CQF Module 1.2 Exercises

1. The function $f(x)$ has the value 3 at $x = 0$. Its first derivative at zero is 1 and its second derivative is -2. Using Taylor series estimate the function's value at $x = 1$.

$$f(1) = f(0) + (1-0)f'(0) + \frac{1}{2}(1-0)^2 f''(0) + \cdots$$

$$f(1) \approx 3 + 1 \times 1 + \frac{1}{2} \times 1^2 \times (-2) = 3.$$

2. The exponential function e^x is special since its gradient is the same as the function itself:

$$\frac{d(e^x)}{dx} = e^x.$$

Use this fact to find the Taylor series expansion of e^x about zero. (Do not stop after the second term but write down the infinite expression).

Let us be ambitious, and first find the general expression for the terms in Taylor series. (This isn't a proof that Taylor series works... but it is a start.)

Assume that

$$f\ (x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots . \tag{1}$$

Put $x = 0$ into this expression:

$$f\ (0) = a_0 + a_1 0 + a_2 0^2 + \cdots + a_n 0^n + \cdots = a + 0.$$

Therefore

$$a_0 = f\ (0).$$

Now differentiate (1) with respect to x to get

$$f\ '(x) = a_1 + 2a_2 x + \cdots + n a_n x^{n-1} + \cdots \tag{2}$$

We are now using the short-hand notation

$$f\ '(x) = \frac{df}{dx}.$$

Again, put $x = 0$ into expression (2):

$$f\ '(0) = a_1 + 2a_2 0 + \cdots + n a_n 0^{n-1} + \cdots = a_1.$$

Therefore
$$a_1 = f\,'(0).$$
Now differentiate (2) to get
$$f\,''(x) = 2a_2 + \cdots + n(n-1)a_n x^{n-2} + \cdots . \tag{3}$$
Again, put $x = 0$ into expression (3):
$$f\,''(0) = 2a_2 + \cdots + n(n-1)a_n 0^{n-2} + \cdots = 2a_2,$$
so that
$$a_2 = \frac{1}{2} f\,''(0).$$
Keep differentiating the series and substituting in zero to find that
$$a_n = \frac{1}{n!} f\,^{(n)}(0).$$
This gives us the general series expansion
$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} f^{(n)}(0).$$
Notes:

1. We have demonstrated two different notations for the nth derivative, $f\,^{(n)}$ and $d^n f/dx^n$.

2. The expansion does not have to be about $x = 0$. In general
$$f(x) = \sum_{n=0}^{\infty} \frac{(x-x_0)^n}{n!} f\,^{(n)}(x_0).$$
Now to the specific example in the question.
Since
$$\frac{d(e^x)}{dx} = e^x$$
we know that
$$\frac{d^n(e^x)}{dx^n} = e^x.$$
Therefore
$$\frac{d^n(e^x)}{dx^n}(0) = e^0 = 1.$$
and so
$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

3. Find the Taylor series expansions of the following functions about $x = 0$ (*by first using a Binomial expansion in part a) and then considering how the function in part b) is related to that in part a)*).

a)

$$f(x) = \frac{1}{1-x}.$$

b)

$$f(x) = \ln(1+x).$$

Binomial expansion gives

$$f\ \ (x) = (1-x)^{-1} = 1 + x + x^2 + x^3 + \text{.........} + x^n = \sum_{n=0}^{\infty} x^n$$

Note: Taylor series give us an expansion for functions (in terms of polynomials) in the neighbourhood of a point $x = a$. When $a = 0$, then we call it a Maclaurin Series, and if the function is already a polynomial, then the method is called the Binomial Expansion, i.e.

$$(1+nx)^p = 1 + pnx + \frac{p\,(p-1)}{2!}\,(nx)^2 + + \frac{p\,(p-1)\,(p-2)}{3!}\,(nx)^3 + O\left(x^4\right),$$

so Maclarin and Binomial are just special cases of Taylor series.

(b) Using above

$$\frac{1}{1+x} = \frac{1}{1-(-)\,x} = (1+x)^{-1} = 1 - x + x^2 - x^3 + \text{.........} + (-1)^n\ x^n = \sum_{n=0}^{\infty} (-1)^n\ x^n \qquad (**)$$

Now

$$\log\left(1+x\right) = \int_0^x \frac{1}{1+t}dt$$

so we integrate each term in the series for $(1+x)^{-1}$ giving

$$\begin{aligned} \frac{1}{1+x} &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}\text{.........} + (-1)^{n+1}\,\frac{x^n}{n} = \sum_{n=1}^{\infty} (-1)^{n+1}\,\frac{x^n}{n} \\ &= \sum_{n=0}^{\infty} (-1)^n\,\frac{x^{n+1}}{n+1} \end{aligned}$$

This final result is what we would have expected as it is the integral of the power series given by $(**)$ above.

4. Find the limiting behaviour of

$$\left(1+\frac{x}{n}\right)^{nt}$$

as $n \to \infty$.

$$\begin{aligned}
& \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{nt} \\
\equiv\ & \lim_{n\to\infty}\exp\left[nt\left(\log\left(1+\frac{x}{n}\right)\right)\right] \\
=\ & \lim_{n\to\infty}\exp\left[nt\left(\frac{x}{n}-\frac{x^2}{2n^2}+\frac{x^3}{3n^3}-\frac{x^4}{4n^4}+......\right)\right] \\
=\ & \lim_{n\to\infty}\exp\left[t\left(x-\frac{x^2}{2n}+\frac{x^3}{3n^2}-\frac{x^4}{4n^3}+......\right)\right] \\
=\ & \lim_{n\to\infty}\exp\left[t\left(x\right)\right]=\exp\left(tx\right)
\end{aligned}$$

5. Evaluate the following

$$\int\frac{1}{1+e^{-x}}\,dx\ ; \qquad \int_3^4\frac{3x^3-18x^2+29x-4}{(x+1)\,(x-2)^3}\,dx\ ; \qquad \int_0^4\frac{x}{\sqrt{x^2+9}}\,dx\ ; \qquad \int e^x\cos x\,dx$$

$$\int\frac{1}{1+e^{-x}}\,dx$$

Multiply both numerator and denominator by e^x to give $\int\frac{e^x}{e^x+1}\,dx \equiv \int\frac{f\,'(x)}{f(x)}\,dx$ and hence

$$\int\frac{1}{1+e^{-x}}\,dx=\ln\left|1+e^x\right|+K$$

$$\int_3^4\frac{3x^3-18x^2+29x-4}{(x+1)\,(x-2)^3}\,dx$$

The integrand here is of the form $\frac{P(x)}{Q(x)}$, where both numerator and denominator are polynomials in x. However, because degree of $Q(x)$ >degree of $P(x)$ we can express this as a partial fraction. So

$$\frac{3x^3-18x^2+29x-4}{(x+1)\,(x-2)^3}\equiv\frac{A}{(x+1)}+\frac{B}{(x-2)}+\frac{C}{(x-2)^2}+\frac{D}{(x-2)^3}$$

which gives

$$A=2;\quad B=1;\quad C=-3;\quad D=2,$$

hence we have

$$\begin{aligned}\int \frac{3x^3-18x^2+29x-4}{(x+1)(x-2)^3} &\equiv \int \frac{2}{(x+1)}+\int \frac{1}{(x-2)}-\int \frac{3}{(x-2)^2}+\int \frac{2}{(x-2)^3} \\ &= 2\ln|x+1|+\ln|x-2|+\frac{3}{(x-2)}-\frac{1}{(x-2)^2}.\end{aligned}$$

We can simplify this to obtain

$$\ln(x+1)^2|x-2|+\frac{3x-7}{(x-2)^2}.$$

So the definite integral

$$\begin{aligned}\int_3^4 \frac{3x^3-18x^2+29x-4}{(x+1)(x-2)^3}\,dx &= \ln(x+1)^2|x-2|+\frac{3x-7}{(x-2)^2}\bigg|_3^4=\left(\ln 50+\tfrac{5}{4}\right)-(\ln 16+2) \\ &= \ln\tfrac{25}{8}-\tfrac{3}{4}\end{aligned}$$

$$\int_0^4 \frac{x}{\sqrt{x^2+9}}\,dx$$

We note that the integrand is of the form $g\ (u\ (x))\ u'(x)$, where

$$u\ (x)=x^2+9;\quad g(\ u\ (x))=\frac{1}{2}(\ u\ (x))^{-1/2}.$$

So we can use a substitution, $u=x^2+9\to du=2x\ dx\ \therefore$

$$\begin{aligned}\int \frac{x}{\sqrt{x^2+9}}\,dx &\equiv \int x\ u^{-1/2}\frac{du}{2x}=\frac{1}{2}\int u^{-1/2}\ du=u^{1/2} \\ &= \sqrt{x^2+9}\Big|_0^4=2\end{aligned}$$

We could alternatively have changed the limits in $x\in[0,4]$ to limits in $u\in[25,9]$ when we performed the substitution and not reverted back to x. This makes no difference and is a personal choice.

$$\int e^x\cos x\ dx$$

We begin by writing $I=\displaystyle\int e^x\cos x\ dx$ and using integration by parts (twice). Put

$$\begin{aligned}v &= e^x\to v'=e^x \\ u' &= \cos x\to u=\sin x\end{aligned}$$

so

$$I = e^x \sin x - \int e^x \sin x \; dx.$$

Now use integration by parts for the second time on the integral $\int e^x \sin x \; dx$ by putting

$$\begin{aligned} v &= e^x \to v' = e^x \\ u' &= \sin x \to u = -\cos x \end{aligned}$$

and we get

$$\begin{aligned} \int e^x \sin x \; dx &= -e^x \cos x + \int e^x \cos x \; dx \\ &= -e^x \cos x + I \end{aligned}$$

therefore

$$\begin{aligned} I &= e^x \sin x - (-e^x \cos x + I) = e^x \sin x + e^x \cos x - I \\ \therefore \quad I &= \int e^x \cos x \; dx = \frac{1}{2} e^x (\sin x + \cos x) + C. \end{aligned}$$

6. Solve the following

(a) $x^2 \dfrac{dy}{dx} + 5xy + 3x^5 = 0$

Upon inspection it is clear that the ODE is not of the separable kind. If we re-write as

$$y' + \frac{5}{x} y = -3x^3$$

where $y' \equiv \dfrac{dy}{dx}$, we note that this is a linear equation, i.e. $y' + P(x)\, y = Q(x)$. So $P(x) = \frac{5}{x}$, to give an integrating factor $I\ (x) = \exp \int \frac{5}{x} dx = x^5$. Multiply the ODE throughout by $I\ (x)$ to obtain

$$x^5 y' + 5x^4 y = -3x^8.$$

The right hand side can be written as $\dfrac{d}{dx}(Iy) = \dfrac{d}{dx}\left(x^5 y\right)$, so we have

$$\frac{d}{dx}\left(x^5 y\right) = -3x^8$$

which is now separable. Integrating both sides yields

$$x^5 y = -\frac{1}{3} x^9 + K$$

which gives the general solution

$$y = -\frac{1}{3}x^4 + Kx^{-5}.$$

(b) $\dfrac{dy}{dx} + 2xy - \exp\left(-x^2\right) = x; \qquad y = 1 \;\text{ when }\; x = 0$

By writing this ODE as $y' + 2xy = x + \exp\left(-x^2\right)$, which is linear with $P(x) = 2x$; $Q(x) = x + \exp\left(-x^2\right)$. Multiply ODE by integrating factor $I\ (x) = \exp\int 2x\ dx = \exp\left(x^2\right)$, to give

$$\begin{aligned}
\exp\left(x^2\right)y' + 2x\exp\left(x^2\right)y &= x\exp\left(x^2\right) + 1 \\
\frac{d}{dx}\left(y\exp\left(x^2\right)\right) &= x\exp\left(x^2\right) + 1 \\
y\exp\left(x^2\right) &= \underbrace{\int x\exp\left(x^2\right)}_{\text{By substitution } u=x^2} + x + C \\
y &= \frac{1}{2} + x\exp\left(-x^2\right) + C\exp\left(-x^2\right)
\end{aligned}$$

which is the general solution (GS). We are given an initial condition, which allows us to determine the constant of integration C and hence obtain a particular solution (PS). Substituting $x = 0$ & $y = 1$ into the GS above gives $C = 1/2$, therefore the PS becomes

$$y = \frac{1}{2} + x\exp\left(-x^2\right) + \frac{1}{2}\exp\left(-x^2\right).$$

(c) $y'' + 3y' + 2y = 0$ is a 2nd order homogeneous equation with constant coefficients, hence a solution of the form $y = \exp(\lambda x)$ exists. We need to determine the values of λ. the Auxillary Equation (A.E) is

$$\lambda^2 + 3\lambda + 2 = 0 \longrightarrow (\lambda + 1)(\lambda + 2) = 0 \longrightarrow \lambda = -1,\ -2$$

Hence the complimentary function becomes

$$y = A\exp(-x) + B\exp(-2x)$$

where A and B are arbitrary constants.

7. Using a trinomial random walk and Taylor series, derive the backward Kolmogorov partial differential equation.

$$\frac{\partial p}{\partial t} + c^2\frac{\partial^2 p}{\partial y^2} = 0$$

for the transition density function $p(y, t\,; y', t')$. Show that

$$\frac{1}{2c\sqrt{2\pi(t'-t)}}\exp\left(-\frac{(y'-y)^2}{4c^2(t'-t)}\right)$$

satisfies the backward Kolmogorov equation.
We will work with the same trinomial random walk given in the lecture notes where y is the value of our random variable. δt is a time step. α is a probability. δy is the size of the move in y.

We can relate the probability of being at y at time t to the probability of being at the three states at time $t+\delta t$ by

$$\begin{aligned} p(y,t\ ;y',t') &= \alpha\, p(y+\delta y,t+\delta t;y',t') \\ &\quad +(1-2\alpha)p(y,t+\delta t\ ;y',t')+\alpha\ p(y-\delta y,t+\delta t\ ;y',t') \end{aligned} \tag{4}$$

We can expand in Taylor series as follows:

$$p(y+\delta y,t+\delta t;y',t') = p(y,t;y',t')+\delta t\frac{\partial p}{\partial t}+\delta y\frac{\partial p}{\partial y}+\frac{1}{2}\delta y^2\frac{\partial^2 p}{\partial y^2}+\cdots,$$

$$p(y-\delta y,t+\delta t;y',t') = p(y,t;y',t')+\delta t\frac{\partial p}{\partial t}-\delta y\frac{\partial p}{\partial y}+\frac{1}{2}\delta y^2\frac{\partial^2 p}{\partial y^2}+\cdots$$

and

$$p(y,t+\delta t;y',t') = p(y,t;y',t')+\delta t\frac{\partial p}{\partial t}+\cdots.$$

Equation (4) becomes

$$p=\alpha\left(p+\delta t\frac{\partial p}{\partial t}+\delta y\frac{\partial p}{\partial y}+\frac{1}{2}\delta y^2\frac{\partial^2 p}{\partial y^2}+\cdots\right)$$

$$+(1-2\alpha)\left(p+\delta t\frac{\partial p}{\partial t}+\cdots\right)+\alpha\left(p+\delta t\frac{\partial p}{\partial t}-\delta y\frac{\partial p}{\partial y}+\frac{1}{2}\delta y^2\frac{\partial^2 p}{\partial y^2}+\cdots\right).$$

Many of these terms cancel leaving

$$\frac{\partial p}{\partial t}+\frac{\alpha\ \delta y^2}{\delta t}\frac{\partial^2 p}{\partial y^2}+\cdots=0.$$

Introducing

$$\frac{\alpha\ \delta y^2}{\delta t}=c^2,$$

and letting both δt and δy tend to zero gives the desired result.

Let us see whether

$$p=\frac{1}{2c\sqrt{\pi(t'-t)}}\exp\left(-\frac{(y'-y)^2}{4c^2(t'-t)}\right) \tag{5}$$

is a solution of this equation.

From (5)

$$\frac{\partial p}{\partial t} = \frac{1}{2c\sqrt{\pi}}\left(\frac{1}{2(t'-t)^{3/2}}\right)\exp(\cdots)+\frac{1}{2c\sqrt{\pi}}\left(\frac{1}{(t'-t)^{1/2}}\right)\left(-\frac{(y'-y)^2}{4c^2(t'-t)^2}\right)\exp(\cdots).$$

Also from (5)

$$\frac{\partial p}{\partial y} = \frac{1}{2c\sqrt{\pi}}\left(\frac{1}{(t'-t)^{1/2}}\right)\left(\frac{(y'-y)}{2c^2(t'-t)}\right)\exp(\cdots) \tag{6}$$

and

$$\begin{aligned}\frac{\partial^2 p}{\partial y^2} \quad = \quad & \frac{1}{2c\sqrt{\pi}}\left(\frac{1}{(t'-t)^{1/2}}\right)\left(\frac{1}{2c^2(t'-t)}\right)\exp(\cdots)+ \\ & \frac{1}{2c\sqrt{\pi}}\left(\frac{1}{(t'-t)^{1/2}}\right)\left(\frac{(y'-y)^2}{4c^4(t'-t)^4}\right)\exp(\cdots).\end{aligned} \tag{7}$$

When (6) and (7) are substituted into the backward Kolmogorov equation each of the two terms from (6) cancels with one from (7).