to 2.

[5]The forward par swap rate is the swap rate that gives a forward swap a value of zero today.

4 Model Specification and Estimation

The differences between models in the time-homogeneous Gaussian HJM class come from the number of factors that is assumed and the functional shape of the volatility function of each factor, which is a function of the time to maturity *T-t* of the forward rate. We choose to analyze two types of specifications for the volatility function in the Gaussian HJM-class:

(I) Parametric One-Factor Models
$$\sigma_1(T-t) = \gamma_1 e^{-\gamma_2(T-t)}(1+\gamma_3(T-t))$$

(II) PCA One-, Two- and Three-Factor Models
$$\begin{aligned}\sigma_1(T-t) &= \lambda_1 \cdot g_1(T-t)\\ \sigma_2(T-t) &= \lambda_2 \cdot g_2(T-t)\\ \sigma_3(T-t) &= \lambda_3 \cdot g_3(T-t)\end{aligned}$$

The choice for these models is largely inspired by models that are proposed and analyzed in the existing literature on interest rate models. The parametric one-factor model (I) is proposed by Amin and Morton (1994) and Mercurio and Moraleda (1996). In its general form, it implies a humped volatility structure if $\gamma_2 > \gamma_3$. We will also analyze two special cases of this model. First, if γ_2 and γ_3 are equal to zero, the constant volatility model of Ho and Lee (1986) is obtained. If γ_3 is equal to zero, the Generalized Vasicek (1977) model is obtained, or equivalently, the one-factor model of Hull and White (1990). Amin and Morton (1994) are not able to estimate the general specification (I), and only estimate restricted versions of this specification.

The one-factor PCA model is obtained if λ_2 and λ_3 are equal to zero, and the two-factor PCA is obtained when λ_3 is zero. The functions $g_1(T-t)$, $g_2(T-t)$ and $g_3(T-t)$ in the PCA models will be estimated using principal components analysis (PCA). Below we will explain how the parameters λ_1, λ_2 and λ_3 are estimated. The use of principal components analysis to estimate HJM-volatility functions was proposed initially by Heath, Jarrow and Morton (1990), and has been applied to interest rate data by, for example, Littermann and Scheinkman (1991), Knez, Litterman and Scheinkman (1994), Moraleda and Vorst (1996) and Buhler et al. (1999). Below we will further discuss estimation of these models.

By analyzing these two classes of models, we can first of all determine which specification gives the best description of the *variances* of forward interest rates. Hereby we extend Amin and Morton (1994), by also analyzing models based on PCA-estimates. Furthermore, by analyzing PCA models with either one, two and three factors, we can analyze which specification gives the best description of the *correlations* between forward interest rates of different maturities.

If one takes the mentioned models literally, the parameters in the specifications above should be constant over time, at least if the model under consideration is the correct model. Then one could use, for example, the Generalized Methods of Moments (see Hansen (1982)) using the price restrictions for caps and swaptions to estimate the parameters and test the overidentifying restrictions. This is exactly the procedure followed by Flesaker (1993). Following Amin and Morton (1994) and Buhler et al. (1999), we use a different estimation approach: both in the cases of option-based estimation and interest-rate-based estimation, we do not restrict the parameters to be constant over the entire valuation period from 1995 to 1999, as there is some evidence for time-varying interest rate volatility in the literature[6]. One could try to model time-varying volatility using the stochastic-volatility approach (Hull-White (1987)). However, the specification and estimation of such models is more difficult than for the models proposed here. For example, in stochastic volatility models, one must specify the risk premium associated with the stochastic volatility. Also, as noted by Amin and Morton (1994), using a constant volatility model with market-implied or time-varying volatility parameters is a good approximation of a stochastic volatility model, as long as the options that are analyzed are not far from at-the-money. This is because the valuation formula for an at-the-money option price in a stochastic volatility model has a similar form as the valuation formula for an option price in a constant volatility model, with the constant volatility parameter replaced by the expected volatility over the lifetime of the option, as shown by Hull and White (1987).

4.1 Interest-Rate-Based Estimation of Volatility Functions

We will use the term interest-rate-based estimation for all estimation strategies that are based on interest rate data only. There are several possibilities to estimate the volatility functions using historical interest rate data. In principle, one would like to use the most efficient estimator for each model. However, for HJM models, that are in principle not stationary, a maximum likelihood estimator is not easily constructed. Then, because we have a particular application of the models in mind, namely the pricing and hedging of caps and swaptions, for which variances and covariances of interest rates are important determinants, it is natural to consider estimators that somehow use the information in variances and covariances of interest rates.

Above, we argued why we want to use parameter estimates that vary over time. For the interest-rate-based estimation methods, we therefore use a rolling horizon for the data-period that is used for estimation, to account for the time-varying behaviour of interest rate volatility. In line with Buhler et al. (1999), we use a rolling horizon of 9 months[7].

For the model-class (II), we use principal components analysis to estimate the functions $g_1(T-t)$, $g_2(T-t)$ and $g_3(T-t)$. The approach uses the fact that for Gaussian HJM models, the covariance matrix of instantaneous forward rate changes is given by

[6]See e.g. Ball and Torous (1999).

[7]We have also used exponential smoothing to give recent observations a higher weight. This does not improve the pricing of caps and swaptions in general. This is in contrast with the results of Bali and Karagozoglu (1999) for Eurodollar futures options.

$$Cov[df(t,T_i), df(t,T_j)] = \sum_{k=1}^{K} \sigma_k(T_i - t)\sigma_k(T_j - t)\,dt. \tag{8}$$

By approximation, this relationship also holds for forward rate changes over small time periods, in our case weekly changes[8]. We choose a finite number of forward rate maturities, construct a covariance matrix of forward rate changes for these forward rate maturities and determine the first three principal components of this covariance matrix. This renders estimates of the volatility functions $g_i(T-t)$ at the forward rate maturities that are used, and we linearly interpolate between these points to obtain the entire volatility function. This approach implies that the volatility function of the one-factor model is the same as the volatility function of the first factor of a two- or three-factor model. We use a set of 3-month forward rates with forward rate maturities from 0 up to 11 years, with quarterly intervals. We construct weekly changes for these forward rates, and consequently perform principal components analysis on the covariance matrix of these forward rate changes of the 9 months prior to the valuation date.

For the models with parametric volatility functions in specification (I), a principal component analysis is not directly applicable. Therefore, we choose a different approach that is approximately based on the same information as used with principal components analysis. More precisely, we use the generalized method of moments (GMM, Hansen (1982)) to estimate the parameters of the volatility functions, using both variances and covariances of forward rate changes as moment restrictions, which are given in equation (8). More precisely, the following moment restrictions are used: the variances of forward rate changes with forward maturities of 3 months, 1, 3, 5, 7 and 10 years, and the covariance of the change in the forward rate with 3-month forward maturity with the changes in forward rates with forward maturities of 1, 3, 5, 7 and 10 years. This yields 11 moment restrictions. Again, a rolling horizon of 9 months is used for this GMM estimation.

4.2 Option-Based Estimation of Volatility Functions

A different way to estimate the parameters of the volatility functions of the different models is to use the cross-section of derivative price data. We shall call this estimation strategy option-based estimation. As opposed to the interest-rate-based estimation method, that is in some sense backward-looking, the option-based estimated parameters reflect the market expectations that are present in option prices and are thus forward-looking.

For the models with parametric volatility functions in (I), we estimate the parameters by minimizing the sum of squared differences between observed prices of caps and swaptions and the prices for caps and

[8]This approximate relationship is only exact if the drift of forward rate changes is equal to zero. For weekly forward rate changes, the drift term is very small relative to the volatility of forward rate changes.

swaptions implied by the model[9]. This minimization is performed for each trading day separately, and this can thus lead to parameter estimates that differ from day to day.

For the models with PCA volatility functions, option-based estimation is less trivial. To obtain a parsimonious specification of the volatility function and facilitate the interpretation of the factors, we choose to maintain the shape of the volatility functions, as estimated with principal components analysis based on the last 9 months. Thus, the volatility function of each factor $g_i(T-t)$ is multiplied by a single parameter λ_i. These parameters are then chosen to minimize the weighted sum of squared pricing errors at each trading day. Hence, the shape of each factor volatility function is the same as for interest-rate-based estimation, and only the volatility of the factor itself can be different for option-based estimation.

5 Estimated Volatilities and Correlations

5.1 Interest-Rate-Based Estimation

In table 5, we provide information on the parameter estimates and in figure 2 we graph the volatilities of forward rates implied by the models. The parametric models are statistically not rejected based on these moment restrictions, which is not surprising, given the small sample of 39 weeks that is used for each GMM estimation. For the Hull-White model, the average estimate for the mean-reversion parameter γ_2 is small and negative. This is the result of the humped shape of the variance of forward rate changes. Given a humped volatility structure, a low volatility parameter γ_1 and a negative estimate for the mean-reversion parameter γ_2 will roughly give the same fit as a high volatility parameter and a positive mean-reversion estimate. This is confirmed by the parameter estimates for the Mercurio-Moraleda model, which are such that the volatility structure for this model is humped shaped, as shown in figure 2. In this figure, it is also shown that the humped shape for the forward rate volatilities in the Mercurio-Moraleda model is very different from the flat shapes of the Ho-Lee and Hull-White models.

In figure 1, we plot the average over time of the volatility functions for the first three factors. The shapes for these three factors can be interpreted as *level*, *steepness* and *curvature*. These shapes are also found by, for example, Littermann and Scheinkmann (1991). In table 6, some summary statistics of the estimated volatility functions are given. As we use a rolling horizon, the estimated volatility functions change weekly, but the shapes of these volatility functions are quite constant over time. On average, the first three factors explain about 97.8% of the variation in forward interest rates. The first factor explains 83.7%, the second factor 10.1% and the third factor 4.0%. In figure 2 we graph the volatilities of forward

[9]Because we observe much more swaption than cap prices, namely 56 vs. 7, we put a higher weight on caps than swaptions. To be precise, all swaptions with a fixed option maturity (there are 9 different option maturities) have a total weight of one, and each cap has a weight of one.

rates for different forward rate maturities, which again reveal a humped volatility structure. The shape of the hump is however very different from the humped shape that is implied by the Mercurio-Moraleda model. Also, in figure 3 we plot the correlation of a spot 3-month interest rate with forward rates of different maturities, for the two- and three-factor PCA models. This graph shows that the difference between the correlations of the two- and three-factor models is quite large. Hence, although the third factor only explains 4.0% of the total variation in forward rates, it strongly affects correlations between interest rates.

5.2 Option-Based Estimation

For the parametric models, the option-based parameter estimates are given in table 7. For the Ho-Lee model, the option-based parameter estimate is slightly higher than the interest-rate-based parameter estimate. For the Hull-White model, the mean-reversion parameter is now slightly positive on average, although the average is not significantly different from zero. For the Mercurio-Moraleda model, all average parameter estimates are significantly different from zero. In most cases, the standard deviations of the time-series of parameter estimates are a little higher for option-based estimation, compared to interest-rate-based estimation.

In figure 2, we plot the average forward rate volatilities for the three models. The Ho-Lee and Hull-White model again imply (almost) flat term structures of volatility, whereas the Mercurio-Moraleda model again implies a humped shaped volatility curve. For option-based estimation, the shape of the hump is however very different from the hump implied by interest-rate-based estimation. In particular, option-based estimation leads to much lower estimates for the volatilities of long-maturity forward rates than interest-rate-based estimation.

In table 8 we present the estimates for the parameters that multiply the factor volatility functions $g_i(T-t)$, for both the one-, two- and three-factor PCA models. It follows that the average estimates for the first, second and third factor are significantly different from zero. The average parameter estimates are not very far from one, which is the value that corresponds to interest-rate-based estimation. The parameter estimates for the first factor are in all cases not very volatile. However, the time series of parameter estimates for the second and third factor have a much larger standard deviation. Because the first factor primarily determines volatilities of interest rates, whereas the second and third factor mainly change the correlations between interest rates, it follows that the option-based estimates for the correlations are less stable over time than the option-based volatility estimates.

In figure 2, the average forward rate volatilities are graphed for the option-based estimated PCA models. For the interest-rate-based estimates of the PCA models, the volatilities of forward rates increase with the number of factors. For the option-based estimates, this is not necessarily the case, and the forward rate volatilities for the one-, two- and three-factor model are quit close to each other. Also, the option-based estimates are on average lower than the interest-rate-based estimates. Again, the shape of the volatility hump implied by the PCA models is very different from the hump implied by the Mercurio-Moraleda model, because the long-maturity forward rates have much higher volatilities in the PCA models.

In figure 3, we plot the average correlation of the 3-month spot interest rate with forward rates of different forward maturities, for the two- and three-factor models. The differences between the correlations implied by the two- and three-factor models are large. Also, the option-based correlation estimates are almost always lower than the interest-rate-based estimates. Hence, the correlations implicit in swaption prices are on average lower than the estimates from historical interest rate data. For the three-factor model, the correlation structure does not look very plausible, because one would expect that correlations decrease if the difference in maturity between the two forward interest rates increases.

6 Conditional Prediction of Derivative Prices

6.1 Comparison of Models

To determine the value at risk of a portfolio in interest rate derivatives, one needs to determine the probability distribution of the underlying term structure at future dates, and given each of these possible term structures, the values of the interest rate derivatives must be determined. In this section, we will focus on the second step in this risk calculation procedure, and we will denote this as the conditional prediction of derivative prices. This procedure is based on Amin and Morton (1994), who refer to this as pricing options with lagged volatility.

To measure how well a given model conditionally predicts derivative prices, our procedure is as follows. First, at day 1, we estimate the parameters of a model given information up to this day, either from historical interest rate movements or the cross-section of derivative prices. Then, at day $1+\tau$, we value the caps and swaptions using these parameter values and the term structure at day $1+\tau$ and compare the implied prices of the caps and swaptions with the observed prices. This procedure is then repeated for all days in the dataset. We will choose τ equal to 10 days, reflecting the risk horizon often used in bank risk management.

Notice that this provides a fair comparison between the option-based estimation method and interest-rate-based estimation method, because we compare the out-of-sample fit of derivative prices. If we would compare the fit on derivative prices at the same day as the day at which parameters are estimated (day 1), option-based estimated models would always have a better fit than interest-rate-based estimated models. Also, if there are measurement errors in the derivative price data, and if these measurement errors are uncorrelated over time, analyzing conditional predictions enables us to detect whether models are overfitted to noise. For completeness, we also present the pricing results for caps and swaptions at the estimation day (τ equal to zero) in tables 9 and 10.

In table 11, we present the prediction results for caps. Almost all models underprice caps on average. The three-factor PCA model, combined with option-based parameter estimation, has the lowest absolute prediction errors, which are on average around 8%, and this average is significantly different from zero. These sizes of percentage pricing errors are smaller than those reported by Buhler et al. (1999) and Amin and Morton (1994). For each model, option-based estimation leads to lower absolute prediction errors than interest-rate-based estimation. The differences in prediction errors between option-based and interest-rate-based estimation are in most cases statistically significant, but economically not very large: the difference in average absolute prediction errors is largest for the Hull-White model, and equal to 3.7% of the price. In general, not all implied estimated models outperform the interest-rate-based estimated models. The interest-rate-based estimated PCA models outperform both the interest-rate-based and option-based estimated Ho-Lee and Hull-White models. This result is caused by the fact that these latter two models are not able to provide a humped volatility structure.

In table 12, we give the results of a pairwise comparison of the models, on the basis of cap prediction errors. We compute the differences of absolute prediction errors of each pair of models and test whether the mean of this difference is equal to zero. It follows that the three-factor PCA model, combined with option-based estimation, has significantly lower prediction errors than all other models. For the subset of interest-rate-based estimated models, the two-factor PCA model has the lowest prediction errors, but the difference with the three-factor PCA model is not large and also not significant.

In figure 4a, we plot the average and average absolute cap prediction errors for the three-factor PCA model and option-based estimation. It is clear that there are maturity effects in these pricing errors. The 1-year cap is overpriced, all other caps are underpriced. The average absolute size of the prediction errors is almost constant over all caps. Therefore, all caps contribute to the significant mispricing of caps reported in table 11.

In tables 13 and 14, we give the prediction results for swaptions. For the interest-rate-based estimated models, the one-factor Ho-Lee model has the lowest prediction errors. The option-based estimated models all statistically outperform the interest-rate-based estimated models for swaptions, and the difference in prediction errors is quite large for all models, and much larger than the differences that were found for caps. In case of option-based estimation, the one-factor Hull-White model has the lowest absolute prediction errors, which are on average equal to 8.5%. The difference in prediction errors with the other models is significantly different from zero, except for the three-factor PCA model. In fact, the Hull-White model, the Mercurio-Moraleda model and the three-factor PCA model have prediction errors that are very close to each other.

The fact that a model that does not contain a humped volatility structure comes out as best for swaptions, implies that the humped volatility structure is much less present in the prices of swaptions, which also follows from the fact that the implied estimated Ho-Lee model yields smaller prediction errors than the one-factor PCA model.

In figures 4b and 4c, we plot the prediction errors of the three-factor PCA model. It follows that swaptions with short option or short swap maturities have the highest prediction errors. Also, swaptions with short swap maturities are largely overpriced on average.

For the joint prediction of cap and swaption prices, the three-factor PCA model, combined with option-based estimation, has the best performance, as it is the only model that is not significantly outperformed by any other model in predicting cap or swaption prices, whereas the model outperforms any other model either in predicting cap prices or predicting swaption prices, or both.

Finally, we note that for most models we find average underpricing of caps and average overpricing of swaptions. A priori, one would expect such a result only for one-factor models, because swaption prices are determined by a combination of interest rates that are not perfectly correlated, whereas cap prices are determined only by variances of single interest rates. Hence, a one-factor model will most likely underestimate variances and overestimate covariances of interest rates, leading to overpricing of swaptions and underpricing of caps. However, our results for the multi-factor models indicate that this explanation only holds to a small extent, as the multi-factor models also on average underprice caps and overprice swaptions. Future research has to indicate whether other models can explain this under- and overpricing, or whether this effect can be exploited to construct trading strategies to gain abnormal returns.

6.2 Volatility and Correlation Effects

Above, we analyzed for both one-factor and multi-factor models the prediction of cap and swaption prices. In this subsection, we will further analyze the difference between one-factor and multi-factor PCA models, and decompose this difference into differences in volatility structures and differences in correlations.

If extra factors are added to the one-factor PCA model, two things change, namely the volatility of forward rates and the correlations between these forward rates. It is important to decompose the effect (on cap and swaption prices) of adding a factor into these two components, because the real added value of an extra factor is determined by the size of the correlation effect. This is because, for the class of Gaussian HJM models, the volatilities of forward rates that are implied by a multi-factor model can in all cases also be obtained from a one-factor model. If we take the multi-factor model given in equation (3), the one-factor model that has the same volatilities of forward rates as the multi-factor model in (3) is given by

$$\begin{aligned} df(t,T) &= \mu(t,T)\,dt + \nu(T-t)\,dZ(t), \\ \nu(T-t) &= \sqrt{\sum_{i=1}^{K} \sigma_i(T-t)^2} \end{aligned} \qquad (9)$$

where *Z(t)* is a standard Brownian Motion. Because we focus on the PCA models, we refer to the model in (9) as the one-factor PCA model with K principal components. The only difference between models (3) and (9) is the correlation structure between forward rates. Hence, to identify the correlation and volatility effects, we compare three models. The *volatility effect* of adding a principal component is measured by the difference between the one-factor model with one principal component and the one-factor model with two or three principal components. The *correlation effect* is defined by the difference between the one-factor PCA model with two (or three) principal components and the two-factor (or three-factor) PCA model.

It is important to note that we do not re-estimate the parameters γ in model class (II) for the one-factor PCA models with multiple principal components. This implies that, both for option-based estimation and interest-rate-based estimation, the one-factor model with two (or three) principal components generates exactly the same volatilities of forward rates as the two-factor (or three-factor) model. This in turn implies that the prediction errors for caps are exactly the same for these two models, because cap prices only depend on forward rate volatilities. The only difference between these models will thus be the pricing of swaptions.

In table 15 we present the average absolute prediction errors for swaptions. If we increase the variance of forward rates in a one-factor model (by adding principal components to the one-factor model), the absolute prediction errors for swaptions increase. This is because in the one-factor model with one principal component, swaptions are already overpriced, as shown in table 13, so that increasing the variance of forward rates in a one-factor model leads to even higher overpricing of swaptions. Hence, the volatility effect is negative for swaptions. This negative volatility effect is compensated by the correlation effect, which is around 2.2% of the price for two-factor models and around 3.7% of the price for three-factor models, in case of option-based estimation. In other words, the absolute pricing errors decrease with 3.7% of the price, if we allow interest rates to be non-perfectly correlated, while keeping the variance of forward interest rates at the same level. In case of interest-rate-based estimation, the correlation effect is somewhat smaller. This is consistent with the results in figure 3, where it is shown that option-based estimation leads to lower correlations than interest-rate-based estimation.

7 Accuracy of Hedging Caps and Swaptions

Besides the pricing of interest-rate derivatives, a second standard application of term structure models is the hedging of derivative instruments. In this section, we empirically investigate the size of hedging errors of simple delta-hedging strategies, and we analyze the differences between the hedging errors of the several models. In particular, we focus again on the differences between one-factor and multi-factor models.

The setup is as follows. At each trading day, we have estimated a model using either option-based or interest-rate-based estimation. We calculate for each cap and swaption the deltas, implied by the model under consideration, with respect to certain hedge instruments, and construct for each derivative instrument a delta-hedged portfolio. Then, we compute the change in the value of this hedge portfolio after 10 trading days, using the observed prices for the hedge instruments and the derivatives[10]. This procedure is repeated at each trading day. This gives us 230 (partly overlapping) time-series observations on hedging errors, for each cap and swaption. We measure the accuracy of a hedging strategy by calculating for each cap and swaption the ratio of the standard deviation of these 230 hedging errors and the standard deviation of 10-day changes in an unhedged investment in the particular derivative instrument. This latter standard deviation is model independent. The ratio of standard deviations thus measures how much of the variability in the derivative instrument is removed by a delta-hedging strategy. If we would hedge continuously using the correct model, the hedge-portfolio would have zero variance. Because we only hedge discretely, and because the models we analyze are approximations to reality, we observe a positive hedging error variance.

We implement two hedging strategies, factor hedging and tenor hedging. *Factor hedging* is based on the fact that in an *N*-factor model, *N* different instruments (and the money-market account) are theoretically sufficient to exactly replicate every derivative instrument in continuous time. Furthermore, the same *N* instruments can be used for all derivatives. If the model correctly describes interest rate movements, the choice of hedging instruments is irrelevant. We choose a zero-coupon bond with 6 months maturity as the hedge instrument for all one-factor models, zero-coupon bonds with maturities of 6 months and 10 years for all two-factor models and zero-coupon bonds with maturities of 6 months, 3 years and 10 years for all three-factor models. This choice of instruments is inspired by our results for the PCA models and the fact that (i) the first factor of an interest rate model is often chosen to be related to the short interest rate, (ii) the second factor is often associated with the spread between a long and short maturity interest rate, and (iii) the third factor is related to the curvature of the term structure of interest rates (see, for example, Andersen and Lund (1997), Longstaff and Schwartz (1992) and Boudoukh et al. (1997)). Canabarro (1995) examines hedging accuracy for one-factor models in simulated two-factor economies, using a single instrument to hedge all interest rate options. He finds that, in a two-factor economy, 'the hedging accuracy of extended one-factor models is poor'.

For factor hedging, the hedge instruments are the same for all caps and swaptions, and depend only on the number of factors in the model. For *tenor hedging* the converse is true; the hedge instruments are different for each derivative, but independent of the number of factors of the model. The hedge instruments for each cap or swaption are zero-coupon bonds with maturities that correspond to all *tenor dates* relevant to the particular derivative. For example, for a 2-year cap, that consists of 7 quarterly caplets, we use as hedge instruments zero-coupon bonds with maturities of 3 months, 6 months, ..., 2 years. For a 1-year option on a 5 year swap with yearly payments, we use zero-coupon bonds with maturities of 1 year, 2

[10]We observe the prices of at-the-money caps and swaptions at each trading day, so that after 10 trading days, we do not observe the price of a cap or swaption with the at-the-money strike rate of the starting date. To be able to calculate the price of a cap or swaption after these 10 trading days, we assume that there is no implied Black volatility smile, i.e. we assume that the observed implied Black volatility for a cap or a swaption is the same for all strike rates. As the deviations from the at-the-money strike rate in 10 days are not very large, this assumption seems reasonable and not in favour of any particular model.

years,...., 6 years as hedge instruments. In the appendix, we give the formulas that lead to the hedge ratios that we use for factor and tenor hedging.

Thus, factor hedging represents a hedge strategy that makes use of as few instruments as formally necessary, whereas tenor hedging probably comes more close to hedging as it is performed in practice for large books of derivative instruments. Of course, other choices for the hedge instruments are possible, but for almost all other choices the number of hedge instruments will lie between the number of hedge instruments of factor hedging and the number of instruments of tenor hedging.

In table 16, we present the results of the factor hedging strategy. For factor hedging, the number of factors is of great importance; for the three-factor PCA model we find that factor hedging leads to a reduction in standard deviation of around 65% for both caps and swaptions, whereas the one-factor models lead, at best, to a standard deviation reduction of 46% for caps and 36% for swaptions. Hence, because of non-parallel movements of the term structure, it does not suffice to use only one hedge instrument to hedge all possible term structure movements. This result is in line with the results of Canabarro (1995). The difference between the interest-rate-based and option-based estimated models is not very large in all cases, although the overall hedging results for option-based estimation are slightly better.

If we focus on the results for one-factor models, the results also show that for caps a model with a humped volatility structure performs a little better than models with constant or declining volatility functions, whereas for swaptions the converse is true. This is in line with the results of the conditional prediction in the previous section.

In figure 5, we graph the ratios of standard deviation for all caps and swaptions. For caps, adding the 10-year bond as hedge instrument improves the hedge results for the long-maturity caps, and consequently adding the 3-year bond to the hedge instruments improves the hedge for caps with intermediate maturities, as could be expected. For swaptions, a similar result is present. For the one-factor PCA model, with the 6-month bond as hedge instrument, the reduction in variability is largest for swaptions with short total maturities. Adding the 10-year bond as hedge instrument improves the hedge results for swaptions with long total maturities, and, consequently, adding the 3-year bond reduces primarily the variance of hedge portfolios for swaptions with intermediate maturities.

For all models and all instruments, there is still considerable variation left in the hedge portfolio. Besides the fact that we only hedge discretely, one important cause of the non-zero standard deviation is the vega-effect; after 10 days the (implied) volatility of the cap or swaption has changed. To calculate how large this vega-effect is, we have recalculated the hedge results, but now we value the portfolio after 10 days (the end of the hedging period) using the same implied Black volatilities that were used to value the portfolio at the beginning of the hedge. It follows from this analysis that, on average, the reduction in standard deviation is for caps 10% higher and for swaptions 6% higher, than without this vega-correction. In tables 2 and 4 can be seen that, on average, the volatility of implied cap volatilities is larger than the volatility of implied swaption volatilities, which is thus consistent with our findings for the size of the vega-effect.

In table 17 we give the results for tenor hedging. In contrast to factor hedging, there are hardly any differences between the models and estimation methods, although the variance reduction is a little larger for interest-rate-based estimated models. It turns out that the hedge ratios for tenor hedging, which are given in equation (A.1) and (A.6) in the appendix, are not very sensitive to changes in parameter values. For example, doubling the Ho-Lee parameter from 0.01 to 0.02 on a yearly basis (i.e., doubling the volatility of all forward rates), leads to an average change in the hedge ratios of less than 1%. For the hedge ratios that correspond to factor hedging, such a change in the parameter value leads to changes in hedge ratios between 5% and 20%. Thus, we conclude that the choice of the number of hedge instruments is more important than the particular model and estimation method that is used.

In figure 5, we give the tenor hedging results per cap and swaption, for the option-based estimated 3-factor PCA model. The results for the other models are quite similar. For the 7-year and 10-year cap, factor hedging with a three-factor model leads to larger variance reductions than the tenor hedging method, which implies that tenor hedging leads to almost no variance reduction for caplets with long maturities. This can be explained as follows. With tenor hedging, a caplet with a maturity of 9.75 years on a 3-month interest rate is hedged with two zero-coupon bonds with maturities of 9.75 and 10 years, by taking a long position in one bond and a short position in the other. However, as the difference between these two bond prices is essentially the 3-month forward rate with a forward maturity of 9.75 years, and because long-maturities forward rates are estimated (using equation (7)) with quite a high error variance, a large part of the movements in this hedge portfolio is estimation error and thus uncorrelated with movements in the derivative price. As the hedge instruments that are used for factor hedging are more robust to this estimation error, factor hedging outperforms tenor hedging for long-maturity caps. Still, for all caps tenor hedging leads to hedge reductions that are, on average, a little larger than for factor hedging.

For swaptions, tenor hedging largely improves the hedge results in comparison to the hedge results for factor hedging with a three-factor model, indicating that three hedge instruments are not sufficient to hedge all swaptions accurately. In figure 5, it can be seen that for swaptions with long option maturities, estimation error for the underlying instruments plays a role, although the effect is less strong than for caps.

Finally, we also calculated the size of the vega-effect for tenor hedging, and the size of this effect is around 16 percent points for caps and 13 percent points for swaptions. In other words, the ratios of standard deviations in table 17 are 16% and 13% lower for respectively caps and swaptions, if we correct for the vega-effect.

8 Concluding Remarks

The goal of this paper is to provide an empirical analysis and comparison of several one-factor and multi-factor term structure models, and, in particular, to analyze the importance of a term structure model with multiple factors for the pricing and hedging of interest rate derivatives. In contrast to most previous research, we have used data on derivative instruments to compare the models, namely caps and swaption prices. Because of the large variety in option maturities and swap maturities, the prices of these instruments potentially contain much information.

We compare the models by analyzing for each model the prediction of caps and swaption prices and the accuracy of hedging caps and swaptions. It has been claimed in the literature that, for the pricing of swaptions, one-factor models are too restrictive because they imply perfect correlation between interest rates of different maturities. We find that, on average, a three-factor model gives the best prediction of cap and swaption prices, but the differences with one-factor models are both economically and statistically not very large. We also find that parameter estimation on the basis of option prices, as applied by Amin and Morton (1994), leads to better prediction of derivative prices than estimation on the basis of interest rate changes, which is done by Buhler et al. (1999). In almost all cases, caps are underpriced by both one-factor and multi-factor models, whereas swaptions are overpriced by these models. This over- and underpricing is persistent over almost 5 years of derivative price data.

A hedging analysis reveals that, if one uses as many hedge instruments as the number of factors in the model, which are the same for all caps and swaptions, we find large differences between the one-factor and multi-factor models; the reduction in derivative price variability due to delta-hedging with the three-factor model is almost twice as large as for the one-factor models. However, if we use for every cap and swaption a set of hedge instruments that covers all maturities at which the cap or swaption pays out, the differences between the one-factor and multi-factor models disappear. Hence, the choice for the number of hedge instruments and the maturities of these hedge instruments is more important than the particular model choice.

An obvious extension of this work would be to explicitly include stochastic volatility in the setup of the model (see e.g. Andersen and Lund (1997) and Andreasen, Dufresne and Shi (1996)). In this paper, we implicitly include stochastic volatility by re-estimating the models at each trading day. It would be interesting to analyze whether a rich enough stochastic volatility model is able to generate prices for caps and swaptions that are as accurate as the prices generated by the models in this paper, without re-estimating this model daily.

Appendix: Factor and Tenor Hedging

In this appendix we discuss the hedging of swaptions in Gaussian time-homogeneous HJM models. As a caplet is an option on a one-period swap, this discussion also applies to caps. The results in this appendix are all derived by Brace and Musiela (1994).

In equations (5) and (6), the price of a payer swaption, that gives the right to enter a swap at time T with fixed rate k, where the swap has payment dates $T_1, T_2,.., T_n$, is given. To simplify notation we assume here that the payment dates are equally spaced and define the daycount fraction as the time between payment dates $\delta = T_{i+1} - T_i$. We also define $C_i = k\delta,\ i=1,....,n-1$ and $C_n = 1+k\delta$. For the case of Gaussian HJM models, Brace and Musiela (1994) derive for the partial derivatives of a swaption price $V(t)$ with respect to zero-coupon bond prices $P(t,T)$ that

$$\begin{aligned} \frac{\partial V(t)}{\partial P(t,T)} &= N_K(A^c), \\ \frac{\partial V(t)}{\partial P(t,T_i)} &= -C_i N_K(A^c + \gamma_i) \end{aligned} \tag{A.1}$$

where $\gamma_i,\ i=1,...,n$ are defined in equation (6), and where

$$\begin{aligned} N_K(B) &= \int_B \phi_K(x)dx, \\ A &= \{x \in \mathbb{R}^K : P(t,T)\,\phi_K(x) - \sum_{i=1}^{n} C_i P(t,T_i)\,\phi_K(x+\gamma_i) \le 0\} \end{aligned} \tag{A.2}$$

$A^c \in \mathbb{R}^K$ is defined as the complement of A. As before, $\phi_K(x)$ is the density function of the K-dimensional standard normal distribution.

The swaption price $V(t)$ satisfies the following stochastic differential equation (under the equivalent martingale measure Q with the money market account as numeraire asset)

$$dV(t) = r(t)V(t)dt - \sum_{i=1}^{K} [\sum_{j=1}^{n} \frac{\partial V(t)}{\partial P(t,T_j)} P(t,T_j)\Sigma_i(t,T_j) + \frac{\partial V(t)}{\partial P(t,T)} P(t,T)\Sigma_i(t,T)]\,dW_i(t) \tag{A.3}$$

where

$$\Sigma_i(t,T) = \int_t^T \sigma_i(T-s)\,ds, \quad i=1,\ldots,K \tag{A.4}$$

Also, bond prices satisfy the following SDE in Gaussian time-homogeneous HJM models

$$dP(t,T) = P(t,T)\left[r(t)\,dt - \sum_{i=1}^{K} \Sigma_i(t,T)\,dW_i(t)\right] \tag{A.5}$$

Equations (A.3) and (A.5) show that the hedge-ratios in equation (A.1) can directly be used to hedge the swaption with $n+1$ zero-coupon bonds with maturities T and T_1, T_2,.., T_n, because

$$dV(t) = \ldots.dt + \frac{\partial V(t)}{\partial P(t,T)}dP(t,T) + \sum_{j=1}^{n} \frac{\partial V(t)}{\partial P(t,T_j)}dP(t,T_j) \tag{A.6}$$

This is exactly the hedge strategy of *tenor hedging*. Note also that in a K-factor model, the K Brownian Motions in equation (A.1) can be substituted by K zero-coupon bond prices of different maturities, using equation (A.6). This procedure leads directly to the hedge portfolio that is used for *factor hedging*.

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Table 1. Fit of Money-Market Rates and Swap Rates.

This table gives the results of fitting the smooth discount function in (8) to money-market rates and swap rates, both in relative terms as in basispoints, for weekly data from January 1994 until June 1999.

	Average Relative Error	Average Absolute Relative Error	Average of Daily Maximal Errors
Money Market Rates	-0.04% (-0.2 bp)	0.46% (2.3 bp)	0.77% (3.9 bp)
Swap Rates	0.02 % (0.1 bp)	0.25% (1.2 bp)	0.47% (2.4 bp)

Table 2. Statistics Cap Implied Black Volatilities.

Averages and standard deviations are calculated from 232 weekly observations on implied volatilities of caps.

Maturity in Years	Average	Standard Deviation
1	15.2%	4.3%
2	18.9%	4.9%
3	19.6%	4.5%
4	19.4%	3.8%
5	19.1%	3.3%
7	18.3%	2.7%
10	17.5%	2.4%

Table 3. Averages of Swaption Implied Volatilities.

Averages are calculated from 232 weekly observations on implied volatilities of swaptions. All maturities are expressed in years.

Maturity	1 (Swap)	2 (Swap)	3 (Swap)	4 (Swap)	5 (Swap)	7 (Swap)	10 (Swap)
0.85	16.7%	17.0%	16.8%	16.7%	16.6%	15.9%	15.3%
0.25	16.5%	16.9%	16.6%	16.4%	16.0%	15.8%	15.0%
0.50	16.5%	16.8%	16.6%	16.4%	16.2%	15.6%	14.8%
1.00	17.8%	17.2%	16.7%	16.1%	15.8%	15.2%	14.4%
1.50	17.5%	16.9%	16.3%	15.9%	15.6%	14.9%	
2.00	17.1%	16.6%	15.9%	15.7%	15.3%	14.8%	
3.00	16.7%	16.0%	15.7%	15.3%	14.9%	14.3%	
4.00	16.5%	15.8%	15.2%	14.4%	14.1%		
5.00	16.2%	15.8%	15.1%	14.6%	14.2%		

Table 4. Standard Deviations of Swaption Implied Volatilities.

Standard deviations are calculated from 232 weekly observations on implied volatilities of swaptions. All maturities are expressed in years.

Maturity	1 (Swap)	2 (Swap)	3 (Swap)	4 (Swap)	5 (Swap)	7 (Swap)	10 (Swap)
0.85	5.6%	4.1%	3.9%	3.6%	3.4%	3.0%	2.6%
0.25	4.6%	3.7%	3.4%	3.1%	2.8%	2.6%	2.2%
0.50	3.5%	3.1%	2.9%	2.6%	2.4%	2.1%	1.8%
1.00	3.0%	2.6%	2.3%	2.1%	1.9%	1.7%	1.5%
1.50	2.6%	2.3%	2.0%	1.9%	1.8%	1.6%	
2.00	2.3%	2.1%	1.9%	1.8%	1.8%	1.6%	
3.00	1.9%	1.8%	1.7%	1.6%	1.6%	1.5%	
4.00	1.7%	1.8%	1.6%	1.5%	1.5%		
5.00	1.6%	1.6%	1.3%	1.4%	1.4%		

Table 5. GMM Interest-Rate-Based Estimation Results.

The table reports average parameter estimates for 1-factor models, based on GMM with moment restrictions as described in section 5. A rolling horizon of 9 months is used. The volatility parameters are all expressed in percentages on an annual basis.

	1-Factor Ho-Lee Model	1-Factor Hull-White	1-Factor Mercurio-Moraleda
Avg. Estimate γ_1	0.975%	0.950%	0.681%
Standard Deviation γ_1	0.169%	0.254%	0.193%
Avg. t-ratio γ_1	19.11	18.56	15.32
Avg. Estimate γ_2	-	-0.009	0.191
Standard Deviation γ_2	-	0.031	0.091
Avg. t-ratio γ_2	-	2.22	2.45
Avg. Estimate γ_3	-	-	0.850
Standard Deviation γ_3	-	-	0.531
Avg. t-ratio γ_3	-	-	1.89
Avg. J-statistic (and avg. p-value)	4.15 (0.98)	4.11 (0.94)	3.90 (0.93)

Table 6. Results of Principal Components Analysis.

The table reports the average estimates and standard deviations of the factor volatility functions at certain forward rate maturities. The factor volatility functions are estimated using principal components analysis, with a rolling horizon of 9 months. The averages and standard deviations are calculated from the 282 resulting daily observations on the factor volatility functions. Parameters are all expressed in percentages on an annual basis.

Forward Maturity	Factor 1 Average	Factor 1 St. Devation	Factor 2 Average	Factor 2 St. Devation	Factor 3 Average	Factor 3 St. Devation
3 months	0.52%	0.17%	-0.18%	0.14%	0.21%	0.11%
1 year	1.12%	0.32%	-0.30%	0.25%	0.30%	0.11%
5 years	0.98%	0.14%	0.01%	0.12%	-0.15%	0.08%
10 years	0.98%	0.17%	0.37%	0.40%	0.10%	0.22%

Table 7. Option-Based Estimation Results Parametric Models.

Averages and standard deviations are taken over 282 daily estimates. The p-value refers to testing whether the average is different from zero. For the parameter γ_1 the parameter sign is not identified, implying that, if the true parameter value is equal to zero, the average has asymptotically a truncated normal distribution.

	1-Factor Ho-Lee Model	1-Factor Hull-White	1-Factor Mercurio-Moraleda
Avg. Estimate γ_1	0.993%	1.057%	0.863%
Standard Deviation γ_1	0.166%	0.322%	0.421%
p-value γ_1	0	0	0
Avg. Estimate γ_2	-	0.017	0.248
Standard Deviation γ_2	-	0.069	0.086
p-value γ_2	-	0.403	0
Avg. Estimate γ_3	-	-	0.879
Standard Deviation γ_3	-	-	0.737
p-value γ_3	-	-	0

Table 8. Implied Parameter Estimates for PCA Models.

Averages and standard deviations are taken over 282 daily estimates. The p-value refers to testing whether the average is larger than zero. If the true parameter value is equal to zero, the average has asymptotically a truncated normal distribution.

	Average	Standard Deviation	P-value
1-Factor Model, Factor 1 Parameter	0.989	0.114	0.000
2-Factor Model, Factor 1 Parameter	0.912	0.104	0.000
2-Factor Model, Factor 2 Parameter	0.996	0.665	0.000
3-Factor Model, Factor 1 Parameter	0.869	0.100	0.000
3-Factor Model, Factor 2 Parameter	0.901	0.689	0.000
3-Factor Model, Factor 3 Parameter	1.203	0.620	0.000

Table 9. Pricing Results for Cap Prices.

The table provides statistics on the cap pricing errors, which are defined as model price minus observed price, divided by the observed price. Averages are based on 232 time-series observations, for 7 caps per day.

	Interest-Rate-Based Estimation *Average*	Interest-Rate-Based Estimation *Average Absolute*	Option-Based Estimation *Average*	Option-Based Estimation *Average Absolute*
Ho-Lee Model	-8.83%	16.67%	-7.55%	14.21%
Hull-White Model	-11.90%	16.86%	-6.42%	12.70%
Mercurio-Moraleda Model	-9.79%	13.20%	-7.32%	10.26%
1-Factor PCA	-9.36%	12.28%	-11.54%	12.10%
2-Factor PCA	-4.02%	9.57%	-8.32%	9.11%
3-Factor PCA	1.28%	9.72%	-5.34%	6.83%

Table 10. Pricing Results for Swaption Prices.

The table provides statistics on swaption pricing errors, which are defined as model price minus observed price, divided by the observed price. Averages are based on 232 time-series observations, for 56 swaptions per day.

	Interest-Rate-Based Estimation *Average*	Interest-Rate-Based Estimation *Average Absolute*	Option-Based Estimation *Average*	Option-Based Estimation *Average Absolute*
Ho-Lee Model	2.22%	12.33%	3.55%	8.65%
Hull-White Model	0.63%	13.01%	2.88%	7.24%
Mercurio-Moraleda Model	15.01%	19.34%	4.33%	7.34%
1-Factor PCA	10.23%	14.27%	7.47%	9.50%
2-Factor PCA	14.14%	16.34%	5.78%	8.36%
3-Factor PCA	16.17%	17.90%	3.28%	7.23%

Table 11. 10-day Prediction Results for Cap Prices.

The table provides statistics on the 10-day conditional cap prediction errors, which are defined as model prediction minus observed price, divided by the observed price. The conditional predictions are calculated as described in the text. Averages and standard errors are based on 232 time-series observations, for 7 caps per day.

	Interest-Rate-Based Estimation *Average*	Interest-Rate-Based Estimation *Average Absolute*	Option-Based Estimation *Average*	Option-Based Estimation *Average Absolute*
Ho-Lee Model	-8.57% (2.07%)	16.61% (2.37%)	-7.01% (1.53%)	14.60% (2.05%)
Hull-White Model	-11.54% (2.05%)	16.89% (2.38%)	-5.81% (1.02%)	13.19% (1.81%)
Mercurio-Moraleda Model	-9.41% (2.09%)	13.24% (2.19%)	-6.71% (1.21%)	10.79% (1.71%)
1-Factor PCA	-8.95% (2.07%)	12.40% (2.04%)	-10.98% (1.72%)	11.92% (1.79%)
2-Factor PCA	-3.60% (1.56%)	9.82% (1.55%)	-7.74% (1.29%)	9.47% (1.41%)
3-Factor PCA	1.66% (1.55%)	10.15% (1.55%)	-4.75% (0.85%)	7.81% (1.15%)

Table 12. Pairwise Model Comparison: Predicting Cap Prices, 10-day Horizon.

Models I-VI are historically estimated models: I is Ho-Lee, II is Hull-White Model, III is the Mercurio-Moraleda model, IV, V and VI are 1/2/3-factor PCA models. Models VII-XII are impliedly estimated models, in same order as historically estimated models. The table contains t-ratios for the difference in average absolute pricing errors of pairs of models, i.e. the average absolute pricing error of model i (in row i) minus the average absolute pricing error of model j (in column j). The t-ratios are calculated allowing for a general cross-correlation structure of prediction errors, and corrected for heteroskedasticity and 10th degree autocorrelation using Newey-West (1987).

	I	II	III	IV	V	VI	VII	VIII	IX	X	XI
II	0.5										
III	-3.7	-5.8									
IV	-4.2	-4.7	-1.4								
V	-5.7	-5.7	-4.3	-3.7							
VI	-4.7	-4.6	-2.5	-1.8	0.5						
VII	-2.3	-2.1	1.1	1.7	4.2	3.9					
VIII	-3.1	-3.2	-0.1	0.6	3.1	3.2	-3.0				
IX	-4.7	-4.5	-2.0	-1.2	1.0	0.8	-5.6	-5.1			
X	-4.6	-4.2	-1.3	-0.5	2.6	1.9	-3.7	-1.7	2.1		
XI	-5.4	-5.1	-3.1	-2.3	-0.4	-0.7	-5.9	-5.2	-3.9	-3.9	
XII	-5.7	-5.7	-4.2	-3.3	-2.3	-2.6	-6.1	-6.4	-5.8	-5.0	-43

Table 13. 10-day Prediction Results for Swaption Prices.

The table provides statistics on the 10-day conditional swaption prediction errors, which are defined as model prediction minus observed price, divided by the observed price. The conditional predictions are calculated as described in the text. Results are based on 282 weekly observations, for 56 swaptions per day.

	Interest-Rate-Based Estimation *Average*	Interest-Rate-Based Estimation *Average Absolute*	Option-Based Estimation *Average*	Option-Based Estimation *Average Absolute*
Ho-Lee Model	2.29% (1.83%)	12.58% (1.93%)	3.90% (1.01%)	9.80% (1.42%)
Hull-White Model	0.76% (2.00%)	13.24% (2.03%)	3.21% (0.71%)	8.54% (1.19%)
Mercurio-Moraleda Model	15.15% (3.58%)	19.58% (3.64%)	4.69% (0.91%)	8.81% (1.34%)
1-Factor PCA	10.35% (2.46%)	14.62% (2.52%)	7.78% (1.22%)	10.54% (1.49%)
2-Factor PCA	14.27% (2.74%)	16.79% (2.79%)	6.12% (1.04%)	9.67% (1.36%)
3-Factor PCA	16.28% (2.97%)	18.34% (3.01%)	3.64% (0.71%)	8.70% (1.23%)

Table 14. Pairwise Model Comparison: Predicting Swaption Prices, 10-day Horizon.

Models I-VI are historically estimated models: I is Ho-Lee, II is Hull-White Model, III is the Mercurio-Moraleda Model, IV, V and VI are 1/2/3-factor PCA models. Models VII-XI are impliedly estimated models, in same order as historically estimated models. The table contains t-ratios for the difference in average absolute pricing errors of pairs of models, i.e. the average absolute pricing error of model i (in row i) minus the average absolute pricing error of model j (in column j). The t-ratios are calculated allowing for a general cross-correlation structure of prediction errors, and corrected for heteroskedasticity and 10th degree autocorrelation using Newey-West(1987).

j i	I	II	III	IV	V	VI	VII	VIII	IX	X	XI
II	1.6										
III	3.4	3.2									
IV	1.7	1.1	-4.1								
V	2.9	2.4	-3.0	4.8							
VI	3.4	3.0	-1.3	5.1	4.7						
VII	-2.7	-3.2	-3.9	-2.8	-3.6	-3.9					
VIII	-3.6	-4.0	-4.1	-3.3	-3.9	-4.2	-3.3				
IX	-3.4	-3.8	-4.0	-3.2	-3.9	-4.2	-2.4	2.6			
X	-2.0	-2.6	-3.6	-2.4	-3.3	-4.4	2.4	4.7	4.3		
XI	-2.8	-3.4	-3.9	-2.9	-3.6	-3.7	-0.5	3.6	2.9	-3.3	
XII	-3.5	-4.0	-4.1	-3.3	-3.9	-3.9	-3.4	0.9	-0.5	-4.7	-4.0

Table 15. Volatility and Correlation Effects.

The table contains average absolute 10-day prediction errors for swaptions, for PCA models with one-, two- and three factors. For the one-factor PCA model, three versions are presented, namely models with either one, two or three principal components determining the volatility function of this model. Then, the volatility effect is given by the difference between the one-factor model with two or three components, and the one-factor model with one principal component. The correlation effect is equal to the difference between the two-factor (or three-factor) model and the one-factor model with two (or three) principal components.

	One-Factor PCA Model	Two-Factor PCA Model	Three-Factor PCA Model
Interest-Rate-Based Estimation	*Average Absolute Swaption Prediction Error*	*Average Absolute Swaption Prediction Error*	*Average Absolute Swaption Prediction Error*
One Principal Component	14.62%	-	-
Two Principal Components	18.13%	16.79%	-
Three Principal Components	21.45%	-	18.34%
Option-Based Estimation	*Average Absolute Swaption Prediction Error*	*Average Absolute Swaption Prediction Error*	*Average Absolute Swaption Prediction Error*
One Principal Component	10.54%	-	-
Two Principal Components	11.87%	9.67%	-
Three Principal Components	12.44%	-	8.70%

Table 16. Results for Factor Hedging of Caps and Swaptions.

For each cap and swaption, the ratio of the standard deviation of 10-day changes in the hedge portfolio and the standard deviation of 10-day changes in the unhedged portfolio are calculated. The table presents the averages of these ratios over all caps and all swaptions. The hedge instrument for 1-factor models is a 6-month zero-coupon bond, for 2-factor models 6-month and 10-year zero-coupon bonds, and for 3-factor models 6-month, 3-year and 10-year zero-coupon bonds.

Model	Interest-Rate-Based Estimation *Caps*	Option-Based Estimation *Caps*	Interest-Rate-Based Estimation *Swaptions*	Option-Based Estimation *Swaptions*
Ho-Lee Model	55.46%	55.45%	64.04%	64.00%
Hull-White Model	55.52%	56.92%	64.18%	64.91%
Mercurio- Moraleda Model	55.01%	54.89%	65.58%	65.22%
1-Factor PCA	53.96%	53.90%	65.96%	65.86%
2-Factor PCA	53.86%	53.83%	58.80%	58.76%
3-Factor PCA	34.74%	34.66%	35.51%	35.44%

Table 17. Results for Tenor Hedging of Caps and Swaptions.

For each cap and swaption, the ratio of the standard deviation of 10-day changes in the hedge portfolio and the standard deviation of 10-day changes in the unhedged portfolio are calculated. The table presents the averages of these ratios over all caps and all swaptions. The hedge instruments are zero-coupon bonds that correspond to all dates that are relevant for the particular derivative, so that the number of hedge-instruments is equal to number of tenor dates of each derivative instrument.

	Interest-Rate-Based Estimation *Caps*	Option-Based Estimation *Caps*	Interest-Rate-Based Estimation *Swaptions*	Option-Based Estimation *Swaptions*
Ho-Lee Model	34.70%	34.65%	31.04%	31.02%
Hull-White Model	34.70%	34.64%	31.03%	30.98%
Mercurio- Moraleda Model	34.69%	34.62%	31.08%	31.02%
1-Factor PCA	34.66%	34.61%	31.11%	31.01%
2-Factor PCA	34.58%	34.58%	31.13%	30.99%
3-Factor PCA	34.53%	34.56%	31.14%	30.97%

Figure 1. PCA Volatility functions. Volatility functions for the first, second and third factor of the PCA models. The volatility functions are estimated using principal components analysis on 3-month forward rate changes with different forward maturities, using a rolling horizon of 9 months. The figure gives the average volatility functions over all valuation dates.

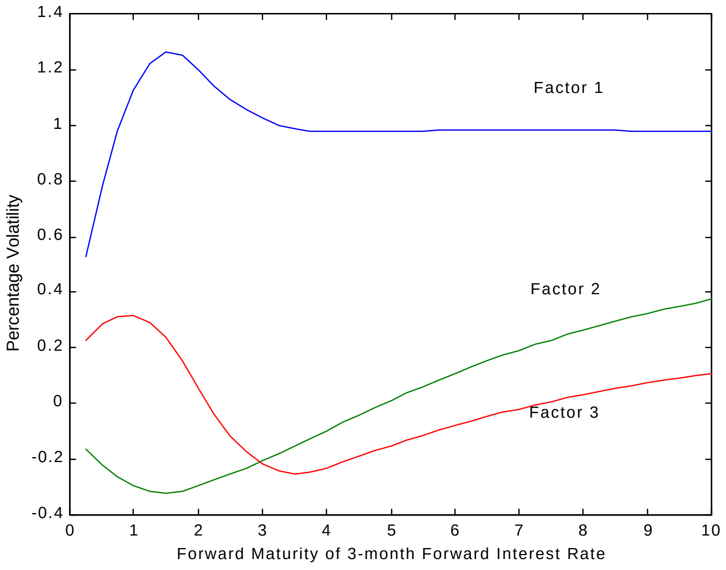


Figure 2. Estimated Forward Rate Volatilities. The plots contain the average volatilities of forward rates of different forward rate maturities implied by parametric models (left) and PCA models (right). The upper graphs correspond to interest-rate-based estimation, the lower graphs correspond to option-based estimation. The forward rate volatilities are obtained by averaging the weekly volatility estimates from January 1995 until June 1999.

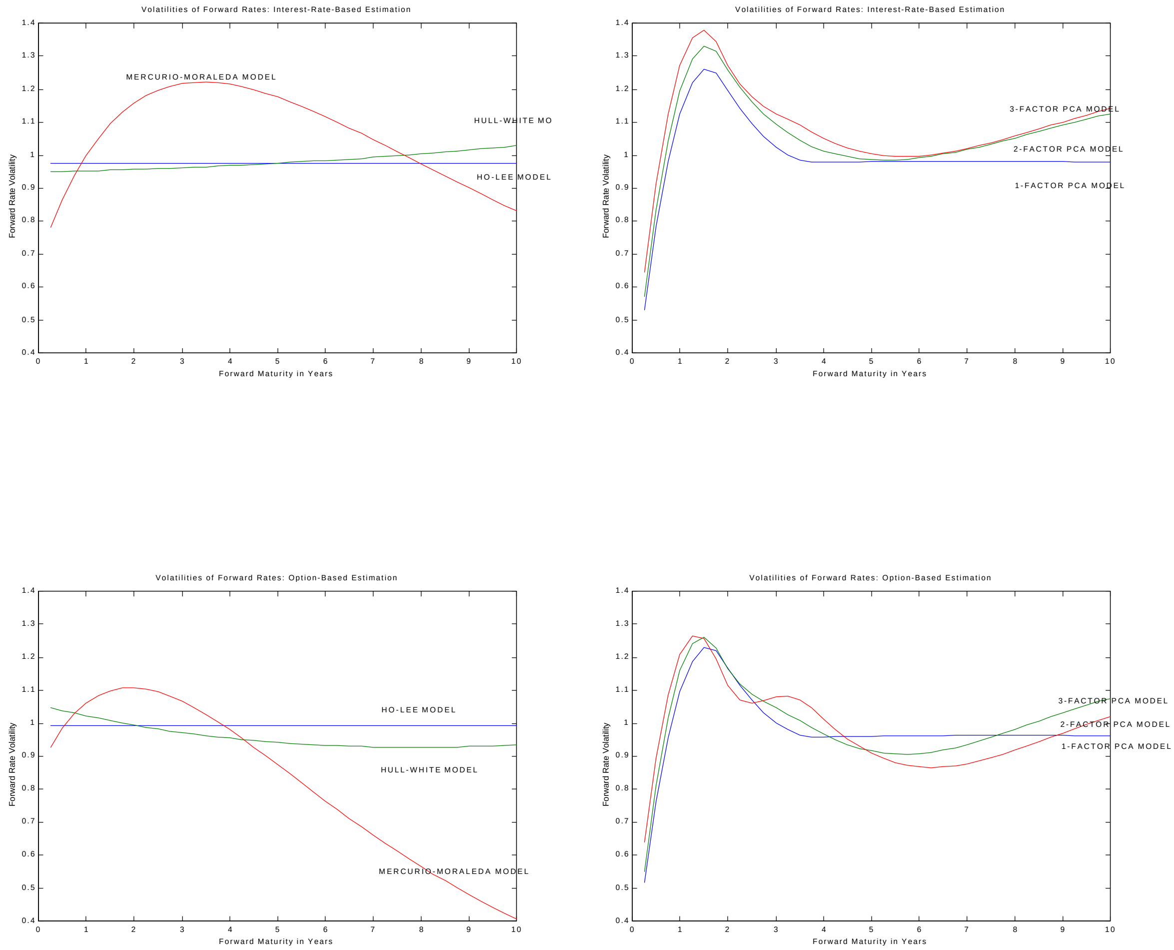


Figure 3. Estimated Forward Rate Correlations. The plot contains the average correlations of the 3-month spot interest rate with 3-month forward rates of different forward rate maturities, implied by PCA models. The correlations are obtained by averaging the weekly correlation estimates, calculated using the covariance matrix in equation (7), from January 1995 until June 1999.

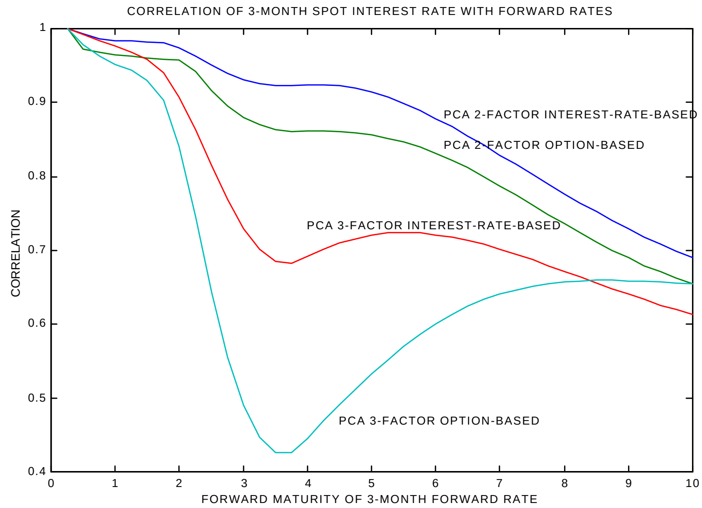


Figure 4a. Cap Prediction Errors. Average and average absolute 10-day prediction errors for caps: Three-factor PCA model with option-based estimation.

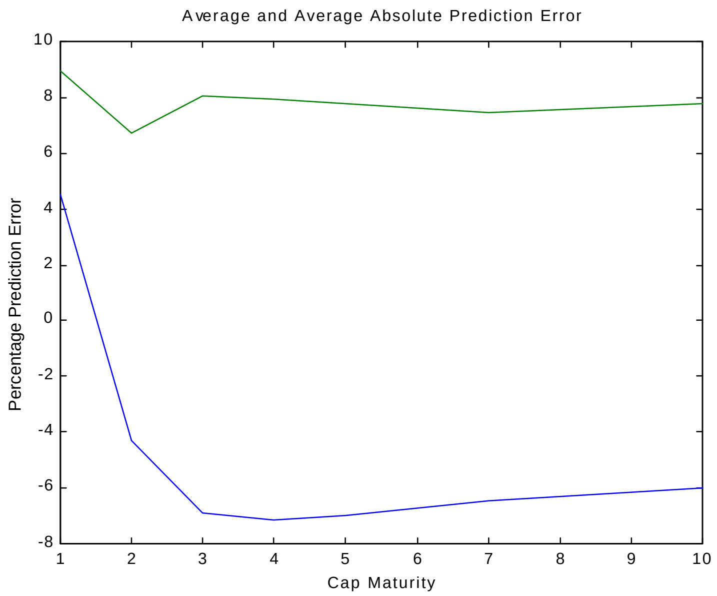
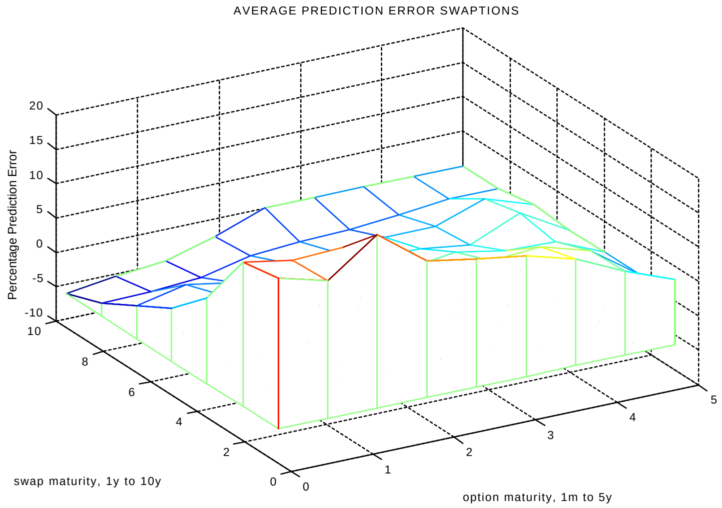


Figure 4b-c. Swaption Prediction Errors. Average and average absolute 10-day prediction errors for swaptions: Three-factor PCA model with option-based estimation.



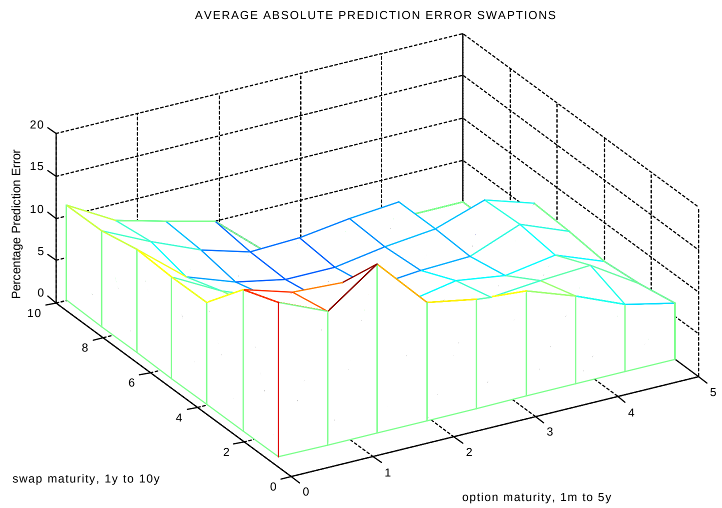


Figure 5a. Cap Hedge Results. Ratios of standard deviations of 10-day changes in hedged and unhedged portfolios, for caps and for factor hedging and tenor hedging. All results are for the one-, two- and three-factor PCA models and option-based estimation.

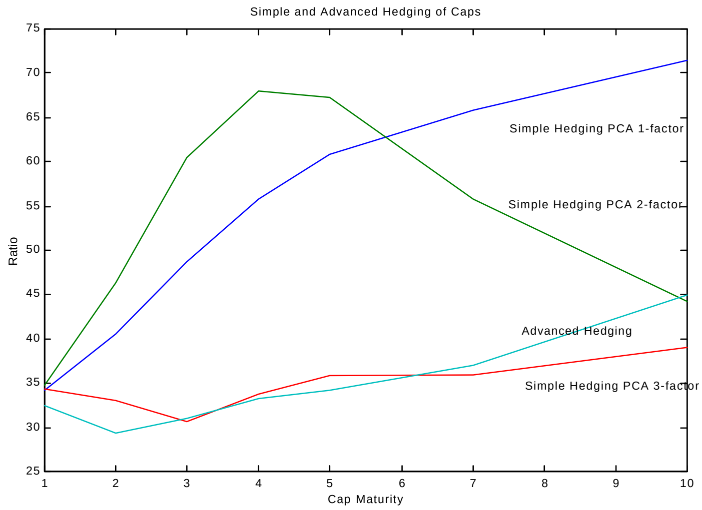


Figure 5b-e. Swaption Hedge Results. Ratios of standard deviations of 10-day changes in hedged and unhedged portfolios, for swaptions, and for factor hedging and tenor hedging. All results are for the one-, two- and three-factor PCA models and option-based estimation.