Optimal Stochastic Recovery for Base Correlation

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Abstract

On the back of monoline protection unwind and positive gamma hunting, spreads of the senior tranches of the CDX investment-grade index have widened so much that they went beyond the maximum spread levels that standard Gaussian Copula model can allow. Indeed, using a 40% recovery assumption and 100% default correlation gives a spread of $39bp$ on the $5Y$ $[30 - 100]$ tranche while the market price was $55bp$ on June, 27^{th} 2008 with a CDX.IG9 ref of $148bp$.

An other challenge is the emergence in the market of a new super senior tranche $[60 - 100]$ that used to have no value (under the 40% recovery assumption) and that starts showing up on the screens initially at $5bp$ then $10bp$ and lately at low to mid twenties.

We present in this document an enhancement to the standard copula Model by introducing an optimal stochastic recovery specification.

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1 Introduction

1.1 CDO market after the credit crunch

On the back of monoline protection unwind and positive gamma hunting, spreads of the senior tranches of the CDX investment-grade index have widened so much that they went beyond the maximum spread levels that standard Gaussian Copula model can allow. Indeed, using a 40% recovery assumption and 100% default correlation gives a spread of $39bp$ on the $5Y$ $[30-100]$ tranche while the market price was $55bp$ on June, 27^{th} 2008 with a CDX.IG9 ref of $148bp$.

Correlation Traders & Quants have then focused on how to enhance the credit correlation model to get it working again. A quick fix of the problem is to lower (mark down) the recovery rate and then to allow for losses to potentially hit the top part of the capital structure even if there is a few number of corporates defaulting.

The key challenge of this approach is that Index and Single name traders are still using the 40% recovery rate assumptions for their PV calculation and more importantly for upfront calculation when trading out of the money CDS. The second challenge is the emergence in the market of a new super senior tranche $[60-100]$ that used to have no value (under the 40% recovery assumption) and that starts showing up on the screens initially at $5bp$ then $10bp$ and lately at mid to low twenties. The same fix mentioned above can be applied again here. In fact, with a 10% recovery level and using a 95% correlation level, the $5Y$ $[60-100]$ spread is $30bp$ with a CDX.IG9 ref of $148bp$.

The third issue that appeared in summer'07 (i.e. before the two other problems) is negative deltas on senior mezz tranches (typically the $[15-30]$). This is due to the steep base correlation curve.

In this document, we will present **an extension to the standard base correlation & Gaussian copula framework by allowing the recovery to be random between 0% and 100% while maintaining Single Name CDS valuation properties**: i.e. our Model is fully consistent with using a constant 40% recovery rate for Single Name CDS. In this model, the recovery rate is positively correlated to the Gaussian copula factor which implies that recovery is negatively correlated to the number of defaults (i.e. if when the systemic factor is low, the macro-economy is bad and hence recovery rate should be low)

We do believe that the industry is considering similar approaches [5], but our choice of the dependency between recovery and the systemic factor has a major advantage of making the model behaving as a recovery mark down while having all the benefits of a true stochastic recovery model.

The rest of the paper is organized as follows. We will briefly recap the current problems in CDO tranches market in the first section. In the second section, we will review the well-known gaussian copula model and introduce our recovery model. In section 3 we will discuss calibration results on CDX and we will analyze the model behavior: we will look into model delta and Gammas for all CDX tranches. In section 4 we will have a close look to the super duper tranche: $[60-100]$.

1.2 Previous Work on Recovery Modeling for CDO

Rating agency models and the Credit Metrics [97] model had the recovery modeled by a beta distribution matching historical recovery average and standard deviation, as their inputs is based on historical data, and their implementation is based on a Monte-Carlo simulation, this does not lead to any calibration or tractability problem. Practitioners quickly noticed that recovery randomness diversifies away as the number of names in a portfolio increases if there is no recovery correlation, and that therefore, correlated recovery would be needed to have a material impact on a typical 100 names CDO pricing.

In early 2004, correlated recovery re-emerged as a research theme with copula models featuring correlated recovery tested for super-senior tranches. However, recovery correlation fitted to historical parameters is not able to explain all the skew observed on supersenior prices, and the emphasis was put on modeling default dependence rather than modeling recovery correlation. The development of a standardized CDO market later in 2004 led to the adoption of base correlation with fixed recovery. Although, it appeared from 2005 that the market implied loss surface was not consistent with an assumption of recovery of 40%, and the methodology was never suited to high supersenior pricing due to delta becoming negative for investment grade tranches above 40%, these problems were not perceived as material as they affected tranches that had near zero spread, and would generally not be traded.

The CDO market changes since 2007 have brought back the issues of pricing supersenior to focus. The paper published by [5] highlights the fact that the distribution of a name's recovery only makes sense as the distribution of that recovery conditional on that name's default, and that the calibration of the recovery model is of prime importance for market implied probability pricing (there is no closed form calibration of expected recovery for a beta distributed recovery and Gaussian copula of default). With this in mind, the most tractable specification of recovery distribution is chosen, where any discrete distribution conditional on name default can be specified and calibrated.

In the following, rather than specifying the recovery distribution exogeneously, we use one parameter family to describe the recovery uncertainty. We believe that this choice is justified because this one parameter family is specifically adapted to the pricing of options on the loss with a Gaussian copula for default correlation, and will therefore result in an optimal calibration range and minimizes base correlation skew.

2 Model Specification

2.1 One factor Default Copula

In the rest of this paper, we will be using the following notations. We assume that we have n issuers.

- τ_i : default time of issuer i

- $\overline{R}$: stripping recovery (assumed to be 40%)
- p_i : Probability of issuer i defaulting before time T (this is extracted from CDS market in the usual way i.e. assuming a constant $\overline{R}$ recovery)
- R_i : recovery of issuer i upon default
- w_i : issuer weight in the portfolio (weight is $\frac{1}{n}$ for an equally weighted portfolio)

Using a one factor default copula model means that there is a uniform random variable U conditional on which the default events are independent.

The copula is then defined by the functional form of the default probability conditionally on U

$$g\left(p_i, u\right) = \mathbb{P}\left(\tau_i < T | U = u\right)$$

Conditionally on U, the default events are independent, and their probability is given by $g\left(p_i, u\right)$. The portfolio loss distribution conditionally on U is the distribution of a sum of independent variables taking the values:

- 0 with probability $1 - g\left(p_i, u\right)$
- $w_i\left(1 - R_i\right)$ with probability $g\left(p_i, u\right)$

The one factor Gaussian copula model corresponds to the following construction:

The default event for issuer i is modelled by a Gaussian variable (asset value) X_i being lower to a certain threshold c_i at date T. The correlation of the asset values is introduced by mapping the systemic factor to a standard normal variable and setting the thresholds as:

$$\begin{aligned} Z &= \mathcal{N}^{-1}\left(U\right) \\ X_i &= \sqrt{\rho} Z + \sqrt{1-\rho}\varepsilon_i \\ \tau_i < t &\cong X_i < \mathcal{N}^{-1}\left(P_i(T)\right) \\ Z \text{ and } &\ \varepsilon_i \text{ are i.i.d} \sim \mathcal{N}\left(0, 1\right) \end{aligned}$$

this corresponds to the following conditional probability formula

$$g_\rho\left(p_i, u\right) = \mathcal{N}\left(\frac{\mathcal{N}^{-1}\left(p_i\right) - \sqrt{\rho} z}{\sqrt{1-\rho}}\right) \tag{1}$$

The properties of the loss distribution in this model are as follows:

1. Loss is between 0 and $\overline{R}$ thus the $[60\% - 100\%]$ tranche has zero value within this framework.
2. For all strikes, base tranche[1] expected loss is a monotonic function of correlation
3. Loss variance conditional on Z tends to 0 as the number of names increases (a result of the defaults events being conditionally independent), so that the variance of the loss is introduced by default correlation

[1] tranche of the form [K-100]

2.2 Stochastic Recovery specification

As mentioned above, one needs to review recovery assumptions to be able to calibrate senior tranches and to have a non zero value in the super duper tranche $[60\% - 100\%]$. To allow for recoveries to be lower than 40%, we model the recovery as a deterministic function of the copula factor Z:

$$(1 - R_i(z)) = \left(1 - \widetilde{R}\right) \frac{g_\rho(\widetilde{p}_i, z)}{g_\rho(p_i, z)} \tag{2}$$

the copula of default event is kept unchanged, so the formula for conditional default probability is:

$$\mathbb{E}\left[1_{\tau_i < t} | Z = z\right] = g_\rho(p_i, z)$$

hence, the conditional expected loss is:

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[(1 - R_i) 1_{\tau_i < t} | Z = z\right] &= \left(1 - \widetilde{R}\right) \frac{g_\rho(\widetilde{p}_i, z)}{g_\rho(p_i, z)} g_\rho(p_i, z) \\
&= \left(1 - \widetilde{R}\right) g_\rho(\widetilde{p}_i, z)
\end{aligned}$$

This equation shows that this stochastic recovery model should behave like a recovery mark down from $\overline{R}$ to $\widetilde{R}$ although the recovery has no variance in a recovery mark down approach. The loss variance introduced by stochastic recovery is compensated by the variance introduced by marking down the recovery. Furthermore, this parameterisation ensures that the conditional loss, which is what we care about when pricing tranches which are options on loss has a smooth form still controlled by the function family g_ρ.

To be consistent with single name and index pricing[2], the recovery model needs to be calibrated such that the expected recovery conditional on default is the same as mid recovery. This yields the following calibration equation:

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[R_i | \tau_i < T\right] &= \overline{R}, \text{ for any maturity } T \\
\mathbb{E}^{\mathbb{Q}}\left[(1 - R_i) 1_{\tau_i < T}\right] &= \left(1 - \overline{R}\right) p_i \\
&= \mathbb{E}^{\mathbb{Q}}\left[\left(1 - \widetilde{R}\right) \frac{g_\rho(\widetilde{p}_i, z)}{g_\rho(p_i, z)} g_\rho(p_i, z)\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[\left(1 - \widetilde{R}\right) g_\rho(\widetilde{p}_i, z)\right] \\
&= \left(1 - \widetilde{R}\right) \widetilde{p}_i
\end{aligned}$$

Then:

$$\widetilde{p}_i = \frac{1 - \overline{R}}{1 - \widetilde{R}} p_i.$$

We see that the calibration equation ends up simply linear. It should be noted that this calibration ensures that both the protection leg and the premium

[2] CDX.IG9 UF could be different by as mush as 75k\$ for a 100M\$ contract between 0% recovery and 40% recovery

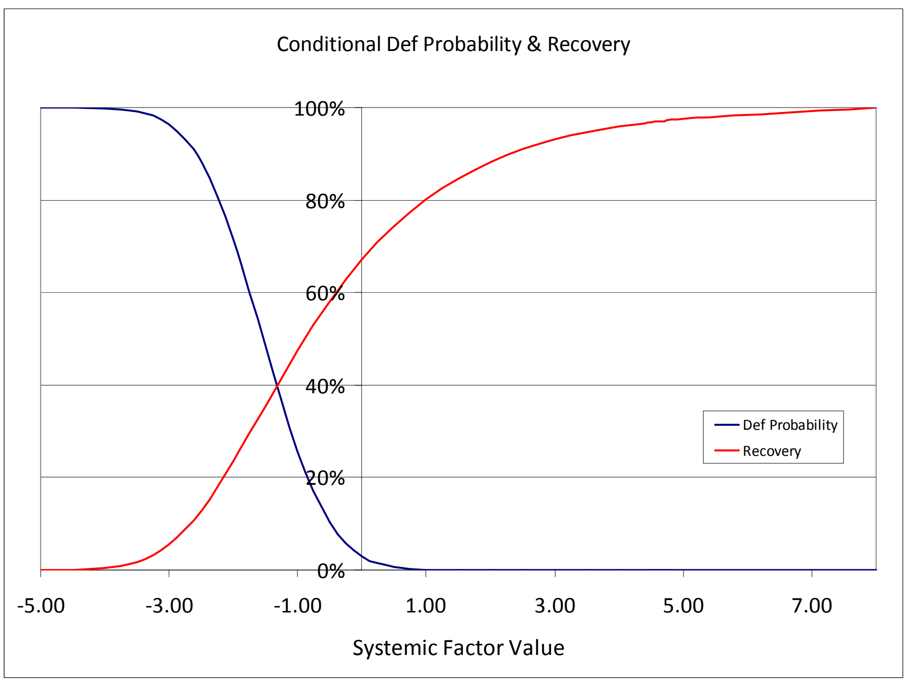
leg value of a CDS will be consistent with CDS pricing with recovery $\overline{R}$, since the calibration ensures

$$\begin{aligned}\mathbb{E}\left[(1-R)\,1_{\tau_i<t}\right] &= \left(1-\overline{R}\right)p_i \\ \mathbb{E}\left[1_{\tau_i<t}\right] &= p_i\end{aligned}$$

this contrasts with the recovery mark down where the default probability being lower $\mathbb{E}\left[1_{\tau_i<t}\right] = \widetilde{p}_i$ results in higher premium leg value than the $\overline{R}$ case, and premium leg values are slightly different. However, differences in premium leg do not lead to big PV differences provided expected loss have been calibrated to the same market value, so both models are expected to behave very similarly anyway.

2.3 Recovery Distribution

The recovery and probability conditional on the factor Z have the following shape.



The recovery is very low when the factor is low i.e. when probability of having too many companies defaulting is high. One can show that the recovery varies between 100% and $\widetilde{R}$. The chart above has been obtained for $5Y$ spread of $150bp$,$\rho = 60\%$, $\widetilde{R} = 0\%$

When default correlation tends to 0%, the two curves tend to be flat, and the recovery is fixed. When the correlation tends to 100%, the two curve are steep functions, and the recovery density has a dirac at $\widetilde{R}$ and 100%, with weights such that the recovery conditional on default is $\overline{R}$.

2.4 Price Calibration Range

In a one factor copula setting, since we expect have lower recovery in extreme loss scenarios, the sigmoid shape of the recovery as a function of the copula factor makes sense. The specificity of the parametrization proposed is that the recovery function steepness varies with the default copula correlation. One consequence is that the model is able to fit the widest range of expected tranche losses, varying from the one where $R = 40\%$ and $\rho = 0\%$ to prices where $R = Rmin$ and $\rho = 100\%$. If we chose instead a specification where the recovery steepness does not depend on the Gaussian copula parameter, say some arbitrary sigmoid function of Z then the model would have a reduced calibration range, because, for $\rho = 0\%$ the stochastic recovery would impose a loss variance floor that results on a Expected Tranche Loss higher than in the previous model, and for $\rho = 100\%$ the loss function would still be smoother than the step function we obtain for recovery mark down, resulting in a lower Expected Tranche Loss maximum value.

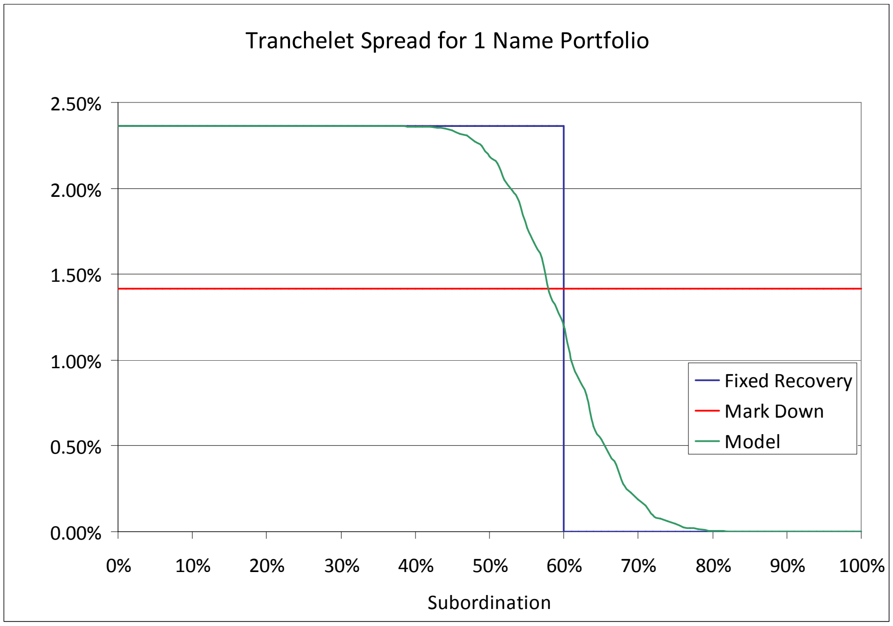
In practice, this extended calibration range means that this stochastic recovery specification, just as recovery mark down, leads to less base correlation skew when the model is calibrated to market prices.

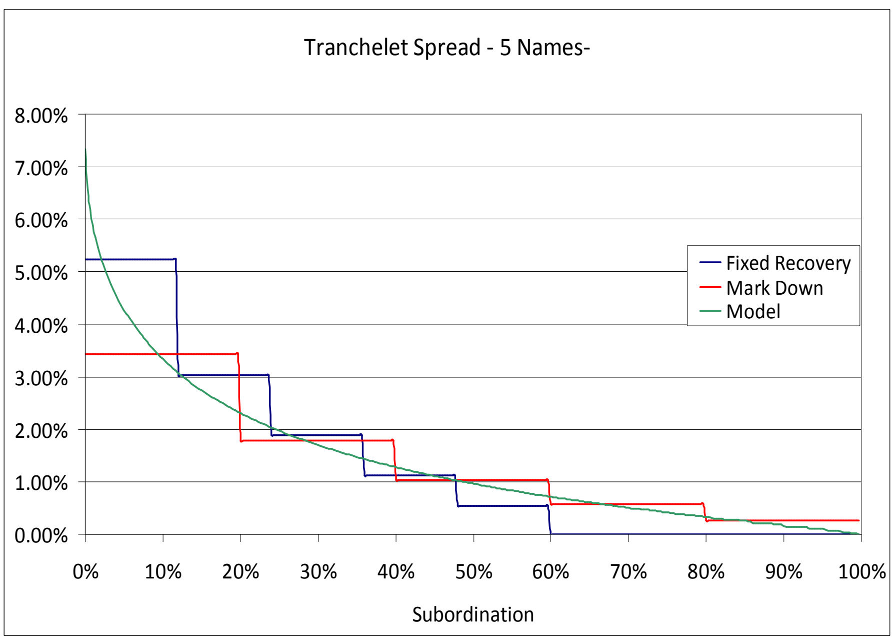
2.5 Cumulative Loss Distribution

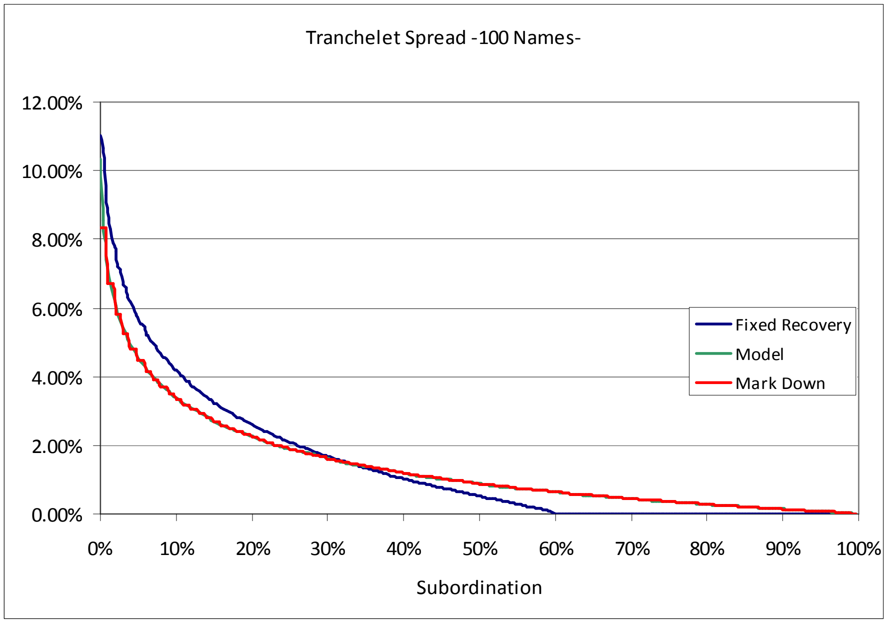
For a large pool portfolio, the stochastic recovery proposed and recovery mark down are exactly equivalent. For a discrete portfolio using the same default correlation, the loss distribution obtained always has more variance in the recovery mark down case.

In the following, we propose to visualize the base tranche slope $\frac{1}{T}\frac{\partial \mathbb{E}^{\mathbb{Q}}(\min[L_T,K])}{\partial K}$ which corresponds to the loss cumulative distribution. The scaling factor T means that the value can also be interpreted as an approximation of the tranchelet spread.

We show results for portfolio with $1, 5$ and 100 names to illustrate the convergence of the stochastic recovery model to recovery mark down and the difference with the standard case (Fixed recovery of $\overline{R}$).


Tranchelet Spread for 1 Name Portfolio
2.50%
2.00%
1.50%
1.00%
0.50%
0.00%
0%
10%
20%
30%
40%
50%
60%
70%
80%
90%
100%
Subordination
Fixed Recovery
Mark Down
Model


Tranchelet Spread - 5 Names-
8.00%
7.00%
6.00%
5.00%
4.00%
3.00%
2.00%
1.00%
0.00%
0%
10%
20%
30%
40%
50%
60%
70%
80%
90%
100%
Subordination
Fixed Recovery
Mark Down
Model



Due to the quality of the convergence for 100 names, this stochastic recovery model and the recovery mark down lead to very similar base correlations surfaces.

3 Model behavior

3.1 Base Correlation Calibration Results

We look at calibration results for Itrx S9 and CDX.IG9 on June 27^{th} 2008.Market data used is :

ITRAXX S9	**5Y Ref 105bp**	**7Y Ref 112bp**	**10Y Ref 117bp**
0-3	33% 1/4 +500bp	40% 1/4+500bp	45% 5/8+500bp
3-6	396.5bp	510.5bp	644bp
6-9	250bp	306.5bp	384bp
9-12	171bp	199bp	236.5bp
12-22	83.5bp	101.5bp	122.5bp

and

CDX.IG9	5Y Ref 148bp	7Y Ref 146bp	10Y Ref 143bp
0-3	58% 3/4+500bp	64% 3/8+500bp	67% 7/8+500bp
3-7	566bp	678bp	767bp
7-10	307.5bp	377.5bp	428.5bp
10-15	169bp	205.5bp	225.5bp
15-30	101bp	103bp	107.5bp

We calibrated the std model (using a 40% recovery), our Model with $\widetilde{R}$ equal to 0% and by marking down recovery to zero as well. This gives a correlation curve on 5Y Itrx.S9 as

Strike	Fixed Recovery	Mark Down	Model
3%	52.17%	42.67%	43.82%
6%	65.45%	54.30%	54.39%
9%	72.82%	60.43%	60.10%
12%	75.58%	65.13%	64.56%
22%	94.09%	78.27%	77.37%

and on 5Y CDX.IG9 as

Strike	Fixed Recovery	Mark Down	Model
3%	50.03%	39.57%	36.40%
7%	74.33%	61.57%	55.74%
10%	83.16%	68.90%	62.98%
15%	96.01%	78.86%	72.99%
30%	N/A	94.17%	88.57%

The base correlations obtained when marking down recovery from 40% to 0% and re-stripping are much lower. The base correlations obtained with our stochastic recovery model are similar to the ones obtained by recovery mark down. There are two reasons why the recovery mark down and our model base correlations are different

1. stripping CDS with a different recovery results in a slightly different timing of expected loss for each CDS

2. the convergence of stochastic recovery to recovery mark down is slower when portfolio is very heterogeneous, which is the case for the CDX.IG9

To isolate the 2 effects, we run another calibration with stripping done at 40% and probabilities adjusted to $\widetilde{p}_i = p_i \frac{1-\overline{R}_i}{1-\widetilde{R}}$ internally:

Strike	Correlation
3%	33.61%
7%	54.64%
10%	62.22%
15%	72.52%
30%	88.74%

This shows that super senior base correlation difference is due mainly to the first cause, whereas the difference in equity tranches is due to the latter one.

3.2 Tranches Leverage

On 5Y Itrx, leverage is quite similar across models

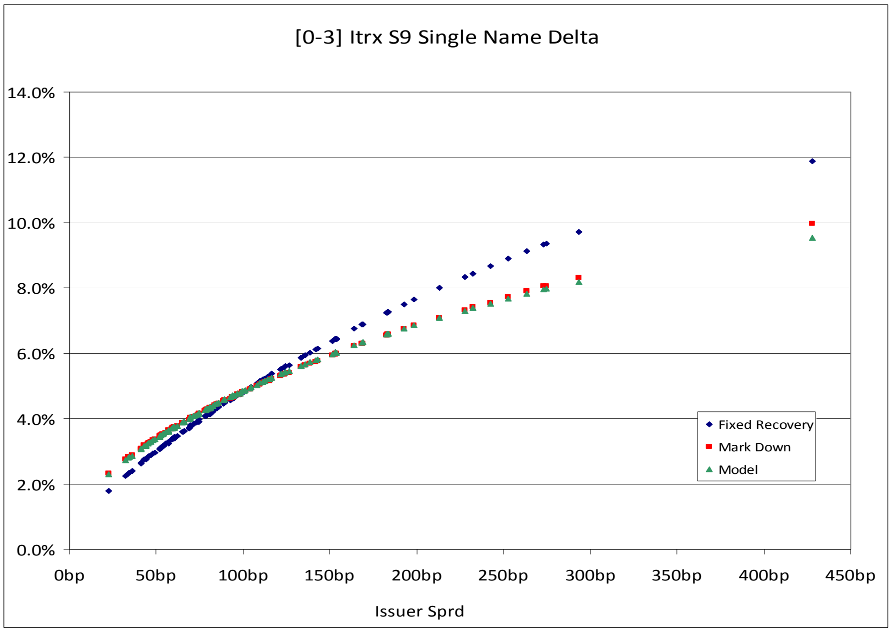
ITRAXX S9	Fixed Recovery	Mark Down	Model
0-3	6.04	5.97	5.99
3-6	3.68	3.63	3.65
6-9	2.49	2.47	2.48
9-12	1.79	1.80	1.81
12-22	0.99	0.97	0.97
22-100	0.64	0.66	0.64

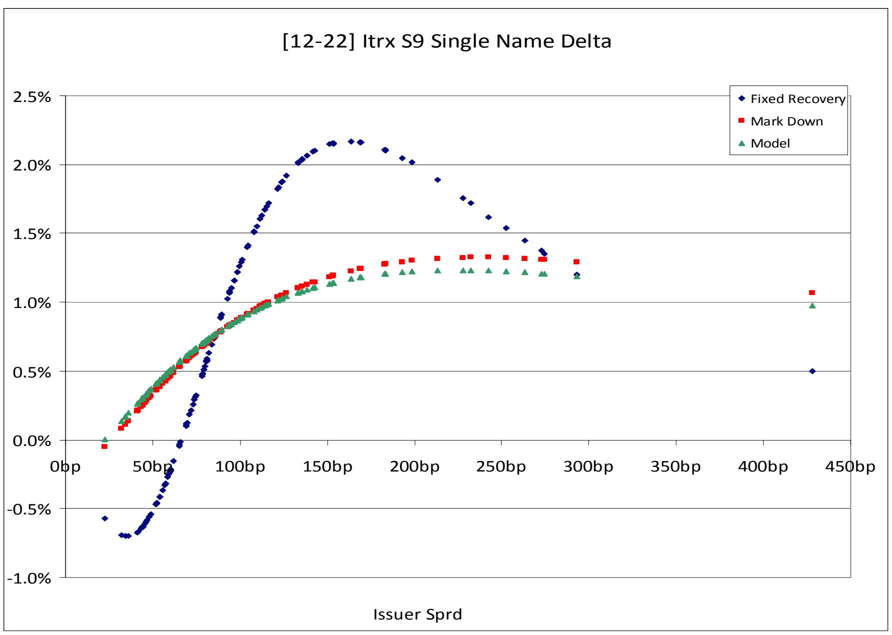
and on 5Y CDX, Fixed recovery at 40% can not calibrate market prices on $[15-30]$ and as a result the $[30-100]$ is not calibrated either.

CDX.IG9	Fixed Recovery	Mark Down	Model
0-3	3.12	3.12	3.51
3-7	2.67	2.67	2.99
7-10	1.55	1.55	1.89
10-15	0.49	0.49	1.11
15-30	1.55	1.55	0.64
30-100	0.69	0.68	0.80

3.3 Single Name delta and Idiosyncratic Gamma

When looking to how deltas are split across names, one can notice that delta as a function of spread in our model is much more flat on Equity and introduce less discrimination on senior tranches. This is due to the fact that correlation is lower in our model.





We can see that our Model (as well as recovery mark down), corrects the negative delta problem, smooths the deltas (and thus Idiosyncratic Gamma). Similar results can be found for CDX and for senior tranches as well.

3.4 Model Parameter choice

One has to consider the following, when setting $\widetilde{R}$:

1. there is a maximum value that one can use to be able to calibrate senior tranches.

2. when increasing the $\widetilde{R}$, correlation curve tends to be steeper, thus this may lead to negative deltas on some tranches and may also introduce arbitrage when looking to trachelet spreads. we present hereafter, the 5Y CDX.IG9 correl for various values of $\widetilde{R}$ at strike3% and correl slope per unit of strike.

Strike	$\tilde{\mathbf{R}} = \mathbf{0\%}$	$\tilde{\mathbf{R}} = \mathbf{10\%}$	$\tilde{\mathbf{R}} = \mathbf{15\%}$
3.00%	33.61%	36.26%	37.66%
7%	5.26	5.42	5.56
10%	2.53	2.63	2.67
15%	2.06	2.13	2.18
30%	1.06	1.11	1.15

4 Super Duper Tranche: $[60-100]$

On the back of monoline protection unwind and positive Gamma hunting, senior tranche spreads have widened so much. The risk allocated[3] to the $[30-100]$ has reached a high of 25% of total portfolio risk. The $[60-100]$ tranche risk allocation has reached almost 10% and this tranche is currently pricing in the low twenties which what used to be the $5Y$ Itraxx main Index one year ago!.

As explained by [6], a combination of $[15-30]$ and $[30-100]$ tranches can lead to a minimum model-free spread for the $[60-100]$. Indeed, the tranche value of $[30-100]$ is the sum of $[30-60]$ tranche value and $[60-100]$. The model-free spread of the $[30-60]$ should be less or equal to $[15-30]$. so setting the $[30-60]$ to its maximum spread, leads to the minimum spread of the $[60-100]$. Mathematically, this gives

$$\begin{aligned}(100\%-30\%)\, S^{[30-100]} DV01^{[30-100]} &= (60\%-30\%)\, S^{[30-60]} DV01^{[30-60]} \\ &\quad + (100\%-60\%)\, S^{[60-100]} DV01^{[60-100]}\end{aligned}$$

then

$$S^{[60-100]} = \frac{70.S^{[30-100]} DV01^{[30-100]} - 30.S^{[30-60]} DV01^{[30-60]}}{40 DV01^{[60-100]}}$$

if we assume that $DV01$ are all the same, this simplifies to

$$S_{\min}^{[60-100]} = \frac{70.S^{[30-100]} - 30.S^{[30-60]}}{40}$$

[3]this is defined as Tranche Expected Loss multiplied by its size and divided by PTF expected Loss

Based on COB of 27^{th} of June 2008 where the mid market of $[15-30]$ was $101bp$ and $[30-100]$ of $55bp$, the minimum spread is $20bp$.

On the other hand, one can derive the maximum spread by having the minimum value in $[30-60]$ which should be equal to $[60-100]$. This gives a maximum spread equal to the $[30-100]$ spread. For instance, the maximum spread is $55bp$. An other maximum spread[4] can be calculated using a zero recovery value and 100% default correlation. For instance, this gives $52bp$.

5 Conclusion

We presented in this document a flexible and tractable stochastic recovery model that can allow for senior tranches calibration and that has a similar behavior as a recovery mark down approach. Further research is needed to look into more details how this will interact with bespoke mapping methods for bespoke CDO pricing and risk management. An other area of further research is how to deal with Fixed recovery deals. In this model, a fair fixed recovery trade has to set the "fixed recovery" value to a high value on Equity and lower value on senior tranches.

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[5] M. Krekel (2008): Pricing distressed CDOs with Base Correlation and Stochastic Recovery

[6] Paola Lamedica (2008): The Bermuda Triangle of Super Senior Risk, Structured Credit Strategy

[4]this is useful for bespoke portfolios where thereis no market price for the other tranches