Modeling of Variance and Volatility Swaps for Financial Markets with Stochastic Volatilities

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Abstract

A new probabilistic approach is proposed to study variance and volatility swaps for financial markets with underlying asset and variance that follow the Heston (1993) model. We also study covariance and correlation swaps for the financial markets. As an application, we provide a numerical example using $S\&P60$ Canada Index to price swap on the volatility.

1 Introduction

In the early 1970's, Black and Scholes (1973) made a major breakthrough by deriving pricing formulas for vanilla options written on the stock. The Black-Scholes model assumes that the volatility term is a constant. This assumption is not always satisfied by real-life options as the probability distribution of an equity has a fatter left tail and thinner right tail than the lognormal distribution (see Hull (2000)), and the assumption of constant volatility σ in financial model (such as the original Black-Scholes model) is incompatible with derivatives prices observed in the market.

The above issues have been addressed and studied in several ways, such as:

(i) Volatility is assumed to be a deterministic function of the time: $\sigma \equiv \sigma(t)$ (see Wilmott et al. (1995)); Merton (1973) extended the term structure of volatility to $\sigma := \sigma_t$ (deterministic function of time), with the implied volatility for an option of maturity T given by $\hat{\sigma}_T^2 = \frac{1}{T}\int_0^T \sigma_u^2 du$;

(ii) Volatility is assumed to be a function of the time and the current level of the stock price $S(t)$: $\sigma \equiv \sigma(t, S(t))$ (see Hull (2000)); the dynamics of the stock price satisfies the following stochastic differential equation:

$$dS(t) = \mu S(t)dt + \sigma(t, S(t))S(t)dW_1(t),$$

where $W_1(t)$ is a standard Wiener process;

(iii) The time variation of the volatility involves an additional source of randomness, besides $W_1(t)$, represented by $W_2(t)$, and is given by

$$d\sigma(t) = a(t, \sigma(t))dt + b(t, \sigma(t))dW_2(t),$$

where $W_2(t)$ and $W_1(t)$ (the initial Wiener process that governs the price process) may be correlated (see Buff (2002), Hull and White (1987), Heston (1993));

(iv) The volatility depends on a random parameter x such as $\sigma(t) \equiv \sigma(x(t))$, where $x(t)$ is some random process (see Elliott and Swishchuk (2002), Griego and Swishchuk (2000), Swishchuk (1995), Swishchuk (2000), Swishchuk et al. (2000));

(v) Another approach is connected with stochastic volatility, namely, uncertain volatility scenario (see Buff (2002)). This approach is based on the uncertain volatility model developed in Avellaneda et al. (1995), where a concrete volatility surface is selected among a candidate set of volatility surfaces. This approach addresses the sensitivity question by computing an upper bound for the value of the portfolio under arbitrary candidate volatility, and this is achieved by choosing the local volatility $\sigma(t, S(t))$ among two extreme values $\sigma_{\min}$ and $\sigma_{\max}$ such that the value of the portfolio is maximized locally;

(vi) The volatility $\sigma(t, S_t)$ depends on $S_t := S(t + \theta)$ for $\theta \in [-\tau, 0]$, namely, stochastic volatility with delay (see Kazmerchuk, Swishchuk and Wu (2002));

In the approach (i), the volatility coefficient is independent of the current level of the underlying stochastic process $S(t)$. This is a deterministic volatility model, and the special case where σ is a constant reduces to the well-known Black-Scholes model that suggests changes in stock prices are lognormal distributed. But the empirical test by Bollerslev (1986) seems to indicate otherwise. One explanation for this problem of a lognormal model is the possibility that the variance of $\log(S(t)/S(t-1))$ changes randomly. This motivated the work of Chesney and Scott (1989), where the prices are analyzed for European options using the modified Black-Scholes model of foreign currency options and a random variance model. In their works the results of Hull and White (1987), Scott (1987) and Wiggins (1987) were used in order to incorporate randomly changing variance rates.

In the approach (ii), several ways have been developed to derive the corresponding Black-Scholes formula: one can obtain the formula by using stochastic calculus and, in particular, the Ito's formula (see Øksendal (1998), for example).

A generalized volatility coefficient of the form $\sigma(t, S(t))$ is said to be *level-dependent*. Because volatility and asset price are perfectly correlated, we have only one source of randomness given by $W_1(t)$. A time and level-dependent volatility coefficient makes the arithmetic more challenging and usually precludes the existence of a closed-form solution. However, the *arbitrage argument* based on portfolio replication and a completeness of the market remain unchanged.

The situation becomes different if the volatility is influenced by a second "non-tradable" source of randomness. This is addressed in the approach (iii), (iv) and (v) we usually obtains a *stochastic volatility model*, which is general enough to include the deterministic model as a special case. The concept of stochastic volatility was introduced by Hull and White (1987), and subsequent development includes the work of Wiggins (1987), Johnson and Shanno (1987), Scott (1987), Stein and Stein (1991) and Heston (1993). We also refer to Frey (1997) for an excellent survey on level-dependent and stochastic volatility models. We should mention that the approach (iv) is taken by, for example, Griego and Swishchuk (2000).

Hobson and Rogers (1998) suggested a new class of non-constant volatility models, which can be extended to include the aforementioned level-dependent model and share many characteristics with the stochastic volatility model. The volatility is non-constant and can be regarded as an endogenous factor in the sense that it is defined in terms of the *past behaviour* of the stock price. This is done in such a way that the price and volatility form a multi-dimensional Markov process.

A stock's volatility is the simplest measure of its risk less or uncertainty. Formally, the volatility σ_R is the annualized standard deviation of the stock's returns during the period of interest, where the subscript R denotes the observed or "realized" volatility.

The easy way to trade volatility is to use volatility swaps, sometimes called realized volatility forward contracts, because they provide pure exposure to volatility(and only to volatility).

Volatility swaps are forward contracts on future realized stock volatility, variance swaps are similar contract on variance, the square of the future volatility, both these instruments provide an easy way for investors to gain exposure to the future level of volatility.

Demeterfi, K., Derman, E., Kamal, M., and Zou, J. (1999) explained the properties and the theory of both variance and volatility swaps. They derived an analytical formula for theoretical fair value in the presence of realistic vilatility skews, and pointed out that volatility swaps can be replicated by dynamically trading the more straightforward variance swap.

Javaheri A, Wilmott, P. and Haug, E. G. (2002) discussed the valuation and hedging of a GARCH(1,1) stochastic volatility model. They used a general and exible PDE approach to determine the first two moments of the realized variance in a continuous or discrete context. Then they approximate the expected realized volatility via a convexity adjustment.

Brockhaus and Long (2000) provided an analytical approximation for the valuation of volatility swaps and analyzed other options with volatility exposure.

Working paper by Théoret, Zabré and Rostan (2002) presented an analytical solution for pricing of volatility swaps, proposed by Javaheri, Wilmott and Haug (2002). They priced the volatility swaps within framework of GARCH(1,1) stochastic volatility model and applied the analytical solution to price a swap on volatility of the $S\&P60$ Canada Index (5-year historical period: 1997−2002).

In the paper we propose a new probabilistic approach to the study of stochastic volatility model (Section 3), Heston (1993) model, to model variance and volatility swaps (Section 2). The Heston asset process has a variance σ_t^2 that follows a Cox, Ingersoll & Ross (1985) process. We find some analytical close forms for expectation and variance of the realized both continuously (Section 3.4) and discrete sampled variance (Section 3.5), which are needed for study of variance and volatility swaps, and price of pseudo-variance, pseudo-volatility, the problems proposed by He & Wang (2002) for financial markets with deterministic volatility as a function of time. This approach was first applied to the study of stochastic stability of Cox-Ingersoll-Ross process in Swishchuk and Kalemanova (2000).

The same expressions for $E[V]$ and for $Var[V]$ (like in present paper) were obtained by Brockhaus & Long (2000) using another analytical approach. Most articles on volatility products focus on the relatively straightforward variance swaps. They take the subject further with a simple model of volatility swaps.

We also study covariance and correlation swaps for the securities markets with two underlying assets with stochastic volatilities (Section 4).

As an application of our analytical solutions, we provide a numerical example using $S\&P60$ Canada Index to price swap on the volatility (Section 5).

2 Variance and Volatility Swaps

Volatility swaps are forward contracts on future realized stock volatility, variance swaps are similar contract on variance, the square of the future volatility, both these instruments provide an easy way for investors to gain exposure to the future level of volatility.

A stock's volatility is the simplest measure of its risk less or uncertainty. Formally, the volatility $\sigma_R(S)$ is the annualized standard deviation of the stock's returns during the period of interest, where the subscript R denotes the observed or "realized" volatility for the stock S.

The easy way to trade volatility is to use volatility swaps, sometimes called realized volatility forward contracts, because they provide pure exposure to volatility (and only to volatility) (see Demeterfi, K., Derman, E., Kamal, M., and Zou, J. (1999)).

A stock *volatility swap* is a forward contract on the annualized volatility. Its payoff at expiration is equal to

$$N(\sigma_R(S) - K_{vol}),$$

where $\sigma_R(S)$ is the realized stock volatility (quoted in annual terms) over the life of contract,

$$\sigma_R(S) := \sqrt{\frac{1}{T} \int_0^T \sigma_s^2 ds},$$

σ_t is a stochastic stock volatility, K_{vol} is the annualized volatility delivery price, and N is the notional amount of the swap in dollar per annualized volatility point. The holder of a volatility swap at expiration receives N dollars for every point by which the stock's realized volatility σ_R has exceeded the volatility delivery price K_{vol}. The holder is swapping a fixed volatility K_{vol} for the actual (floating) future volatility σ_R. We note that usually $N = \alpha I$, where α is a converting parameter such as 1 per volatility-square, and I is a long-short index (+1 for long and −1 for short).

Although options market participants talk of volatility, it is variance, or volatility squared, that has more fundamental significance (see Demeterfi, K., Derman, E., Kamal, M., and Zou, J. (1999)).

A variance swap is a forward contract on annualized variance, the square of the realized volatility. Its payoff at expiration is equal to

$$N(\sigma_R^2(S) - K_{var}),$$

where $\sigma_R^2(S)$ is the realized stock variance (quoted in annual terms) over the life of the contract,

$$\sigma_R^2(S) := \frac{1}{T} \int_0^T \sigma_s^2 ds,$$

K_{var} is the delivery price for variance, and N is the notional amount of the swap in dollars per annualized volatility point squared. The holder of variance swap at expiration receives N dollars for every point by which the stock's realized variance $\sigma_R^2(S)$ has exceeded the variance delivery price K_{var}.

Therefore, pricing the variance swap reduces to calculating the realized volatility square.

Valuing a variance forward contract or swap is no different from valuing any other derivative security. The value of a forward contract F on future realized variance with strike price K_{var} is the expected present value of the future payoff in the risk-neutral world:

$$F = E\left\{e^{-rT}\left(\sigma_R^2(S) - K_{var}\right)\right\},$$

where r is the risk-free discount rate corresponding to the expiration date T, and E denotes the expectation.

Thus, for calculating variance swaps we need to know only $E\{\sigma_R^2(S)\}$, namely, mean value of the underlying variance.

To calculate volatility swaps we need more. From Brockhaus-Long (2000) approximation (which is used the second order Taylor expansion for function $\sqrt{x}$) we have (see also Javaheri *et al* (2002), p.16):

$$E\left\{\sqrt{\sigma_R^2(S)}\right\} \approx \sqrt{E\{V\}} - \frac{Var\{V\}}{8E\{V\}^{3/2}},$$

where $V := \sigma_R^2(S)$ and $\frac{Var\{V\}}{8E\{V\}^{3/2}}$ is the convexity adjustment.

Thus, to calculate volatility swaps we need both $E\{V\}$ and $Var\{V\}$.

The realised continuously sampled variance is defined in the following way:

$$V := Var(S) := \frac{1}{T} \int_0^T \sigma_t^2 dt.$$

The realised discrete sampled variance is defined as follows:

$$Var_n(S) := \frac{n}{(n-1)T} \sum_{i=1}^n \log^2 \frac{S_{t_i}}{S_{t_{i-1}}},$$

where we neglected by $\frac{1}{n}\sum_{i=1}^n \log \frac{S_{t_i}}{S_{t_{i-1}}}$ since we assume that the mean of the returns is of the order $\frac{1}{n}$ and can be neglected. The scaling by $\frac{n}{T}$ ensures that these quantities annualized (daily) if the maturity T is expressed in years (days).

$Var_n(S)$ is unbiased variance estimation for σ_t. It can be shown that (see Brockhaus & Long (2000))

$$V := Var(S) = \lim_{n \to +\infty} Var_n(S).$$

Realised discrete sampled volatility is given by:

$$Vol_n(S) := \sqrt{Var_n(S)}.$$

Realised continuously sampled volatility is defined as follows:

$$Vol(S) := \sqrt{Var(S)} = \sqrt{V}.$$

The expressions for V, $Var_n(S)$ and $Vol(S)$ are used for calculation of variance and volatility swaps.

3 Variance and Volatility Swaps for Heston Model of Securities Markets

3.1 Stochastic Volatility Model

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be probability space with filtration $\mathcal{F}_t, t \in [0, T]$.

Assume that underlying asset S_t in the risk-neutral world and variance follow the following model, Heston (1993) model:

$$\begin{cases} \dfrac{dS_t}{S_t} = r_t dt + \sigma_t dw_t^1 \\ d\sigma_t^2 = k(\theta^2 - \sigma_t^2)dt + \gamma \sigma_t dw_t^2, \end{cases} \qquad (1)$$

where r_t is deterministic interest rate, σ_0 and θ are short and long volatility, $k > 0$ is a reversion speed, $\gamma > 0$ is a volatility (of volatility) parameter, w_t^1 and w_t^2 are independent standard Wiener processes.

The Heston asset process has a variance σ_t^2 that follows Cox-Ingersoll-Ross (1985) process, described by the second equation in (1).

If the volatility σ_t follows Ornstein-Uhlenbeck process (see, for example, Øksendal (1998)), then Ito's lemma shows that the variance σ_t^2 follows the process described exactly by the second equation in (1).

3.2 Explicit Expression for σ_t^2

In this section we propose a new probabilistic approach to solve the equation for variance σ_t^2 in (1) explicitly, using change of time method (see Ikeda and Watanabe (1981)).

Define the following process:

$$v_t := e^{kt}(\sigma_t^2 - \theta^2). \qquad (2)$$

Then, using Ito formula (see Øksendal (1995)) we obtain the equation for v_t :

$$dv_t = \gamma e^{kt}\sqrt{e^{-kt}v_t + \theta^2}dw_t^2. \qquad (3)$$

Using change of time approach to the general equation (see Ikeda and Watanabe (1981))

$$dX_t = \alpha(t, X_t)dw_t^2,$$

we obtain the following solution of the equation (3):

$$v_t = \sigma_0^2 - \theta^2 + \tilde{w}^2(\phi_t^{-1}),$$

or (see (2)),

$$\sigma_t^2 = e^{-kt}(\sigma_0^2 - \theta^2 + \tilde{w}^2(\phi_t^{-1})) + \theta^2, \qquad (4)$$

where $\tilde{w}^2(t)$ is an $\mathcal{F}_t$-measurable one-dimensional Wiener process, ϕ_t^{-1} is an inverse function to ϕ_t:

$$\phi_t = \gamma^{-2}\int_0^t \left\{ e^{k\phi_s}(\sigma_0^2 - \theta^2 + \tilde{w}^2(s)) + \theta^2 e^{2k\phi_s} \right\}^{-1} ds.$$

3.3 Properties of Processes $\tilde{\omega}^2(\phi_t^{-1})$ and σ_t^2

The properties of $\tilde{w}^2(\phi_t^{-1}) := b(t)$ are the following:

$$Eb(t) = 0; \qquad (5)$$

$$E(b(t))^2 = \gamma^2 \left\{ \frac{e^{kt}-1}{k}(\sigma_0^2 - \theta^2) + \frac{e^{2kt}-1}{2k}\theta^2 \right\}; \qquad (6)$$

$$Eb(t)b(s) = \gamma^2 \left\{ \frac{e^{k(t\wedge s)}-1}{k}(\sigma_0^2 - \theta^2) + \frac{e^{2k(t\wedge s)}-1}{2k}\theta^2 \right\}, \qquad (7)$$

where $t \wedge s := \min(t, s)$.

Using representation (4) and properties (5)–(7) of $b(t)$ we obtain the properties of σ_t^2. Straightforward calculations give us the following results:

$$E\sigma_t^2 = e^{-kt}(\sigma_0^2 - \theta^2) + \theta^2;$$

$$\begin{aligned} E\sigma_t^2\sigma_s^2 = \gamma^2 e^{-k(t+s)}\{ &\frac{e^{k(t\wedge s)}-1}{k}(\sigma_0^2 - \theta^2) \\ &+ \frac{e^{2k(t\wedge s)}-1}{2k}\theta^2\} + e^{-k(t+s)}(\sigma_0^2 - \theta^2)^2 \\ &+ e^{-kt}(\sigma_0^2 - \theta^2)\theta^2 + e^{-ks}(\sigma_0^2 - \theta^2)\theta^2 + \theta^4. \end{aligned} \qquad (8)$$

3.4 Valuing Variance and Volatility Swaps

From formula (8) we obtain mean value for V:

$$\begin{aligned} E\{V\} &= \frac{1}{T}\int_0^T E\sigma_t^2 dt \\ &= \frac{1}{T}\int_0^T \{e^{-kt}(\sigma_0^2 - \theta^2) + \theta^2\}dt \\ &= \frac{1 - e^{-kT}}{kT}(\sigma_0^2 - \theta^2) + \theta^2. \end{aligned} \qquad (9)$$

The same expression for $E[V]$ may be found in Brockhaus and Long (2000).

Substituting $E[V]$ from (9) into formula

$$F = e^{-rT}(E\{\sigma_R^2(S)\} - K_{var}) \qquad (10)$$

we obtain the value of the variance swap.

Variance for V equals to:

$$Var(V) = EV^2 - (EV)^2.$$

From (9) we have:

$$(EV)^2 = \frac{1 - 2e^{-kT} + e^{-2kT}}{k^2T^2}(\sigma_0^2 - \theta^2)^2 + \frac{2(1 - e^{-kT})}{kT}(\sigma_0^2 - \theta^2)\theta^2 + \theta^4. \qquad (11)$$

Second moment may found as follows using formula (8):

$$EV^2 = \frac{1}{T^2}\int_0^T\int_0^T E\sigma_t^2\sigma_s^2 dtds$$

$$= \frac{\gamma^2}{T^2} \int_0^T \int_0^T e^{-k(t+s)} \left\{ \frac{e^{k(t \wedge s)} - 1}{k} (\sigma_0^2 - \theta^2) + \frac{e^{2k(t \wedge s)} - 1}{2k} \theta^2 \right\} dt ds$$
$$+ \frac{1 - 2e^{-kT} + e^{-2kT}}{k^2 T^2} (\sigma_0^2 - \theta^2)^2 + \frac{2(1 - e^{-kT})}{kT} (\sigma_0^2 - \theta^2)\theta^2 + \theta^4. \tag{12}$$

Taking into account (11) and (12) we obtain:

$$Var(V) = EV^2 - (EV)^2$$
$$= \frac{\gamma^2}{T^2} \int_0^T \int_0^T e^{-k(t+s)} \left\{ \frac{e^{k(t \wedge s)} - 1}{k} (\sigma_0^2 - \theta^2) + \frac{e^{2k(t \wedge s)} - 1}{2k} \theta^2 \right\} dt ds.$$

After calculations the last expression we obtain the following expression for variance of V:

$$Var(V) = \frac{\gamma^2 e^{-2kT}}{2k^3 T^2} [(2e^{2kT} - 4e^{kT}kT - 2)(\sigma_0^2 - \theta^2) \tag{13}$$
$$+ (2e^{2kT}kT - 3e^{2kT} + 4e^{kT} - 1)\theta^2].$$

Similar expression for $Var[V]$ may be found in Brockhaus and Long (2000).

Substituting EV from (9) and $Var(V)$ from (13) into formula

$$F = \{e^{-rT}(E\{\sigma_R(S)\} - K_{var})\} \tag{14}$$

with

$$E\{\sigma_R(S)\} = E\{\sqrt{\sigma_R^2(S)}\} \approx \sqrt{E\{V\}} - \frac{Var\{V\}}{8E\{V\}^{3/2}},$$

we obtain the value of volatility swap.

3.5 Calculation of $E\{V\}$ in Discrete Case

The realised discrete sampled variance:

$$Var_n(S) := \frac{n}{(n-1)T} \sum_{i=1}^n \log^2 \frac{S_{t_i}}{S_{t_{i-1}}},$$

where we neglected by $\frac{1}{n} \sum_{i=1}^n \log \frac{S_{t_i}}{S_{t_{i-1}}}$ for simplicity reason only. We note that

$$\log \frac{S_{t_i}}{S_{t_{i-1}}} = \int_{t_{i-1}}^{t_i} (r_t - \sigma_t^2/2)dt + \int_{t_{i-1}}^{t_i} \sigma_t dw_t^1.$$

$$E\{Var_n(S)\} = \frac{n}{(n-1)T} \sum_{i=1}^n E\{\log^2 \frac{S_{t_i}}{S_{t_{i-1}}}\}.$$

$$E\left\{\log^2 \frac{S_{t_i}}{S_{t_{i-1}}}\right\} = \left(\int_{t_{i-1}}^{t_i} r_t dt\right)^2 - \int_{t_{i-1}}^{t_i} r_t dt \int_{t_{i-1}}^{t_i} E\sigma_t^2/2 dt + \frac{1}{4} \int_{t_{i-1}}^{t_i} \int_{t_{i-1}}^{t_i} E\sigma_t^2 \sigma_s^2 dt ds$$
$$- E\left(\int_{t_{i-1}}^{t_i} \sigma_t^2/2 dt \int_{t_{i-1}}^{t_i} \sigma_t dw_t^1\right) + \int_{t_{i-1}}^{t_i} E\sigma_t^2 dt.$$

We know the expressions for $E\sigma_t^2$ and for $E\sigma_t^2\sigma_s^2$, and the fourth expression is equal to zero. Hence, we can easily calculate all the above expressions and, hence, $E\{Var_n(S)\}$ and variance swap in this case.

Remark 1. Some expressions for price of the realised discrete sampled variance $Var_n(S) := \frac{n}{(n-1)T} \sum_{i=1}^n \log^2 \frac{S_{t_i}}{S_{t_{i-1}}}$, (or pseudo-variance) were obtained in the Proceedings of the 6th PIMS Industrial Problems Solving Workshop, PIMS IPSW 6, UBC, Vancouver, Canada, May 27-31, 2002. Editor: J. Macki, University of Alberta, Canada, June, 2002, pp.45–55.

4 Covariance and Correlation Swaps for Two Assets with Stochastic Volatilities

4.1 Definitions of Covariance and Correlation Swaps

Option dependent on exchange rate movements, such as those paying in a currency different from the underlying currency, have an exposure to movements of the correlation between the asset and the exchange rate, this risk may be eliminated by using covariance swap.

A *covariance swap* is a covariance forward contact of the underlying rates S^1 and S^2 which payoff at expiration is equal to

$$N(Cov_R(S^1, S^2) - K_{cov}),$$

where K_{cov} is a strike price, N is the notional amount, $Cov_R(S^1, S^2)$ is a covariance between two assets S^1 and S^2.

Logically, a *correlation swap* is a correlation forward contract of two underlying rates S^1 and S^2 which payoff at expiration is equal to:

$$N(Corr_R(S^1, S^2) - K_{corr}),$$

where $Corr(S^1, S^2)$ is a realized correlation of two underlying assets S^1 and S^2, K_{corr} is a strike price, N is the notional amount.

Pricing covariance swap, from a theoretical point of view, is similar to pricing variance swaps, since

$$Cov_R(S^1, S^2) = 1/4\{\sigma_R^2(S^1 S^2) - \sigma_R^2(S^1/S^2)\}$$

where S^1 and S^2 are given two assets, $\sigma_R^2(S)$ is a variance swap for underlying assets, $Cov_R(S^1, S^2)$ is a realized covariance of the two underlying assets S^1 and S^2.

Thus, we need to know variances for $S^1 S^2$ and for S^1/S^2 (see Section 4.2 for details). Correlation $Corr_R(S^1, S^2)$ is defined as follows:

$$Corr_R(S^1, S^2) = \frac{Cov_R(S^1, S^2)}{\sqrt{\sigma_R^2(S^1)}\sqrt{\sigma_R^2(S^2)}},$$

where $Cov_R(S^1, S^2)$ is defined above and $\sigma_R^2(S^1)$ in Section 3.4.

Given two assets S_t^1 and S_t^2 with $t \in [0, T]$, sampled on days $t_0 = 0 < t_1 < t_2 < \ldots < t_n = T$ between today and maturity T, the log-return each asset is:

$$R_i^j := \log(\frac{S_{t_i}^j}{S_{t_{i-1}}^j}), \quad i = 1, 2, \ldots, n, \quad j = 1, 2.$$

Covariance and correlation can be approximated by

$$Cov_n(S^1, S^2) = \frac{n}{(n-1)T} \sum_{i=1}^{n} R_i^1 R_i^2$$

and

$$Corr_n(S^1, S^2) = \frac{Cov_n(S^1, S^2)}{\sqrt{Var_n(S^1)}\sqrt{Var_n(S^2)}},$$

respectively.

4.2 Valuing of Covariance and Correlation Swaps

To value covariance swap we need to calculate the following

$$F = \{e^{-rT}(ECov(S^1, S^2) - K_{cov}\}. \tag{15}$$

To calculate $ECov(S^1, S^2)$ we need to calculate $E\{\sigma_R^2(S^1 S^2) - \sigma_R^2(S^1/S^2)\}$ for a given two assets S^1 and S^2.

Let $S_t^i, i = 1, 2$, be two strictly positive Ito's processes given by the following model

$$\begin{cases} \frac{dS_t^i}{S_t^i} = \mu_t^i dt + \sigma_t^i dw_t^i, \\ d(\sigma^i)_t^2 = k^i(\theta_i^2 - (\sigma^i)_t^2)dt + \gamma^i \sigma_t^i dw_t^j, \quad i = 1, 2, \quad j = 3, 4, \end{cases} \tag{16}$$

where $\mu_t^i, i = 1, 2$, are deterministic functions, $k^i, \theta^i, \gamma^i, i = 1, 2$, are defined in similar way as in (1), standard Wiener processes $w_t^j, j = 3, 4$, are independent, $[w_t^1, w_t^2] = \rho_t dt$, ρ_t is deterministic function of time, $[,]$ means the quadratic covariance, and standard Wiener processes $w_t^i, i = 1, 2$, and $w_t^j, j = 3, 4$, are independent.

We note that

$$d \ln S_t^i = m_t^i dt + \sigma_t^i dw_t^i, \tag{17}$$

where

$$m_t^i := (\mu_t^i - \frac{(\sigma_t^i)^2}{2}), \tag{18}$$

and

$$Cov_R(S_T^1, S_T^2) = \frac{1}{T}\left[\ln S_T^1, \ln S_T^2\right]$$
$$= \frac{1}{T}\left[\int_0^T \sigma_t^1 dw_t^1, \int_0^T \sigma_t^2 dw_t^2\right] = \frac{1}{T}\int_0^T \rho_t \sigma_t^1 \sigma_t^2 dt. \tag{19}$$

Let us show that

$$\left[\ln S_T^1, \ln S_T^2\right] = \frac{1}{4}\left([\ln(S_T^1 S_T^2)] - [\ln(S_T^1/S_T^2)]\right). \tag{20}$$

Remark first that

$$d \ln(S_t^1 S_t^2) = (m_t^1 + m_t^2)dt + \sigma_t^+ dw_t^+, \tag{21}$$

and

$$d \ln(S_t^1/S_t^2) = (m_t^1 - m_t^2)dt + \sigma_t^- dw_t^-, \tag{22}$$

where

$$(\sigma_t^\pm)^2 := (\sigma_t^1)^2 \pm 2\rho_t \sigma_t^1 \sigma_t^2 + (\sigma_t^2)^2, \tag{23}$$

and

$$dw_t^\pm := \frac{1}{\sigma_t^\pm}(\sigma_t^1 dw_t^1 \pm \sigma_t^2 dw_t^2). \tag{24}$$

Processes $w_t^\pm$ in (24) are standard Wiener processes by Levi-Kunita-Watanabe theorem and $\sigma_t^\pm$ are defined in (23).

In this way, from (21) and (22) we obtain that

$$[\ln(S_t^1 S_t^2)] = \int_0^t (\sigma_s^+)^2 ds = \int_0^t ((\sigma_s^1)^2 + 2\rho_t \sigma_s^1 \sigma_s^2 + (\sigma_s^2)^2)ds, \tag{25}$$

and

$$[\ln(S_t^1/S_t^2)] = \int_0^t (\sigma_s^-)^2 ds = \int_0^t ((\sigma_s^1)^2 - 2\rho_t \sigma_s^1 \sigma_s^2 + (\sigma_s^2)^2)ds. \tag{26}$$

From (20), (25) and (26) we have directly formula (20):

$$[\ln S_T^1, \ln S_T^2] = \frac{1}{4}([\ln(S_T^1 S_T^2)] - [\ln(S_T^1/S_T^2)]). \tag{27}$$

Thus, from (27) we obtain that (see (20) and Section 4.1))

$$Cov_R(S^1, S^2) = 1/4(\sigma_R^2(S^1 S^2) - \sigma_R^2(S^1/S^2)).$$

Returning to the valuation of the covariance swap we have

$$F = E\{e^{-rT}(Cov(S^1, S^2) - K_{cov}\} = \frac{1}{4}e^{-rT}(E\sigma_R^2(S^1 S^2) - E\sigma_R^2(S^1/S^2) - 4K_{cov}).$$

The problem now has reduced to the same problem as in the Section 3, but instead of σ_t^2 we need to take $(\sigma_t^+)^2$ for $S^1 S^2$ and $(\sigma_t^-)^2$ for S^1/S^2 (see (23)), and proceed with the similar calculations as in Section 3.

Remark 2. The results of the Sections 2-4 were presented on the Sixth Annual Financial Econometrics Conference "Estimation of Diffusion Processes in Finance", Friday, March 19, 2004, Centre for Advanced Studies in Finance, University of Waterloo, Waterloo, Canada (*Abstract online: http://arts.uwaterloo.ca/ACCT/finance/fec6.htm*).

5 Numerical Example: S&P60 Canada Index

In this section, we apply the analytical solutions from Section 3 to price a swap on the volatility of the S&P60 Canada index for five years (January 1997-February 2002).

These data were kindly presented to author by Raymond Théoret (Université du Québec à Montréal, Montréal, Québec, Canada) and Pierre Rostan (Analyst at the R&D Department of Bourse de Montréal and

Université du Québec à Montréal, Montréal, Québec, Canada). They calibrated the GARCH parameters from five years of daily historic *S&P*60 Canada Index (from January 1997 to February 2002) (see working paper "Pricing volatility swaps: Empirical testing with Canadian data" by R. Theoret, L. Zabre and P. Rostan (2002)).

In the end of February 2002, we wanted to price the fixed leg of a volatility swap based on the volatility of the S&P60 Canada index. The statistics on log returns *S&P*60 Canada Index for 5 year (January 1997-February 2002) is presented in Table 1:

TABLE 1: STATISTICS ON LOG RETURNS S&P60 CANADA INDEX

Series:	LOG RETURNS *S&P*60 CANADA INDEX
Sample:	1 1300
Observations:	1300
Mean	0.000235
Median	0.000593
Maximum	0.051983
Minimum	−0.101108
Std. Dev.	0.013567
Skewness	−0.665741
Kurtosis	7.787327

From the histogram of the S&P60 Canada index log returns on a 5-year historical period (1,300 observations from January 1997 to February 2002) it may be seen leptokurtosis in the histogram. If we take a look at the graph of the S&P60 Canada index log returns on a 5-year historical period we may see volatility clustering in the returns series. These facts indicate about the conditional heteroscedasticity. A GARCH(1,1) regression is applied to the series and the results is obtained as in the next Table 2:

This table allows to generate different input variables to the volatility swap model.

We use the following relationship

$$\theta = \frac{V}{dt},$$

$$k = \frac{1 - \alpha - \beta}{dt},$$

$$\gamma = \alpha\sqrt{\frac{\xi - 1}{dt}},$$

to calculate the following discrete GARCH(1,1) parameters:

ARCH(1,1) coefficient $\alpha = 0.060445$;
GARCH(1,1) coefficient $\beta = 0.927264$;

TABLE 2: ESTIMATION OF THE GARCH(1,1) PROCESS

Dependent Variable: Log returns of S&P60 Canada Index Prices				
Method: ML-ARCH				
Included Observations: 1300				
Convergence achieved after 28 observations				
-	**Coefficient**:	**Std. error**:	**z-statistic**:	**Prob.**
C	0.000617	0.000338	1.824378	0.0681
Variance Equation				
C	2.58E-06	3.91E-07	6.597337	0
ARCH(1)	0.060445	0.007336	8.238968	0
GARCH(1)	0.927264	0.006554	141.4812	0
R-squared	−0.000791	Mean dependent var	-	0.000235
Adjusted R-squared	−0.003108	S.D. dependent var	-	0.013567
S.E. of regression	0.013588	Akaike info criterion	-	-5.928474
Sum squared resid	0.239283	Schwartz criterion	-	-5.912566
Log likelihood	3857.508	Durbin-Watson stat	-	1.886028

the Pearson kurtosis (fourth moment of the drift-adjusted stock return) $\xi = 7.787327$;
long volatility $\theta = 0.05289724$;
$k = 3.09733$;
$\gamma = 2.499827486$;
a short volatility σ_0 equals to 0.01;
Parameter V may be found from the expression $V = \frac{C}{1-\alpha-\beta}$, where $C = 2.58 \times 10^{-6}$ is defined in Table 2. Thus, $V = 0.00020991$;
$dt = 1/252 = 0.003968254$.

Now, applying the analytical solutions (9) and (13) for a swap maturity T of 0.91 year, we find the following values:

$$E\{V\} = \frac{1 - e^{-kT}}{kT}(\sigma_0^2 - \theta^2) + \theta^2 = .3364100835,$$

and

$$Var(V) = \frac{\gamma^2 e^{-2kT}}{2k^3 T^2}\left[(2e^{2kT} - 4e^{kT}kT - 2)(\sigma_0^2 - \theta^2) + (2e^{2kT}kT - 3e^{2kT} + 4e^{kT} - 1)\theta^2\right] = .0005516049969.$$

The convexity adjustment $\frac{Var\{V\}}{8E\{V\}^{3/2}}$ is equal to .0003533740855.
If the non-adjusted strike is equal to 18.7751%, then the adjusted strike is equal to

$$18.7751\% - 0.03533740855\% = 18.73976259\%.$$

This is the fixed leg of the volatility swap for a maturity $T = 0.91$.

Repeating this approach for a series of maturities up to 10 years we obtain the following plot (see Appendix, Figure 2) of *S&P*60 Canada Index Volatility Swap.

Figure 1 (see Appendix) illustrates the non-adjusted and adjusted volatility for the same series of maturities.

6 Conclusions

In this paper, we presented a new probabilitic approach, based on changing of time method, to study variance and volatility swaps for financial markets with underlying asset and variance that follow the Heston (1993) model.

We obtained the formulas for variance and volatility swaps which are similar to those in the paper by Brockhaus & Long (2000), obtained by another analytical approach.

We also studied covariance and correlation swaps for the financial markets with two underlying assets with stochastic volatilities.

As an application of our analytical solutions, we provided a numerical example using *S&P*60 Canada Index to price swap on the volatility.
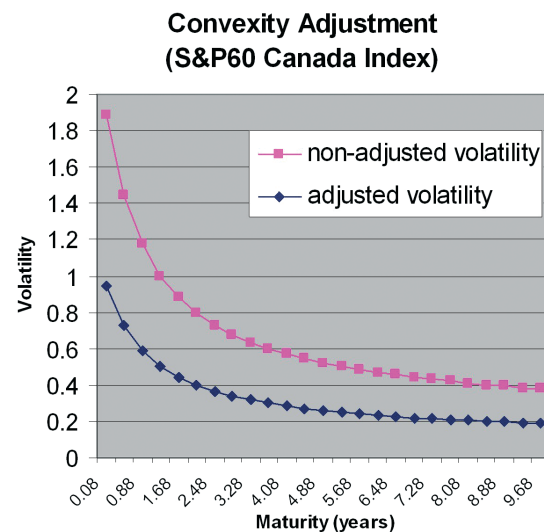
7 Appendix: Figures.
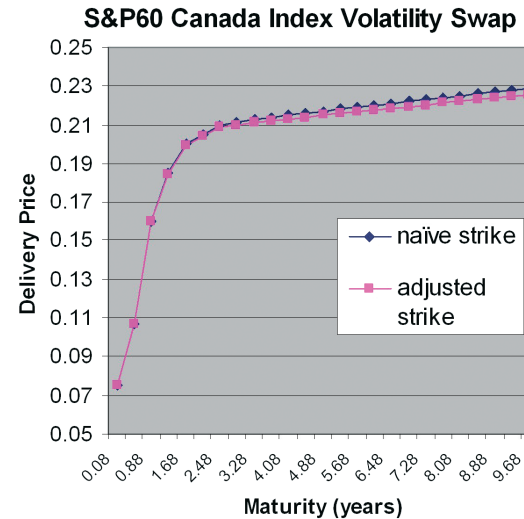

Figure 1: Convexity Adjustment

Figure 2: *S&P*60 Canada Index Volatility Swap

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