CQF Module 3.2 The Greeks

1. Use put-call parity to find the relationships between the deltas(Δ), gammas(Γ), vegas($vega$), thetas(Θ), rhos(ρ) of European call and put options.

2. Show that for a delta-neutral portfolio of options on a non-dividend paying stock, Π,

$$\Theta + \frac{1}{2}\sigma^2 S^2 \Gamma = r\Pi.$$

3. Show that

$$\frac{\partial \Delta}{\partial \sigma} = \frac{\partial\ vega}{\partial S}, \quad \frac{\partial \Gamma}{\partial \sigma} = \frac{\partial^2\ vega}{\partial S^2}, \quad \frac{\partial \Theta}{\partial \sigma} = \frac{\partial\ vega}{\partial t}, \quad \frac{\partial \Delta}{\partial r} = \frac{\partial \rho}{\partial S}.$$

4. The Black–Scholes formula for a European call option $C(S,t)$ is given by

$$C(S,t) = S\exp(-D(T-t))N(d_1) - E\exp(-r(T-t))N(d_2).$$

Show that the Speed of this option $\left(\frac{\partial \Gamma}{\partial S}\right)$ is given by

$$\text{Speed} = \frac{\partial^3 C}{\partial S^3} = -\frac{\Gamma}{S}\left(1 + \frac{d_1}{\sigma\sqrt{T-t}}\right)$$

You do not need to prove the result for Γ.

5. Consider a delta-neutral portfolio of derivatives, Π. For a small change in the price of the underlying asset, δS, over a short time interval, δt, show that the change in the portfolio value, $\delta\Pi$, satisfies

$$\delta\Pi = \Theta\delta t + \frac{1}{2}\Gamma\delta S^2$$

where $\Theta = \frac{\partial \Pi}{\partial t}$ and $\Gamma = \frac{\partial^2 \Pi}{\partial S^2}$.

6. (a) By differentiating the Black–Scholes equation with respect to σ, show that the vega of an option, $vega$, satisfies the differential equation

$$\frac{\partial vega}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 vega}{\partial S^2} + rS\frac{\partial vega}{\partial S} - r vega + \sigma S^2 \Gamma = 0$$

where $\Gamma = \partial^2 V / \partial S^2$. What is the final condition (payoff) for this PDE?

(b) Similarly, find the PDE satisfied by ρ, the sensitivity of the option value to the interest rate.