CQF Final Project Workshop

Project Outline

This project is mainly about implementing two major numeric methods.

1. **Part1: Implement HJM**. Estimate volatility and risk neutral drift for instantaneous forward rate via PCA, then evolve forward curve by using MC simulation to price simple interest rate option.

2. **Part2: Finite Difference Method**. Use explicit FD method to price a statically hedged binary option with uncertain volatility model, and then optimize this value.

Project report should include. . .

- Introduce the model and methodology that you are going to use.
- Discuss the details of how to implement your model.
- Present result via tables and figures, give sensible representation if possible.
- Describe how to use your software if it is not obvious.
- Test accuracy and convergency of your software.
- Discuss any pros and cons of this project, any improvement you can make

1st Part: HJM model

This project requires you to implement HJM model to price zero coupon bond, interest rate options including caps and floors by Monte Carlo simulation. Before that you should use historical data, with appropriate parametrization, to estimate HJM model with at least 3 factors.

Data

Ideally forward rate is bootstrapped from a set of bond prices. However the cleaned data can be obtained from Bank of England's official website. We must thank BoE because data clean up is quite a heavy work.

This is the address

http://www.bankofengland.co.uk/statistics/yieldcurve/archive.htm

One factor HJM

Unlike many other short rate models, such as Vasicek and CIR, that focus on spot interest rate, the HJM model works on instantaneous forward rate. So its underlying stochastic process is forward rate process of $F(t,T)$ which can be written in terms of the following SDE

$$dF(t,T) = m(t,T)dt + \nu(t,T)dX$$

For the time being the SDE didn't show what probability measure the it is using.

Risk neutral dynamics

one thing crystal clear is that under no arbitrage condition, there exists a restriction on risk-neutral drift of forward rate.

$$m(t,T) = \nu(t,T)\int_t^T \nu(t,s)ds$$

One can see that the risk-neutral drift $m(t,T)$ only depends on volatility of forward rate, so that it is crucial to determine volatility function $\nu(t,T)$ when implementing HJM.

Evolve Forward curve

In addition to modeling on $F(t,T)$ which is one point on the forward curve, in theory the HJM model actually evolves the whole forward curve, i.e., T varies from t to T^*(the longest time horizon you are interested in). However it is certainly impossible in practice, one can only evolve discrete points on the forward curve.

Multi-factor HJM

The HJM model we have introduced so far are one factor model, which implies that all rates are perfectly correlated. To incorporate richer dynamics multi-factor can be adopted. The SDE of n-factor HJM can be written as

$$dF(t,T) = m(t,T)dt + \sum_{i=1}^{n} \nu_i(t,T)dX_i$$

where dX_is are uncorrelated BM, and the risk-neutral drift of $F(t,T)$ is

$$m(t,T) = \sum_{i=1}^{n} \nu_i(t,T) \int_t^T \nu_i(t,s)ds$$

PCA

To implement HJM model, whether one factor or multi-factor, one need to estimate volatility function $\nu(t,T)$. Principal Component Analysis is one of the approaches to achieve this task. It identifies the most prominent factors that determines multi-dimensional stochastic processes by orthogonalization of it historical time series data.

Multi-dimension process

Suppose we are modeling a vector of stochastic process $(F_1(t), \cdots, F_n(t))$ with associated SDEs,

$$
\begin{aligned}
dF_1(t) &= m_1(t)dt + \sigma_1(t)dW_1 \\
&\vdots \\
dF_n(t) &= m_n(t)dt + \sigma_n(t)dW_n
\end{aligned} \tag{1}
$$

where m_i and σ_i are drift and volatility(they can be function of F as well), and $(W_1, \cdots, W_n)$ are n-dimensional standard BM with correlation matrix Σ, i.e.,

$$\Sigma_{ij} = \rho_{ij}$$

Matrix factorization

Since Σ is a symmetric matrix, it can be factorized into the following form

$$\Sigma = V\Lambda V'$$

where V is an orthogonal matrix with Σ's eigenvector in its column, and Λ is a diagonal matrix whose diagonal corresponds to eigenvalues of Σ. So

$$\Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

and $\lambda_1 > \cdots > \lambda_n$.

What's more, since Σ is semi-positive definite, we can have

$$\Sigma = V\Lambda^{\frac{1}{2}}\left(V\Lambda^{\frac{1}{2}}\right)'$$

The diffusions in the block of equations (1) are originally in terms of correlated BM can then be reformed in terms of independent BM like this

$$V\Lambda^{\frac{1}{2}}\mathbf{dX} \tag{2}$$

Where $\mathbf{dX}$ is independent BM vector.

Dimension reduction

If one only chooses the first d factors, expand (2) equation (1) can be rewritten as

$$
\begin{aligned}
dF_1(t) &= m_1(t)dt + \sum_{i=1}^{d} \sqrt{\lambda_i} v_{i1} dX_i \\
\vdots \quad & \quad \vdots \\
dF_n(t) &= m_n(t)dt + \sum_{i=1}^{d} \sqrt{\lambda_i} v_{in} dX_i
\end{aligned}
\tag{3}
$$

where v_{ij} is the jth element in the ith column of V.

Musiela Parametrization

It will be easier to model volatility function at each maturity $\tau = T - t$, i.e.,

$$\nu(t,T) = \bar{\nu}(t,\tau) \quad \text{and} \quad \bar{F}(t,\tau) = F(t,t+\tau)$$

$$\begin{aligned}
d\bar{F}(t,\tau) &= dF(t,t+\tau) \\
&= dF(t,T) + \frac{\partial F(t,T)}{\partial T}dt \\
&= \left(\nu(t,T)\int_t^T \nu(t,s)ds\right)dt + \nu(t,T)dX + \frac{\partial \bar{F}(t,\tau)}{\partial \tau}dt \\
&= \left(\bar{\nu}(t,\tau)\int_0^\tau \bar{\nu}(t,s)ds + \frac{\partial \bar{F}(t,\tau)}{\partial \tau}\right)dt + \bar{\nu}(t,\tau)dX
\end{aligned}$$

2nd Part: FD method

I am not going to tell you more about explicit FD method and uncertain volatility model, let's tackle this project immediately.

Static Hedge

How to hedge a long position binary call option?

Answer: Traditionally to hedge(replicate) binary call is to Create a call spread by short selling a call option with lower strike and buying a call option with higher strike.

Hedging Portfolio

Let's assume binary call has price B and these two vanilla call option has market price C_1 and C_2, to hedge this binary call we hold λ_1 and λ_2 unit of each option respectively.

The value of hedging portfolio today is

$$V = B + \lambda_1 C_1 + \lambda_2 C_2$$

Valuation of Hedging Portfolio

With payoff function of each option, need to calculate hedging portfolio value V using uncertain volatility model.

To implement, 2 major extensions to which need to pay attention in addition to single constant volatility option programme.

Check Gamma Sign

Only need one if-then-else statement.

if $\Gamma > 0$, then

$\sigma = \sigma^-$

else

$\sigma = \sigma^+$

Jump Condition

Value of Hedged Binary Call

Hedging cost is

$$\lambda_1 C_1 + \lambda_2 C_2$$

Long position of a hedged binary call has value

$$B(\lambda_1, \lambda_2) = V(\lambda_1, \lambda_2) - \lambda_1 C_1 + \lambda_2 C_2$$

Optimization

$$\max_{\lambda_1,\lambda_2} B$$

Bid-Ask Spread

Similarly, compute statically hedged short position for 1 unit of this binary call option to obtain bid-ask spread.

Compare it to the bid-ask spread when there is no static hedging.