1)

THE VASICEK MODEL FOR SPOT INTEREST RATE IS DEFINED, BY THE PROCESS

$$dz = \gamma(\bar{z} - z)\,dt + \sigma\,dX$$

STRIP OUT THE DRIFT TERM AND DENOTE IT AS

$$Y(t) = \gamma(\bar{z} - z(t)) \qquad \Big/ * \; e^{\gamma t}$$

$$e^{\gamma t}Y(t) = f(z(t)) = e^{\gamma t}(\gamma(\bar{z} - z(t)))$$

APPLY ITÔ'S LEMMA

$$\frac{\partial f(z(t))}{\partial t} = \gamma e^{\gamma t}Y(t)$$

$$\frac{\partial f(z(t))}{\partial z} = -\gamma e^{\gamma t}$$

$$\frac{\partial^2 f(z(t))}{\partial z^2} = 0$$

$$f(z(t)) - f(z(0)) = \int_0^t \frac{\partial f(z(s))}{\partial s}\,ds +$$

$$+ \int_0^t \frac{\partial f(z(s))}{\partial z}\,d(s) + \frac{1}{2}\int_0^t \frac{\partial^2 f(z(s))}{\partial z^2}\,d(z)(s) =$$

$$= \int_0^t \gamma e^{\gamma s}Y(s)\,ds - \int_0^t \gamma e^{\gamma s}\left(Y(s)\,ds + \sigma\,dX(s)\right) =$$

$$= -\int_0^t \gamma e^{\gamma s}\sigma\,dX(s)$$

We have a recursive expression for $z(t)$ in terms of its previous value, $z(0)$.

$$e^{\gamma t}Y(t) - e^{\gamma 0}Y(0) = -\int_0^t \gamma e^{\gamma s}\sigma\, dX(s)$$

$$\gamma e^{\gamma t}(\bar{z} - z(t)) - \gamma(\bar{z} - z(0)) = -\int_0^t \gamma e^{\gamma s}\sigma\, dX(s)$$

$$-\gamma e^{\gamma t} z(t) = -\gamma \bar{z} e^{\gamma t} + \gamma \bar{z} - \gamma z(0) - \int_0^t \gamma e^{\gamma s}\sigma\, dX(s)$$

$$z(t) = \underbrace{\bar{z}(1 - e^{-\gamma t}) + e^{-\gamma t} z(0)}_{\text{drift}} + \underbrace{\int_0^t e^{-\gamma(t-s)}\sigma\, dX(s)}_{\text{Diffusion}}$$

2)

The fixed interest payments, since they are all known in terms of actual dollar amount, can be seen as the sum of zero coupon bonds

If the fixed rate of interest is r_s then the fixed payments add up to

$$r_s \tau \sum_{i=1}^{N} Z(t;T_i), \quad r_s = \frac{1 - Z(t;T_N)}{\tau \sum_{i=1}^{N} Z(t;T_i)}$$

where

$V = P e^{-y(T-t)}$ - ZCB

$\tau = 0.5$ - time interval (semiannual)

$T = 5$ - maturity of experation

$P = \$1$ - principal

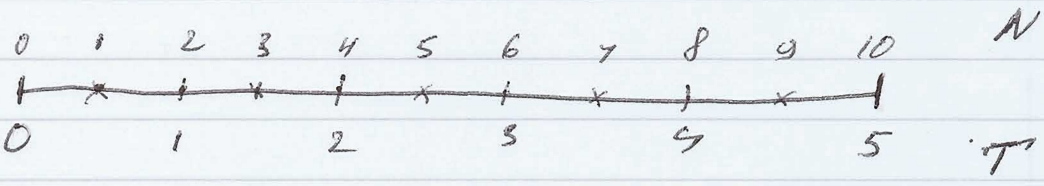
Then

$$r_s \, 0.5 \sum_{i=1}^{10} P \times e^{-y(T_i - t)}$$

$$0.5 \, r_s \sum_{i=1}^{10} 1 \times e^{-5y}, \quad y = -\frac{\log \frac{V_{ZCB}}{P}}{T-t}$$

where

y - yield to maturity



3)

A floorlet is the counterpart of a put on the floating interest rate and pays the amount max($r_f - r_L$,0) at expiry, where we assume that actual floating rate is the underlying spot interest rate, i.e $r_L = r$ and r_f is the floor rate. Typically, a floorlet might be purchased by an investor who has to make a stream of payments based on a floating interest rate such as LIBOR, the London InterBank Offer Rate, and who wishes to protect himself against sharp decrease in interest rates. Thus it follows that a floorlet is an insurance against low rates. These interest rate derivatives can be used individually or combined into portfolios: a portfolio of floorlets being known as a floor.

To price interest rate derivatives, it is necessary to model the behaviour of interest rates. It is usual to assume that the spot interest rate r obeys the stochastic differential equation,

$$dr = u(r, t)dt + w(r, t)dX,$$

where dX is normally distributed with zero mean and variance dt and w is the volatility.
Constructing a risk neutral portfolio leads us to the following partial differential equation (PDE) Bond Pricing Equation for the price $V(r, t)$ of an interest rate derivative,

$$\frac{\partial V}{\partial t} + \frac{w^2}{2}\frac{\partial^2 V}{\partial r^2} + (u - \lambda w)\frac{\partial V}{\partial r} - rV = 0$$

where $\lambda(r, t)$ is the market price of interest rate risk, and $u - \lambda w$ is the risk adjusted drift. This equation is valid for times $t \leq T$, where T is the expiry of the derivative. Considering Vasicek model, for which $u - \lambda w = \eta - \gamma r$ and $w = \sigma$, with η, γ and σ constants rather than functions of time, so that becomes a modified Bond Pricing Equation

$$\frac{\partial V}{\partial t} + \frac{w^2}{2}\frac{\partial^2 V}{\partial r^2} + (\eta - \gamma r)\frac{\partial V}{\partial r} - rV = 0$$

with a final condition on zero –coupon bond

$$V(r, t; T) = 1$$

This model is mean-reverting to a constant level, which is a desirable property for interest rates.
This equation must be solved together with the pay-off at expiry of $V(r, T) = \max(r_f - r, 0)$ for a floorlet, which lead to modified Vasicek equation for floorlet

$$\frac{\partial V}{\partial t} + \frac{w^2}{2}\frac{\partial^2 V}{\partial r^2} + (\eta - \gamma r)\frac{\partial V}{\partial r} = rV - \max(rf - r, 0)$$

Alternatively, we can use a Taylor series solution of the bond pricing equation for short times to expiry.

Substituting this

$$Z(r, t; T) = 1 + a(r)\,(T-t) + \tfrac{1}{2}b(r)\,(T-t)^2 + \ldots$$

into Bond Pricing Equation;

$$-a - 2b\,(T-t) - 3c\,(T-t) + \frac{1}{2}\left(w^2 + 2(T-t)w\frac{\partial w}{\partial t}\right)\left((T-t)\frac{d^2a}{dr^2} + (T-t)^2\frac{d^2b}{dr^2}\right)$$

$$+\left((u-\lambda w) + (T-t)^2\frac{\partial(u-\lambda w)}{\partial t}\right)(T-t)\left(\frac{da}{dr} + (T-t)^2\frac{db}{dr}\right) - r\left(1 + a\,(T-t) + c\,(T-t)^2\right) + \ldots = 0$$

We find by equating powers of (T – t) that;

$a(r) = -r$, $\qquad b(r) = \frac{1}{2}r^2 - \frac{1}{2}r\,(u-\lambda w) \cong r^2 - r\,(u-\lambda w)$

and

$$c(r) = \frac{1}{12}w^2\frac{\partial^2}{\partial r^2}(r^2 - r(u-\lambda w)) - \frac{1}{6}(u-\lambda w)\frac{\partial}{\partial r}(r^2 - r(u-\lambda w) - \frac{1}{3}\frac{\partial}{\partial r}(u-\lambda w) + \frac{1}{6}r^2\,(r - (u-\lambda w))$$

In all of these $u-\lambda w$ and w are evaluated at r and T.

From the Taylor series expression for Z we find that the yield to maturity is given by

$$\frac{Ln(Z(r,t;T))}{T-t} \sim -a + \left(\frac{1}{2}a^2 - b\right)(T-t) + \left(a\,b - c - \frac{1}{3}a^3\right)(T-t)^2 + \ldots$$

After replacing a, b and c the equation is

$$\frac{Ln(Z(r,t;T))}{T-t} \sim -r + \frac{1}{2}(u-\lambda w)\,(T-t) + \ldots \qquad \text{as} \qquad t \to T$$

Now we can use it to calculate the the cashflow of the floorlet with one month LIBOR floating rate.

$$\max(rf - r, 0)$$

We can write this approximately as

$$\max(rf - r - \frac{1}{24}(u - \lambda w), 0)$$

The $\frac{1}{24}$ comes from $\frac{1}{2}$ multiplied by the maturity of the one – month rate measure in years $(\frac{1}{12})$.

After applying Vasicek model, for which $u - \lambda w = \eta - \gamma r$ and $w = \sigma$, we got *the final equation*

$$\max(rf - r - \frac{1}{24}(\eta - \gamma r), 0)$$

4) Once interest rates move the swap will have a non-zero value. This may be positive or negative depending on the direction in which the floating legs move.

The swap can then be closed out resulting in a profit or loss.

If it assumes a principal of \$1

then the reciever of the fixed side, is

$$-1 + Z(t;T_N) + r_s \tau \sum_{i=1}^{N} Z(t;T_i), \text{ where}$$

r_s - fixed rate of interest
τ - time interval b/n payments

Add up all the floating legs, the floating side has value

$$1 - Z(t;T_N)$$

The fixed interest payments, since they are all known in terms of actual dollar, can be seen as the sum of zero coupon bonds.

$$r_s \tau \sum_{i=1}^{N} Z(t;T_i)$$

$$1 - Z(t;T_N) <> r_s \tau \sum_{i=1}^{N} Z(t;T_i)$$

5)

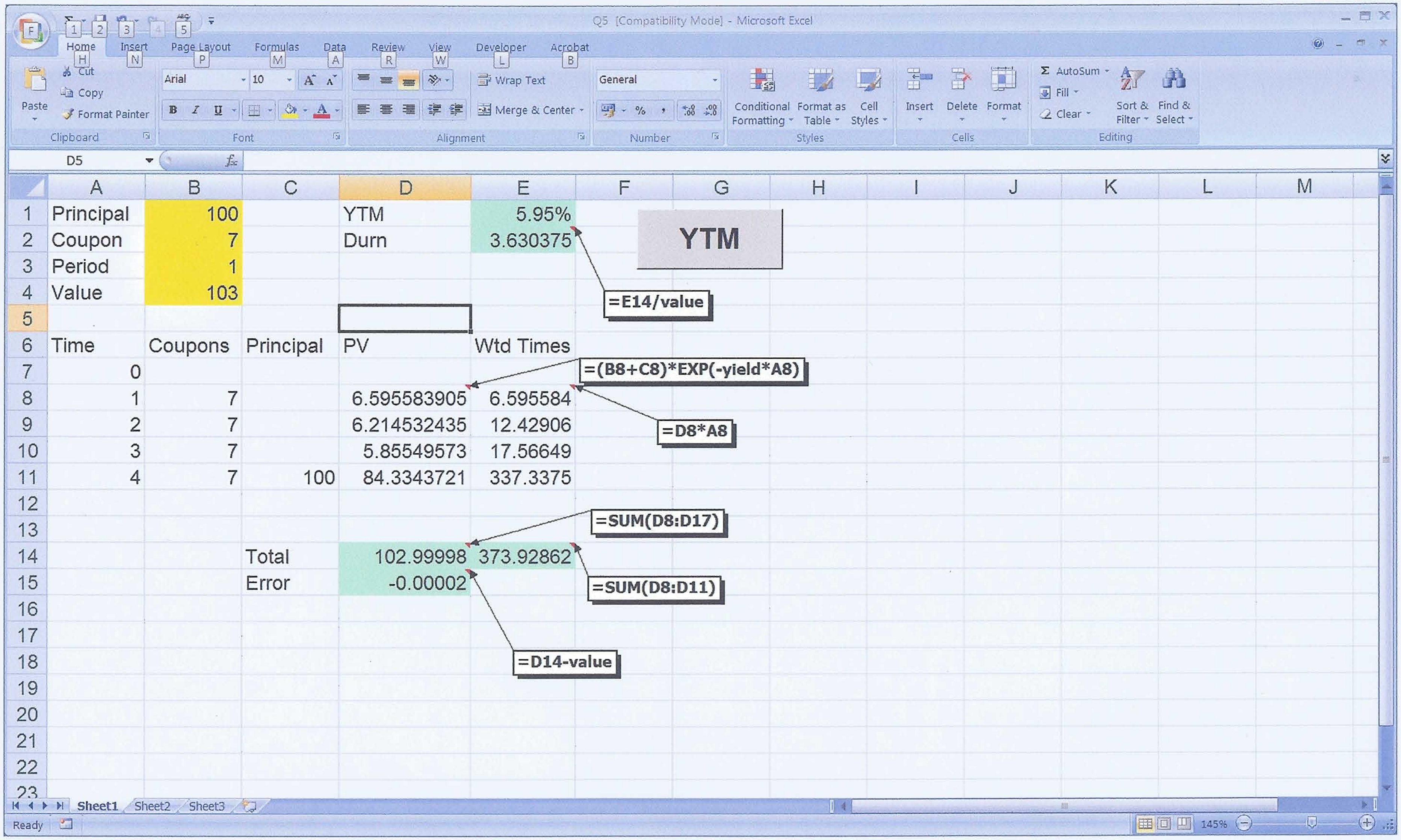
$V = 103$, $P = 100$, $C = 0.07 \times P$, $N = 4$

$$V = P e^{-y(T-t)} + \sum_{i=1}^{N} C_i e^{-y(t_i - t)}$$

so

$$103 = 100 e^{-4y} + \sum_{i=1}^{4} 7 e^{-y(i)}$$

$$y = 0.0595$$



	A	B	C	D	E
1	Principal	100		YTM	5.95%
2	Coupon	7		Durn	3.630375
3	Period	1			
4	Value	103			
5					
6	Time	Coupons	Principal	PV	Wtd Times
7	0				
8	1	7		6.595583905	6.595584
9	2	7		6.214532435	12.42906
10	3	7		5.85549573	17.56649
11	4	7	100	84.3343721	337.3375
12					
13					
14			Total	102.99998	373.92862
15			Error	-0.00002	

6)

CIR model for spot rate r is given by

$$dr = (\eta - \gamma r)\,dt + \sqrt{\alpha r}\,dX \quad , \quad \bar{r} = \frac{\eta}{\gamma}$$

$$\Downarrow$$

$$dr = \gamma(\bar{r} - r)\,dt + \sigma\sqrt{r}\,dX$$

$$Z(r, t:T) = e^{A(t:T) - r B(t:T)}$$

$$\Downarrow$$

$$Z(r, t; T) = A(t;T)\, e^{-B(t;T) r} \quad , \quad \phi = \sqrt{\gamma^2 + 2\sigma^2}$$

$$A(t;T) = \left(\frac{2\phi\, e^{(\gamma+\phi)\frac{(T-t)}{2}}}{(\phi+\gamma)(e^{\phi(T-t)} - 1) + 2\phi} \right)^{\frac{2\gamma\bar{r}}{\sigma^2}}$$

$$B(t;T) = \frac{2(e^{\phi(T-t)} - 1)}{(\phi+\gamma)(e^{\phi(T-t)} - 1) + 2\phi}$$

where

$r = 10\%$

$\sigma = 0.02$

$\bar{r} = \frac{\eta}{\gamma} = 0.1$

$\gamma = 0,1$

$T = 10$

Cox, Ingersoll and Ross Model

RN model $dr = \gamma(\check{r} - r)\,dt - \sigma\,\mathrm{sqrt}(r)\,dX$

γ	0.1		B(t;T)	6.2956
$\check{r}$	0.1		A(t;T)	0.6927
r	10%			
t (nowyr)	0		**Z(r,t;T)**	**0.3691**
T (zeroyr)	10			
zero life	10			
σ	0.02			
ϕ	0.1039			

7)

The objective of CALIBRATION IS TO CHOOSE THE MODEL PARAMETERS IN SUCH A WAY THAT THE MODEL PRICES ARE CONSISTENT WITH THE MARKET PRICES of LIQUID INSTRUMENTS. BECAUSE of THIS NEED TO CORRECTY PRICE THESE INSTRUMENTS, THE IDEA of YIELD CURVE fITTING OR CALLIBRATION HAS become POPULAR.

TO MATCH A THEORETICAL YIELD CURVE TO A MARKET YIELD CURVE REQUIRES A MODEL WITH enough degrees of fREEDOM. THIS IS DONE by MAKING ONE OR MORE PARAMETERS TIME DEPENDENT. THIS fUNCTIONAL DEPENDENCE ON TIME IS THEN CAREfULLY CHOSEN TO MAKE AN OUTPUT of THE MODEL, THE PRICE of ZERO-COUPON Bond, EXACTLY MATCH THE MARKET PRICES FOR THESE INSTRUMENTS.

$$dr = (\eta(t) - \gamma r)\, dt + c\, dX$$

CONSIDER VASICEK EXTENDED HULL AND White MODEL, WE ASSUME γ AND c have been ESTIMATED STATISTICALLY, AND WE CHOOSE $\eta = \eta^*(t)$ AT TIME t^* SO THAT OUR THEORETHICAL AND THE MARKET PRICE of THE bonds COINCIDE.

To fIT THE YIELD CURVE AT TIME t^* We MUST MAKE $\eta^*(t)$ SATISfy

$$A(t^*;T) = -\int_{t^*}^{T} \eta^*(s) B(s;T)\, ds + \frac{c^2}{2\gamma^2}\left(T - t^* + \frac{2}{\gamma} e^{-\gamma(T-t^*)} - \frac{1}{2\gamma} e^{-2\gamma(T-t^*)} - \frac{3}{2\gamma}\right)$$

$$= \log\left(Z_M(t^*;T)\right) + r^* B(t^*,T)$$

This is an integral equation for $\eta^*(t)$, if we are given all of the other parameters and functions, such as the market price of bonds $Z_M(t^*;T)$;

By differentiating the equation twice with respect to T, we get

$$\eta^*(t) = -\frac{\partial^2}{\partial t^2}\log(Z_M(t^*;t)) - \gamma\frac{\partial}{\partial t}\log(Z_M(t^*;t)) + \frac{c^2}{2\gamma}\left(1-e^{-2\gamma(t-t^*)}\right)$$

then the function $A(t;T)$ is

$$A(t;T) = \log\left(\frac{Z_M(t^*;T)}{Z_M(t^*;t)}\right) - B(t;T)\frac{\partial}{\partial t}\log(Z_M(t^*;t))$$
$$-\frac{c^2}{4\gamma^3}\left(e^{-\gamma(T-t^*)} - e^{-\gamma(t-t^*)}\right)^2\left(e^{2\gamma(t-t^*)}-1\right)$$