Taylor Series and Transition Density Functions

In this lecture. . .

- Taylor series
- A trinomial random walk
- Transition density functions
- Our first differential equation
- Similarity solutions

By the end of this lecture you will be able to

- use Taylor series for expanding functions in a power series

- interpret partial differential equations

- transform simple partial differential equations into even simpler ordinary differential equations

Introduction

This lecture is an introduction to some simple ideas and some simple tools that we shall be seeing again and again in our journey through the subject of quantitative finance.

Quantitative finance can be relatively painless if approached correctly.

This lecture shows how an applied mathematician approaches problems with the aim of getting to a useful model or result with the minimum of fuss.

What is Taylor series?

Taylor series is simply a way of approximating a function by a power series.

So a function of x such as $f(x)$ will be approximated by a constant plus a linear plus a quadratic plus a cubic plus . . .

(Why use a power series? The maths is simple!)

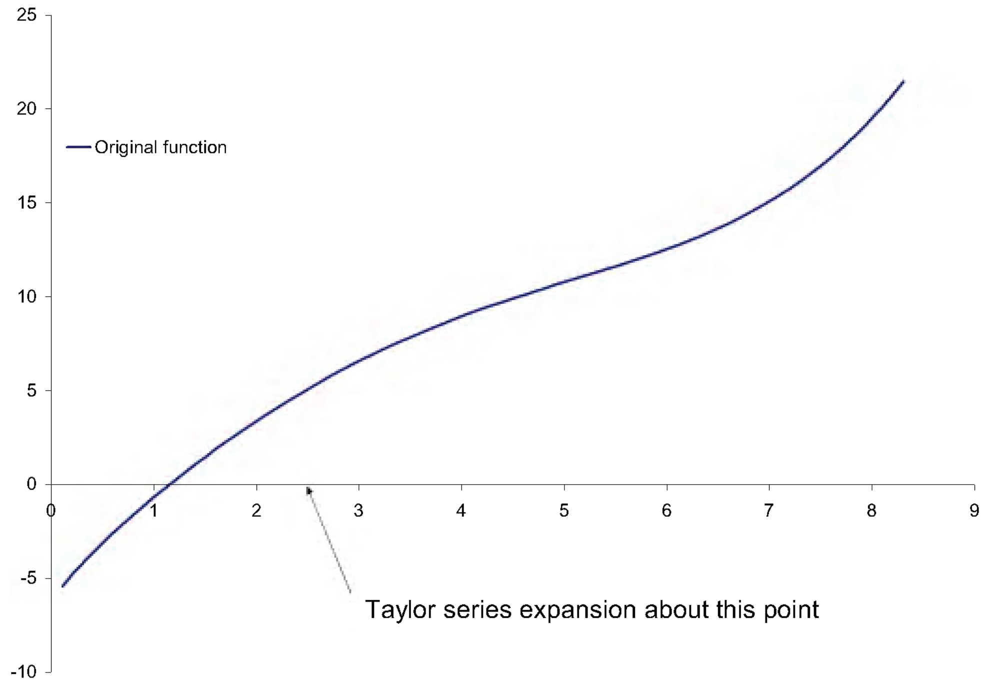
Why do we need Taylor series?

Taylor series is useful if we have a very complicated function to calculate. And a lot of finance theory is based on approximations like Taylor series.

Suppose you have a complicated option that takes seconds (!) to price. You might prefer to have a good approximation that takes milliseconds only.

Pictorially. . .

Here is a plot of some complicated function, $f(x)$. (It doesn't matter what the function is!)



I'd like to approximate this function with something much simpler. A power series! And I'd like it to be a good approximation near the point $x = 2.5$.

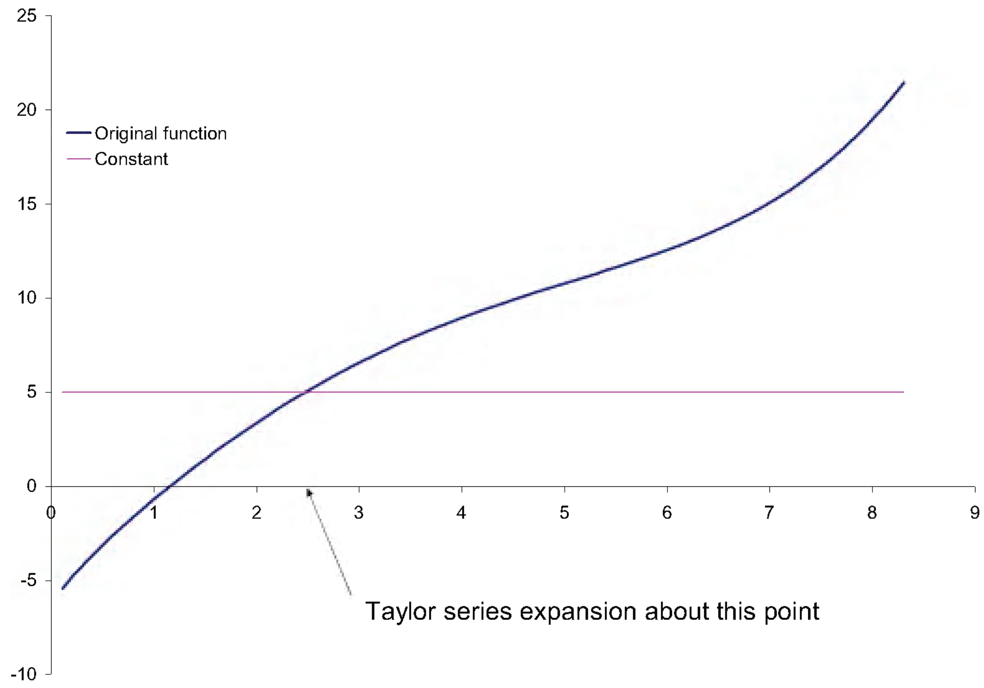
We say "We will expand the function $f(x)$ in a Taylor series about the point $x = 2.5$."

The simplest power series is the rather trivial ‘constant.’

So we could start with

$$f(x) \approx f(2.5).$$

Not the most exciting approximation in the world, but it is ok if we are close to the point 2.5:

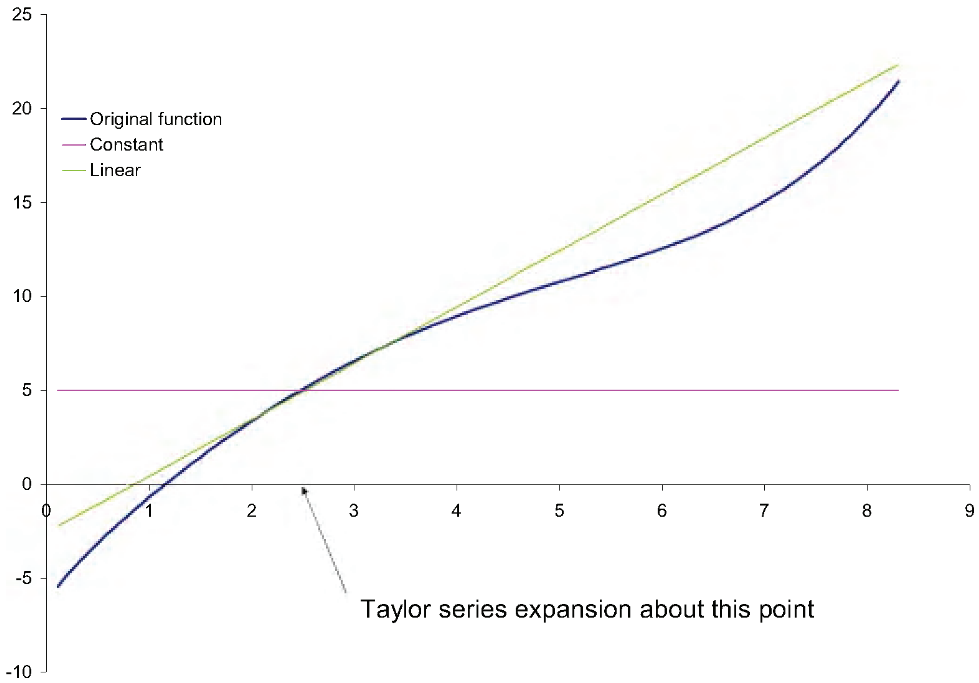


What's the next approximation up? Linear!

We could write

$$f(x) \approx f(2.5) + \text{ constant } \times (x - 2.5).$$

This looks like:



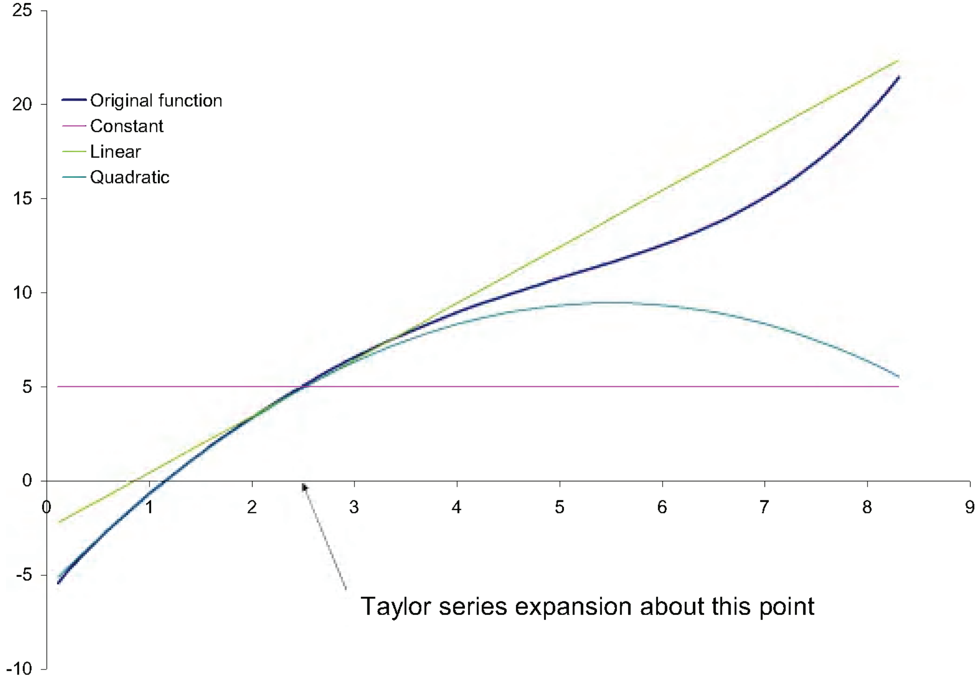
But what is the value of that ‘constant’ ?

The slope of the straight line should match the slope of the function $f(x)$ at the point $x = 2.5$, and this is

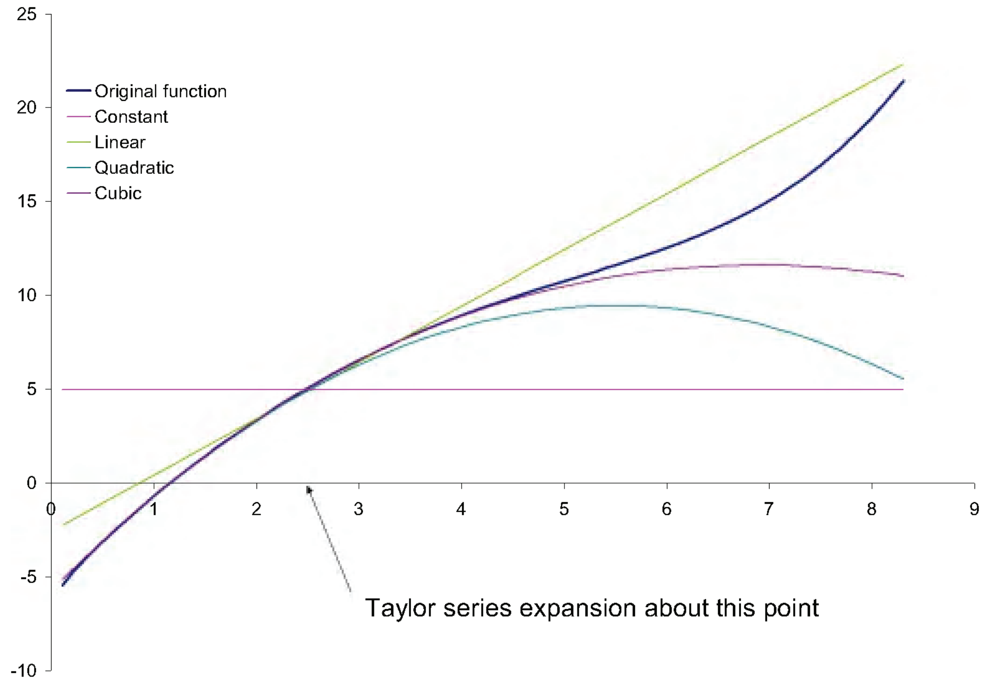
$$\frac{df}{dx} \text{ evaluated at } x = 2.5.$$

And this is therefore the value of the ‘constant.’

A better approximation is a quadratic curve:



And even better is a cubic curve:

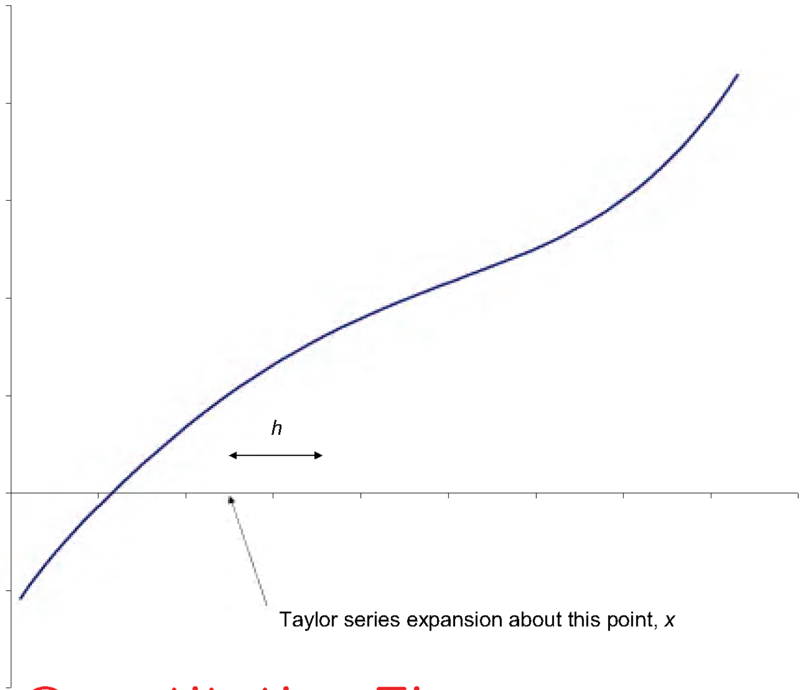


The approximations are only good 'close to' the point 2.5, but the more terms in the approximation the better the approximation.

Before doing the mathematics of all this, I want to change the notation slightly. I want to only use symbols, no ‘2.5’ etc.

“We are going to perform a Taylor series expansion of the function $f(x)$ about the point x.”

Introduce the idea of being a small distance, h, away from the point x:



So we write

$$f(x+h) \approx a + b\,h + c\,h^2 + d\,h^3 + \cdots.$$

I.e. approximate the function around the point x in a power series in the distance from that point, h.

But what are a, b, c, etc.?

The maths...

The Taylor series representation of $f(x+h)$ is the infinite sum

- $$f(x+h) = f(x) + \sum_{i=1}^{\infty} \frac{1}{i!} h^i \frac{d^i f}{dx^i}(x).$$

Taylor series is incredibly useful in derivatives theory.

Note:

1. This is an approximation

2. **The expression on the right only uses information about the function at the point x**

Example 1: Expand e^x in Taylor series about the point zero.

This is a classic question.

Since

$$\frac{d}{dx}\,e^x = e^x$$

all derivatives of e^x with respect to x are e^x and so

$$e^x = 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \cdots + \frac{x^i}{i!} + \cdots.$$

Example 2: Expand $\ln(1+x)$ in Taylor series about the point zero.

Another classic!

$$\frac{d}{dx}\ln(1+x) = \frac{1}{1+x},$$

$$\frac{d^2}{dx^2}\ln(1+x) = \frac{d}{dx}\frac{1}{1+x} = -\frac{1}{(1+x)^2},$$

$$\frac{d^3}{dx^3}\ln(1+x) = \cdots = +\frac{2}{(1+x)^3},$$

etc. Every time you differentiate with respect to x you multiply by what was the power previously. And so...

$$\ln(1+x) = x - \tfrac{1}{2}x^2 + \cdots + (-1)^i \frac{1}{i} x^i + \cdots.$$

Importance in quant finance

In quantitative finance the function that we are interested in, instead of being f, is V, the value of an option.

The independent variable is no longer x but is S, the price of the underlying asset.

From day to day the asset price changes by a small amount. This asset price change is just δS (instead of δx).

So we can use Taylor series for option prices, we just have different names for the variables!

However. . . the value of an option is not only a function of the asset price S but also the time t.

We write $V(S, t)$.

This brings us into the world of *partial* differentiation.

Partial Differential Equations

Think of the function $V(S,t)$ as a surface with coordinates S and t on a horizontal plane. The function V is the height of a hill above sea level with S and t being distances, or map coordinates, in the northerly and westerly directions.

The *partial* derivative of $V(S,t)$ with respect to S is written

$$\frac{\partial V}{\partial S}$$

and is defined as

- $$\frac{\partial V}{\partial S} = \lim_{\delta S \to 0} \frac{V(S+\delta S,t) - V(S,t)}{\delta S}.$$

Note that in this V is only ever evaluated at time t.

In quant finance this derivative is usually called the **delta** of an option, and is denoted by Δ.

This is like measuring the gradient of the function $V(S,t)$ in the S direction along a constant value of t i.e. the slope of our hillside in the northerly direction.

Note also that we are now using a special curly ∂ instead of the roman d.

The partial derivative of $V(S,t)$ with respect to t is similarly defined as

- $$\frac{\partial V}{\partial t} = \lim_{\delta t \to 0} \frac{V(S, t+\delta t) - V(S,t)}{\delta t}$$

and is the slope of the hill in the westerly direction.

In quant finance this derivative is usually called the **theta** of an option, and is denoted by θ.

Higher-order derivatives are defined in the obvious manner.

The Taylor series expansion of the value of an option is then

- $$V(S + \delta S, t + \delta t) \approx$$
$$V(S,t) + \delta t \frac{\partial V}{\partial t} + \delta S \frac{\partial V}{\partial S} + \tfrac{1}{2}(\delta S)^2 \frac{\partial^2 V}{\partial S^2} + \cdots.$$

This series goes on for ever, but only the largest and most important terms have been written down here, those which are required in most financial problems.

In quant finance the derivative $\frac{\partial^2 V}{\partial S^2}$ is usually called the **gamma** of an option, and is denoted by Γ.

Question: Why have I only gone as far as δt yet have included δS^2?

We will use Taylor series expansions in two variables a lot!

Randomness in finance

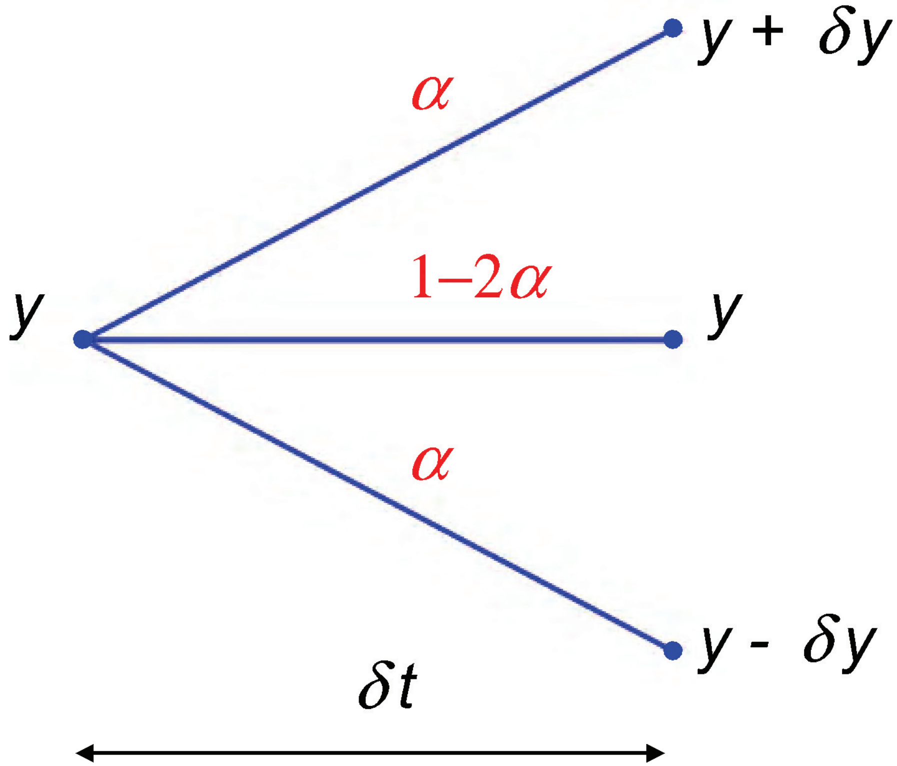
Modern finance theory, especially derivatives theory, is based on the random movement of financial quantities.

We are now going to explore the simple idea of the random walk and see its relationship to differential equations.

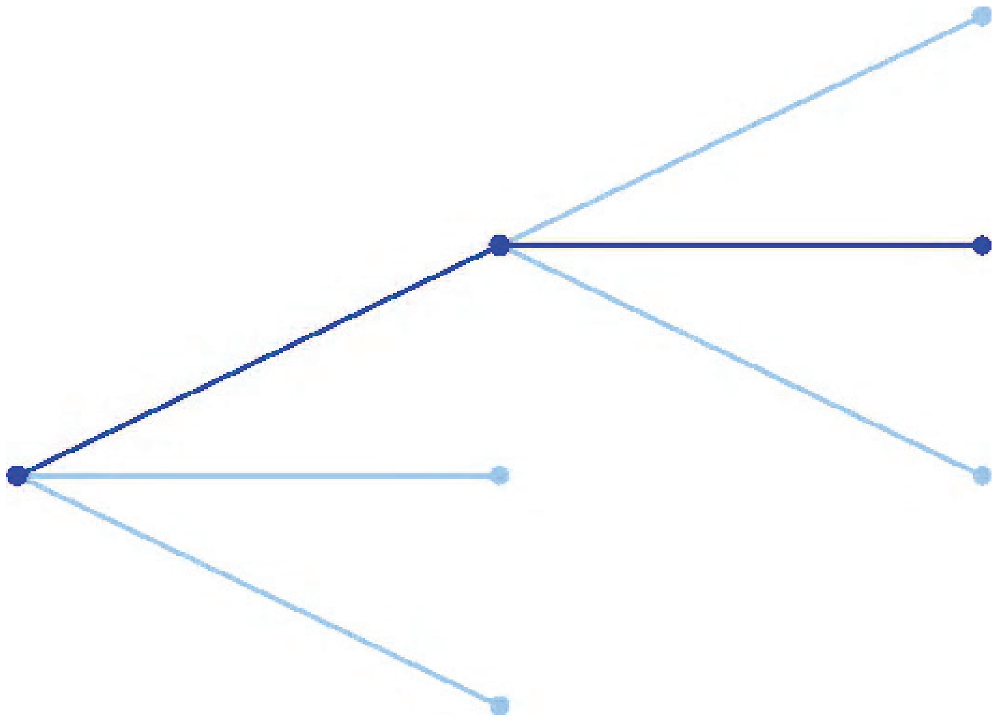
This is achieved via the concept of a **transition probability density function**.

- The trinomial random walk
- The transition probability density function
- An equation for the transition probability density function

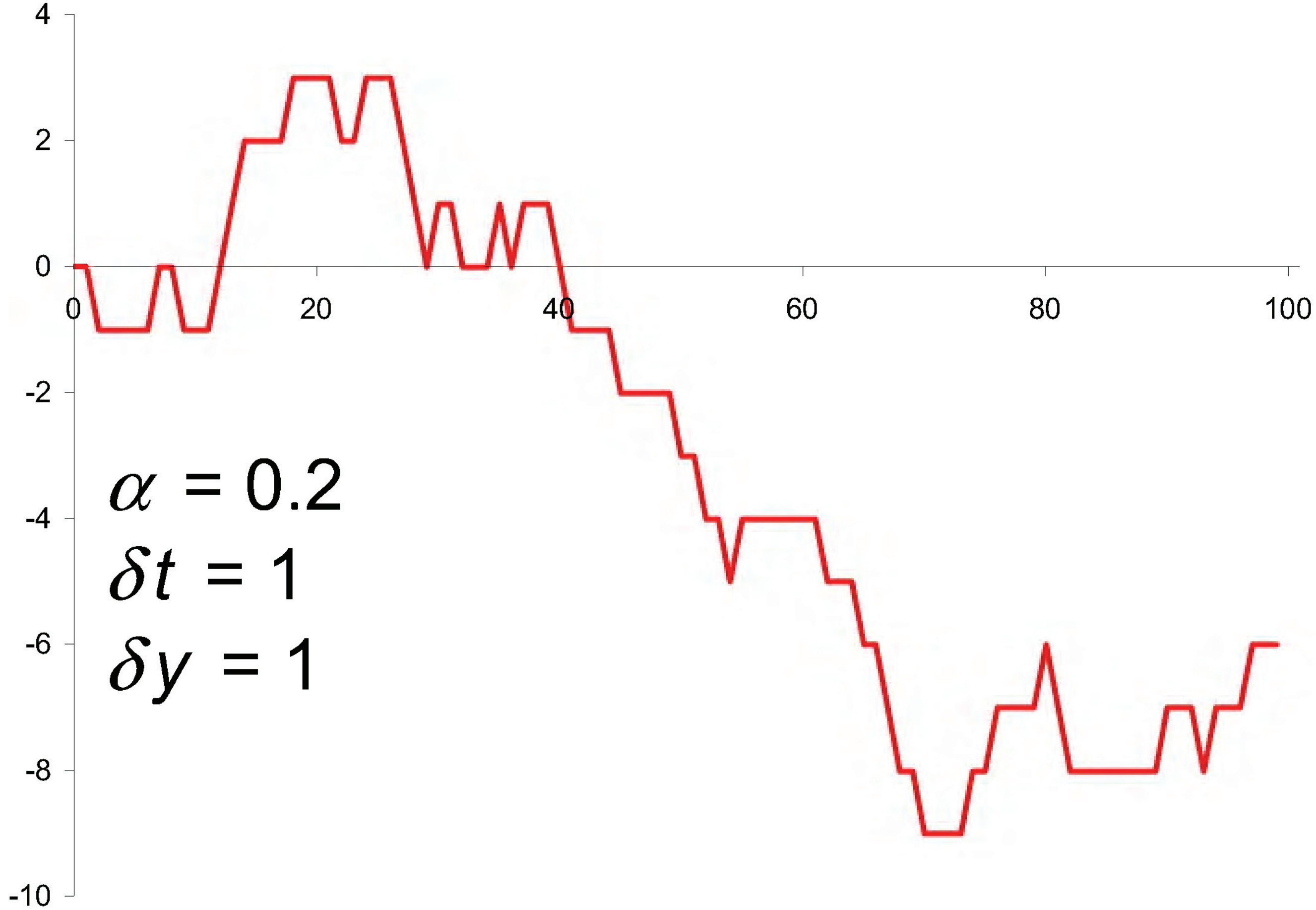
The trinomial random walk



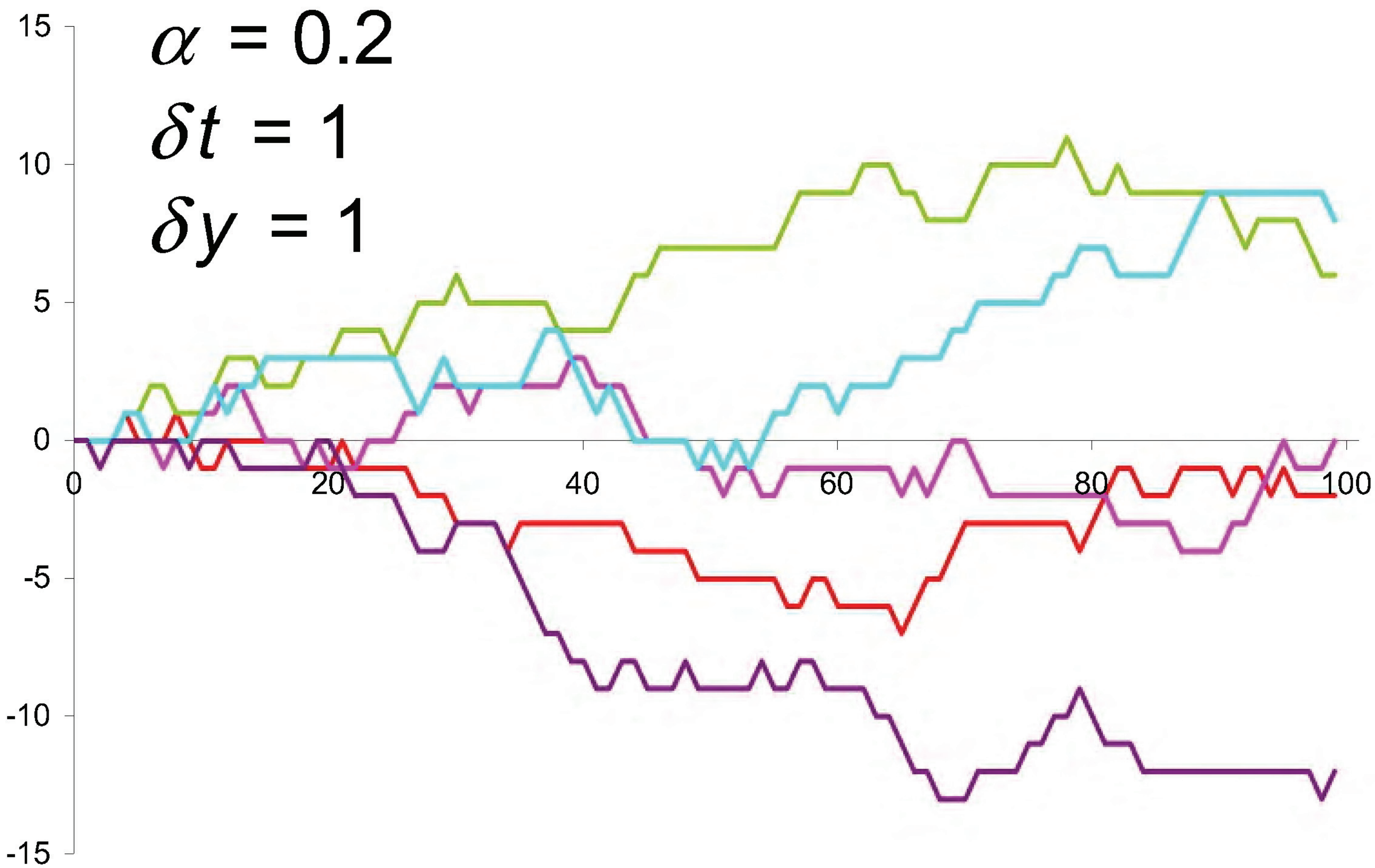
y is the value of our random variable. δt is a time step. α is a probability. δy is the size of the move in y.



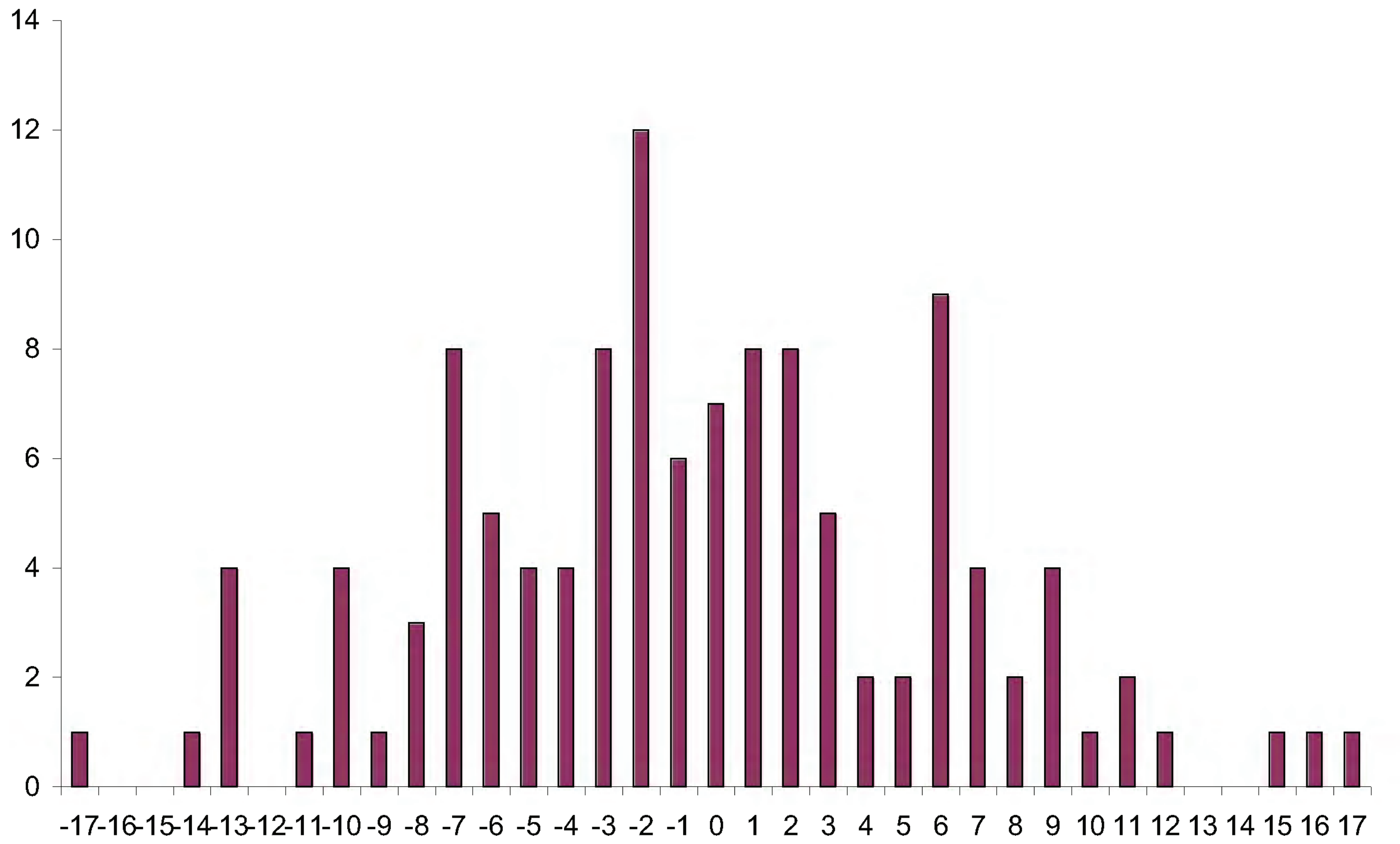
Suppose the top branch is chosen after one time step. After the second there are three places that y could be.



After lots of time steps we might end up with a picture like this. This is a random walk.



Often we are interested in the probabilistic properties of the random walk rather than the outcome of a single realization.



It is very natural to look at the distribution of ys after some time period.

The transition probability density function

To analyze the probabilistic properties of the random walk, we introduce the **transition probability density function** $p(y, t; y', t')$ defined by

$$\text{Prob}(a < y' < b \text{ at time } t' | y \text{ at time } t) = \int_a^b p(y, t; y', t') \, dy'.$$

In words this is "the probability that the random variable y' lies between a and b at time t' in the future, given that it started out with value y at time t."

Warning: The trinomial random walk presented here is ‘discrete’ in the sense that time and random variable only take discrete values.

We will be moving over to a continuous-time and continuous-variable model shortly.

Think of y and t as being current values with y' and t' being future values. The transition probability density function can be used to answer the question,

"What is the probability of the variable y' being in a specified range at time t' in the future given that it started out with value y at time t?"

The transition probability density function $p(y, t; y', t')$ satisfies two equations, one involving derivatives with respect to the future state and time (y' and t') and called the **forward equation**, and the other involving derivatives with respect to the current state and time (y and t) and called the **backward equation**.

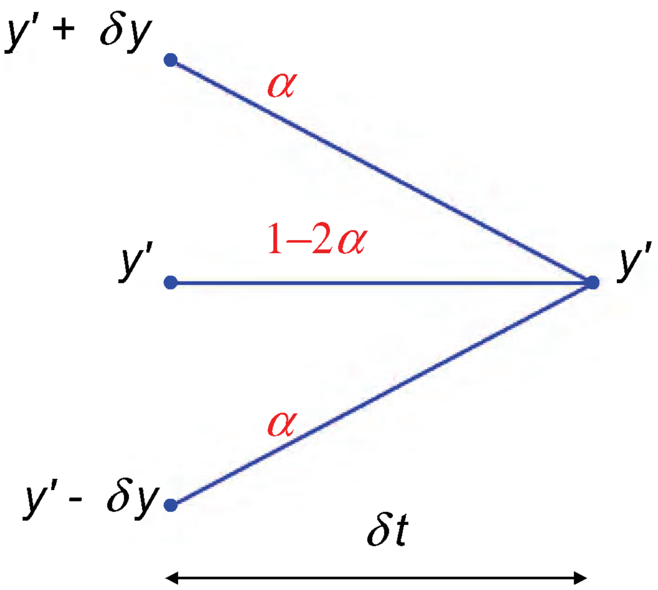
From the trinomial model to the transition probability density function

The variable y can either rise, fall or take the same value after a time step δt. These movements have certain probabilities associated with them.

We are going to assume that the probability of a rise and a fall are both the same, $\alpha < \frac{1}{2}$. (But, of course, this can be generalized. Why would we want to generalize this?)

The forward equation

The variable y takes the value y' at time t', but how did it get there?



In our trinomial walk we can only get to the point y' from the three values $y' + \delta y$, y' and $y' - \delta y$.

The probability of being at y' at time t' is related to the probabilities of being at the previous three values and *moving in the right direction*:

$$p(y,t;y',t') = \alpha\, p(y,t;y'+\delta y,t'-\delta t)$$
$$+(1-2\alpha)p(y,t;y',t'-\delta t) + \alpha\, p(y,t;y'-\delta y,t'-\delta t). \qquad (1)$$

We can easily expand each of the terms in Taylor series about the point y', t'. For example,

$$p(y,t;y'+\delta y,t'-\delta t) \approx p(y,t;y',t') - \delta t\frac{\partial p}{\partial t'} + \delta y\frac{\partial p}{\partial y'} + \tfrac{1}{2}\delta y^2\frac{\partial^2 p}{\partial y'^2} + \cdots,$$

$$p(y,t;y'-\delta y,t'-\delta t) \approx p(y,t;y',t') - \delta t\frac{\partial p}{\partial t'} - \delta y\frac{\partial p}{\partial y'} + \tfrac{1}{2}\delta y^2\frac{\partial^2 p}{\partial y'^2} + \cdots$$

and

$$p(y,t;y',t'-\delta t) \approx p(y,t;y',t') - \delta t\frac{\partial p}{\partial t'} + \cdots.$$

Equation (1) becomes (and we are using shorthand notation $p(y,t;y',t') = p$)

$$p = \alpha(p - \delta t\frac{\partial p}{\partial t'} + \delta y\frac{\partial p}{\partial y'} + \tfrac{1}{2}\delta y^2\frac{\partial^2 p}{\partial y'^2} + \cdots)$$

$$+(1-2\alpha)(p - \delta t\frac{\partial p}{\partial t'} + \cdots)$$

$$+\alpha(p - \delta t\frac{\partial p}{\partial t'} - \delta y\frac{\partial p}{\partial y'} + \tfrac{1}{2}\delta y^2\frac{\partial^2 p}{\partial y'^2} + \cdots).$$

Lots of these terms cancel out!

We are left with

$$\delta t \frac{\partial p}{\partial t'} = \alpha\, \delta y^2 \frac{\partial^2 p}{\partial y'^2} + \cdots .$$

Now we explicitly mention that we are really interested in the continuous limit, as increments in time and y get smaller and smaller.

The above equation only makes sense if

$$\frac{\alpha\, \delta y^2}{\delta t}$$

tends to some finite limit as the time step and the y increment δy go to zero.

Let's define

$$\frac{\alpha\,\delta y^2}{\delta t} = c^2,$$

for some finite, non-zero c.

The final equation is now

$$\frac{\partial p}{\partial t'} = c^2 \frac{\partial^2 p}{\partial y'^2}.$$

This is the **Fokker–Planck** or **forward Kolmogorov equation**. It is a forward parabolic partial differential equation, requiring initial conditions at time t and to be solved for $t' > t$.

This equation is to be used if there is some special state now and you want to know what could happen later. For example, you know the current value of y and want to know the distribution of values at some later date.

Observations:

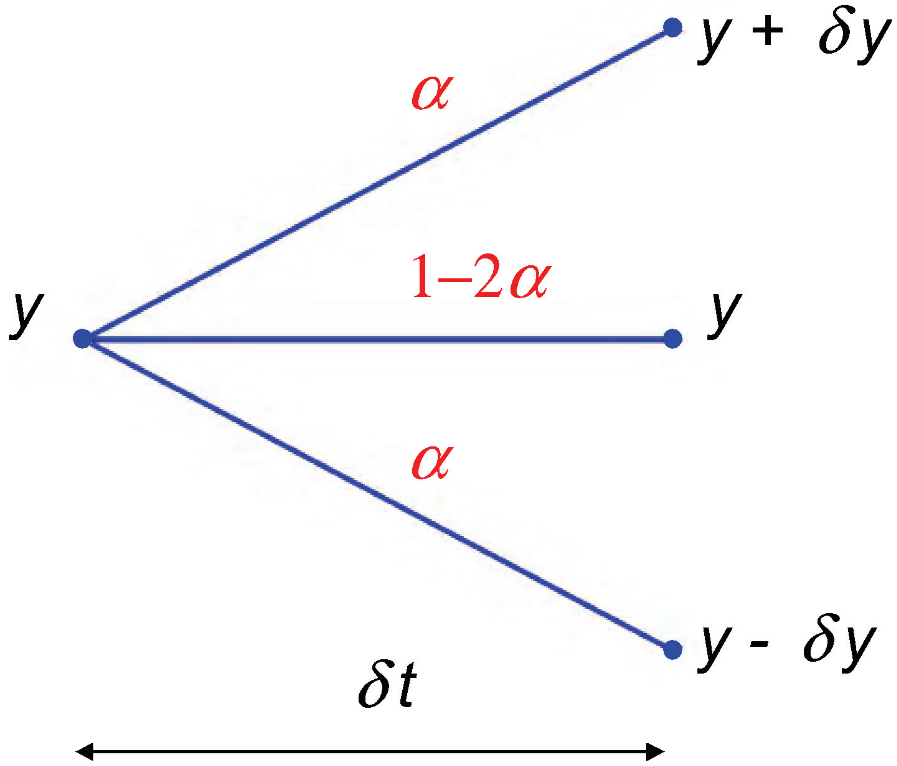
- This is a partial differential equation for p as a function of two independent variables y' and t'.
- It is an example of a diffusion equation.
- y and t are rather like parameters in this problem, think of them as starting quantities for the random walk.
- This is a diffusion equation. You need that special relationship between α, δt and δy to get this equation.
- This is also an example of Brownian motion.
- When we get on to financial applications the quantity c will be related to volatility.

The backward equation

Now we come to find the backward equation. This will be useful if we want to calculate probabilities of reaching a specified final state from various initial states. It will be a backward parabolic partial differential equation requiring conditions imposed in the future, and solved backwards in time.

Whereas the forward equation had independent variable t' and y' the backward equation has variables t and y.

The derivation uses the trinomial random walk directly as drawn here.



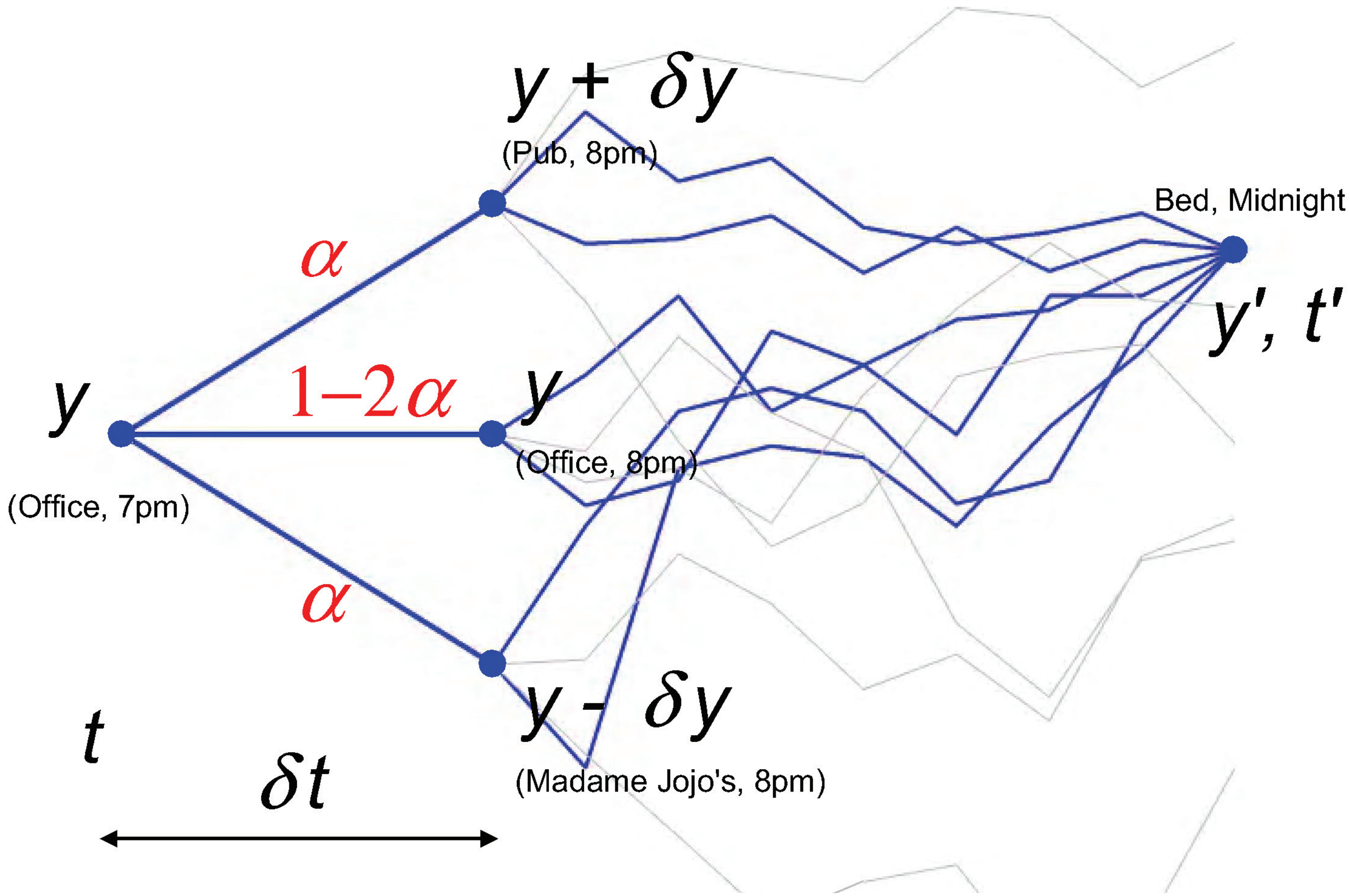
The justification for the relationship between the probabilities of being at the four 'nodes' is more subtle than in the derivation of the forward equation.

So, let's look at a concrete example.

At 7pm you are in the office. (This is the point (y, t).)

At 8pm you will be at one of three places: The Pub; Still at the office; Madame Jojo's. (These are the points $(y + \delta y, t + \delta t)$, $(y, t + \delta t)$ and $(y - \delta y, t + \delta t)$.)

We are going to look at the probability that at midnight you are tucked up in bed. (This is the point (y', t').)


y + δy
(Pub, 8pm)
Bed, Midnight
α
y', t'
y
1–2α
y
(Office, 8pm)
(Office, 7pm)
α
y - δy
t
δt
(Madame Jojo's, 8pm)

Remember that $p(y,t;y',t')$ represents the probability of being at the future point (y',t'), bed at midnight, given that you started at (y,t), the office at 7pm.

You can only get to the bed at midnight via either the pub, the office or Madame Jojo's at 8pm.

What happens after 8pm doesn't matter (you may not even remember!), we are only concerned with the probability that you are in bed at midnight, not how you got there.

In the previous figure there are lots of different paths, only the ones ending up in bed are of interest to us.

In words:

The probability of going from the office at 7pm to bed at midnight is the probability of going to the pub from the office and then to bed at midnight plus the probability of staying in the office and then getting to bed at midnight plus the probability of going to Madame Jojo's from the office and then to bed at midnight.

In symbols we can write this as

$$p(y,t;y',t') = \alpha\, p(y+\delta y, t+\delta t; y', t')$$
$$+(1-2\alpha)p(y, t+\delta t; y', t') + \alpha\, p(y-\delta y, t+\delta t; y', t').$$

The Taylor series expansion leads to the **backward Kolmogorov equation**.

Exercise!

The end result is

$$\frac{\partial p}{\partial t} + c^2 \frac{\partial^2 p}{\partial y^2} = 0.$$

Exactly the same as the forward equation, but with a sign change.

(The sign change makes all the difference between a forward and a backward diffusion equation.)

Warning: More general random walks lead to slightly more complicated forward and backward equations, and their relationship is no longer as simple as a change of sign.

Generalization

We will look at more general random walks in later lectures. But why would we want to consider them?

- Financial quantities are more interesting than the trinomial model here.

- At the very least, equity prices cannot go negative, unlike the variable y here.

- We might need different models for different financial quantities, equities, interest rates,...

Similarity solutions

That was our first partial differential equation.

As a general rule, we are not going to spend much time finding explicit solutions to equations—our emphasis will be on number crunching—but it is well worth looking at the forward diffusion equation in some detail. In particular, it is helpful to solve the equation in a simple case because

- it illustrates a very useful technique, similarity solutions

- it highlights the important role that the normal distribution plays

The equation to be solved is

$$\frac{\partial p}{\partial t'} = c^2 \frac{\partial^2 p}{\partial y'^2}.$$

This equation has an infinite number of solutions. It has different solutions for different **initial conditions** and different **boundary conditions**

The initial condition tells you how the solution starts off. We must specify p as a function of y' at some point in time, t'.

Boundary conditions tell you how the function behaves on specified y' boundaries. Diffusion equations typically need two boundary conditions.

Warning: We are now going to do lots of (very fast) manipulations of functions, including solving ordinary differential equations. Do not panic!

We are now going to find a very simple solution. It is very simple and very special because unlike most solutions of the diffusion it does not depend on *two* independent variable y' and t' but on a combination of them.

Let us seek a solution of the form

$$p = {t'}^a f\left(\frac{y'}{{t'}^b}\right).$$

Here a and b are constants.

Note that f is a function of only the one variable.

From

$$p = t'^a f\left(\frac{y'}{t'^b}\right)$$

we have

$$\frac{\partial p}{\partial y'} = t'^{a-b}\frac{df}{d\xi}$$

where

$$\xi = \frac{y'}{t'^b}.$$

Note: Derivatives of f are ordinary, not partial, since f only has one argument.

Then

$$\frac{\partial^2 p}{\partial y'^2} = t'^{a-2b} \frac{d^2 f}{d\xi^2}.$$

Also

$$\frac{\partial p}{\partial t'} = a t'^{a-1} f(\xi) - b y' t'^{a-b-1} \frac{df}{d\xi}.$$

Let's substitute these into our partial differential equation and see what happens.

$$a t'^{a-1} f(\xi) - b y' t'^{a-b-1} \frac{df}{d\xi} = c^2 t'^{a-2b} \frac{d^2 f}{d\xi^2}.$$

Let's cancel some t's and write $y' = t'^b \xi$:

$$af(\xi) - b\xi \frac{df}{d\xi} = c^2 t'^{-2b+1} \frac{d^2 f}{d\xi^2}.$$

So far we have only been fooling around 'changing variables.' The next step is important.

The left-hand side of this equation is only a function of ξ, whereas the right-hand side depends on both ξ and t'. This is only possible if the right-hand side is also independent of t'.

And this is only possible if $b = \frac{1}{2}$.

If we can solve

$$a f(\xi) - \frac{1}{2}\xi\frac{df}{d\xi} = c^2\frac{d^2 f}{d\xi^2}$$

then we have found a solution of our original equation in the form

$$p = {t'}^{a} f\left(\frac{y'}{\sqrt{t'}}\right).$$

And this isn't just a single solution, it is a whole family of solutions because we can choose the constant a.

However, for our present problem, only one value of a is relevant.

Remember that p represents a probability. That means that its integral must be one:

$$\int_{-\infty}^{\infty} p(y', t')\, dy' = 1 = \int_{-\infty}^{\infty} t'^{a} f\left(\frac{y'}{\sqrt{t'}}\right)\, dy'.$$

Change variables by writing $y' = t'^{1/2} u$ to get

$$t'^{a+1/2} \int_{-\infty}^{\infty} f\,(u)\; du = 1.$$

Conclusion? This is only possible if $a = -\frac{1}{2}$.

Our ordinary differential equation is now

$$-\frac{1}{2}f(\xi) - \frac{1}{2}\xi\frac{df}{d\xi} = c^2\frac{d^2 f}{d\xi^2}.$$

This can be written as

$$-\frac{1}{2}\frac{d\left(\xi\ f(\xi)\right)}{d\xi} = c^2\frac{d^2 f}{d\xi^2}.$$

(That was lucky!)

This can be integrated once to give

$$-\frac{1}{2}\xi\, f(\xi) = c^2 \frac{df}{d\xi}.$$

(There's an arbitrary constant of integration that could go in here but for the answer we want this is zero.)

This can be rewritten as

$$c^2 \frac{d\,(\ln f)}{d\xi} = -\frac{1}{2}\xi$$

. . . and integrated again to give

$$\ln f(\xi) = -\frac{\xi^2}{4c^2} + \text{ an arbitrary constant of integration}$$

or

$$f(\xi) = A \exp\left(-\frac{\xi^2}{4c^2}\right).$$

The constant A is chosen so that the integral of f is one.

Cutting to the chase, and going back to the original t' and y', we have

$$p = \frac{1}{2c\sqrt{\pi\, t'}} \exp\left(-\frac{{y'}^2}{4c^2 t'}\right).$$

Do you recognize this expression?

It is very like

$$\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\phi^2}{2}\right).$$

It is the probability density function for the normal distribution!

So

- y' is normally distributed

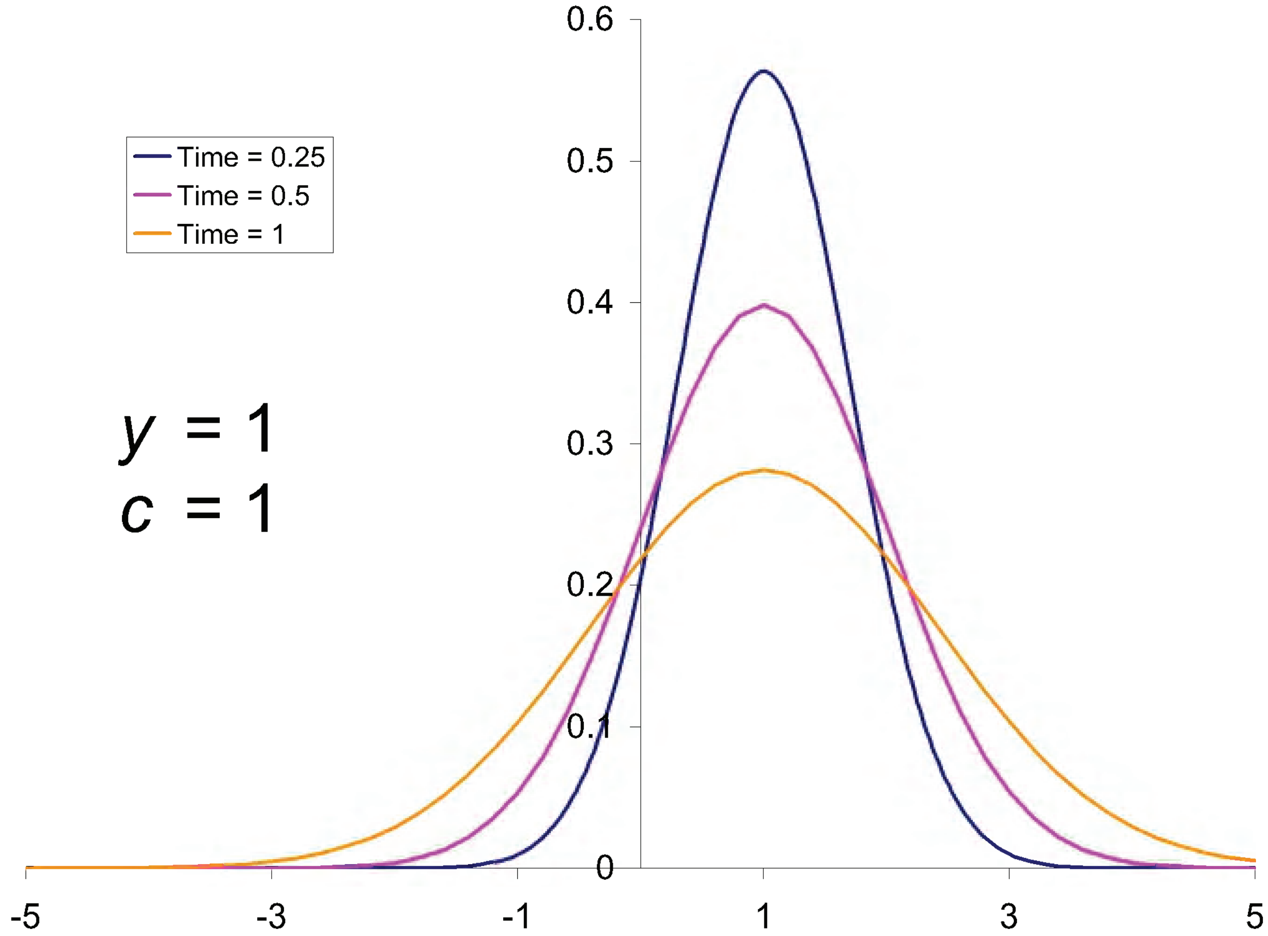
- with mean of zero

- and standard deviation of $c\sqrt{2t'}$

Minor generalization... suppose that y' has value y at time t then we have

$$p(y,t;y',t') = \frac{1}{2c\sqrt{\pi(t'-t)}} \exp\left(-\frac{(y'-y)^2}{4c^2(t'-t)}\right).$$

And this is our transition probability density function for our random walk!


0.6
0.5
0.4
0.3
0.2
0.1
0
-5
-3
-1
1
3
5
Time = 0.25
Time = 0.5
Time = 1
y = 1
c = 1

Summary

Please take away the following important ideas

- Taylor series as a way of finding out values of a function knowing only local information (such as gradients, and higher derivatives). It is an approximation only.

- Random walks have associated differential equations for their probability density functions, and are naturally related to the normal distribution.

- Generally partial differential equations are hard to solve explicitly, but sometimes they can be simplified to ordinary differential equations.