Exercise 5.5
Solution

1. **Synthetic CDO**. A balance sheet synthetic CDO is comprised of the following underlying portfolio:

Assets:	125 single-name CDS
Principal:	0.8 million
Maturity:	5 years
CDS spread:	200 bps
Payments:	Act/360 quarterly in arrears

The CDO is structured with the following capital structure:

Tranche	Attachment point	Expected Loss	Fair Spread	Rating
Senior	7%-10%	0.002%	L+45	AAA
Class A	5%-7%	0.1%	L+70	AA-
Class B	2%-5%	2.3%	L+20	BBB-
Class C	0%-2%	26.27%	Excess spread	NR

(a) which noteholders are long correlation? Which tranche is the most sensitive to changes in default correlation? Why is this?

(b) how concerned are mezzanine noteholders with changes in the level of default correlation?

(c) how many defaults must there be before the Senior note experiences capital loss? Assume 0% recovery. If we assume 40% recovery how much more protection does this afford the Senior noteholder?

(d) How many defaults must there be before the implied rating of the note is downgraded assuming no recovery and downgrade occurs when entire equity tranche is lost?

Answers

(a) The Equity investors are long correlation, as the Equity note value increases with increase in correlation. Higher correlation increases the probability of both less and more defaults. Equity investors are sensitive to even one default so will desire higher probability of fewer defaults. The Senior tranche has a high sensitivity to large changes in correlation. The senior tranche only suffers loss in extreme situations (catastrophic loss) which occur with high correlation and multiple defaults

(b) Mezzanine notes are the least sensitive to correlation and their value is least impacted by changes in correlation, so investors in these notes are less concerned with this parameter when valuing the notes

(c) The Senior note has 7% subordination, therefore it is impacted once the portfolio suffers $0.07*125 = 8.75$ or 9 defaults. If we assume 40% recovery, the loss per default is $0.6 * 0.48$ or 0.48 million. The subordination is 7% of the portfolio or 7,000,000, which is eaten into after $7,000,000/480,000 = 14.58333$ or 15 defaults. Therefore assuming a recovery rate, this affords 6 more defaults as additional protection to the Senior note.

(d) We assume an implied rating downgrade after the first tranche is completely eaten away. This will be after 2% of the portfolio has suffered default, or 2.5 defaults, assuming no recovery value. In practice this will mean 2 defaults.

2. (a) Consider a random default time X that, given default intensity parameter θ, can be modeled as an exponential distribution, i.e.,

$$\text{Prob}(X \le x|\theta) = 1 - e^{-\theta x}.$$

Now assume θ is Gamma distribution, i.e.,

$$\theta \sim \Gamma(\alpha, \beta),$$

and the PDF of θ is $g(\theta)$, where

$$g(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{\alpha-1}e^{-\beta\theta}.$$

Show that marginal distribution of X is

$$F(x) = \text{Prob}(X \leq x) = 1 - \left(1 + \frac{x}{\beta}\right)^{-\alpha}.$$

Hint: Integrate conditional marginal distribution $F(x|\theta)$ w.r.t. θ to find unconditional marginal $F(x)$.

(b) Suppose conditional on θ, there exists two independent and identically distributed default times X_1 and X_2, such that their joint distribution function is $F(X_1, X_2)$, by finding F show that the associated copula function is

$$C(u_1, u_2) = u_1 + u_2 - 1 + \left((1-u_1)^{-\frac{1}{\alpha}} + (1-u_2)^{-\frac{1}{\alpha}} - 1\right)^{-\alpha}.$$

Hint: To find joint distribution you can use the result $F(x_1, x_2) = 1 - \Pr(X_1 > x_1) - \Pr(X_2 > x_2) + \Pr(X_1 > x_1, X_2 > x_2)$, then identify marginal distribution hidden in $F(x_1, x_2)$ hence express it in terms of uniform. This question actually shows that joint distribution function can be expressed as copula function.

Sol:

(a)

$$\begin{aligned}
F(x) &= \int_0^\infty \Pr(X \leq x|\theta)g(\theta)d\theta \\
&= \int_0^\infty (1 - e^{\theta x})\frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{\alpha-1}e^{-\beta\theta}d\theta \\
&= 1 - \int_0^\infty \frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{\alpha-1}e^{-(x+\beta)\theta}d\theta \\
&= 1 - \int_0^\infty \frac{(\beta+x)^\alpha}{\Gamma(\alpha)}\theta^{\alpha-1}e^{-(x+\beta)\theta}d\theta \left(\frac{\beta}{\beta+x}\right)^\alpha \\
&= 1 - \left(1 + \frac{x}{\beta}\right)^{-\alpha}
\end{aligned}$$

(b)

$$\begin{aligned}
\Pr(X_1 > x_1, X_2 > x_2) &= \int_0^\infty \Pr(X_1 > x_1, X_2 > x_2|\theta)g(\theta)d\theta \\
&= \int_0^\infty \Pr(X_1 > x_1|\theta)\Pr(X_2 > x_2|\theta)g(\theta)d\theta \\
&= \int_0^\infty e^{-\theta x_1}e^{-\theta x_2}\frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{\alpha-1}e^{-\beta\theta}d\theta \\
&= \left(\frac{\beta}{\beta + x_1 + x_2}\right)^\alpha \\
&= \left(1 + \frac{x_1}{\beta} + 1 + \frac{x_2}{\beta} - 1\right)^{-\alpha}
\end{aligned}$$

where

$$\left(1 + \frac{x}{\beta}\right)^{-\alpha} = 1 - F(x).$$

so

$$\Pr(X_1 > x_1, X_2 > x_2) = \left((1 - F(x_1))^{-\frac{1}{\alpha}} + (1 - F(x_2))^{-\frac{1}{\alpha}} - 1\right)^{-\alpha}$$

$$\begin{aligned}
F(x_1, x_2) &= 1 - \Pr(X_1 > x_1) - \Pr(X_2 > x_2) + \Pr(X_1 > x_1, X_2 > x_2) \\
&= 1 - (1 - F(x_1)) - (1 - F(x_2)) + \left((1 - F(x_1))^{-\frac{1}{\alpha}} + (1 - F(x_2))^{-\frac{1}{\alpha}} - 1\right)^{-\alpha} \\
&= F(x_1) + F(x_2) - 1 + \left((1 - F(x_1))^{-\frac{1}{\alpha}} + (1 - F(x_2))^{-\frac{1}{\alpha}} - 1\right)^{-\alpha}
\end{aligned}$$

now replace $F(x_i)$ by u_i to obtain the associated copula function.