Stat 761 Winter 2009
Stochastic Processes
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Lecture 12: Martingales, Quadratic Variation, Doob Inequalities: Continuous Time

Outline

⇒ **Martingales: Definition, Examples**

⇒ **Quadratic Variation**

⇒ **Sub- and Supermartingales: Definitions**

⇒ **Doob Inequalities**

1 Martingales in Continuous Time

1.1 Filtrations

A **filtration** $(\mathcal{F}_t)_{t\geq 0}$ in continuous time is a family of σ-algebras $\mathcal{F}_t$ such that $\mathcal{F}_{t_1} \subseteq \mathcal{F}_{t_2}$, for any $0 \leq t_1 \leq t_2 < +\infty$.

The symbol $\mathcal{F}_t^X$ denotes the **information** generated by the stochastic process X_t on the interval $[0, t]$ (record, history, observations, sample path). We write $A \in \mathcal{F}_t^X$, if it is possible to decide whether a given event A has occured or not, based upon observations of the trajectory $\{X_s, 0 \leq s \leq t\}$. We say that Y_t is $\mathcal{F}_t$-**adapted** if it is $\mathcal{F}_t$-measurable.

Examples.

1. Let $A := \{X_s \leq \pi, \forall s \leq 18\}$. Then $A \in \mathcal{F}_{18}^X$, but $A \notin \mathcal{F}_{17}^X$.
2. For the event $A := \{X_{10} > 8\}, A \in \mathcal{F}_s^X$ iff $s \geq 10$.
3. The stochastic variable

$$Z := \int_0^5 X_s ds$$

is in $\mathcal{F}_s^X$ iff $s \geq 5$.

4. If W_t is Wiener process and $M_t := \max_{0\leq s\leq t} W_s$, then M is adapted to the Wiener filtration $\mathcal{F}_t^W$.

5. If W_t is Wiener process and $\tilde{M}_t := \max_{0\leq s\leq t+1} W_s$, then $\tilde{M}$ is not adapted to the Wiener filtration $\mathcal{F}_t^W$.

Consider a probability space $(\Omega, \mathcal{F}, P)$ with filtration $(\mathcal{F}_t)_{t\geq 0}$.

1.2 Martingales

An adapted family M_t of r.v. on this space with $E|M_t| < +\infty$ for all $t \geq 0$ is a **martingale** if for any $s \leq t$

$$E[M_t/\mathcal{F}_s] = M_s.$$

Here, we mean $E \equiv E_P$, expectation under measure P.

Examples.

1. Let W_t be a standard Wiener process adapted to $\mathcal{F}_t$. Then W_t is an $\mathcal{F}_t$-martingale:

$$\begin{aligned} E(W_t/\mathcal{F}_s) &= E(M_t - M_s + M_s/\mathcal{F}_s) \\ &= E(M_t - M_s/\mathcal{F}_s) + M_s = E(M_t - M_s) + M_s \\ &= 0 + M_s = M_s, \end{aligned}$$

where we've used independency increments for W_t and measurability of M_s wrt $\mathcal{F}_s$.

2. Let W_t be a standard Wiener process adapted to $\mathcal{F}_t$. Then $W_t^2 - t$ is an $\mathcal{F}_t$-martingale:

$$\begin{aligned} E(W_t^2 - t/\mathcal{F}_s) &= E(M_t^2 - M_s^2 + M_s^2 - (t-s) - s/\mathcal{F}_s) \\ &= E(M_t^2 - M_s^2/\mathcal{F}_s) + M_s^2 - (t-s) - s \\ &= (t-s) + M_s^2 - (t-s) - s = M_s^2 - s. \end{aligned}$$

3. Compensated Poisson Process. Let N_t be Piosson process with intensity $\lambda > 0$. Then $M_t := N_t - \lambda t$ is a martingale (compensated Poisson process) (we note that $EN_t = \lambda t$):

$$\begin{aligned} E(N_t - \lambda t/\mathcal{F}_s^N) &= E(N_t - N_s + N_s - \lambda t + \lambda s - \lambda s/\mathcal{F}_s^N) \\ &= N_s - \lambda s + E(N_t - N_s - \lambda t + \lambda s) \\ &= N_s - \lambda s, \end{aligned}$$

where we've used independency increments of N_t and that $EN_t = \lambda t$.

4. Compound Poisson process $L_t := \prod_1^{N_t}(1 + U_i)$ is a martingale iff $EU_i = 0$, i.e., the mean values of jumps are equal to zero.

1.3 Sub-and Supermartingales (Semimartingales)

An adapted family X_t of r.v. on this space with $E|X_t| < +\infty$ for all $t \geq 0$ is a **submartingale** if for any $s \leq t$

$$E[M_t/\mathcal{F}_s] \geq M_s.$$

Example. If M_t is an $\mathcal{F}_t$-martingale, then $X_t := M_t + at$ is a submartingale, $a > 0$.

An adapted family X_t of r.v. on this space with $E|X_t| < +\infty$ for all $t \geq 0$ is a **supermartingale** if for any $s \leq t$

$$E[M_t/\mathcal{F}_s] \leq M_s.$$

Example. If M_t is an $\mathcal{F}_t$-martingale, then $X_t := M_t - at$ is a supermartingale, $a > 0$.

2 First Variation, Quadratic Variation, Covariation (or Cross-Variation)

Let $\Pi = \{t_0, t_1, ..., t_n\}$ be a partition of $[0, T]$, i.e., $0 = t_0 \le t_1 \le ... \le t_n = T$. The mesh of the partition is defined to be

$$||\Pi|| = \max_{k=0,...,n-1} (t_{k+1} - t_k).$$

We then define the **first variation of f** to be

$$FV_{[0,T]}(f) := \lim_{||\Pi|| \to 0} \sum_{k=0}^{n-1} |f(t_{k+1}) - f(t_k)|.$$

If f is differentiable, then

$$\sum_{k=0}^{n-1} |f(t_{k+1}) - f(t_k)| = \sum_{k=0}^{n-1} |f'(t_k^*)|(t_{k+1} - t_k)$$

and

$$FV_{[0,T]}(f) = \sum_{k=0}^{n-1} |f'(t_k^*)|(t_{k+1} - t_k) = \int_0^T |f'(t)|dt.$$

In this way, the first variation of f is a length of a curve formed by $f(t)$ on the interval $[0, T]$.

The **quadratic variation** of a function f on an interval $[0, T]$ is

$$< f > (T) := \lim_{||\Pi|| \to 0} \sum_{k=0}^{n-1} |f(t_{k+1}) - f(t_k)|^2.$$

If f is differentiable, then $< f > (T) = 0$, because

$$\begin{array}{rcl} \sum_{k=0}^{n-1} |f(t_{k+1}) - f(t_k)|^2 & = & \sum_{k=0}^{n-1} |f'(t_k^*)|^2 (t_{k+1} - t_k)^2 \\ & =\le & ||\Pi|| \sum_{k=0}^{n-1} |f'(t_k^*)|^2 (t_{k+1} - t_k) \end{array}$$

and

$$\begin{array}{rcl} < f > (T) & \le & \lim_{||\Pi|| \to 0} ||\Pi|| \lim_{||\Pi|| \to 0} \sum_{k=0}^{n-1} |f'(t_k^*)|^2 (t_{k+1} - t_k) \\ & = & \lim_{||\Pi|| \to 0} ||\Pi|| \int_0^T |f'(t)|^2 dt = 0. \end{array}$$

We can see that **quadratic variation** $< B_t >$ of a Wiener process B_t is t.

Important Remark. *The paths of Brownian motion are not differentiable, since $< B_t >= t$ and does not equal to zero.*

Quadratic variation $< M_t >$ of a martingale M_t is defined as such process that $M_t^2 - < M_t >$ is a martingale.

For any two martingales M and N with continuous sample paths the **covariation (or cross-variation)** is defined as follows

$$< M, N >:= \frac{1}{4}[< M + N >_t - < M - N >_t].$$

The process $< M, N >$ is continuous and of bounded variation (difference of two nondecreasing processe). is a martingale. If M, N are independent, then $< M, N >= 0$. M, N are **orthogonal**, if $< M, N >= 0$.

3 Doob Inequalities

1. If $X_t, \quad 0 \leq t \leq T$, is a submartingale, then

$$aP(\sup_{0 \leq t \leq T} X_t \geq a) \leq E|X_T|, \quad a \in R.$$

2. If $X_t, \quad 0 \leq t \leq T$, is a supermartingale, then

$$aP(\sup_{0 \leq t \leq T} X_t \geq a) \leq E|X_0|, \quad a \in R.$$

3. If $X_t, \quad 0 \leq t \leq T$, is a martingale, then

$$aP(\sup_{0 \leq t \leq T} |X_t| \geq a) \leq E|X_T|, \quad a \in R.$$

4. If $M_t, \quad 0 \leq t \leq T$, is a martingale, then

$$E[\sup_{0 \leq t \leq T} M_t^2] \leq 4E[M_T^2].$$

Recommended Textbook: 'A First Course in Stochastic Processes' by S. Karlin and H. Taylor, Academic Press, 2nd ed., 1975.

Recommended Exerscises:

1. 3. Exponential martingale. Use the following relationship $Ee^{\sigma W_t} = e^{\frac{\sigma^2 t}{2}}$, prove that if W_t is a standard Wiener process adapted to $\mathcal{F}_t$, then $e^{\sigma W_t - \frac{\sigma^2 t}{2}}$ is an $\mathcal{F}_t$-martingale

2. Prove Doob inequality 2.