given the Black-Scholes formula

$$S N(d_1) - E e^{-r(T-t)} N(d_2)$$

what is the actual probability of exercise of option?

Proba under $\bar{Q}$

$\underline{\overline{RN}}$ Proba of exercise actual proba of exercise

$$P[S_T > E]$$

$$= P\left[S_0 e^{\sigma X_T + (\mu - \frac{1}{2}\sigma^2)T} > E\right]$$

$$= P\left[\ln \frac{S_0}{E} + (\mu - \frac{1}{2}\sigma^2)T > -\sigma X_T\right]$$

$$X_T \sim N(0, T) \qquad \xi \sim N(0,1)$$

$$= P\left[\ln \frac{S_0}{E} + (\mu - \frac{1}{2}\sigma^2)T > -\sigma\sqrt{T}\,\xi\right]$$

$$= P\left[\frac{\ln \frac{S_0}{E} + (\mu - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} > \xi\right] = N(d_3)$$

just like $N(d_2)$ but for the real drift μ.

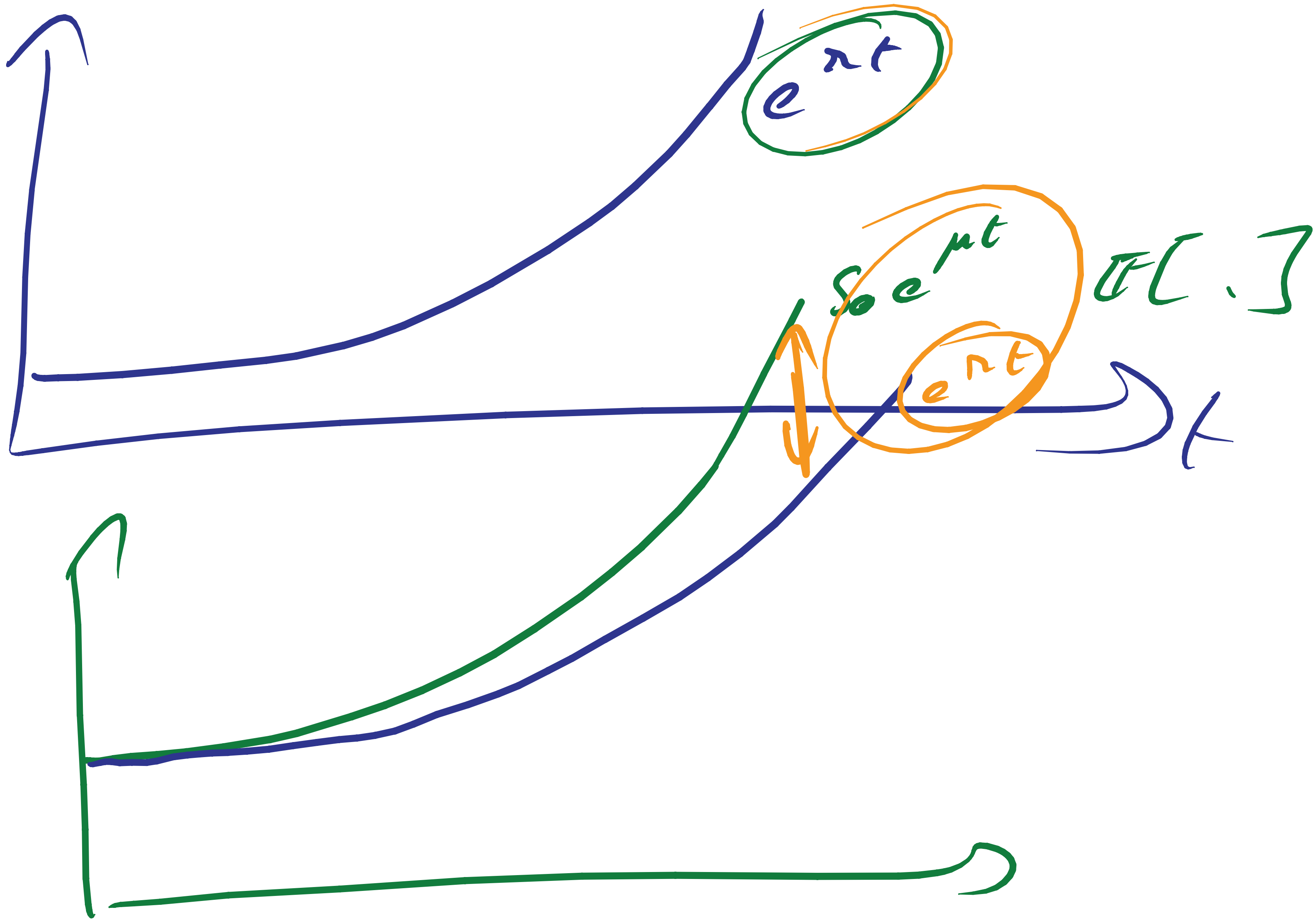
Lecture 24. Radon-Nikodym derivative

$\mathbb{P} \sim \mathbb{Q}$

$$\boxed{\mathbb{Q}(A)} = \int_A \Lambda \, d\mathbb{P}$$

Radon-Nikodym derivative

$$\boxed{\Lambda = \frac{d\mathbb{Q}}{d\mathbb{P}}}$$


e^{rt}
$S_0 e^{\mu t}$
$\mathbb{E}[\,\cdot\,]$
e^{rt}
t

under $\mathbb{P}$

$$\frac{dS^*_t}{S^*_t} = (\mu - r)\,dt + \sigma\, dX_t \quad = (dX^{\mathbb{Q}}_t - \theta(t)\,dt)$$

We want to chose $\mathbb{Q}$ such that $S^*(t)$ is a martingale, i.e. such that $S^*(t)$ under $\mathbb{Q}$ is driftless. under $\mathbb{Q}$

$$\frac{dS^*_t}{S^*_t} = (\mu - r)\,dt + \sigma\,(dX^{\mathbb{Q}}_t - \theta(t)\,dt)$$

$$= \underbrace{(\mu - r - \sigma\,\cancel{\theta(t)})}_{=0 \text{ under } \mathbb{Q}}\,dt + \sigma\, dX^{\mathbb{Q}}_t$$

θ

$$\Rightarrow \boxed{\theta = \frac{\mu - r}{\sigma}}$$

$$\mathbb{E}\left[\mathbb{1}_{\{X\in A\}}\right] = \int_\Omega \mathbb{1}_{\{X\in A\}}\, d\mathbb{P}$$

$$= \int_A d\mathbb{P}$$

$$= P[A]$$