CQF Final Examination For The Award Of Distinction

December 2008

There are three sections and nine questions in this examination.

Full marks can be obtained from complete answers to four questions.

You are required to do at least one question from each section. The fourth can be chosen from any section.

If you answer more than four, your best four will be taken and graded.

The use of calculators is permitted in this examination.

You may assume throughout this examination that dX is an increment in a standard Brownian motion $X(t)$:

$$\begin{aligned} \mathbb{E}[dX] &= 0 \\ \mathbb{E}\left[dX^2\right] &= dt. \end{aligned}$$

Section A

1. a) Consider a random variable Y, that is normally distributed such that $Y \sim N\left(\mu, \sigma^2\right)$. If the Moment Generating Function (MGF) is denoted by $M_Y(\theta)$, show that for this distribution

$$M_Y(\theta) = e^{\theta\mu + \frac{1}{2}\theta^2\sigma^2}.$$

If $X \sim N\left(\mu, \sigma^2\right)$, then we can construct a standard normal $\phi \sim N(0,1)$ by setting $\phi = \dfrac{X-\mu}{\sigma} \Longrightarrow X = \mu + \sigma\phi$.

The MGF is

$$\begin{aligned} M_\theta(X) &= \mathbb{E}\left[e^{\theta x}\right] = \mathbb{E}\left[e^{\theta(\mu+\phi\sigma)}\right] \\ &= e^{\theta\mu}\mathbb{E}\left[e^{\theta\sigma\phi}\right] \end{aligned}$$

So the MGF of X is therefore equal to the MGF of ϕ but with θ replaced by $\theta\sigma$.This is much nicer than trying to calculate the MGF of $X \sim N\left(\mu, \sigma^2\right)$.

$$\begin{aligned} \mathbb{E}\left[e^{\theta\phi}\right] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\theta x}e^{-x^2/2}dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\theta x - x^2/2}dx \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(x^2-2\theta x+\theta^2-\theta^2\right)}dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-\theta)^2+\frac{1}{2}\theta^2}dx \\ &= e^{\frac{1}{2}\theta^2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-\theta)^2}dx \end{aligned}$$

Now do a change of variable - put $u = x - \theta$

$$\mathbb{E}\left[e^{\theta\phi}\right] = e^{\frac{1}{2}\theta^2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}u^2}du = e^{\frac{1}{2}\theta^2}$$

Thus

$$\begin{aligned} M_\theta(X) &= e^{\theta\mu}\mathbb{E}\left[e^{\theta\sigma\phi}\right] \\ &= e^{\theta\mu+\frac{1}{2}\theta^2\sigma^2} \end{aligned}$$

Now define the k^{th} moment m_k of the random variable Y by

$$m_k = \left.\frac{d^k}{d\theta^k}M_Y(\theta)\right|_{\theta=0} \quad ; \quad k = 0, 1, 2, ...$$

Use this to obtain the first four moments of a standard normal random variable $\phi \sim N(0,1)$ and hence calculate the **skew** and **kurtosis** for ϕ.To get the simpler formula for a standard normal distribution put $\mu = 0,\ \sigma = 1$ to get $M_\theta(X) = e^{\frac{1}{2}\theta^2}$.

We can now obtain the first four moments for a standard normal

$$\begin{aligned} m_1 &= \left.\frac{d}{d\theta}e^{\frac{1}{2}\theta^2}\right|_{\theta=0} \\ &= \left.\theta e^{\frac{1}{2}\theta^2}\right|_{\theta=0} = 0 \end{aligned}$$

$$\begin{aligned} m_2 &= \left.\frac{d^2}{d\theta^2}e^{\frac{1}{2}\theta^2}\right|_{\theta=0} \\ &= \left.\left(\theta^2+1\right)e^{\frac{1}{2}\theta^2}\right|_{\theta=0} = 1 \end{aligned}$$

$$\begin{aligned} m_3 &= \left.\frac{d^3}{d\theta^3}e^{\frac{1}{2}\theta^2}\right|_{\theta=0} \\ &= \left.\left(\theta^3+3\theta\right)e^{\frac{1}{2}\theta^2}\right|_{\theta=0} = 0 \end{aligned}$$

$$\begin{aligned} m_4 &= \left.\frac{d^4}{d\theta^4}e^{\frac{1}{2}\theta^2}\right|_{\theta=0} \\ &= \left.\left(\theta^4+6\theta^2+3\right)e^{\frac{1}{2}\theta^2}\right|_{\theta=0} = 3 \end{aligned}$$

$$\begin{aligned} \text{Skew} &= \frac{\mathbb{E}\left[(\phi-\mu)^3\right]}{\sigma^3} = \frac{\mathbb{E}\left[(\phi)^3\right]}{1} = 0 \\ \text{kurtosis} &= \frac{\mathbb{E}\left[(\phi-\mu)^4\right]}{\sigma^3} = \frac{\mathbb{E}\left[(\phi)^4\right]}{1} = 3 \end{aligned}$$

1. b) The random number generator in Excel, RAND() , produces uniformly distributed random variables over $[0,1]$, written Unif$[0,1]$.

Show that if we generate a number N of this random variable then the algorithm

$$\sqrt{\frac{12}{N}}\left(\sum_1^N RAND() - \frac{N}{2}\right) \tag{1}$$

produces a single standardized Normal $\phi \sim N(0,1)$. Expression (1) should be derived. Further, show that (1) is consistent with the Central Limit Theorem. We begin by noting that the mean and variance of each `RAND()` is $\frac{1}{2}$ and $\frac{1}{12}$ respectively. Put $\psi =$ `RAND()`

$\mathbb{E}\left[\sum_{i=1}^{N}\psi_i\right] = \sum_{i=1}^{N}\mathbb{E}\left[\psi_i\right] = \frac{N}{2}$ (but we want a zero mean), so we set

$$\sum_{i=1}^{N} \psi_i - \frac{N}{2}.$$

Now the variance is obtained from $\mathbb{V}\left[\sum\limits_{i=1}^{N} \psi_i - \frac{N}{2}\right] = \sum\limits_{i=1}^{N} \mathbb{V}\left[\psi_i\right] - \mathbb{V}\left[\frac{N}{2}\right] = \sum\limits_{i=1}^{N} \frac{1}{12} = \frac{N}{12}$

This can now be used to obtain a normalising constant - the property used is $\mathbb{V}\left[\alpha x\right] = \alpha^2 \mathbb{V}\left[x\right]$. So

$$\sqrt{\frac{12}{N}} \left[\sum_{i=1}^{N} \psi_i - \frac{N}{2}\right]$$

This can be written as

$$\lim_{N \longrightarrow \infty} \frac{\left[\sum\limits_{i=1}^{N} \psi_i - \frac{N}{2}\right]}{\sqrt{\frac{N}{12}}}.$$

Comparing with the CLT

$$\lim_{N \longrightarrow \infty} \frac{\sum\limits_{i=1}^{N} \psi_i - N\mu}{\sigma\sqrt{N}}$$

where $\mu = 1/2$, $\sigma = \sqrt{1/12}$ makes our approximation consistent with the CLT.

2. In the following, $X(t)$ is a standard Brownian motion.
Part A Which of the following processes are Martingales?

a) $Y(t) = X(t) + 4t$. Intuitively, this cannot be a Martingale since $X(t)$ is a martingale and $4t$ adds some drift. Mathematically, the SDE for $Y(t)$ is:

$$dY(t) = 4dt + dX(t)$$

$Y(t)$ is a Brownian motion with drift. Hence, $Y(t)$ is **not a martingale**.

b) $Y(t) = X^2(t)$. Intuitively, this cannot be a Martingale since $X^2(t) - t$ is a martingale (recalling the *quadratic variation* property of Brownian motions). Mathematically, by Itô applied to the function $f(x) = x^2$, the SDE for the process $Y(t) = f(X(t))$ is given by

$$dY(t) = dt + 2X(t)dX(t)$$

The dynamics of $Y(t)$ has a drift: $Y(t)$ is **not a martingale**

c) $Y(t) = t^2X(t) - 2\int_0^t sX(s)ds$. The easisest way to tackle this proble is by defining a new stochastic process $Z(t) = t^2X(t)$. To check that $Y(t)$ is a martingale it is enough to check that the drift of $Z(t)$ is equal to $2\int_0^t sX(s)ds$.

By Itô applied to the function $f(s, x) = s^2x$, the SDE for the process $Z(t) = f(t, X(t))$ is

$$dZ(t) = 2tX(t)dt + t^2dX(t)$$

with $Z(0) = 0$. Integrating over $[0, t]$,

$$Z(t) = 2\int_0^t sX(s)ds + \int_0^t s^2dX(s)$$

As required, the drift of $Z(t)$ is equal to $2\int_0^t sX(s)ds$. Hence

$$Y(t) = \int_0^t s^2dX(s)$$

and $Y(t)$ **is a martingale**.

d) $Y(t) = X_1(t)X_2(t)$ where $X_1(t)$ and $X_2(t)$ are two standard Brownian motions with correlation ρ so that $dX_1(t)dX_2(t) \to \rho dt$. Does the answer depend on the value of ρ?

By the *Itô product rule*,

$$Y(t) = X_1(t)dX_2(t) + X_2(t)dX_1(t) + \rho dt$$

For $Y(t)$ to be a martingale, its dynamics must be driftless, i.e. we must have $\rho dt = 0$. This is only the case when $\rho = 0$ and the two Brownian motions $X_1(t)$ and $X_2(t)$ are independent. In the general case, when $\rho \neq 0$, $Y(t)$ is not a martingale.

Part B

Define the process $Y(t)$ by the stochastic differential equation

$$dY(t) = f(t, Y(t))dt + dX(t)$$

with $Y(0) = y_0$, and $f(t, Y(t))$ is some given function satisfying the Novikov condition. Consider in addition the exponential martingale $M(t)$ given by

$$M(t) = \exp\left\{-\int_0^t f(s, Y(s))ds - \frac{1}{2}\int_0^t f^2(s, Y(s))dX(s)\right\}$$

Show that the process $Z(t) = M(t)Y(t)$ is a martingale.

This exercise is a special case of an exercise given in the problem sheet for Lecture 4.3.

To get the dynamics of $M(t)$, consider the process $L(t) = \ln M(t)$, then

$$L(t) = -\frac{1}{2}\int_0^t f^2(s, Y(s))ds - \int_0^t f(s, Y(s))dX(s)$$

which implies that

$$dL(t) = -\frac{1}{2}f^2(t, Y(t))dt - f(t, Y(t))dX(t)$$

By Itô, applied to the function $g(x) = e^x$, the dynamics of $M(t) = e^{L(t)}$ is given by

$$dM(t) = -f(t, Y(t))M(t)dX(t)$$

with $M(0) = 0$.

We can now answer our original question. By the *Itô product rule*, the SDE for $Z(t) = Y(t)M(t)$ is

$$\begin{aligned}
dZ(t) &= d(YM)(t) \\
&= Y(t)dM(t) + M(t)dY(t) - f(t, Y(t))M(t)dt \\
&= -f(t, Y(t))Y(t)M(t)dX(t) + M(t)\left(f(t, Y(t))dt + dX(t)\right) - f(t, Y(t))M(t)dt \\
&= M(t)\left(1 - f(t, Y(t))Y(t)\right)dX(t)
\end{aligned}$$

The dynamics of $Z(t)$ is driftless, therefore $Z(t) = Y(t)M(t)$ **is a martingale**.

3. Suppose that the process S evolves according to Geometric Brownian motion

$$dS = \mu S dt + \sigma S dX.$$

Show that

$$d(\log S) = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dX.$$

Now deduce that the expected value of $S(t)$ at time $t > 0$, given S_0 at time zero, is

$$\mathbb{E}\left[S(t)|S_0\right] = S_0 e^{\mu t}.$$

Show that if $V = S^n$, for constant n, then V follows

$$\frac{dV}{V} = n\left(\mu + \tfrac{1}{2}(n-1)\sigma^2\right) dt + n\sigma dX.$$

Hence or otherwise find a general expression for

$$\mathbb{E}\left[S^n(t)|S_0\right].$$

Start with

$$dV = \left(\mu S \frac{dV}{dS} + \frac{1}{2}\sigma^2 S^2 \frac{d^2V}{dS^2}\right) dt + \sigma S \frac{dV}{dS} dX. \tag{3}$$

for $V = \log S$. So $\frac{dV}{dS} = \frac{1}{S}$, $\frac{d^2V}{dS^2} = -\frac{1}{S^2}$. Substitute in to get

$$d((\log S)) = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dX$$

Integrating both sides over 0 and t

$$\int_0^t d((\log S)) = \int_0^t \left(\mu - \frac{1}{2}\sigma^2\right) ds + \int_0^t \sigma dX$$

we obtain (using $X(0)$)

$$\log \frac{S(t)}{S(0)} = \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma X(t)$$

$$S_t = S_0 \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma X(t)\right\}. \tag{4}$$

$$\mathbb{E}\left[S(t)|S_0\right] = S_0 \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right) t\right\} \mathbb{E}\left[e^{\sigma X(t)}\right]$$

with $X(t) \in N(0,t)$

$$\begin{aligned}\mathbb{E}\left[e^{\sigma X(t)}\right] &= \frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}} e^{\sigma x} e^{-x^2/2t} dx \\ &= \left(\frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}} e^{-(x-\sigma t)^2/2t} dx\right) e^{\sigma^2 t/2} \\ &= e^{\sigma^2 t/2}\end{aligned}$$

Therefore

$$\begin{aligned}\mathbb{E}\left[S(t)|\, S_0\right] &= S_0 \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)t\right\} e^{\sigma^2 t/2} \\ &= S_0 e^{\mu t}\end{aligned}$$

$V = S^n$; so performing Itô gives

$$\begin{aligned}dV &= nS^{n-1}dS + \tfrac{1}{2}n(n-1)S^{n-2}dS^2 + ... \\ &= nS^n(\mu dt + \sigma dX) + \tfrac{1}{2}n(n-1)S^{n-2}\sigma^2 S^n dt \\ &= nV\left\{\left(\mu + \tfrac{1}{2}(n-1)\sigma^2\right)dt + \sigma dX\right\} \\ \frac{dV}{V} &= n\left(\mu + \tfrac{1}{2}(n-1)\sigma^2\right)dt + n\sigma dX\end{aligned}$$

Then

$$\begin{aligned}\mathbb{E}\left[S^n(t)|\, S_0\right] &\equiv \mathbb{E}\left[R(t)|\, R_0\right] \\ &= R_0 \exp\left(n\left(\mu + \tfrac{1}{2}(n-1)\sigma^2\right)t\right) \\ &= S_0^n \exp\left(n\left(\mu + \tfrac{1}{2}(n-1)\sigma^2\right)t\right)\end{aligned}$$

(b) We wish to compute $m(t) = \mathbb{E}\left[e^{aX(t)}\right]$, where a is a given constant. First, use Itô to show that

$$m(t) = 1 + \frac{a^2}{2}\int_0^t m(s)ds$$

Next take the derivative with respect to t to deduce an ODE for $m(t)$.

Finally, solve the ODE to evaluate the expectation $m(t)$.

Define $Z(t) = e^{aX(t)}$ so that $m(t) = \mathbf{E}[Z(t)]$. By Itô applied to the function $f(x) = e^{ax}$, the SDE for $Z(t) = f(X(t))$ is

$$dZ(t) = \frac{1}{2}a^2 Z(t)dt + aZ(t)dX(t)$$

Integrating over $[0,t]$, and noting that $Z(0) = 1$, we get

$$Z(t) = 1 + \frac{1}{2}a^2\int_0^t Z(s)ds + a\int_0^t Z(s)dX(s)$$

Taking the expectation, we obtain

$$\mathbf{E}\left[Z(t)\right] = 1 + \frac{a^2}{2}\mathbf{E}\left[\int_0^t Z(s)ds\right]$$

By *Fubini's theorem*, we can move the expectation inside the integral to obtain

$$\begin{aligned} m(t) &= 1 + \frac{a^2}{2}\int_0^t \mathbf{E}\left[Z(s)\right]ds \\ &= 1 + \frac{a^2}{2}\int_0^t m(s)ds \end{aligned}$$

Taking the t-derivative of expression above, we obtain the following differential equation

$$\dot{m}(t) = \frac{a^2}{2}m(t)$$

with initial condition $m(0) = \mathbf{E}\left[Z(0)\right] = 1$

This is a separable ODE and its solution is

$$m(t) = e^{\frac{a^2 t}{2}}$$

and hence

$$m(t) = \mathbf{E}\left[e^{aX(t)}\right] = e^{\frac{a^2 t}{2}}$$

Section B

4. **(a)**

Briefly explain the following:

i. The difference between hedging, speculation and arbitrage.

A trader is *hedging* when he/she has an exposure to the price of an asset and takes a position in a derivative to offset the exposure. In a *speculation* the trader has no exposure to offset - they are betting on the future movements in the price of the asset. *Arbitrage* involves taking a position in two or more different markets to lock in profit.

ii. The difference between writing a call option and buying a put option.

Writing a call option involves selling an option to someone else. It gives a payoff of

$$\min\left(E - S_T,\ 0\right)$$

where E is the strike and S_T the spot price at expiry.

Buying a put option involves buying an option from someone else. It gives a payoff of

$$\max\left(E - S_T,\ 0\right)$$

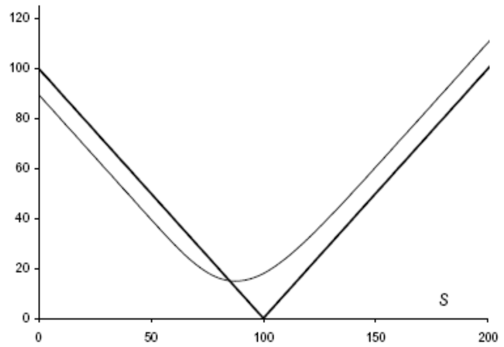
In both cases the potential payoff is $E - S_T$. When you write a call option, the payoff is negative or zero. When you buy a put option, the payoff is zero or positive.

iii. What happens when an investor shorts a share (also discuss dividends).

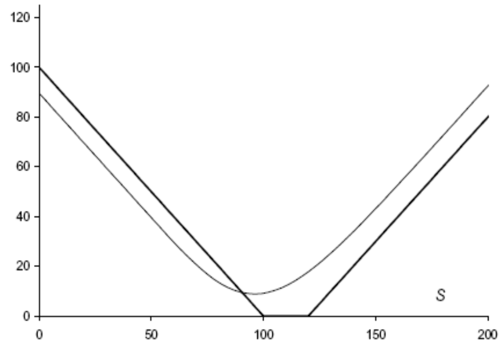
Suppose I wished to short sell some shares, say 1000 shares in China Airways - I would instruct a broker to sell 1000 China Airways shares from his/her portfolio. That is, 1000 shares that someone else owns. The proceeds of the sale would be paid into my account (and what I didn't spend would grow at the risk-free rate). I would then be obliged to pay the broker (and hence owner of shares) any cash flows associated with the shares. If China Airways paid any dividend whilst I was short the shares, then I would be obliged to pay the broker the dividend (which would then be paid to the original owner of the shares). At some point, normally agreed at the time of the initial short sale, I would be obliged to buy back the shares (so the original owner of the sold shares owned them again).

iv. What is the difference between a straddle and strangle?

A straddle consists of a call and a put with the same strike.



A strangle is similar except that the strikes are different.



(b)

A stock is currently worth £40. It is also known that at the end of a six month term this asset price will have risen to £45 or fallen to £38. The risk-free interest rate is 5% per annum.

Using the Binomial Method, what is the value of a six month European call option with a strike price of £40?

Binomial tree for share price is

$$\begin{array}{cc} & 45 \\ 40 & \\ & 38 \end{array}$$

Binomial tree for option price V is

$$\begin{array}{cl} & 5 \ \ (= \max(45-40,0)) \\ V & \\ & 0 \ \ (= \max(38-40,0) \end{array}$$

Now set up a Black-Scholes hedged portfolio, $V - \Delta S$, then binomial tree for its value is

$$\begin{array}{ccc} & & 5 - 45\Delta \\ V - 40\Delta & & \\ & & -38\Delta \end{array}$$

For risk-free portfolio choose Δ such that $5 - 45\Delta = -38\Delta \Rightarrow \Delta = \frac{5}{7}$. So in absence of arbitrage, $V - 40\Delta = \left(e^{-0.05 \times 0.5}\right)(-38\Delta)$, and $V = 2.098$.

(c)

A binary Call option has payoff $H(S - E)$ at expiry $t = T$ and a binary Put has payoff $H(E - S)$ where H is the Heaviside step function defined by

$$\begin{cases} H(Y) = 1 & Y > 0 \\ H(Y) = \frac{1}{2} & Y = 0 \\ H(Y) = 0 & Y < 0 \end{cases}$$

A customer wants to enter into a position which consists of a Binary Call plus a Binary Put with the same exercise price E and expiry time T. If the interest rate is r what is the fair price of this position?

Solution: Payoff for Binary Call + Binary Put

$$\begin{aligned} &= H(S - E) + H(E - S) \\ &= 1 \end{aligned}$$

so value at time $t < T$

$$= \exp(-r(T - t)).$$

5. Consider a Markowitz world with a three asset risky economy where the covariance matrix of expected returns is given by

$$\begin{pmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_2^2 & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_3^2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}.$$

Firstly show that the covariance matrix is strictly positive definite in the sense that

$$\begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_2^2 & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_3^2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} > 0$$

if $\begin{pmatrix} x & y & z \end{pmatrix} \neq \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}$. **(It is sufficient to show that the resulting quadratic form is a sum of perfect squares.)**

Deduce that the inverse of the covariance matrix is

$$\begin{pmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_2^2 & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_3^2 \end{pmatrix}^{-1} = \frac{1}{2} \begin{pmatrix} 5 & -3 & 1 \\ -3 & 3 & -1 \\ 1 & -1 & 1 \end{pmatrix}$$

Assume that the expected returns on the risky assets are respectively,

$$\begin{pmatrix} R_1 \\ R_2 \\ R_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

Using the method of Lagrange optimisation, deduce that the boundary of the opportunity set is given by

$$\sigma_{\sqcap}^2(\rho) = \frac{1}{4}\left(3\rho^2 - 8\rho + 8\right),$$

where ρ is the prescribed level of expected return and $\sigma(\rho)$ is the minimal level of risk corresponding to the level of expected return.

By differentiating $\sigma_{\sqcap}^2$, show that the efficient frontier is given by

$$\sigma_{\sqcap}^2 = \frac{1}{4}\left(3\rho^2 - 8\rho + 8\right) \text{ for } \rho \geq \frac{4}{3},$$

and identify this on a diagram.

Solution:

A symmetric matrix A is positive definite if for all $\underline{x} \neq \underline{0}$, $\underline{x}^{\mathrm{T}} A \underline{x} > 0$. Therefore using the covariance matrix given

$$(x,\ y,\ z)\begin{pmatrix}1&1&0\\1&2&1\\0&1&3\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=(x,\ y,\ z)\begin{pmatrix}x+y\\x+2y+z\\y+3z\end{pmatrix}$$

$= x^2+2y^2+3z^2+2xy+2yz$
$= (x+y)^2+(y+z)^2+2z^2$ (sum of squares)
$\Rightarrow$ for $(x,\ y,\ z)\neq \underline{0}$, the quadratic form above is always positive, hence the covariance matrix is strictly positive definite.

Put $A=\begin{pmatrix}1&1&0\\1&2&1\\0&1&3\end{pmatrix}$

So the determinant is given by $|A|=2$

$$\mathsf{adj}\ A=\begin{pmatrix}5&-3&1\\-3&3&-1\\1&-1&1\end{pmatrix}^{\mathrm{T}}$$

We note A is symmetric, hence $A^{\mathrm{T}}=A$. We can now write the inverse of A

$$A^{-1}=\frac{1}{2}\begin{pmatrix}5&-3&1\\-3&3&-1\\1&-1&1\end{pmatrix}$$

<u>Boundary of the opportunity set</u>:

Create a portfolio $\sqcap=\sum_{i=1}^{3}\lambda_i S_i$

$S_i = \begin{pmatrix}S_1\\S_2\\S_3\end{pmatrix}$ are risky assets and $\lambda_i = \begin{pmatrix}\lambda_1\\\lambda_2\\\lambda_3\end{pmatrix}$ is the allocation vector,

i.e fraction of our total wealth being invested in each asset. So $\sum_{i=1}^{3}\lambda_i=1$.

The expected return is

$$\begin{aligned}R_{\sqcap} &= \text{Expected return} = \underline{\lambda}.\underline{R}\\ &= (\lambda_1,\ \lambda_2\ ,\lambda_3)\,.\,(1,\ 2,\ 3)\\ &= \lambda_1+2\lambda_2+3\lambda_3\\ &= r\end{aligned}$$

.

Now define the risk: $\sigma_{\sqcap}^2 = \underline{\lambda}^T A\underline{\lambda} \Rightarrow \sigma_{\sqcap}^2 = \lambda_1^2 + 2\lambda_2^2 + 3\lambda_3^2 + 2\lambda_1\lambda_2 + 2\lambda_2\lambda_3$ (we have used the working earlier, when we showed that the covariance matrix is +ve definite - with the vector $\underline{x}$ now replaced by the allocation vector $\underline{\lambda}$).

The problem here is to find an allocation vector $\underline{\lambda}$ (for a prescribed level of return r), so that $\sigma_{\sqcap}^2$ is minimised under the restrictions

$$\begin{aligned} f\ (\underline{\lambda}) &= \lambda_1 + 2\lambda_2 + 3\lambda_3 - r = 0 && \text{which comes from } \underline{\lambda}.\underline{R} = r \text{ and} \\ g\ (\underline{\lambda}) &= \lambda_1 + \lambda_2 + \lambda_3 - 1 = 0 && \text{which is our total wealth } \sum_{i=1}^{3} \lambda_i = 1. \end{aligned}$$

Introduce the Lagrange function L and Lagrange multipliers α, β:

$$\begin{aligned} L(\lambda_1,\ \lambda_2,\ \lambda_3,\ \alpha,\ \beta) &= \sigma_{\sqcap}^2 + \alpha g\ (\underline{\lambda}) + \beta f\ (\underline{\lambda}) \quad \Rightarrow \\ L &= \underline{\lambda}^T A\underline{\lambda} + \beta (r - \underline{\lambda}.\underline{R}) + \alpha (1 - \underline{\lambda}.\underline{1}) \end{aligned}$$

where $\underline{1}$ is the unit vector.

Minimise L **with respect to** $\underline{\lambda}$, by differentiating L with respect to a vector quantity, and setting it to zero. We do this like any other differentiation, and note that $\underline{\lambda}^T A\underline{\lambda}$ behaves like $\underline{\lambda}^2 A$. Alternatively we can use the product rule on $\underline{\lambda}^T A\underline{\lambda}$.

$$\begin{aligned} \frac{\partial L}{\partial \underline{\lambda}} &= 2\underline{\lambda}A - \beta\underline{R} - \alpha\underline{1} = 0 = \left(\underline{\lambda}^T A + A\underline{\lambda}\right) - \beta\underline{R} - \alpha\underline{1} \\ &\rightarrow\ 2\underline{\lambda}A = \beta\underline{R} + \alpha\underline{1} \\ &\rightarrow\ \underline{\lambda} = \frac{1}{2}A^{-1}(\alpha\underline{1} + \beta\underline{R}) \\ &\therefore\ \underline{\lambda} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \frac{1}{2}A^{-1}(\alpha\underline{1} + \beta\underline{R}) \\ &= \frac{1}{4}\begin{pmatrix} 5 & -3 & 1 \\ -3 & 3 & -1 \\ 1 & -1 & 1 \end{pmatrix}\left(\alpha\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \beta\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}\right) \\ &= \frac{1}{4}\begin{pmatrix} 3\alpha + 2\beta \\ -\alpha \\ \alpha + 2\beta \end{pmatrix} \end{aligned}$$

OR

Calculate
$$\frac{\partial L}{\partial \lambda_1}=\frac{\partial L}{\partial \lambda_2}=\frac{\partial L}{\partial \lambda_3}=0$$
for which we know

$$L=\lambda_1^2+2\lambda_2^2+3\lambda_3^2+2\lambda_1\lambda_2+2\lambda_2\lambda_3-\alpha\left(\lambda_1+\lambda_2+\lambda_3-1\right)-\beta\left(\lambda_1+2\lambda_2+3\lambda_3-r\right)$$

so
$$\left.\begin{array}{c}\dfrac{\partial L}{\partial \lambda_1}=2\lambda_1+2\lambda_2-\alpha-\beta=0\\ \dfrac{\partial L}{\partial \lambda_2}=4\lambda_2+2\lambda_1+2\lambda_3-\alpha-2\beta=0\\ \dfrac{\partial L}{\partial \lambda_3}=6\lambda_3+2\lambda_2-\alpha-3\beta=0\end{array}\right\}\text{ Solve for }\ \lambda\text{'s in terms of }\ \alpha\ \text{ and }\ \beta$$
which gives

$$\underline{\lambda}=\begin{pmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\end{pmatrix}=\frac{1}{4}\left(\alpha\begin{pmatrix}3\\ -1\\ 1\end{pmatrix}+\beta\begin{pmatrix}2\\ 0\\ 2\end{pmatrix}\right)=\frac{1}{4}\begin{pmatrix}3\alpha+2\beta\\ -\alpha\\ \alpha+2\beta\end{pmatrix}$$
and using these values of $\lambda_1,\ \lambda_2,\ \lambda_3$ allows us to calculate $\sigma_{\Pi}^2\left(\alpha,\ \beta\right)$.
We now minimize L with respect to α & β in turn so
$$\left.\begin{array}{c}\dfrac{\partial L}{\partial \alpha}=0\Rightarrow\underline{\lambda}.\underline{1}=\lambda_1+\lambda_2+\lambda_3=1\\ \dfrac{\partial L}{\partial \beta}=0\Rightarrow\underline{\lambda}.\underline{R}=\lambda_1+2\lambda_2+3\lambda_3=r\end{array}\right\}.$$
So we have
$$\begin{pmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\end{pmatrix}=\frac{1}{4}\begin{pmatrix}3\alpha+2\beta\\ -\alpha\\ \alpha+2\beta\end{pmatrix}\quad\text{and}\quad\left\{\begin{array}{c}\lambda_1+\lambda_2+\lambda_3=1\\ \lambda_1+2\lambda_2+3\lambda_3=r\end{array}\right.$$
which gives us the Lagrange multipliers α and β in terms of r

$$\left.\begin{array}{c}\alpha=4-2r\\ \beta=\frac{3}{2}r-2\end{array}\right\}.$$

Substituting the values of these two parameters in $\underline{\lambda}$ gives

$$\begin{pmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\end{pmatrix}=\frac{1}{4}\begin{pmatrix}8-3r\\ 2r-4\\ r\end{pmatrix},$$

and we can obtain our expression for risk $\sigma_{\sqcap}^2$ by substituting in these λ's and simplifying to give

$$\sigma_{\sqcap}^2(r) = \frac{1}{4}\left(3r^2 - 8r + 8\right).$$

So for varying amounts of return r we can calculate the minimum risk. This is a hyperbola parameterised by r and is the **Boundary of the Opportunity Set.**

Excluding short sales means $\lambda_i \geq 0,\ \ i = 1, 2, 3$
So

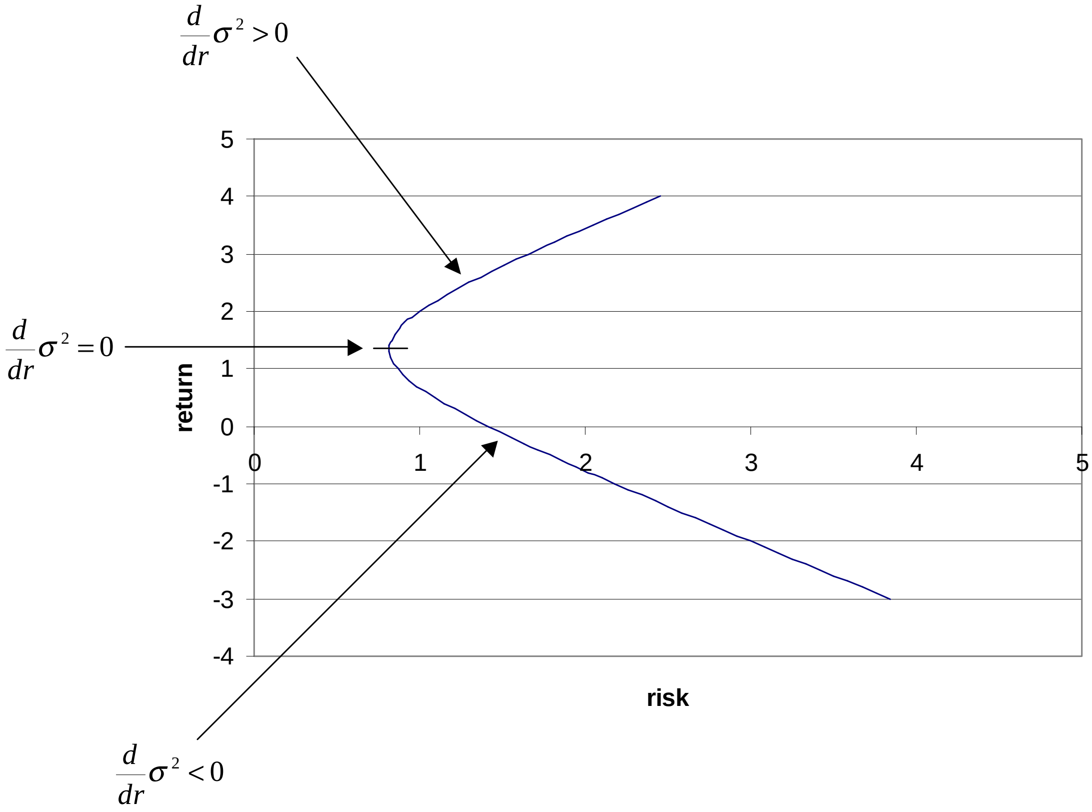
$$\begin{aligned} 8 - 3r &\geq 0 \Rightarrow r \leq \frac{8}{3} \\ 2r - 4 &\geq 0 \Rightarrow r \geq 2 \\ r &\geq 0 \end{aligned}$$

and these three conditions are only satisfied when $2 \leq r \leq \frac{8}{3}$. Thus we can only achieve minimal risk without short sales if

$$2 \leq r \leq \frac{8}{3}.$$

Having obtained $\sigma_{\sqcap}^2(r)$ we can calculate the **efficient frontier**.
For this we require

$$\frac{d}{dr}\sigma_{\sqcap}^2 \geq 0$$



So

$$\begin{aligned}\frac{d}{dr}\sigma_{\sqcap}^2 &= \frac{1}{4}(6r-8)=0\\ &\to r=\frac{4}{3}.\end{aligned}$$

The minimum level of risk corresponding to $r=\frac{4}{3}$ is $\sigma_{\sqcap}^2(4/3)=2/3$. We have the efficient frontier

$$\sigma_{\sqcap}^2=\frac{1}{4}\left(3r^2-8r+8\right) \quad \text{for} \quad r\geq\frac{4}{3}.$$

6. **(a)** Consider the Itô integral of the form

$$\int_0^T f(t, X(t))\, dX(t) = \lim_{N\to\infty} \sum_{i=0}^{N-1} f(t_i, X_i)(X_{i+1} - X_i).$$

The interval $[0, T]$ is divided into N partitions with end points

$$t_0 = 0 < t_1 < t_2 < \; ...... \; < t_{N-1} < t_N = T,$$

where the length of an interval $t_i - t_{i+1}$ tends to zero as $N \to \infty$.

Using Itô's lemma show that

$$3\int_0^T X(t)^2\, dX(t) = X(T)^3 - X(0)^3 - 3\int_0^T X(t)\, dt.$$

Show from the definition of the Itô integral that the result can also be found by initially writing the integral

$$3\int_0^T X^2 dX = \lim_{N\to\infty} 3\sum_{i=0}^{N-1} X_i^2 (X_{i+1} - X_i)$$

Hint: You may use $3b^2(a-b) = a^3 - b^3 - 3b(a-b)^2 - (a-b)^3$**. Full details of all working should be given.**

We use the formula for stochastic integrals

$$\int_0^T \frac{dF}{dX} dX(\tau) = F(X(T)) - F(X(0)) - \frac{1}{2}\int_0^T \frac{d^2F}{dX^2} dt$$

$$\begin{aligned} \frac{dF}{dX} &= X^2 \longrightarrow \frac{d^2F}{dX^2} = 2X \\ F &= \tfrac{1}{3}X^3(t) \end{aligned}$$

$$\begin{aligned} \int_0^T X^2 dX(\tau) &= \tfrac{1}{3}X^3(T) - \tfrac{1}{3}X^3(0) - \frac{1}{2}\int_0^T 2X\, dt \\ 3\int_0^T X^2 dX(\tau) &= X^3(T) - X^3(0) - 3\int_0^T X\, dt \end{aligned}$$

For the stochastic integral from first principles:

$$3\sum_{i=0}^{N-1} X_i^2(X_{i+1} - X_i) = \sum_{i=0}^{N-1} X_{i+1}^3 - \sum_{i=0}^{N-1} X_i^3 - 3\sum_{i=0}^{N-1} X_i^3 (X_{i+1} - X_i)^2 - \sum_{i=0}^{N-1} (X_{i+1} - X_i)^3$$

$$\sum_{i=0}^{N-1} X_{i+1}^3 - \sum_{i=0}^{N-1} X_i^3 = X_N^3 - X_0^3$$
$$= X(T)^3 - X(0)^3$$
$$\lim_{N\to\infty} \sum_{i=0}^{N-1} X_i^3 (X_{i+1} - X_i)^2 \longrightarrow \int_0^T X(t)\,dt$$
$$\lim_{N\to\infty} \sum_{i=0}^{N-1} (X_{i+1} - X_i)^3 \longrightarrow dX^3 \longrightarrow 0$$

hence result.

(b) Consider the diffusion process for the spot rate r which evolves according to the stochastic differential equation

$$dr = -ar dt + b dX.$$

Both a and b are constants. Write down (**not derive**) the forward Fokker-Planck equation for the transition probability density function $p(r', t')$ for this process, where a primed variable refers to a future state/time.

By solving the Fokker-Planck equation which you have obtained, obtain the **steady state** probability distribution $p_\infty(r')$, which is given by

$$p_\infty = \sqrt{\frac{a}{b^2\pi}} \exp\left(-\tfrac{a}{b^2} r'^2\right).$$

For a process governed by the SDE $dy = A(y,t)\,dt + B(y,t)\,dX$, the Fokker-Planck eqn is

$$\frac{\partial p}{\partial t'} = \frac{1}{2}\frac{\partial^2}{\partial y'^2}\left(B(y,t)^2 p\right) - \frac{\partial}{\partial y'}\left(A(y',t)\,p\right)$$

where $A \equiv -ar$, $B \equiv b$ and $y' \equiv r'$ to give

$$\frac{\partial p}{\partial t'} = \frac{1}{2}b^2 \frac{\partial^2 p}{\partial r'^2} + a\frac{\partial}{\partial r'}(r'p)$$

where $p = p(r', t')$ is the transition PDF and the variables refer to future states. In the steady state case, there is no time dependency, hence the Fokker Planck PDE becomes an ODE with

$$\frac{1}{2}b^2 \frac{d^2 p_\infty}{dr^2} + a\frac{d}{dr}(r p_\infty) = 0$$

$p_\infty = p_\infty(r)$. The prime notation and subscript have been dropped simply for convenience at this stage. To solve the steady-state equation:

Integrate both sides wrt r

$$\frac{1}{2}b^2 \frac{dp}{dr} + arp = k$$

where k is a constant of integration and can be calculated from the condtions, that as $r \to \infty$

$$\left\{ \begin{array}{l} \frac{dp}{dr} \to 0 \\ p \to 0 \end{array} \right. \Rightarrow k = 0$$

which gives

$$\frac{dp}{dr} = -\frac{2a}{b^2} rp,$$

a first order variable separable equation. So

$$\begin{aligned} \int \frac{dp}{p} &= -\frac{2a}{b^2} \int r dr \to \\ \ln p &= -\frac{a}{b^2} r^2 + C \ , \qquad C \text{ is arbitrary.} \end{aligned}$$

Rearranging and taking exponentials of both sides to give

$$p = E \exp\left(-\frac{a}{b^2} r^2\right)$$

To find E we know as p_∞ is a PDF so use

$$\begin{aligned} \int_{-\infty}^{\infty} p_\infty \ dr' &= 1 \to \\ E \int_{-\infty}^{\infty} \exp\left(-\frac{a}{b^2} r^2\right) \ dr &= 1 \end{aligned}$$

A few (related) ways to calculate E . Now use the error function, i.e.

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

So put

$$x = \sqrt{\tfrac{a}{b^2}} r \to dx = \sqrt{\tfrac{a}{b^2}} dr$$

which transforms the integral above

$$\frac{Eb}{\sqrt{a}} \int_{-\infty}^{\infty} e^{-x^2} \ dx = 1 \to Eb \sqrt{\frac{\pi}{a}} = 1$$

therefore

$$E = \tfrac{1}{b} \sqrt{\frac{a}{\pi}}.$$

This allows us to finally write the steady-state transition PDF as

$$p_\infty (r') = \tfrac{1}{b} \sqrt{\frac{a}{\pi}} \exp\left(-\frac{a}{b^2} r'^{\ 2}\right).$$

Section C

7.

(a) An asset S follows the lognormal random walk

$$dS = \mu S dt + \sigma S dX$$

and we wish to value a derivative that pays off at expiry T an amount which is a function of the path taken by the asset between time zero and expiry.

Assuming that an option value V thus depends on S, t and a quantity

$$I = \int_0^t f(S, \tau)\, d\tau,$$

where f is a specified function and r the risk free interest rate, derive the pricing equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + f(S,t)\frac{\partial V}{\partial I} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{7}$$

for the function $V(S, I, t)$.

Solution:

$$dI = \int_t^{t+dt} f(S,\tau)\, d\tau = f(S,t)\, dt$$

$$dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}dS^2 + \frac{\partial V}{\partial I}dI + ...$$

where $V = V(S, I, t)$; now $dS^2 = \sigma^2 S^2 dt + O\left(dt^{3/2}\right)$, so

$$dV = \left(\frac{\partial V}{\partial t} + f\frac{\partial V}{\partial I} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right)dt + \frac{\partial V}{\partial S}dS$$

Create risk-free portfolio, i.e. $\Pi = V - \Delta S$, so choose $\Delta = \partial V/\partial S$, at time t. Then for $t \longrightarrow t + dt$

$$\begin{aligned} d\Pi &= \left(\frac{\partial V}{\partial t} + f\frac{\partial V}{\partial I} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right)dt \\ &= r\Pi dt = \left(r\left(V - S\frac{\partial V}{\partial S}\right)\right)dt \text{ for No Arbitrage} \end{aligned}$$

which gives the equation

$$\frac{\partial V}{\partial t} + f\frac{\partial V}{\partial I} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = r\left(V - S\frac{\partial V}{\partial S}\right)$$

(b) For an arithmetic strike Asian call option the payoff at time T is

$$\max\left(S - \tfrac{1}{T}\int_0^T S(\tau)\,d\tau, 0\right)$$

and for a put option the payoff is

$$\max\left(\tfrac{1}{T}\int_0^T S(\tau)\,d\tau - S, 0\right).$$

Write down the corresponding partial differential equation for this call option $V_C(S,I,t)$ and put option $V_P(S,I,t)$, and hence verify that

$$V_C(S,I,t) - V_P(S,I,t) = S\left(1 - \tfrac{1}{rT}\left(1 - e^{-r(T-t)}\right)\right) - \tfrac{1}{T}e^{-r(T-t)}\int_0^t S(\tau)\,d\tau$$

Briefly outline the method of *upwind differencing* when applied to solving equations of the form (7), when using finite difference methods. Why is such a technique necessary when solving (7) numerically?

Solution: For an arithmetic strike option $f(S,t) = S$. So

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + S\frac{\partial V}{\partial I} + rS\frac{\partial V}{\partial S} - rV = 0$$

for both V_C and V_P. Subtract equations for V_C and V_P to give, by writing

$$U = V_C - V_P,$$

$$\frac{\partial U}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 U}{\partial S^2} + S\frac{\partial U}{\partial I} + rS\frac{\partial U}{\partial S} - rU = 0 \qquad (1)$$

and $I(t) = \int_0^t S(\tau)\,d\tau$. Payoff for U is

$$U(S, I(t), t) = S - \frac{I(T)}{T} \qquad (2)$$

Consider the formula given in the question, i.e.

$$V_C(S,I,t) - V_P(S,I,t) = S\left(1 - \tfrac{1}{rT}\left(1 - e^{-r(T-t)}\right)\right) - \tfrac{1}{T}e^{-r(T-t)}I(t)$$

When $t = T$, this satisfies the payoff condition (2). What about (1)?

$$\begin{aligned}
\frac{\partial^2 U}{\partial S^2} &= 0, \ \frac{\partial U}{\partial S} = \left(1 - \tfrac{1}{rT}\left(1 - e^{-r(T-t)}\right)\right) \\
\frac{\partial U}{\partial I} &= -\tfrac{1}{T}e^{-r(T-t)}; \ \frac{\partial U}{\partial t} = \tfrac{rS}{rT}e^{-r(T-t)} - \tfrac{r}{T}e^{-r(T-t)}I(t)
\end{aligned}$$

For upwinding we are essentially looking for a few lines from the following:

For this reason a one-sided difference must be used

If the coefficient $f(S, I, t)$ changes sign then the choice of difference must reflect this, upwind differencing *must* be used.

Here is a possible choice for the difference:

$$\begin{aligned} \text{if } f(S,I,t) &\geq 0 \text{ then} \\ f(S,I,t)\tfrac{\partial V}{\partial I}(S,I,t) &= f^m_{n,j+\frac{1}{2}} \tfrac{V^m_{n,j+1}-V^m_{n,j}}{\delta I} \end{aligned}$$

but if

$$\begin{aligned} \text{if } f(S,I,t) &< 0 \text{ then} \\ f(S,I,t)\tfrac{\partial V}{\partial I}(S,I,t) &= f^m_{n,j-\frac{1}{2}} \tfrac{V^m_{n,j}-V^m_{n,j-1}}{\delta I}. \end{aligned}$$

8. (a) Briefly explain how the Explicit Finite Difference Method can be used to price American call options.

(b) Consider a perpetual American **put** option $V(S)$, which satisfies the Euler problem

$$\frac{1}{2}\sigma^2 S^2 \frac{d^2V}{dS^2} + (r-D)\, S\frac{dV}{dS} - rV = 0, \quad S^* < S < \infty, \tag{1}$$

$$V(S) = E - S, \quad 0 \le S \le S^*,$$

$$V(S^*) = E - S^*, \quad \frac{dV}{dS}(S^*) = -1, \quad \lim_{S\to\infty} V(S) \to 0,$$

where $S \geq 0$ is the spot price, $E > 0$ is the strike, $S^* > 0$ is the optimal exercise boundary, $\sigma > 0$ is the constant volatility, $r > 0$ is the constant interest rate and D is the dividend yield.
Show that $V(S) = S^\alpha$ is a solution of the differential equation above provided that

$$\frac{1}{2}\sigma^2\alpha^2 + \left(r - D - \frac{1}{2}\sigma^2\right)\alpha - r = 0$$

Show that one of the roots of this equation, α^-, is always negative. Hence deduce that for $S > S^*$,

$$V(S) = -\frac{S^*}{\alpha^-}\left(\frac{S}{S^*}\right)^{\alpha^-}, \quad S^* = \frac{\alpha^-}{\alpha^- - 1}E.$$

Show that

$$\sigma^2\alpha^- + \left(r - D - \frac{1}{2}\sigma^2\right) = -\sqrt{\left(r - D - \frac{1}{2}\sigma^2\right) + 2r\sigma^2} < 0.$$

Solution: To solve differential equation given by (1) look for a solution of the form

$$\begin{aligned} V(S) &= S^\alpha \longrightarrow \\ V'(S) &= \alpha S^{\alpha-1} \longrightarrow V''(S) = \alpha(\alpha-1)S^{\alpha-2} \end{aligned}$$

Substitute in to (1) to get for $S \neq 0$:

$$\tfrac{1}{2}\sigma^2\alpha^2 + (r - D - \tfrac{1}{2}\sigma^2)\alpha - r = 0 \tag{A}$$

Roots are

$$\begin{aligned} \alpha^\pm &= -\frac{\left(r - D - \frac{1}{2}\sigma^2\right)}{\sigma^2} \pm \frac{1}{\sigma^2}\sqrt{(r - D - \tfrac{1}{2}\sigma^2)^2 + 2r\sigma^2} \\ &= \frac{1}{\sigma^2}\left[\pm\sqrt{(r - D - \tfrac{1}{2}\sigma^2)^2 + 2r\sigma^2} - \left(r - D - \tfrac{1}{2}\sigma^2\right)\right] \end{aligned}$$

and it follows from $2r\sigma^2 > 0$ that

$$\sqrt{(r - D - \tfrac{1}{2}\sigma^2)^2 + 2r\sigma^2} > \left| r - D - \tfrac{1}{2}\sigma^2 \right|$$

Thus root

$$\alpha^- = \frac{1}{\sigma^2}\left[-\sqrt{(r - D - \tfrac{1}{2}\sigma^2)^2 + 2r\sigma^2} - \left(r - D - \tfrac{1}{2}\sigma^2\right) \right]$$

is always negative and independent of sign of $(r - D - \frac{1}{2}\sigma^2)$.

We now look for solution $V(S) = cS^\alpha$ $(c > 0)$ for $S \geq S^*$, using following conditions

$$\left. \frac{dV}{dS} \right|_{S=S^*} = -1 \qquad \text{(a)}$$

$$V(S^*) = E - S^* \qquad \text{(b)}$$

$$\frac{dV}{dS} = c\alpha S^{\alpha - 1} \qquad \text{(c)}$$

(c) can only be -ve for $\alpha < 0$, $\therefore \alpha = \alpha^-$.

Further it can be deduced from (a) and (c)

$$c\alpha^- (S^*)^{\alpha^- - 1} = -1 \Rightarrow c = -\frac{1}{\alpha^- (S^*)^{\alpha^-}} S^*$$

therefore

$$V(S) = -\frac{1}{\alpha^- (S^*)^{\alpha^-}} S^*.S^{\alpha^-} \Longrightarrow$$

$$V(S) = -\frac{S^*}{\alpha^-}\left(\frac{S}{S^*}\right)^{\alpha^-}.$$

From the payoff (b) we get

$$c(S^*)^{\alpha^-} = E - S^* = V(S^*)$$

and hence

$$-\frac{S^*}{\alpha^-} = E - S^* \longrightarrow S^*\left(1 - 1/\alpha^-\right) = E = S^*\left(\frac{\alpha^- - 1}{\alpha^-}\right)$$

$\therefore$

$$S^* = \frac{\alpha^-}{\alpha^- - 1}E.$$

From calculating α^- we know

$$\alpha^- = \frac{1}{\sigma^2}\left[-\sqrt{(r - D - \tfrac{1}{2}\sigma^2)^2 + 2r\sigma^2} - \left(r - D - \tfrac{1}{2}\sigma^2\right) \right] \Longrightarrow$$

$$\sigma^2\alpha^- + (r - D - \tfrac{1}{2}\sigma^2) = -\sqrt{(r - D - \tfrac{1}{2}\sigma^2)^2 + 2r\sigma^2} < 0 \qquad \text{(B)}$$

Numerical Scheme: The *no-arbitrage* argument tells us that the value of the option V can not be less than the payoff $P(S,t)$ during that time period, so

$$V \geq P(S,t).$$

Consider the time interval T in which the option may be exercised. As we step backwards in time, the option value is computed. If this price is less than the payoff during T, it is set equal to the payoff. So at each time step, we solve the explicit scheme to obtain the option price $\overline{V}$. Then check the condition $\overline{V} < P(S,t)$? If this is true then the option price $V = P(S,t)$, else $V = U$.

This strategy of checking the early exercise constraint is called the *cutoff* method. Thus the explicit FDM can be expressed in compact form as

$$\begin{aligned} U_n^{m-1} &= \alpha_n V_{n-1}^m + \beta_n V_n^m + \gamma_n V_{n+1}^m \\ V_n^{m-1} &= \begin{cases} U_n^{m-1} & \text{if } U_n^{m-1} \geq P_n^{m-1} \\ P_n^{m-1} & \text{if } U_n^{m-1} < P_n^{m-1} \end{cases} \end{aligned}$$

where P_n^m is the FDA for the payoff at $(S,\ t)$, i.e. P_n^m =Payoff($n\delta S,\ m\delta t$) as opposed to simply P_n^M at time $t = T$. So we have a time dependent payoff function, defined (at each time step) for the life of the option at the time the contract is written.

9. **Part A**
Consider the following mean-reverting Ornstein-Uhlenbeck process U_t, which satisfies the stochastic differential equation

$$dU_t = -\theta U_t dt + \sigma dX.$$

The drift θ and diffusion σ are constant. Show that by using an integrating factor method,

$$U_t = \alpha \exp(-\theta t) + \sigma \left(X_t - \theta \int_0^t \exp(\theta(s-t)) X_s ds \right)$$

where $U(0) = \alpha$.
We use the product rule by writing

$$\begin{aligned} d(I_t U_t) &= I_t dU_t + U_t dI_t \\ &= \exp(\theta t)(-\theta U_t dt + \sigma dX_t) + \theta U_t \exp(\theta t) dt \\ &= \sigma \exp(\theta t) dX_t. \end{aligned}$$

and integrating over $[0, t]$ gives

$$U_t = \alpha \exp(-\theta t) + \sigma \int_0^t \exp[\theta(s-t)] dX_s \tag{9.1}$$

where $U(0) = \alpha = U_0$.

By using integration by parts, i.e. $\int v \; du = uv - \int u \; dv$ we can simplify (9.1).

$$\begin{aligned} u &= X_s \\ v &= \exp(\theta(s-t)) \rightarrow dv = \theta \exp(\theta(s-t)) ds \end{aligned}$$

Therefore

$$\int_0^t \exp(\theta(s-t)) dX_s = X_t - \gamma \int_0^t \exp(\theta(s-t)) X_s ds$$

and we can write (9.1) as

$$U_t = \alpha \exp(-\theta t) + \sigma \left(X_t - \theta \int_0^t \exp(\theta(s-t)) X_s ds \right)$$

Part B
The two factor interest rate model with the Bond Pricing Equation (BPE) is

given by

$$\frac{\partial V}{\partial t}+\frac{1}{2}w^2\frac{\partial^2 V}{\partial r^2}+\rho wq\frac{\partial^2 V}{\partial r\partial l}+\frac{1}{2}q^2\frac{\partial^2 V}{\partial l^2}+\left(u-\lambda_r w\right)\frac{\partial V}{\partial r}+\left(p-\lambda_l q\right)\frac{\partial V}{\partial l}-rV=0,$$

where the two state variables evolve according to

$$\begin{aligned} dr &= udt+wdX_1 \\ dl &= pdt+qdX_2. \end{aligned}$$

The Brownian motions are correlated with $\mathbb{E}\left[dX_1 dX_2\right]=\rho dt$.

Given that $u-\lambda_r w=0=p-\lambda_l q$ and $w=q=\sqrt{a+br+cl}$, where $a,\ b$ and c are constants, derive a set of first order equations and boundary conditions for $A,\ B$ and C such that a bond V is of the form

$$V=\exp\left(A\left(t;T\right)-rB\left(t;T\right)-lC\left(t;T\right)\right),$$

is a solution of the BPE with redemption value

$$V\left(r,l,T;T\right)=1.$$

You are not required to solve these equations.

Solution: The information given reduces the BPE to

$$\frac{\partial V}{\partial t}+\frac{1}{2}\left(a+br+cl\right)\frac{\partial^2 V}{\partial r^2}+\rho\left(a+br+cl\right)\frac{\partial^2 V}{\partial r\partial l}+\frac{1}{2}\left(a+br+cl\right)\frac{\partial^2 V}{\partial l^2}=rV.$$

We are given $V=\exp\left(A\left(t\right)+rB\left(t\right)+lC\left(t\right)\right)\longrightarrow$

$$\begin{aligned} \dot{V} &= \left(\dot{A}\left(t\right)+r\dot{B}\left(t\right)+l\dot{C}\left(t\right)\right)V \\ V_r &= BV\longrightarrow V_{rr}=B^2V \\ V_l &= CV\longrightarrow V_{ll}=C^2V \\ V_{rl} &= BCV \end{aligned}$$

and substitute in BPE to give

$$\left(\dot{A}\left(t\right)+r\dot{B}\left(t\right)+l\dot{C}\left(t\right)\right)+\frac{1}{2}B^2\left(a+br+cl\right)+\rho BC\left(a+br+cl\right)+\frac{1}{2}C^2\left(a+br+cl\right)=r$$

and now equation coefficients of $O\left(1\right),\ O\left(r\right),\ O\left(l\right)$ to give in turn the following ODE's

$$\begin{aligned} \dot{A}\left(t\right)+\frac{1}{2}B^2a+\rho BCa+\frac{1}{2}aC^2 &= 0 \\ \dot{B}\left(t\right)+\frac{1}{2}B^2b+\rho BCb+\frac{1}{2}bC^2 &= 1 \\ \dot{C}\left(t\right)+\frac{1}{2}B^2c+\rho BCc+\frac{1}{2}cC^2 &= 0 \end{aligned}$$

which are solved together with the final condition

$$A\left(T\right)=B\left(T\right)=C\left(T\right)=0.$$