FINA 276 OPTIONS - PAPER: GREEKS DERIVATION FROM BLACK-SCHOLES FORMULA

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Abstract. We derive formulae for Greeks, then use them to analyse European call and put option price sensitivity to input parameters. Only non-dividend paying stocks are considered.

1. Black-Scholes Option Price Formula

Option price formulae (non-dividend paying stocks) are given in equation 12.20 of [1] for call option:

$$c = S_0 N(d_1) - Ke^{-rT} N(d_2) \tag{1.1}$$

and in equation 12.21 for put option

$$p = Ke^{-rT} N(-d_2) - S_0 N(-d_1) \tag{1.2}$$

where

$$d_1 = \frac{ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} \tag{1.3}$$

$$d_2 = \frac{ln(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} = d1 - \sigma\sqrt{T} \tag{1.4}$$

Here, $N(x)$ is a cumulative normal distribution function, which is defined as

$$N(x) = \int_{-\infty}^{x} Z(t)dt \tag{1.5}$$

where

$$Z(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \tag{1.6}$$

see [2] equations 26.2.1 and 26.2.2.

In next five sections we shall get the formulae for five Greeks: delta, theta, gamma, vega and rho.

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2. Delta

2.1. **Call Option Delta.**

Definition 1. *Delta is defined by the following equation (see 14.4 in* [1]*)*

$$\Delta = \frac{\partial c}{\partial S} \tag{2.1}$$

Substituting the price with 1.1

$$\Delta = \frac{\partial}{\partial S}\left\{SN(d_1) - Ke^{-rT}N(d_2)\right\} \tag{2.2}$$

$$= N(d_1) + S\frac{\partial N(d_1)}{\partial S} - Ke^{-rT}\frac{\partial N(d_2)}{\partial S} \tag{2.3}$$

Let's evaluate the last term in the above equation using 2.14 for $\frac{\partial N(d_2)}{\partial S}$

$$Ke^{-rT}\frac{\partial N(d_2)}{\partial S} = Ke^{-rT}\frac{Z(d_1)(S/K)e^{rT}}{\sigma\sqrt{T}S} = \frac{Z(d_1)}{\sigma\sqrt{T}} \tag{2.4}$$

then put the obtained formula back in 2.3 using also 2.13 for $\frac{\partial N(d_1)}{\partial S}$

$$\Delta = N(d_1) + S\frac{Z(d_1)}{\sigma\sqrt{T}S} - \frac{Z(d_1)}{\sigma\sqrt{T}}$$

$$= N(d_1) + \frac{Z(d_1)}{\sigma\sqrt{T}} - \frac{Z(d_1)}{\sigma\sqrt{T}}$$

Finally,

$$\Delta = N(d_1) \tag{2.5}$$

2.2. **Put Option Delta.**

Definition 2. *Delta is defined by the following equation (see 14.4 in* [1]*)*

$$\Delta = \frac{\partial p}{\partial S} \tag{2.6}$$

Substituting the price with 1.2

$$\Delta = \frac{\partial}{\partial S}\left\{Ke^{-rT}N(-d_2) - SN(-d_1)\right\} \tag{2.7}$$

$$= -N(-d_1) - S\frac{\partial N(-d_1)}{\partial S} + Ke^{-rT}\frac{\partial N(-d_2)}{\partial S} \tag{2.8}$$

Let's evaluate the last term in the above equation using 2.16 for $\frac{\partial N(-d_2)}{\partial S}$

$$Ke^{-rT}\frac{\partial N(-d_2)}{\partial S} = -Ke^{-rT}\frac{Z(-d_1)\frac{S}{K}e^{rT}}{\sigma\sqrt{T}S} = -\frac{Z(-d_1)}{\sigma\sqrt{T}} \tag{2.9}$$

then put the obtained formula back in 2.8 using also 2.15 for $\frac{\partial N(-d_1)}{\partial S}$

$$\Delta = -N(-d_1) + S\frac{Z(-d_1)}{\sigma\sqrt{T}S} - \frac{Z(-d_1)}{\sigma\sqrt{T}} = -N(-d_1)$$

Finally, using the equation $N(x) + N(-x) = 1$, which can be derived from 26.2.5 and 26.2.6 in [2], we get

$$\Delta = N(d_1) - 1 \tag{2.10}$$

2.3. **Auxilliary expressions.** Let's obtain forumale for $\frac{\partial N(d_1)}{\partial S}$ and $\frac{\partial N(d_2)}{\partial S}$.

$$\frac{\partial d_1}{\partial S} = \frac{\partial}{\partial S}\frac{ln(S/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} \tag{2.11}$$

$$\frac{\partial d_1}{\partial S} = \frac{1}{\sigma\sqrt{T}}\frac{\partial}{\partial S}ln(S/K) = \frac{1}{\sigma\sqrt{T}}\frac{1}{(S/K)}\frac{1}{K} = \frac{1}{\sigma\sqrt{T}S} \tag{2.12}$$

Therefore,

$$\frac{\partial N(d_1)}{\partial S} = N'(d1)\frac{\partial d_1}{\partial S} = Z(d_1)\frac{\partial d_1}{\partial S} = \frac{Z(d_1)}{\sigma\sqrt{T}S} \tag{2.13}$$

and also,

$$\frac{\partial N(d_2)}{\partial S} = N'(d_2)\frac{\partial d_2}{\partial S} = N'(d_2)\frac{\partial[d_1 - \sigma\sqrt{T}]}{\partial S} = N'(d_2)\frac{\partial d_1}{\partial S} = \frac{Z(d_2)}{\sigma\sqrt{T}S}$$

Finally, using 8.3

$$\frac{\partial N(d_2)}{\partial S} = \frac{Z(d_1)(S/K)e^{rT}}{\sigma\sqrt{T}S} \tag{2.14}$$

Now, let's obtain forumale for $\frac{\partial N(-d_1)}{\partial S}$ and $\frac{\partial N(-d_2)}{\partial S}$. Using 2.12 we can derive the following

$$\frac{\partial N(-d_1)}{\partial S} = -N'(-d1)\frac{\partial d_1}{\partial S} = -Z(-d_1)\frac{\partial d_1}{\partial S} = -\frac{Z(-d_1)}{\sigma\sqrt{T}S} \tag{2.15}$$

Similarly,

$$\frac{\partial N(-d_2)}{\partial S} = -N'(-d_2)\frac{\partial d_2}{\partial S} = -N'(-d_2)\frac{\partial[d_1 - \sigma\sqrt{T}]}{\partial S} = -N'(-d_2)\frac{\partial d_1}{\partial S} = -\frac{Z(-d_2)}{\sigma\sqrt{T}S}$$

then, using 8.4

$$\frac{\partial N(-d_2)}{\partial S} = -\frac{Z(-d_1)(S/K)e^{rT}}{\sigma\sqrt{T}S} \tag{2.16}$$

3. Gamma

Both call and put options have the same expression for Γ. We'll prove this statement in next two subsections.

$$\Gamma = \frac{Z(d_1)}{\sigma\sqrt{T}S} \tag{3.1}$$

3.1. **Call Option Gamma.**

Definition 3. *Gamma is defined by equation (see 14.6 in* [1]*)*

$$\Gamma = \frac{\partial}{\partial S}\left(\frac{\partial c}{\partial S}\right) \tag{3.2}$$

Hence, using definition of Δ (2.1) and its value (2.5),

$$\Gamma = \frac{\partial \Delta}{\partial S} = \frac{\partial N(d_1)}{\partial S} \tag{3.3}$$

Finally, use 2.13 to substitute $\frac{\partial N(d_1)}{\partial S}$

$$\Gamma = \frac{\partial N(d_1)}{\partial S} = \frac{Z(d_1)}{\sigma\sqrt{T}S} \tag{3.4}$$

3.2. **Put Option Gamma.**

Definition 4. *Gamma is defined by equation (see 14.6 in* [1]*)*

$$\Gamma = \frac{\partial}{\partial S}\left(\frac{\partial p}{\partial S}\right) \tag{3.5}$$

Hence, using definition of Δ (2.6) and its value (2.10),

$$\Gamma = \frac{\partial \Delta}{\partial S} = \frac{\partial (N(d_1) - 1)}{\partial S} = \frac{\partial N(d_1)}{\partial S} \tag{3.6}$$

You can see that put option has the same Γ as call options (look at equation 3.3).

4. Theta

In 14.5 of [1], Θ is defined as a "the rate of change of the value of the portfolio with respect to the passage of time with all else remaining the same".

4.1. **Call Option Theta.**

Definition 5. *We re-define* Θ *as follows*

$$\Theta = -\frac{\partial c}{\partial T} \tag{4.1}$$

Note the sign. In Black-Scholes formula 1.1 the option price is expressed as a function of time to maturity T. Since we want to measure sensitivity with respect to passing (or current) time t, then the sign must be negative, because $\delta t = -\delta T$.

Let's substitute c in 4.1 by Black-Scholes formula 1.1

$$\Theta = -\frac{\partial}{\partial T}\left\{SN(d_1) - Ke^{-rT}N(d_2)\right\} \tag{4.2}$$

$$= -\left\{S\frac{\partial N(d_1)}{\partial T} - K\left(N(d_2)\frac{\partial}{\partial T}e^{-rT} + e^{-rT}\frac{\partial}{\partial T}N(d_2)\right)\right\} \tag{4.3}$$

$$= -\left\{S\frac{\partial N(d_1)}{\partial T} - K\left(-re^{-rT}N(d_2) + e^{-rT}\frac{\partial}{\partial T}N(d_2)\right)\right\} \tag{4.4}$$

$$= -\left\{S\frac{\partial N(d_1)}{\partial T} - Ke^{-rT}\left(-rN(d_2) + \frac{\partial}{\partial T}N(d_2)\right)\right\} \tag{4.5}$$

Use 4.29 and 4.28 for $\frac{\partial N(d_1)}{\partial T}$ and $\frac{\partial N(d_2)}{\partial T}$

$$= -\left\{-Z(d_1)(\frac{d_2}{2T} - \frac{r}{\sigma\sqrt{T}})S - Ke^{-rT}\left(-rN(d_2) - Z(d_1)(S/K)e^{rT}(\frac{d_1}{2T} - \frac{r}{\sigma\sqrt{T}})\right)\right\} \tag{4.6}$$

$$= -\left\{-Z(d_1)(\frac{d_2}{2T} - \frac{r}{\sigma\sqrt{T}})S + Z(d_1)(\frac{d_1}{2T} - \frac{r}{\sigma\sqrt{T}})S + Ke^{-rT}rN(d_2)\right\} \tag{4.7}$$

$$= -\left\{Z(d_1)\frac{d_1 - d_2}{2T}S + Ke^{-rT}rN(d_2)\right\} \tag{4.8}$$

Finally,

$$\Theta = -\left\{Z(d_1)S\frac{\sigma}{2\sqrt{T}} + Kre^{-rT}N(d_2)\right\} \tag{4.9}$$

4.2. **Put Option Theta.** Just like for call options in see 4.1 let's define Θ as follows

Definition 6.

$$\Theta = -\frac{\partial p}{\partial T} \tag{4.10}$$

Let's substitute p in 4.10 by Black-Scholes formula 1.2

$$\Theta = -\frac{\partial}{\partial T}\left\{-SN(-d_1) + Ke^{-rT}N(-d_2)\right\} \tag{4.11}$$

$$= -\left\{-S\frac{\partial N(-d_1)}{\partial T} + K\left(N(-d_2)\frac{\partial}{\partial T}e^{-rT} + e^{-rT}\frac{\partial}{\partial T}N(-d_2)\right)\right\} \tag{4.12}$$

$$= -\left\{-S\frac{\partial N(-d_1)}{\partial T} + K\left(-re^{-rT}N(-d_2) + e^{-rT}\frac{\partial}{\partial T}N(-d_2)\right)\right\} \tag{4.13}$$

$$= -\left\{-S\frac{\partial N(-d_1)}{\partial T} + Ke^{-rT}\left(-rN(-d_2) + \frac{\partial}{\partial T}N(-d_2)\right)\right\} \tag{4.14}$$

Use 4.31 and 4.30 for $\frac{\partial N(-d_1)}{\partial T}$ and $\frac{\partial N(-d_2)}{\partial T}$

$$= -\left\{-Z(d_1)(\frac{d_2}{2T} - \frac{r}{\sigma\sqrt{T}})S + Ke^{-rT}\left(-rN(-d_2) + Z(d_1)\frac{S}{K}e^{rT}(\frac{d_1}{2T} - \frac{r}{\sigma\sqrt{T}})\right)\right\} \tag{4.15}$$

$$= -\left\{-Z(d_1)(\frac{d_2}{2T} - \frac{r}{\sigma\sqrt{T}})S + Z(d_1)(\frac{d_1}{2T} - \frac{r}{\sigma\sqrt{T}})S - Ke^{-rT}rN(-d_2)\right\} \tag{4.16}$$

$$= -\left\{Z(d_1)\frac{d_1 - d_2}{2T}S - Ke^{-rT}rN(-d_2)\right\} \tag{4.17}$$

Finally,

$$\Theta = -\left\{ Z(d_1) S \frac{\sigma}{2\sqrt{T}} - K r e^{-rT} N(-d_2) \right\} \tag{4.18}$$

4.3. **Auxilliary expressions.** Let's obtain formulae for $\frac{\partial N(d_1)}{\partial T}$ and $\frac{\partial N(d_2)}{\partial T}$. We shall need expressions for $\frac{\partial d_1}{\partial T}$ and $\frac{\partial d_2}{\partial T}$

$$\frac{\partial d_1}{\partial T} = \frac{\partial}{\partial T} \frac{ln(S/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} \tag{4.19}$$

$$= \frac{1}{\sigma\sqrt{T}} \frac{\partial}{\partial T} \left\{ ln(\frac{S}{K}) + (r + \frac{\sigma^2}{2})T \right\} + \left\{ ln(\frac{S}{K}) + (r + \frac{\sigma^2}{2})T \right\} \frac{\partial}{\partial T} \frac{1}{\sigma\sqrt{T}} \tag{4.20}$$

$$= \frac{1}{\sigma\sqrt{T}}(r + \frac{\sigma^2}{2}) + \left\{ ln(\frac{S}{K}) + (r + \frac{\sigma^2}{2})T \right\} \left(-\frac{1}{2} \right) \frac{1}{\sigma T\sqrt{T}} \tag{4.21}$$

$$= \frac{1}{2} \frac{1}{\sigma\sqrt{T}}(r + \frac{\sigma^2}{2}) - \frac{1}{2} \frac{ln(\frac{S}{K})}{\sigma T\sqrt{T}} = -\frac{1}{2T} \frac{ln(\frac{S}{K}) - (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \tag{4.22}$$

$$= -\frac{1}{2T} \frac{ln(\frac{S}{K}) + (r - \frac{\sigma^2}{2})T - 2rT}{\sigma\sqrt{T}} = -\frac{d_2}{2T} + \frac{r}{\sigma\sqrt{T}} \tag{4.23}$$

$$\frac{\partial d_1}{\partial T} = -\frac{d_2}{2T} + \frac{r}{\sigma\sqrt{T}} \tag{4.24}$$

and

$$\frac{\partial d_2}{\partial T} = \frac{\partial}{\partial T}(d_1 - \sigma\sqrt{T}) = \frac{\partial d_1}{\partial T} - \sigma\frac{\partial\sqrt{T}}{\partial T} = \frac{\partial d_1}{\partial T} - \frac{1}{2}\frac{\sigma}{\sqrt{T}} \tag{4.25}$$

$$= -\frac{d_2}{2T} + \frac{r}{\sigma\sqrt{T}} - \frac{1}{2}\frac{\sigma}{\sqrt{T}} = -\frac{d_1 - \sigma\sqrt{T}}{2T} - \frac{1}{2}\frac{\sigma}{\sqrt{T}} + \frac{r}{\sigma\sqrt{T}} = -\frac{d_1}{2T} + \frac{r}{\sigma\sqrt{T}} \tag{4.26}$$

$$\frac{\partial d_2}{\partial T} = -\frac{d_1}{2T} + \frac{r}{\sigma\sqrt{T}} \tag{4.27}$$

Finally, using 4.24 and 4.27 [1]

$$\frac{\partial N(d_2)}{\partial T} = N'(d_2)\frac{\partial d_2}{\partial T} = -Z(d_2)\left\{ \frac{d_1}{2T} - \frac{r}{\sigma\sqrt{T}} \right\} = -Z(d_1)\frac{S}{K}e^{rT}\left\{ \frac{d_1}{2T} - \frac{r}{\sigma\sqrt{T}} \right\} \tag{4.28}$$

$$\frac{\partial N(d_1)}{\partial T} = N'(d_1)\frac{\partial d_1}{\partial T} = -Z(d_1)\left\{ \frac{d_2}{2T} - \frac{r}{\sigma\sqrt{T}} \right\} \tag{4.29}$$

Let's obtain formulae for $\frac{\partial N(-d_1)}{\partial T}$ and $\frac{\partial N(-d_2)}{\partial T}$.

$$\frac{\partial N(-d_2)}{\partial T} = -N'(-d_2)\frac{\partial d_2}{\partial T} = Z(-d_2)\left\{ \frac{d_1}{2T} - \frac{r}{\sigma\sqrt{T}} \right\} = Z(d_1)\frac{S}{K}e^{rT}\left\{ \frac{d_1}{2T} - \frac{r}{\sigma\sqrt{T}} \right\} \tag{4.30}$$

[1] Note, that we think that in [3] a partial derivative $\frac{\partial d_1}{\partial T}$ is evaluated incorrectly to $-\frac{d_2}{2T}$. When last time we checked, i.e. on March 20, 2007, this error was not corrected yet. We believe that they mistakingly substituted a fraction in 4.22 by $d2$. Subsequently, a partial derivative $\frac{\partial d_2}{\partial T}$ is also evaluated incorrectly. However, these errors "cancel" each other, and the final expression for Θ is correct.

$$\frac{\partial N(-d_1)}{\partial T} = -N'(-d_1)\frac{\partial d_1}{\partial T} = Z(d_1)\left\{\frac{d_2}{2T} - \frac{r}{\sigma\sqrt{T}}\right\} \tag{4.31}$$

5. Vega

Both call and put options have the same Vega. First, we'll show the equation, then we'll prove it in next two subsections.[2]

$$\Upsilon = Z(d_1)S\sqrt{T} \tag{5.1}$$

5.1. Call Option Vega.

Definition 7. *Vega is defined as follows (see 14.8 in* [1]*).*

$$\Upsilon = \frac{\partial c}{\partial \sigma} \tag{5.2}$$

Substitute c with its formula 1.1

$$\Upsilon = \frac{\partial}{\partial \sigma}\left\{SN(d_1) - Ke^{-rT}N(d_2)\right\} \tag{5.3}$$

$$= S\frac{\partial N(d_1)}{\partial \sigma} - Ke^{-rT}\frac{\partial}{\partial \sigma}N(d_2) \tag{5.4}$$

Now, using 5.21 and 5.20 for $\frac{\partial N(d_1)}{\partial \sigma}$ and $\frac{\partial N(d_2)}{\partial \sigma}$

$$= -SZ(d_1)\frac{d_2}{\sigma} - Ke^{-rT}(-Z(d_1))\left\{\frac{S}{K}e^{rT}\frac{d_1}{\sigma}\right\} \tag{5.5}$$

$$= -SZ(d_1)\frac{d_2}{\sigma} + Z(d_1)S\frac{d_1}{\sigma} = SZ(d_1)\frac{d_1 - d_2}{\sigma} = Z(d_1)S\sqrt{T} \tag{5.6}$$

5.2. Put Option Vega.

Definition 8. *Vega is edfined as follows (see 14.8 in* [1]*).*

$$\Upsilon = \frac{\partial c}{\partial \sigma} \tag{5.7}$$

Substitute p with its formula 1.2

$$\Upsilon = \frac{\partial}{\partial \sigma}\left\{-SN(-d_1) + Ke^{-rT}N(-d_2)\right\} \tag{5.8}$$

$$= -S\frac{\partial N(-d_1)}{\partial \sigma} + Ke^{-rT}\frac{\partial}{\partial \sigma}N(-d_2) \tag{5.9}$$

Now, using 5.23 and 5.22 for $\frac{\partial N(-d_1)}{\partial \sigma}$ and $\frac{\partial N(-d_2)}{\partial \sigma}$

$$= -SZ(d_1)\frac{d_2}{\sigma} + Ke^{-rT}Z(d_1)\left\{\frac{S}{K}e^{rT}\frac{d_1}{\sigma}\right\} \tag{5.10}$$

[2] I didn't know which greek letter to use for Vega and chose upsilon.

$$= -SZ(d_1)\frac{d_2}{\sigma} + Z(d_1)S\frac{d_1}{\sigma} = SZ(d_1)\frac{d_1 - d_2}{\sigma} = Z(d_1)S\sqrt{T} \tag{5.11}$$

5.3. **Auxilliary expressions.** Let's obtain forumale for $\frac{\partial N(d_1)}{\partial \sigma}$ and $\frac{\partial N(d_2)}{\partial \sigma}$.

First, let'get $\frac{\partial d_1}{\partial \sigma}$ and $\frac{\partial d_2}{\partial \sigma}$

$$\frac{\partial d_1}{\partial \sigma} = \frac{\partial}{\partial \sigma}\frac{ln(S/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} \tag{5.12}$$

$$= \frac{1}{\sigma\sqrt{T}}\frac{\partial}{\partial \sigma}\left\{ln(\frac{S}{K}) + (r + \frac{\sigma^2}{2})T\right\} + \left\{ln(\frac{S}{K}) + (r + \frac{\sigma^2}{2})T\right\}\frac{\partial}{\partial \sigma}\frac{1}{\sigma\sqrt{T}} \tag{5.13}$$

$$= \frac{1}{\sigma\sqrt{T}}\sigma T + \left\{ln(\frac{S}{K}) + (r + \frac{\sigma^2}{2})T\right\}\left(-\frac{1}{\sigma^2}\right)\frac{1}{\sqrt{T}} \tag{5.14}$$

$$= -\left\{-\frac{1}{\sigma^2\sqrt{T}}\sigma^2 T\right\} + \left\{ln(\frac{S}{K}) + (r + \frac{\sigma^2}{2})T\right\}\left(-\frac{1}{\sigma^2}\right)\frac{1}{\sqrt{T}} \tag{5.15}$$

$$= \left\{ln(\frac{S}{K}) + (r - \frac{\sigma^2}{2})T\right\}\left(-\frac{1}{\sigma^2}\right)\frac{1}{\sqrt{T}} \tag{5.16}$$

$$= -\frac{1}{\sigma}\frac{ln(\frac{S}{K}) + (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \tag{5.17}$$

Therefore,

$$\frac{\partial d_1}{\partial \sigma} = -\frac{d_2}{\sigma} \tag{5.18}$$

and

$$\frac{\partial d_2}{\partial \sigma} = \frac{\partial}{\partial \sigma}(d_1 - \sigma\sqrt{T}) = \frac{\partial d_1}{\partial \sigma} - \sqrt{T} = -\frac{d_2}{\sigma} - \sqrt{T} = -\frac{d_2 + \sigma\sqrt{T}}{\sigma} = -\frac{d_1}{\sigma} \tag{5.19}$$

Finally, using 5.18 and 5.19

$$\frac{\partial}{\partial \sigma}N(d_2) = N'(d_2)\frac{\partial d_2}{\partial \sigma} = -Z(d_2)\frac{d_1}{\sigma} = -Z(d_1)(S/K)e^{rT}\frac{d_1}{\sigma} \tag{5.20}$$

$$\frac{\partial}{\partial \sigma}N(d_1) = N'(d_1)\frac{\partial d_1}{\partial \sigma} = -Z(d_1)\frac{d_2}{\sigma} \tag{5.21}$$

Let's obtain forumale for $\frac{\partial N(-d_1)}{\partial \sigma}$ and $\frac{\partial N(-d_2)}{\partial \sigma}$ using 5.18 and 5.19.

$$\frac{\partial}{\partial \sigma}N(-d_2) = -N'(-d_2)\frac{\partial d_2}{\partial \sigma} = Z(d_2)\frac{d_1}{\sigma} = Z(d_1)(S/K)e^{rT}\frac{d_1}{\sigma} \tag{5.22}$$

$$\frac{\partial}{\partial \sigma}N(-d_1) = -N'(-d_1)\frac{\partial d_1}{\partial \sigma} = Z(d_1)\frac{d_2}{\sigma} \tag{5.23}$$

6. Rho

6.1. **Call Option Rho.**

Definition 9. *Rho is defined by equation (see 14.9 in* [1]*)*

$$rho = \frac{\partial c}{\partial r} \tag{6.1}$$

Substitute c with its formula 1.1

$$rho = \frac{\partial}{\partial r}\left\{SN(d_1) - Ke^{-rT}N(d_2)\right\} \tag{6.2}$$

$$= S\frac{\partial N(d_1)}{\partial r} - K\left(N(d_2)\frac{\partial}{\partial r}e^{-rT} + e^{-rT}\frac{\partial}{\partial r}N(d_2)\right)$$

$$= S\frac{\partial N(d_1)}{\partial r} - Ke^{-rT}\left(-TN(d_2) + \frac{\partial}{\partial r}N(d_2)\right) \tag{6.3}$$

Using 6.15 and 6.16 for $\frac{\partial N(d_1)}{\partial r}$ and $\frac{\partial N(d_2)}{\partial r}$

$$= SZ(d_1)\frac{\sqrt{T}}{\sigma} - Ke^{-rT}\left(-TN(d_2) + Z(d_1)\frac{S}{K}e^{rT}\frac{\sqrt{T}}{\sigma}\right) \tag{6.4}$$

$$= SZ(d_1)\frac{\sqrt{T}}{\sigma} + Ke^{-rT}TN(d_2) - SZ(d_1)\frac{\sqrt{T}}{\sigma} \tag{6.5}$$

Finally,

$$rho = Ke^{-rT}TN(d_2) \tag{6.6}$$

6.2. **Put Option Rho.**

Definition 10. *Rho is defined by equation (see 14.9 in* [1]*)*

$$rho = \frac{\partial p}{\partial r} \tag{6.7}$$

Substitute p with its formula 1.2

$$rho = \frac{\partial}{\partial r}\left\{-SN(-d_1) + Ke^{-rT}N(-d_2)\right\} \tag{6.8}$$

$$= -S\frac{\partial N(-d_1)}{\partial r} + K\left(N(-d_2)\frac{\partial}{\partial r}e^{-rT} + e^{-rT}\frac{\partial}{\partial r}N(-d_2)\right)$$

$$= -S\frac{\partial N(-d_1)}{\partial r} + Ke^{-rT}\left(-TN(-d_2) + \frac{\partial}{\partial r}N(-d_2)\right) \tag{6.9}$$

Using 6.17 and 6.18 for $\frac{\partial N(-d_1)}{\partial r}$ and $\frac{\partial N(-d_2)}{\partial r}$

$$= SZ(d_1)\frac{\sqrt{T}}{\sigma} + Ke^{-rT}\left(-TN(-d_2) - Z(d_1)\frac{S}{K}e^{rT}\frac{\sqrt{T}}{\sigma}\right) \tag{6.10}$$

$$= SZ(d_1)\frac{\sqrt{T}}{\sigma} - Ke^{-rT}TN(-d_2) - SZ(d_1)\frac{\sqrt{T}}{\sigma} \tag{6.11}$$

Finally,

$$rho = -Ke^{-rT}TN(-d_2) \tag{6.12}$$

6.3. **Auxilliary expressions.** Let's obtain forumale for $\frac{\partial N(d_1)}{\partial r}$ and $\frac{\partial N(d_2)}{\partial r}$. First, we need $\frac{\partial d_1}{\partial r}$ and $\frac{\partial d_2}{\partial r}$

$$\frac{\partial d_1}{\partial r} = \frac{\partial}{\partial r}\frac{ln(S/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} = \frac{\sqrt{T}}{\sigma} \tag{6.13}$$

$$\frac{\partial d_2}{\partial r} = \frac{\partial}{\partial \sigma}(d_1 - \sigma\sqrt{T}) = \frac{\partial d_1}{\partial \sigma} = \frac{\sqrt{T}}{\sigma} \tag{6.14}$$

Finally,

$$\frac{\partial}{\partial r}N(d_1) = N'(d_1)\frac{\partial d_1}{\partial r} = Z(d_1)\frac{\sqrt{T}}{\sigma} \tag{6.15}$$

then using 8.3

$$\frac{\partial}{\partial r}N(d_2) = N'(d_2)\frac{\partial d_2}{\partial r} = Z(d_2)\frac{\sqrt{T}}{\sigma} = Z(d_1)\frac{S}{K}e^{rT}\frac{\sqrt{T}}{\sigma} \tag{6.16}$$

Let's obtain forumale for $\frac{\partial N(-d_1)}{\partial r}$ and $\frac{\partial N(-d_2)}{\partial r}$ using equations for $\frac{\partial d_1}{\partial r}$ and $\frac{\partial d_2}{\partial r}$.

$$\frac{\partial}{\partial r}N(-d_1) = -N'(-d_1)\frac{\partial d_1}{\partial r} = -Z(d_1)\frac{\sqrt{T}}{\sigma} \tag{6.17}$$

then using 8.3

$$\frac{\partial}{\partial r}N(-d_2) = -N'(-d_2)\frac{\partial d_2}{\partial r} = -Z(d_2)\frac{\sqrt{T}}{\sigma} = -Z(d_1)\frac{S}{K}e^{rT}\frac{\sqrt{T}}{\sigma} \tag{6.18}$$

7. Summary and sensitivity analysis

7.1. **Call Option Sensitivity.** Let's summarize the obtained results (2.5, 3.1, 4.9, 5.1 and 6.6) in a Table 1.

Table 1. Greeks for non-dividend paying call options

Greek	Definition	Formula	Sign
Delta (Δ)	$\frac{\partial c}{\partial S}$	$N(d_1)$	≥ 0
Gamma (Γ)	$\frac{\partial}{\partial S}\left(\frac{\partial c}{\partial S}\right)$	$Z(d_1)/(\sigma\sqrt{T}S)$	≥ 0
Theta (Θ)	$-\frac{\partial c}{\partial T}$	$-\left\{Z(d_1)S\frac{\sigma}{2\sqrt{T}} + Kre^{-rT}N(d_2)\right\}$	≤ 0
Vega	$\frac{\partial c}{\partial \sigma}$	$Z(d_1)S\sqrt{T}$	≥ 0
Rho	$\frac{\partial c}{\partial r}$	$Ke^{-rT}TN(d_2)$	≥ 0

The last column shows the sign of the Greek. For example, Θ has a non-positive value at any combination of its parameters. It's easy to see where these signs come from if you remember that $N(x)$, $Z(x)$ and e^x are positive at any x.

Now we have almost all inputs to prove the call option part of the sensitivity Table 2. These results can be explained by using Taylor series expansion (see 9.12 in [2]). Consider the first order term of the Taylor series for asset price changes:

$$c_{S_0+\delta S} = c_0 + \frac{\partial c}{\partial S} \cdot \delta S + ... = c_0 + \Delta \cdot \delta S + ...$$

TABLE 2. Call option price sensitivity to inputs

Input	Impact	Explanation
Asset price, S	+	$\Delta \geq 0$
Time (current), $t = T_0 - T$	-	$\Theta \leq 0$
Time to maturity, T	+	$-\Theta \geq 0$
Volatility, σ	+	$Vega \geq 0$
Risk free rate, r	+	$Rho \geq 0$

In order to analyze a sensitivity to changing time, let's suppose that the current time is t_0, option value at time $t_0 + \delta t$ is

$$c_{t_0+\delta t} = c_{t_0} + \frac{\partial c}{\partial t} \cdot \delta t + ...$$

Knowing that $\delta t = -\delta T$ we have $\frac{\partial c}{\partial t} = -\frac{\partial c}{\partial T}$. Therefore,

$$c_{t_0+\delta t} = c_{t_0} - \frac{\partial c}{\partial T} \cdot \delta t + ... = c_{t_0} + \Theta \cdot \delta t + ...$$

i.e. as time passes by European call option value tends to decrease because Θ is a negative number. On the other hand, if time to maturity increases, then option value should increase too.

Sensitivity to volatility and risk free rate changes can be analyzed similarly. Increased volatility and risk free rate should increase European call option price.

7.2. **Put Option Sensitivity.** Just like for call options, let's summarize our findings (2.10, 3.1, 4.18, 5.1 and 6.12) in a Table 3.

TABLE 3. Greeks for non-dividend paying put options

Greek	Definition	Formula	Sign
Delta (Δ)	$\frac{\partial p}{\partial S}$	$N(d_1) - 1$	≤ 0
Gamma (Γ)	$\frac{\partial}{\partial S}\left(\frac{\partial p}{\partial S}\right)$	$Z(d_1)/(\sigma\sqrt{T}S)$	≥ 0
Theta (Θ)	$-\frac{\partial c}{\partial T}$	$-\left\{Z(d_1)S\frac{\sigma}{2\sqrt{T}} - Kre^{-rT}N(-d_2)\right\}$	???
Vega	$\frac{\partial p}{\partial \sigma}$	$Z(d_1)S\sqrt{T}$	≥ 0
Rho	$\frac{\partial p}{\partial r}$	$-Ke^{-rT}TN(-d_2)$	≤ 0

Our analysis is similar to call option sensitivity analysis. In fact, the second and fourth rows are exactly the same. Let's look at the differences.

First of all, Δ is non-positive for put options. It's easy to see why if you recall that $N(x)$ is the cumulative normal distribution function, i.e. it can't be greater than 1. Hence, $N(x) - 1 \leq 0$ for any given x.

Second, Rho is non-positive for put options. Look at the expression for Rho, all its factors (K, e^x, T and $N(x)$) are positive, therefore Rho can not be positive.

The issue is with Θ, because its expression has a subtraction inside $Z(d_1)S\frac{\sigma}{2\sqrt{T}} - Kre^{-rT}N(-d_2)$. Depending on the inputs, it can be positive or negative.

Table 4. Put option price sensitivity to inputs

Input	Impact	Explanation
Asset price, S	-	$\Delta \leq 0$
Time (current), $t = T_0 - T$	???	$\Theta \geq 0 \bigcup \Theta \leq 0$
Time to maturity, T	???	$\Theta \geq 0 \bigcup \Theta \leq 0$
Volatility, σ	+	$Vega \geq 0$
Risk free rate, r	-	$Rho \leq 0$

Now we have almost all inputs to prove the put option part of the sensitivity Table 4. The results can be explained by using Taylor series expansion (see 9.12 in [2]). Consider the first order of the Taylor series for asset price changes:

$$c_{S_0+\delta S} = c_0 + \frac{\partial c}{\partial S} \cdot \delta S + ... = c_0 + \Delta \cdot \delta S + ...$$

Put option has non-positive Δ. Therefore, increase in stock price should decrease the option price. Sensitivity to volatility and risk free rate changes can be analyzed similarly. Increased volatility and decreased risk free rate should increase European put option price.

Due to the fact that Θ can be positive or negative, put option price's sensitivity to time to maturity is uncertain in general case, its sign depends on inputs.

8. Common expressions

By definition (see 26.1 in [2])

$$N'(x) = Z(x) \tag{8.1}$$

Hence,

$$N''(x) = Z'(x) = -\frac{x}{\sqrt{2\pi}} e^{-x^2/2} \tag{8.2}$$

Let's show the relation between $Z(d_2)$ and $Z(d_1)$.

$$Z(d_2) = \frac{1}{\sqrt{2\pi}} e^{-(d_1 - \sigma\sqrt{T})^2/2} = \frac{1}{\sqrt{2\pi}} e^{-d_1^2/2 - (\sigma^2 T - 2d_1\sigma\sqrt{T})/2}$$

Remember that $Z(d_1) = \frac{1}{\sqrt{2\pi}} e^{-d_1^2/2}$ and substitute it in the above formula

$$= Z(d_1)e^{-(\sigma^2 T - 2d_1\sigma\sqrt{T})/2}$$

then expand d_1

$$= Z(d_1)e^{-\sigma^2 T/2 + \frac{ln(S/K)+(r+\sigma^2/2)T}{\sigma\sqrt{T}}\sigma\sqrt{T}} = Z(d_1)e^{-\sigma^2 T/2 + ln(S/K)+(r+\sigma^2/2)T}$$

$$= Z(d_1)\left\{e^{ln(S/K)}e^{rt}e^{-\frac{\sigma^2 T}{2}+\frac{\sigma^2 T}{2}}\right\} = Z(d_1)(S/K)e^{rT}$$

Finally,

$$Z(d_2) = Z(d_1)(S/K)e^{rT} \tag{8.3}$$

Now, using this last equation we can get the following relation between $Z(-d_2)$ and $Z(-d_1)$.

$$Z(-d_2) = Z(d_2) = Z(d_1)(S/K)e^{rT} = Z(-d_1)(S/K)e^{rT} \tag{8.4}$$

Conclusion 1. *We derived five Greeks from Black-Scholes equation for European call and put options with non-divident paying stocks. We analyzed the sensitivity of option prices to changes in input parameters, and were able to figure out the sign of the impact of such changes in asset prices, volatility and risk free rate. We identified that decreasing time to maturity should increase call option price. However, for put options it could either decrease or increase the option price depending on the input parameters.*

We didn't look at the sensitivity to strike price and dividends changes, because it was out of the scope of this work. The focus of this excersize was to derive Greeks from Black-Scholes formula.

References

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