CQF 1.3 Probability & Introduction to Stochastic Calculus

Throughout this problem sheet, you may assume that X is a Brownian Motion (Weiner Process) and dX is its increment.

1. Let ϕ be a random variable which follows a standardised normal distribution, i.e. $\phi \sim N(0,1)$. If $\mathbb{E}[X]$ and $\mathbb{V}[X]$ are used to denote the Expectation and Variance of x in turn, calculate

(a) $\mathbb{E}\left[\phi^2\right]$

(b) $\mathbb{E}[\psi]$

(c) $\mathbb{V}[\psi]$

where $\psi = \sqrt{dt}\phi$. dt is a small time-step.

(a) Var$[\phi]$ =E$\left[\phi^2\right]$ −E$[\phi]^2 = 1$ from the definition of $N(0,1)$, E$[\phi] = 0$, so E$\left[\phi^2\right] = 1$

(b) E$[\psi]$ =E$\left[\sqrt{dt}\phi\right] = \sqrt{dt}E[\phi]$, because dt is not a RV and we also know that E$[\phi]$, therefore E$[\psi] = 0$.

(c) Var$[\psi]$ =E$\left[\psi^2\right]$ −E$[\psi]^2 \rightarrow$ E$\left[dt\ \phi^2\right]$ −E$[\psi]^2 \Rightarrow$ Var$[\psi] = dt$E$\left[\phi^2\right] = dt$

2. Consider the probability density function $p(x)$

$$p(x) = kx^2 \exp\left(-\lambda x^2\right)\ , \qquad -\infty < x < \infty,$$

where $\lambda\,(> 0)$ and k are both constants. Show that

$$k = \frac{2\lambda^{3/2}}{\sqrt{\pi}}.$$

Deduce that the odd moments of $p(x)$ are all zero, i.e.,

$$E\left[x^{2n+1}\right] = 0, \qquad n = 0,\ 1,\ 2, ...$$

We are given the PDF $p(x)$ where

$$p(x) = kx^2 \exp\left(-\lambda x^2\right)\ , \qquad -\infty < x < \infty,$$

We know $\displaystyle\int_{\mathbb{R}} p(x)\,dx = 1$. Hence $\displaystyle k\int_{-\infty}^{\infty} x^2 \exp\left(-\lambda x^2\right) dx = 1.$

Using the substitution $u = \sqrt{\lambda}x \rightarrow du = \sqrt{\lambda}dx$ which gives

$$\begin{aligned}\lambda^{-3/2}k\int_{-\infty}^{\infty} u^2 \exp\left(-u^2\right) du &= 1 \rightarrow \\ \lambda^{-3/2}kI &= 1\end{aligned}$$

We know from the standardised Normal Distribution

$$\int_{-\infty}^{\infty} \exp\left(-x^2\right) dx = \sqrt{\pi}.$$

We solve for I by first writing

$$I = \int_0^{\infty} u\left(u\exp\left(-u^2\right)\right) du$$

and note from integration by substitution that

$$\int u\exp\left(-u^2\right) du = -\tfrac{1}{2}\exp\left(-u^2\right).$$

Now solve for I by performing integration by parts

$$\begin{aligned}
\int_{-\infty}^{\infty} u\left(u\exp\left(-u^2\right)\right) du &= -\frac{u}{2}\exp\left(-u^2\right)\Big|_{-\infty}^{\infty} + \tfrac{1}{2}\int_{-\infty}^{\infty}\exp\left(-u^2\right) du \\
&= 0 + \frac{1}{2}\sqrt{\pi} \quad \text{using the first result obtained in this question}
\end{aligned}$$

$$\begin{aligned}
\lambda^{-3/2}k\frac{1}{2}\sqrt{\pi} &= 1 \rightarrow \tfrac{\sqrt{\pi}}{2}\lambda^{-3/2}k = 1 \rightarrow \\
k &= \frac{2\lambda^{3/2}}{\sqrt{\pi}}
\end{aligned}$$

1. Now consider the odd moments of $p(x)$. The n^{th} order moment is defined as.,

$$\mathbf{E}\left[x^n\right] = \int_{-\infty}^{\infty} x^n \ p(x) \ dx \quad ; \qquad \text{where } \ n = 2p+1 \ \text{ for } n = 0, \ 1, \ 2, \ ...$$

which gives

$$\frac{2\lambda^{3/2}}{\sqrt{\pi}}\int_{-\infty}^{\infty} x^{2p+1}\, x^2\exp\left(-\lambda x^2\right) \ dx = \frac{2\lambda^{3/2}}{\sqrt{\pi}}\int_{-\infty}^{\infty} x^{2p+3} \ \exp\left(-\lambda x^2\right) \ dx \equiv k\int_{-a}^{a} f \ (x)\, dx$$

and we know that $f \ (x) = x^{2p+3} \ \exp\left(-\lambda x^2\right)$ is an odd function , hence the odd moments are trivially zero.

3. Using the formula below for stochastic integrals, for a function $F\left(X\left(\tau\right), \tau\right)$,

$$\int_0^t \frac{\partial F}{\partial X} dX\left(\tau\right) = F\left(X\left(t\right), t\right) - F\left(X\left(0\right), 0\right) - \int_0^t \left(\frac{\partial F}{\partial \tau} + \frac{1}{2}\frac{\partial^2 F}{\partial X^2}\right) d\tau$$

show that we can write

a. $\int_0^t X(\tau)\, dX(\tau) = \frac{1}{2}X^2(t) - \frac{1}{2}t$

b. $\int_0^t \tau dX(\tau) = tX(t) - \int_0^t X(\tau)\, d\tau$

c. $\int_0^t X^2(\tau)\ \ dX(\tau) = \frac{1}{3}X^3(t) - \int_0^t X(\tau)\, d\tau$

a. $\int_0^t (X(\tau) + \tau)\, dX(\tau)$

$$\frac{\partial F}{\partial X} = X(t) + t \longrightarrow F(X(t)) = \frac{1}{2}X^2(t) + tX(t) \longrightarrow \frac{\partial F}{\partial t} = X(t)$$

and $\frac{\partial^2 F}{\partial X^2} = 1$, therefore

$$\int_0^t (X(\tau) + \tau)\, dX(\tau) = \frac{1}{2}X^2(t) + tX(t) - \int_0^t \left(X(t) + \frac{1}{2}\right) d\tau$$

b. $\int_0^t \tau dX(\tau)$

$$\frac{\partial F}{\partial X} = t \longrightarrow F(X(t), t) = tX(t) \Rightarrow \frac{\partial^2 F}{\partial X^2} = 0 \text{ and } \frac{\partial F}{\partial t} = X(t)$$

substituting all of these terms in to the formula

$$\begin{aligned} \int_0^t \tau dX(\tau) &= tX(t) - 0 - \int_0^t \left(X(\tau) + \frac{1}{2}.0\right) d\tau \\ &= tX(t) - \int_0^t X(\tau)\, d\tau \end{aligned}$$

c. $\int_0^t X^2(\tau)\, dX(\tau)$

$$\frac{dF}{dX} = X^2(t) \longrightarrow F(X(t)) = \frac{1}{3}X^3(t) \longrightarrow \frac{d^2F}{dX^2} = 2X(t)$$

$$\int_0^t X^2(\tau)\, dX(\tau) = \frac{1}{3}X^3(t) - \int_0^t X(\tau)\, d\tau$$

hence result.

4. Use Itô's lemma to obtain a SDE for each of the following functions:

(a) $f\ (X) = X^n$

(b) $y\,(X) = \exp\,(X)$

(c) $g\,(X) = \ln X$

(d) $h\,(X) = \sin X + \cos X$

(a) $f\ (X) = X^n \Rightarrow f\ '\,(X) = nX^{\ n-1} \Rightarrow f\ ''\,(X) = n\,(n-1)\,X^{\ n-2}$ and

$$df = \underbrace{\frac{1}{2}n\,(n-1)\,X^{\ n-2}}_{\text{growth rate}}dt + nX^{\ n-1}dX$$

(b) $y\ (X) = \exp(X) \Rightarrow y\ '\,(X) = \exp(X) = y\ ''\,(X)$ therefore

$$dy = \underbrace{\frac{1}{2}\exp(X)}_{\text{growth rate}}dt + \exp(X)dX$$

(c)

$$\begin{aligned} g\,(X) &= \log\,(X) \Rightarrow\ g\ '\,(X) = \frac{1}{X} \Rightarrow g\ ''\,(X) = -\frac{1}{X^2}\ \text{, hence} \\ dg &= \frac{1}{X}dX - \frac{1}{2X^2}dt \quad \text{growth rate } = -\frac{1}{2X^2} \end{aligned}$$

(d) $h\,(X) = \sin X + \cos X \Rightarrow h\ '\,(X) = \cos X - \sin X \Rightarrow$
$h\ ''\,(X) = -\sin X - \cos X$

$$dh = \underbrace{-\frac{1}{2}\,(\sin X + \cos X)}_{\text{growth rate}}dt + (\cos X - \sin X)\,dX$$