# Further Mathematical Methods: 1

In this lecture ...

- Further first order differential equations
  - Exact equation
  - Bernouilli equation
  - Homogeneous equations
- Further Complex Numbers
  - De Moivres Theorem and applications

# Exact Equation

We start by stating a result from calculus: Given a function $G(x, y)$ the total change (or *differential*) denoted $dG$ is defined as

$$dG = \frac{\partial G}{\partial x} dx + \frac{\partial G}{\partial y} dy$$

An equation of the form

$$M(x, y)\, dx + N(x, y)\, dy = 0 \tag{1}$$

is called an **Exact equation**.

Any $1^{\text{st}}$ order equation can be written in the form (1), where $M$, $N$ are functions of $x$ & $y$.

For example $\frac{dy}{dx} = x$ becomes $x\,dx - dy = 0$, so $M(x,y) = x$ and $N(x,y) = -1$.

**Definition**: The equation $M dx + N dy = 0$ is exact (or **Perfect**) if $\exists$ a function $G(x,y)$ s.t. (such that) the differential $dG = M dx + N dy$

The definition above is also a theorem, but we are not interested in the proof.

However, another result which follows from this (called a Corollary) is

**Corollary**: If $M(x,y)\,dx + N(x,y)\,dy = 0$ is exact then $\exists$ $G(x,y)$ s.t.

$M(x,y)\,dx + N(x,y)\,dy = dG = 0 \therefore G(x,y) =$ constant and this is the solution of the original equation (1).

This is now used to solve equations of type (1).

**Example:** $(2x + 3y)\,dx + (3x - y)dy = 0$

So $M = 2x + 3y \qquad N = 3x - y$. Is this equation exact?

$\dfrac{\partial M}{\partial y} = 3 = \dfrac{\partial N}{\partial x}$ so equation is exact.

So $\exists\ G\,(x, y)$ s.t. $dG = \dfrac{\partial G}{\partial x}dx + \dfrac{\partial G}{\partial y}dy \equiv (2x + 3y)\,dx + (3x - y)dy$

$\therefore$

$$\left.\begin{array}{ll} \dfrac{\partial G}{\partial x} = 2x + 3y & \text{(A)} \\ \dfrac{\partial G}{\partial y} = 3x - y & \text{(B)} \end{array}\right\}$$

Integrate (A) wrt $x$ keeping $y$ fixed. Similarly Integrate (B) wrt $y$ keeping $x$ fixed.

$$\int \frac{\partial G}{\partial x} dx = G = x^2 + 3xy + \varphi(y) \tag{2}$$

$$\int \frac{\partial G}{\partial y} dy = G = 3xy - \frac{1}{2}y^2 + \psi(x) \tag{3}$$

$$(2) \equiv (3)$$

$$\therefore x^2 + 3xy + \varphi(y) \equiv 3xy - \frac{1}{2}y^2 + \psi(x)$$

These are identical if $\varphi(y) + \frac{1}{2}y^2 = \psi(x) - x^2 = c$ (recall $F(x) = H(y) \Rightarrow$ each side constant)

$\therefore \psi(x) = c + x^2$ (we have a choice of choosing either)

$\therefore G(x, y) = x^2 + 3xy - \frac{1}{2}y^2 + c$

Solution is $G =$ constant (from earlier corollary)

$\Rightarrow$ GS is $x^2 + 3xy - \frac{1}{2}y^2 = c$

# Reducible To Exact Form

Unless we are fairly lucky or the problem is particularly straight forward, most equations will not be exact. That is equations of type (1) will have

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}.$$

We can use an *integrating factor* (I.F) approach to convert the equation to exact form. If

$$\frac{M_y - N_x}{N} = f(x)$$

then we multiply (1) by the I.F $\mu(x)$, where

$$\mu(x) = \exp\left(\int \frac{M_y - N_x}{N} dx\right).$$

If

$$\frac{N_x - M_y}{M} = g(y)$$

then the I.F $\mu(y)$, is

$$\mu(y) = \exp\left(\int \frac{N_x - M_y}{M} dy\right).$$

**Example:** Consider the IVP $xdx + (x^2y + 4y)dy = 0, \;\; y(4) = 0$

Clearly this equation is not exact because $\dfrac{\partial M}{\partial y} = 0 \neq \dfrac{\partial N}{\partial x} = 2xy.$

Look at (first)

$$\begin{aligned}\frac{M_y - N_x}{N} &= \frac{-2xy}{x^2y + 4y} \\ &= \frac{-2x}{x^2 + 4}\end{aligned}$$

which is a function of $x$ alone. So I.F is

$$\begin{aligned}\mu(x) &= \exp\left(-\int \frac{2x}{x^2+4}dx\right)\\ &= \frac{1}{x^2+4}\end{aligned}$$

which we multiply the differential equation with to get the exact equation

$$\left(\frac{x}{x^2+4}\right)dx + ydy = 0$$

So $\exists\ G(x,y)$ s.t. $dG = \frac{\partial G}{\partial x}dx + \frac{\partial G}{\partial y}dy \equiv \left(\frac{x}{x^2+4}\right)dx + ydy$

$\therefore$

$$\left.\begin{aligned}\frac{\partial G}{\partial x} &= \frac{x}{x^2+4} && \text{(C)}\\ \frac{\partial G}{\partial y} &= y && \text{(D)}\end{aligned}\right\}$$

As with the previous example integrate (C) wrt $x$ keeping $y$ fixed, and integrate (D) wrt $y$ keeping $x$ fixed.

$$G = \frac{1}{2} \ln \left| x^2 + 4 \right| + \varphi(y) \tag{4a}$$

$$G = \frac{1}{2} y^2 + \psi(x) \tag{4b}$$

$$(4a) \equiv (4b)$$

$$\therefore \frac{1}{2} \ln \left| x^2 + 4 \right| + \varphi(y) \equiv \frac{1}{2} y^2 + \psi(x)$$

Identical if $\varphi(y) - \frac{1}{2}y^2 = \psi(x) - \frac{1}{2}\ln\left|x^2+4\right| = c$

$\therefore$Let us choose $\psi(x) = \frac{1}{2}\ln\left|x^2+4\right| + c$

$\therefore G(x,y) = \frac{1}{2}y^2 + \frac{1}{2}\ln\left|x^2+4\right| + c$

Solution is $G =$constant

$\Rightarrow \frac{1}{2}y^2 + \frac{1}{2}\ln\left|x^2+4\right| = c.$

We can tidy this up multiplying through by 2 and taking exponentials

$$\begin{aligned} \exp\left(y^2 + \ln\left|x^2+4\right|\right) &= C \\ \exp\left(y^2\right)\left(x^2+4\right) &= K \end{aligned}$$

which is the general solution. Now use initial condition to determine $K$. When $x = 4$, $y = 0$ gives $K = 20$. Hence the particular solution becomes

$$e^{y^2}\left(x^2 + 4\right) = 20.$$

## Bernoulli Equation

This an ODE of the form

$$y' + P(x)\,y = Q(x)\,y^n \tag{5}$$

and is nonlinear due to the term $y^n$, but for $n = 0, 1$ (5) is linear. In the case $n \geq 2$, divide (5) through by $y^n$, to obtain

$$\frac{1}{y^n}y' + P(x)\,\frac{1}{y^{n-1}} = Q(x) \tag{6}$$

Now let $z = \dfrac{1}{y^{n-1}}$ then

$$\frac{dz}{dx} = \frac{d}{dx}\left(y^{-n+1}\right) = \frac{d}{dy}\left(y^{-n+1}\right)\frac{dy}{dx}$$

$$\frac{dz}{dx} = \frac{-(n-1)}{y^n}\frac{dy}{dx} \tag{7}$$

Rearranging (7) gives $\frac{1}{y^n}y' = \frac{-1}{(n-1)}z'$ so (6) becomes

$$\frac{-1}{(n-1)}z' + P(x)\,z = Q(x)$$

Then multiplying through by $-(n-1)$ gives

$$z'(x) + \widehat{P}(x)z = \widehat{Q}(x)$$

where $\widehat{P}(x) = -(n-1)\,P(x)$, $\widehat{Q}(x) = -(n-1)\,Q(x)$.

**Example:**

Solve the equation

$$y' + 2xy = xy^3$$

This can be written as $\frac{1}{y^3}y' + 2x\frac{1}{y^2} = x$, i.e. $n = 3$, therefore put $z = \frac{1}{y^2}$,so

$$z' = -\frac{2}{y^3}y'$$

which can be re-written as $\frac{1}{y^3}y' = -\frac{1}{2}z' \therefore -\frac{1}{2}z' + 2xz = x$ , or

$$z' - 4xz = -2x \tag{8}$$

which is linear with $P = -4x$; $Q = -2x$.

$$\text{I.F} = R(x) = \exp\left(-4\int x\,dx\right) = \exp\left(-2x^2\right)$$

and multiply through (8) by $\exp\left(-2x^2\right)$

$$\therefore\ \exp\left(-2x^2\right)\left(z' - 4xz\right) = -2x\exp\left(-2x^2\right)$$

$$\text{Then } \frac{d}{dx}\left(z\exp\left(-2x^2\right)\right) = -2x\exp\left(-2x^2\right)$$

$$z \exp\left(-2x^2\right) = -2 \int x \exp\left(-2x^2\right)\ dx + c,$$

we integrate rhs by substitution : put $u = 2x^2$

$$z \exp\left(-2x^2\right) = \frac{1}{2} \exp\left(-2x^2\right) + c$$

$z = \frac{1}{2} + c \exp\left(2x^2\right)$ and we know $z = \frac{1}{y^2}$, so the GS becomes

$$\frac{1}{y^2} = \frac{1}{2} + c \exp\left(2x^2\right).$$

# Homogeneous Equation

**Definition:** A function $f(x,y)$ is **homogeneous of degree** $k$ if

$$f(tx,ty) = t^k f(x,y)$$

**Example** $f(x,y) = \sqrt{\left(x^2+y^2\right)}$

$$\begin{aligned} f(tx,ty) &= \sqrt{\left[(tx)^2+(ty)^2\right]} \\ &= t\sqrt{\left[\left(x^2+y^2\right)\right]} \\ &= tf(x,y) \end{aligned}$$

So $f$ is homogeneous of degree one.

**Example** $f(x,y) = \dfrac{x+y}{x-y}$ then

$$\begin{aligned} f(tx,ty) &= \frac{tx+ty}{tx-ty} \\ &= t^0\left(\frac{x+y}{x-y}\right) \\ &= t^0 f(x,y) \end{aligned}$$

So $f$ is homogeneous of degree zero.

**Example** $f(x,y) = x^2 + y^3$

$$\begin{aligned} f(tx,ty) &= (tx)^2 + (ty)^3 \\ &= t^2x^2 + t^3y^3 \\ &\neq t^k\left(x^2+y^3\right) \end{aligned}$$

for any $k$. So $f$ is not homogeneous.

**Definition** The differential equation $\frac{dy}{dx} = f(x, y)$ is said to be *homogeneous* when $f(x, y)$ is homogeneous of degree $k$ for some $k$.

## Method of Solution

Put $y = vx$ where $v$ is some (as yet) unknown function. Hence we have

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}(vx) = x\frac{dv}{dx} + v\frac{dx}{dx} \\ &= v'x + v \end{aligned}$$

Hence

$$f(x, y) = f(x, vx)$$

Now $f$ is homogeneous of degree $k$ – so

$$f(t\xi, t\eta) = t^k f(\xi, \eta) \ \forall \ \xi, \ \eta$$

so

$$f\left(x\xi, x\eta\right) = x^k f\left(\xi, \eta\right) \quad \forall\ \xi,\ \eta$$

put $\xi = 1, \quad \eta = v$

$$f\left(x.1, x.v\right) = x^k f\left(1, v\right)$$

The differential equation now becomes

$$v'x + v = x^k f\left(1, v\right)$$

which is not always solvable - the method may not work. But when $k = 0$ (homogeneous of degree zero) then $x^k = 1$.

Hence

$$v'x + v = f\left(1, v\right)$$

or

$$x\frac{dv}{dx} = f\left(1, v\right) - v$$

which is separable, i.e.

$$\int \frac{dv}{f\left(1, v\right) - v} = \int \frac{dx}{x} + c$$

and the method is guaranteed to work.

**Example**

$$\frac{dy}{dx} = \frac{y - x}{y + x}$$

First we check:

$$\frac{ty - tx}{ty + tx} = t^0 \left( \frac{y - x}{y + x} \right)$$

which is homogeneous of degree zero. So put $y = vx$

$$v'x + v = f(x, yx) = \frac{vx - x}{vx + x} = \frac{v - 1}{v + 1} = f(1, v)$$

therefore

$$\begin{aligned} v'x &= \frac{v-1}{v+1} - v \\ &= \frac{-\left(1 + v^2\right)}{v+1} \end{aligned}$$

and the D.E is now separable

$$\begin{aligned}
\int \frac{v+1}{v^2+1}dv &= -\int \frac{1}{x}dx \\
\int \frac{v}{v^2+1}dv + \int \frac{1}{v^2+1}dv &= -\int \frac{1}{x}dx \\
\frac{1}{2}\ln\left(1+v^2\right) + \arctan v &= -\ln x + c \\
\frac{1}{2}\ln x^2\left(1+v^2\right) + \arctan v &= c
\end{aligned}$$

Now we turn to the original problem, so put $v = \dfrac{y}{x}$

$$\frac{1}{2}\ln x^2\left(1+\frac{y^2}{x^2}\right) + \arctan\left(\frac{y}{x}\right) = c$$

which simplifies to

$$\frac{1}{2}\ln\left(x^2+y^2\right) + \arctan\left(\frac{y}{x}\right) = c.$$

# Equation Reducible to Homogeneous Form

The equation

$$\frac{dy}{dx} = \frac{ax + by + c}{Ax + By + C}$$

is not homogeneous in its current form.

## Method: Put

$$\begin{aligned} x &= X + h \\ y &= Y + k \end{aligned}$$

where $h,\ k$ are solutions of

$$\begin{aligned} ah + bk + c &= 0 \\ Ah + Bk + C &= 0 \end{aligned}$$

i.e. the geometric interpretation of the above is that $(h, k)$ is the intersection of the lines $ah + bk + c = 0$ and $Ah + Bk + C = 0$. Obviously $(h, k)$ exists provided the lines are not parallel. Then

$$\frac{dy}{dx} = \frac{d\,(Y + k)}{d\,(X + h)} = \frac{dY}{dX}$$

so

$$\begin{aligned} \frac{dY}{dX} &= \frac{a\,(X + h) + b\,(Y + k) + c}{A\,(X + h) + B\,(Y + k) + C} \\ &= \frac{aX + bY + (ah + bk + c)}{AX + BY + (Ah + Bk + C)} \end{aligned}$$

which becomes (from using the earlier expressions)

$$\frac{dY}{dX} = \frac{aX + bY}{AX + BY}$$

and is homogeneous of degree zero. Now set $Y = VX$ and proceed as outlined earlier.

**Example**

$$y' = \frac{2x + y - 1}{x + 2y + 1}$$

put $x = X + h, \quad y = Y + k$ where

$$\left.\begin{array}{l} 2h + k - 1 = 0 \\ h + 2k + 1 = 0 \end{array}\right\}$$

hence $h = 1,\ k = -1$ and $x = X + 1, \quad y = Y - 1$

$$\frac{dY}{dX} = \frac{2X + Y}{X + 2Y}$$

making the equation homogeneous of degree zero, so we put $Y = VX$

$$\begin{aligned}
V'X + V &= \frac{2X + VX}{X + 2VX} = \frac{2 + V}{1 + 2V} \\
V'X &= \frac{2 + V}{1 + 2V} - V \\
X\frac{dV}{dX} &= \frac{2\left(1 - V^2\right)}{1 + 2V}
\end{aligned}$$

which is a separable equation.

$$\int \frac{1 + 2V}{1 - V^2} = 2\int \frac{dX}{X}$$

For the left hand side using a partial fraction approach gives

$$\frac{1 + 2V}{(1 - V)(1 + V)} \equiv \frac{3/2}{1 - v} + \frac{-1/2}{1 + V}$$

hence

$$\begin{aligned}
\int \left( \frac{3/2}{1-V} + \frac{-1/2}{1+V} \right) dV &= 2\int \frac{dX}{X} \\
-\frac{3}{2}\ln(1-V) - \frac{1}{2}\ln(1+V) &= 2\ln X + c \\
\frac{3}{2}\ln(1-V) + \frac{1}{2}\ln(1+V) + 2\ln X &= k \\
\ln(1-V)^{3/2}(1+V)^{1/2}X^2 &= k \\
(1-V)^{3/2}(1+V)^{1/2}X^2 &= C
\end{aligned}$$

Now use $V = \dfrac{Y}{X}$ :

$$\begin{aligned}
\left(1 - \frac{Y}{X}\right)^{3/2} \left(1 + \frac{Y}{X}\right)^{1/2} X^2 &= C \\
(X-Y)^{3/2}(X+Y)^{1/2} &= C \\
(X-Y)^3(X+Y) &= K
\end{aligned}$$

and we know $X = x - 1, \quad Y = y + 1$ so the general solution becomes

$$(x - y - 2)^3 \, (x + y) = \text{constant}$$

# Special Case

The lines

$$\begin{aligned} ah + bk + c &= 0 \\ Ah + Bk + C &= 0 \end{aligned}$$

are parallel.

**Example:**

$$\frac{dy}{dx} = \frac{2x + y - 3}{4x + 2y - 1}$$

lines here are parallel with slope of $-2$. The denominator of the right hand side can be written as $2(2x + y) - 1$ so try a substitution of the form $u = 2x + y$, i.e. $y = u - 2x \longrightarrow$

$$\frac{dy}{dx} = \frac{du}{dx} - 2$$

and the differential equation becomes

$$y' = u' - 2 = \frac{u-3}{2u-1}$$

which in terms of the new variable becomes

$$\begin{aligned} u' &= \frac{u-3}{2u-1} + 2 \\ &= \frac{5u-5}{2u-1} \end{aligned}$$

which is separable. We present the working in full to show the integration step

$$\begin{aligned}
\int \frac{2u-1}{5u-5} du &= \int dx \\
\frac{1}{5}\int \left(2 + \frac{1}{u-1}\right) du &= x + c \\
\frac{1}{5}\left(2u + \ln\left(u-1\right)\right) &= x + c
\end{aligned}$$

Now to return to original variables, put $u = y + 2x$ to get the final form

$$\frac{1}{5}\left(2y + 4x + \ln\left(y + 2x - 1\right)\right) = x + c$$

which is the general solution.

# Complex Numbers

For any $z$, the expression

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \quad \cos z = \frac{e^{iz} + e^{-iz}}{2} \quad \text{and} \quad \tan z = \frac{\sin z}{\cos z}$$

defines the generalised circular functions, and

$$\sinh z = \frac{e^{z} - e^{-z}}{2}, \quad \cosh z = \frac{e^{z} + e^{-z}}{2} \quad \text{and} \quad \tanh z = \frac{\sinh z}{\cosh z}$$

the generalised hyperbolic function.

Using Euler's formula with positive and negative components we have

$$\begin{aligned} e^{i\theta} &= \cos\theta + i\sin\theta \\ e^{-i\theta} &= \cos\theta - i\sin\theta \end{aligned}$$

Adding gives

$$2\cos\theta = e^{i\theta} + e^{-i\theta} \Rightarrow \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

and subtracting gives

$$2i \sin\theta = e^{i\theta} - e^{-i\theta} \Rightarrow \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

We can extend these results to consider other functions:

$$\begin{aligned} \cos\mathrm{ec}\ z &= \frac{1}{\sin z}, \quad \sec = \frac{1}{\cos z}, \quad \cot z = \frac{1}{\tan z} \\ \cosh\mathrm{ec}\ z &= \frac{1}{\sinh z}, \quad \sec = \frac{1}{\cosh z}, \quad \cot z = \frac{1}{\tanh z} \end{aligned}$$

We can also obtain a relationship between circular and hyperbolic functions:

$$\sin(iz) = \frac{1}{2i}\left(e^{-z} - e^{z}\right)$$

we know $1/i = -i$ hence

$$\sin(iz) = -i.\frac{1}{2}\left(e^{-z} - e^{z}\right) = i.\frac{1}{2}\left(e^{z} - e^{-z}\right)$$

so

$$\sin(iz) = i\sinh z.$$

Similarly it can be shown that

$$\begin{aligned}
\sinh(iz) &= i\sin z\\
\cos(iz) &= \cosh z\\
\cosh(iz) &= \cos z\\
\sinh(iz) &= i\sin z
\end{aligned}$$

**Example:**

Let $z = x+iy$ be any complex number, find all the values for which $\cosh z = 0$.

We use the hyperbolic identity

$$\cosh(a+b) = \cosh a \cosh b + \sinh a \sinh b$$

to give

$$\begin{aligned} \cosh z &= \cosh\left(x+iy\right) = \cosh x \cosh iy + \sinh x \sinh iy \\ &= \cosh x \cos y + i \sinh x \sin y \end{aligned}$$

i.e.

$$\cosh x \cos y + i \sinh x \sin y = 0$$

so equating real and imaginary parts we have two equations

$$\cosh x \cos y = 0$$

$$\sinh x \sin y = 0$$

From the first we know that $\cosh x \neq 0$ so we require $\cos y = 0 \Rightarrow y = \frac{\pi}{2} + n\pi \quad \forall n \in \mathbb{Z}$.

Putting this in the second equation gives

$$\sinh x \sin(2n+1)\frac{\pi}{2} = 0$$

where

$$\sin(2n+1)\frac{\pi}{2} = \cos n\pi = (-1)^n$$

so

$$\sinh x = 0$$

which has the solution $x = 0$. Therefore the solution to our equation $\cosh z = 0$ is

$$z_n = i(2n+1)\frac{\pi}{2}, \quad n \in \mathbb{Z}$$

# De Moivres Theorem

$$\begin{aligned}(\cos\theta + i\sin\theta)^n &= \left(e^{i\theta}\right)^n \\ &= e^{in\theta} \\ &= \cos n\theta + i\sin n\theta\end{aligned}$$

Similarly

$$(\cos\theta + i\sin\theta)^{-n} = \cos n\theta - i\sin n\theta.$$

It is quite common to write $\cos\theta + i\sin\theta$ as $cis$.

If

$$z = e^{i\theta} = \cos\theta + i\sin\theta \quad \text{then} \quad \frac{1}{z} = e^{-i\theta} = \overline{z} = \cos\theta - i\sin\theta.$$

So

$$\begin{aligned} \cos\theta &= \operatorname{Re} z = \frac{1}{2}(z+\overline{z}) = \frac{1}{2}\left(z+\frac{1}{z}\right) \\ \sin\theta &= \operatorname{Im} z = \frac{1}{2i}(z-\overline{z}) = \frac{1}{2i}\left(z-\frac{1}{z}\right). \end{aligned}$$

Also $z^n = e^{in\theta} \longrightarrow$

$$\begin{aligned} z^n + z^{-n} &= (\cos n\theta + i\sin n\theta) + (\cos n\theta - i\sin n\theta) \\ &= 2\cos n\theta \end{aligned}$$

$\therefore$rearranging gives

$$\cos n\theta = \frac{1}{2}\left(z^n + \frac{1}{z^n}\right).$$

Similarly

$$\sin n\theta = \frac{1}{2}\left(z^n - \frac{1}{z^n}\right)$$

# Finding Roots of Complex Numbers

Consider a number $w$, which is an $n^{\text{th}}$ root of the complex number $z$. That is, if $w^n = z$, and hence we can write

$$w = z^{1/n}.$$

We begin by writing in polar/mod-arg form

$$z = r\left(\cos\theta + i\sin\theta\right).$$

hence

$$z^{1/n} = r^{1/n}\left(\cos\theta + i\sin\theta\right)^{1/n}$$

and then by DMT we have

$$z^{1/n} = r^{1/n}\left(\cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n}\right) \quad k = 0, 1, ....., n-1.$$

Any other values of $k$ would lead to repetition.

This method is particularly useful for obtaining the $n-$ roots of unity. This requires solving the equation

$$z^n = 1.$$

There are only two real solutions here, $z = \pm 1$, which corresponds to the case of even values of $n$. If $n$ is odd, then there exists one real solution, $z = 1$. Any other solutions will be complex. Unity can be expressed as

$$1 = \cos 2k\pi + i \sin 2k\pi$$

which is true for all $k \in \mathbb{Z}$. So $z^n = 1$ becomes

$$r^n \left(\cos n\theta + i \sin\left(n\theta\right)\right) = \cos 2k\pi + i \sin 2k\pi.$$

The modulus and argument for $z = 1$ is one and zero, in turn. Equating the modulus and argument of both sides gives the following equations

$$r^n = 1 \quad \text{and} \quad n\theta = 2k\pi$$

Therefore

$$\begin{aligned} z &= \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n} = 1 \\ &= \exp\left(\frac{2k\pi i}{n}\right) \quad k = 0, ..., n-1 \end{aligned}$$

If we set $\omega = \exp\left(\frac{2k\pi i}{n}\right)$ then the $n-$ roots of unity are $1, \omega, \omega^2, ....., \omega^{n-1}$.

These roots can be represented geometrically as the vertices of an $n-$ sided regular polygon which is inscribed in a circle of radius 1 and centred at the origin. Such a circle which has equation given by $|z| = 1$ and is called the *unit circle*.

More generally the equation

$$|z - z_0| = R$$

represents a circle centred at $z_0$ of radius $R$. If $z_0 = a + ib$, then

$$\begin{aligned} |z - z_0| &= |(x, y) - (a, b)| \\ &= |(x - a) + i(y - b)| \end{aligned}$$

and

$$
\begin{aligned}
|(x-a)+i\,(y-b)|^2 &= R^2 \\
(x-a)^2+(y-b)^2 &= R^2
\end{aligned}
$$

which is the cartesian form for a circle, centred at $(a, b)$ with radius $R$.

# Applications

**Example 1**

Calculate the indefinite integral $\int \cos^4\theta \, d\theta$.

We begin by expressing $\cos^4\theta$ in terms of $\cos n\theta$ (for different $n$).

$$\begin{aligned}
\cos\theta &= \frac{1}{2}\left(z+\frac{1}{z}\right) \Rightarrow 2^4\cos^4\theta = \left(z+\frac{1}{z}\right)^4 \therefore \\
2^4\cos^4\theta &= z^4 + 4z^3\frac{1}{z} + 6z^2\frac{1}{z^2} + 4z\frac{1}{z^3} + \frac{1}{z^4} \quad \text{using Pascals triangle} \\
&= z^4 + 4z^2 + 6 + 4\frac{1}{z^2} + \frac{1}{z^4} \\
&= \left(z^4 + \frac{1}{z^4}\right) + 4\left(z^2 + \frac{1}{z^2}\right) + 6
\end{aligned}$$

We know

$$\tfrac{1}{2}\left(z^n + \frac{1}{z^n}\right) = \cos n\theta$$

$$2^4 \cos^4\theta = 2.\tfrac{1}{2}\left(z^4 + \frac{1}{z^4}\right) + 4.2.\tfrac{1}{2}\left(z^2 + \frac{1}{z^2}\right) + 6$$

hence

$$\begin{aligned} 2^4 \cos^4\theta &= 2\cos 4\theta + 8\cos 2\theta + 6 \\ \cos^4\theta &= \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3) \therefore \end{aligned}$$

Now integrating

$$\begin{aligned} \int \cos^4\theta d\theta &= \frac{1}{8}\int (\cos 4\theta + 4\cos 2\theta + 3)\ d\theta \\ &= \frac{1}{32}\sin 4\theta + \frac{1}{4}\sin 2\theta + \frac{3}{8}\theta + K \end{aligned}$$

**Example 2**

As another application , express $\cos 4\theta$ in terms of $\cos^n \theta$.

We know from De Moivres theorem that

$$\cos 4\theta = \mathrm{Re}\,(\cos 4\theta + i \sin 4\theta)$$

So

$$\cos 4\theta = \mathrm{Re}\,(\cos \theta + i \sin \theta)^4\,,$$

and put $c \equiv \cos\theta, \quad is \equiv i \sin\theta,$ to give

$$\begin{aligned}
\cos 4\theta &= \mathrm{Re}\left(c^4 + 4c^3\,(is) + 6c^2\,(is)^2 + 4c\,(is)^3 + (is)^4\right) \\
\cos 4\theta &= \mathrm{Re}\left(c^4 + i4c^3 s - 6c^2 s^2 - i4cs^3 + s^4\right) \\
\cos 4\theta &= c^4 - 6c^2 s^2 + s^4
\end{aligned}$$

Now $s^2 = 1 - c^2$, $\therefore$

$$\begin{aligned}\cos 4\theta &= c^4 - 6c^2\left(1 - c^2\right) + \left(1 - c^2\right)^2 = 8c^4 - 8c^2 + 1 \Rightarrow \\ \cos 4\theta &= 8\cos^4\theta - 8\cos^2\theta + 1.\end{aligned}$$

**Example 3**

Find the square roots of $-1$ , i.e. solve $z^2 = -1$. The complex number $-1$ has a modulus of one and argument $\pi$, so

$$-1 = \cos\left(\pi + 2k\pi\right) + i\sin\left(\pi + 2k\pi\right).$$

Hence,

$$\begin{aligned}(-1)^{1/2} &= \left(\cos\left(\pi + 2k\pi\right) + i\sin\left(\pi + 2k\pi\right)\right)^{1/2} \\ &= \cos\left(\frac{\pi + 2k\pi}{2}\right) + i\sin\left(\frac{\pi + 2k\pi}{2}\right)\end{aligned}$$

for $k = 0, 1$ :

$$(-1)^{1/2} = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right) = 0 + i$$

$$(-1)^{1/2} = \cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right) = 0 - i$$

Therefore the square roots of $-1$ are $z_0 = i$ and $z_1 = -i$.

**Example 4**

Find the fifth roots of $-1$ , i.e. solve $z^5 = -1$. The complex number $-1$ has a modulus of one and argument $\pi$, so

$$\begin{aligned} (-1)^{1/5} &= (\cos(\pi + 2k\pi) + i\sin(\pi + 2k\pi))^{1/5} \\ &= \cos\left(\frac{\pi + 2k\pi}{5}\right) + i\sin\left(\frac{\pi + 2k\pi}{5}\right) \end{aligned}$$

for $k = 0, 1, 2, 3, 4$ :

$$z_0 = \cos\left(\frac{\pi}{5}\right) + i\sin\left(\frac{\pi}{5}\right)$$

$$z_1 = \cos\left(\frac{3\pi}{5}\right) + i\sin\left(\frac{3\pi}{5}\right)$$

$$z_2 = \cos(\pi) + i\sin(\pi)$$

$$z_3 = \cos\left(\frac{7\pi}{5}\right) + i\sin\left(\frac{7\pi}{5}\right)$$

$$z_4 = \cos\left(\frac{9\pi}{5}\right) + i\sin\left(\frac{9\pi}{5}\right)$$

**Example 5**

Find all $z \in \mathbb{C}$ such that $z^3 = 1 + i$. So we wish to find the cube roots of $(1 + i)$. The argument of this complex number is $\theta = \arctan 1 = \pi/4$. The modulus of $(1+i)$ is $r=\sqrt{2}$. We can express $(1+i)$ compactly in $r\exp(i\theta)$ as

$$1+i=\sqrt{2}\exp\left(i\frac{\pi}{4}\right)$$

So

$$(1+i)^{1/3}=2^{1/6}\exp\left(i\frac{\pi(8k+1)}{12}\right)$$

for $k=0,1,2$.

$$z_0=2^{1/6}\exp\left(i\frac{\pi}{12}\right)$$

$$z_1=2^{1/6}\exp\left(i\frac{9\pi}{12}\right)$$

$$z_2=2^{1/6}\exp\left(i\frac{17\pi}{12}\right)$$

**Example 6**: We can apply Euler's formula to integral problems. Consider the earlier example

$$\int e^x \cos x dx$$

which was simplified using the integration by parts method. We know $\operatorname{Re} e^{i\theta} = \cos\theta$, so the above becomes

$$\begin{aligned}
\int e^x \operatorname{Re} e^{ix} dx &= \int \operatorname{Re} e^{(i+1)x} dx = \operatorname{Re} \tfrac{1}{1+i} e^{(i+1)x} \\
&= e^x \operatorname{Re} \tfrac{1}{1+i} \left(e^{ix}\right) = e^x \operatorname{Re} \tfrac{1-i}{(1+i)(1-i)} \left(e^{ix}\right) \\
&= \tfrac{1}{2} e^x \operatorname{Re} (1-i) \left(e^{ix}\right) = \tfrac{1}{2} e^x \operatorname{Re} \left(e^{ix} - i e^{ix}\right) \\
&= \tfrac{1}{2} e^x \operatorname{Re} (\cos x + i \sin x - i \cos x + \sin x) \\
&= \tfrac{1}{2} e^x (\cos x + \sin x)
\end{aligned}$$

# Functions

**Polynomial Functions:** A polynomial function of $z$ has the form

$$f(z) = a_0 + a_1 z + a_2 z^2 + O\left(z^3\right) = \sum_{n=0}^{\infty} a_n z^n$$

and is of degree $n$. The domain is the set $\mathbb{C}$ of all complex numbers. So for example a 3rd degree polynomial is $2 - z + a_2 z^2 + 3z^3$.

**Rational Functions:** A rational function has the form

$$R(z) = \frac{P_1(z)}{P_2(z)}$$

where $P_1,\ P_2$ are polynomials. The domain is the set $\mathbb{C}-$zeroes of $P_2(z)$. For example

$$\begin{aligned} f(z) &= \frac{2z+3}{z^2 - 3z + 2} \\ &= \frac{2z+3}{(z-1)(z-2)} \end{aligned}$$

and domain is $\mathbb{C} - \{1, 2\}$.

**Exponential Function:** $\exp(z) = e^z = e^{x+iy} = e^x e^{iy}$.

$\operatorname{Re} e^z : u(x, y) = e^x \cos y$

$\operatorname{Im} e^z : v(x, y) = e^x \sin y$

$|\exp z| = e^x$ and $y$ is the argument.

**Power Series:**

$$
\begin{aligned}
\exp(\pm z) &= 1 \pm z + \frac{1}{2!}z^2 \pm \frac{1}{3!}z^3 + \ldots\ldots = \sum_{n=0}^{\infty} (-1)^n \frac{z^n}{n!} \\
\sinh z &= z + \frac{1}{3!}z^3 + \frac{1}{5!}z^5 + \ldots\ldots = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} \\
\cosh z &= 1 + \frac{1}{2!}z^2 + \frac{1}{4!}z^4 + \ldots\ldots = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} \\
\sin z &= z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 - \ldots\ldots = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \\
\cos z &= 1 - \frac{1}{2!}z^2 + \frac{1}{4!}z^4 - \ldots\ldots = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}
\end{aligned}
$$