Assume that an asset price S evolves according to the SDE

$$\frac{dS}{S} = (\mu - D)\,dt + \sigma dX$$

where μ and σ are constants. In addition S pays out a continuous dividend stream equal to $DS\ dt$ during the infinitesimal time interval dt, where D the dividend yield is constant.

Now suppose a European option is written on this asset with the properties that at expiry the holder receives the asset and prior to expiry the option pays a continuous cash flow $C(S,t)\ dt$ during each time interval of length dt. The value V of the option satisfies the following Black-Scholes equation

$$\begin{aligned} \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D)\,S\frac{\partial V}{\partial S} - rV &= -C(S,t)\,, \\ V(S,T) &= S \end{aligned}$$

Suppose that $C(S,t)$ has the form $C(S,t) = f\ (t)\,S$. By writing $V = \phi(t)\,S$ find an expression for $V(S,t)$, and hence show that the delta of the derivative security is

$$\Delta(S,t) = \exp(-D(T-t)) + \int_t^T \exp(-D(\tau - t))\ f\ (\tau)\,d\tau$$

$\boxed{\text{Solution:}}$

Writing $C(S,t) = f\ (t)\,S$ gives

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D)\,S\frac{\partial V}{\partial S} - rV = -f\ (t)\,S$$

and we now use the transformation $V = \phi(t)\,S$ to convert to an ode which is a function of t alone.

$$\frac{\partial V}{\partial t} = \phi'(t)\,S\ ;\quad \frac{\partial V}{\partial S} = \phi(t)\ ;\quad \frac{\partial^2 V}{\partial S^2} = 0$$

For the final condition we know

$$\begin{aligned} V(S,T) &= S \equiv \phi(T)\,S \\ &\Longrightarrow \phi(T) = 1 \end{aligned}$$

So the original problem reduces to

$$\begin{aligned}\frac{d\phi}{dt}+(r-D)\,\phi-r\phi &= -f\,(t)\\ &\longrightarrow \quad \frac{d\phi}{dt}-D\phi=-f\end{aligned}$$

which is a first order linear equation (i.e. integarting factor method). I.F is

$$\exp\left(-Dt\right)$$

so the ode becomes

$$\begin{aligned}e^{-Dt}\frac{d\phi}{dt}-D\phi e^{-Dt} &= -fe^{-Dt}\\ \frac{d}{dt}\left(e^{-Dt}\phi\right) &= -fe^{-Dt}\\ \int_t^T d\left(e^{-D\tau}\phi\left(\tau\right)\right) &= -\int_t^T f\left(\tau\right)e^{-D\tau}d\tau\\ \left(e^{-D\tau}\phi\left(\tau\right)\right)\Big|_t^T &= -\int_t^T f\left(\tau\right)e^{-D\tau}d\tau\\ e^{-DT}\phi\left(T\right)-e^{-Dt}\phi\left(t\right) &= -\int_t^T f\left(\tau\right)e^{-D\tau}d\tau\end{aligned}$$

and we know $\phi\left(T\right)=1,$ hence

$$\begin{aligned}e^{-DT}-e^{-Dt}\phi\left(t\right) &= -\int_t^T f\left(\tau\right)e^{-D\tau}d\tau\\ e^{-Dt}\phi\left(t\right) &= e^{-DT}+\int_t^T f\left(\tau\right)e^{-D\tau}d\tau\\ \phi\left(t\right) &= e^{-D(T-t)}+\int_t^T f\left(\tau\right)e^{-D(\tau-t)}d\tau\end{aligned}$$

So the option price $V\left(S,t\right)=\phi\left(t\right)S$ and $\Delta\left(S,t\right)=\dfrac{\partial V}{\partial S}=\phi\left(t\right)=$

$$e^{-D(T-t)}+\int_t^T f\left(\tau\right)e^{-D(\tau-t)}d\tau$$