

## CQF Exercises 4.2 (Solutions)

1. The bond pricing equation, derived in class is given by

$$\frac{\partial V}{\partial t} + \frac{1}{2}\omega^2 \frac{\partial^2 V}{\partial r^2} + (u - \lambda\omega) \frac{\partial V}{\partial r} - rV = 0.$$

A bond has payoff at maturity  $t = T$  of one unit, i.e.

$$V(r, T) = 1$$

Solve the above equation for  $V(r, T)$  given that  $\omega$  is constant and  $(u - \lambda\omega) = 1$ .

[Hint: we know the solution has the form  $V(r, t) = \exp(A(t) - rB(t))$ .]

**Solution:**

Now  $V = Z = e^{A-rB}$ . We know  $Z(r, T; T) = 1 \Rightarrow$

$$\exp(A(T; T) - rB(T; T)) = 1.$$

This will only happen when  $A(T; T) - rB(T; T) = 0 \Rightarrow A(T; T) = B(T; T) = 0$

$$Z_t = \left( \dot{A} - r\dot{B} \right) Z \quad Z_r = -BZ, \quad Z_{rr} = B^2 Z \quad \text{where } \dot{\cdot} \equiv \frac{d}{dt}$$

Substituting in the BPE gives

$$\begin{aligned} \Rightarrow \dot{A} - r\dot{B} + \frac{1}{2}\omega^2 B^2 - B - r &= 0 \\ \Rightarrow \left( \dot{A} + \frac{1}{2}\omega^2 B^2 - B \right) - r \left( \dot{B} + 1 \right) &= 0 \end{aligned}$$

Now have two equations,

$$\begin{aligned} \dot{B} + 1 &= 0 \\ \dot{A} + \frac{1}{2}\omega^2 B^2 - B &= 0 \end{aligned}$$

$$\begin{aligned} \frac{dB}{dt} &= -1 \rightarrow \int_t^T dB = - \int_t^T d\tau \\ &\rightarrow \underbrace{B(T; T) - B(t; T)}_{=0} = -(T - t) \\ \therefore B(t; T) &= (T - t) \end{aligned}$$

Now the second equation becomes

$$\begin{aligned}
\dot{A} &= -\frac{1}{2}\omega^2 B^2 + B \\
\frac{dA}{dt} &= -\frac{1}{2}\omega^2 (T-t)^2 + (T-t) \rightarrow \\
\int_t^T dA &= -\frac{1}{2}\omega^2 \int_t^T (T-\tau)^2 d\tau + \int_t^T (T-\tau) d\tau \\
\underbrace{A(T;T) - A(t;T)}_{=0} &= -\frac{1}{2}\omega^2 \int_t^T (T-\tau)^2 d\tau + \int_t^T (T-\tau) d\tau \\
\Rightarrow A &= \frac{\omega^2}{2} \int_t^T (T-\tau)^2 d\tau - \int_t^T (T-\tau) d\tau \\
&= \frac{\omega^2}{6} (T-t)^3 - \frac{1}{2} (T-t)^2
\end{aligned}$$

2. The interest rate  $r$  is assumed to be satisfied by a SDE  $dr = dX$ . By hedging with a bond of different maturity derive the bond pricing equation. Consider a one-factor risk-neutral world in which the spot rate,  $r$ , evolves according to the SDE

$$\frac{\partial V}{\partial t} + \frac{1}{2} \frac{\partial^2 V}{\partial r^2} - a(r, t) \frac{\partial V}{\partial r} - rV = 0,$$

where  $a(r, t)$  is an arbitrary function. Assuming  $a$  is a function of  $t$  only and a bond has payoff at maturity  $t = T$  of one unit, i.e.

$$V(r, T; T) = 1$$

find a solution of the form

$$V(r, t) = \exp(A(t) + rB(t))$$

where  $A(t)$  can be written as

$$A(t) = - \int_t^T \left[ a(s)(s-T) + \beta(s-T)^2 \right] ds$$

and determine the constant  $\beta$ .

**Solution:**

one-factor risk neutral world spot rate:  $dr = dX$

Construct a hedged portfolio:

$$\Pi = V_1 - \Delta V_2 \quad \text{where } Z = \text{zero-coupon bond price}$$

Change in portfolio  $d\Pi$

$$d\Pi = dV_1 - \Delta dV_2$$

with Itô's Lemma:

$$\begin{aligned} \Rightarrow d\Pi &= \left( \frac{\partial V_1}{\partial t} + \frac{1}{2} \frac{\partial^2 V_1}{\partial r^2} \right) dt + \frac{\partial V_1}{\partial r} dX \\ &\quad - \Delta \left( \frac{\partial V_2}{\partial t} + \frac{1}{2} \frac{\partial^2 V_2}{\partial r^2} \right) dt - \Delta \frac{\partial V_2}{\partial r} dX \end{aligned}$$

Choose  $\Delta$  in a way that the risk in  $d\Pi$  vanishes  $\Rightarrow \Delta = \frac{\partial V_1 / \partial r}{\partial V_2 / \partial r}$  and hence  $d\Pi = r\Pi dt$  for no arbitrage. This gives

$$\left( \frac{\partial V_1}{\partial t} + \frac{1}{2} \frac{\partial^2 V_1}{\partial r^2} - rV_1 \right) \bigg/ \frac{\partial V_1}{\partial r} = \left( \frac{\partial V_2}{\partial t} + \frac{1}{2} \frac{\partial^2 V_2}{\partial r^2} - rV_2 \right) \bigg/ \frac{\partial V_2}{\partial r}$$

LHS depends on  $T_1$ , and RHS on  $T_2$ , so both sides must equal a function that is independent of  $T$ , call this  $a(r, t)$ . Dropping subscripts

$$\frac{\partial V}{\partial t} + \frac{1}{2} \frac{\partial^2 V}{\partial r^2} - a(r, t) \frac{\partial V}{\partial r} - rV = 0$$

If  $a = a(t)$  and  $V(r, t; T) = 1$ , then substitution of  $V(r, t) = e^{A(t) + rB(t)}$  in BPE gives:

$$\dot{A} + \frac{1}{2} \dot{B}^2 - aB - r + r\dot{B} = 0$$

which is true for all  $r$ . Gives two equations  $\dot{B} = 1$ ;  $\dot{A} + \frac{1}{2} \dot{B}^2 - aB = 0$

$$\dot{B} = 1 \implies B = t - T$$

$$\begin{aligned} \dot{A} &= a(t)(t - T) - \frac{1}{2}(t - T)^2 \implies \\ \int_t^T dA &= \int_t^T \left( a(s)(s - T) - \frac{1}{2}(s - T)^2 \right) ds \\ A(t) &= - \left( \int_t^T a(s)(s - T) ds - \frac{1}{6}(s - T)^3 \bigg|_t^T \right) \\ A(t) &= - \int_t^T [a(s)(s - T)] ds - \frac{(t - T)^3}{6} \end{aligned}$$

hence  $\beta = -1/2$ .

3. What final condition (payoff) should be applied to the bond pricing equation for a swap, cap, floor, zero-coupon bond and a bond option?

**Solution:**

Final condition for a swap:

$$V(r, T) = (r - r_s) P,$$

where  $r_s$  is the fixed rate and  $P$  is the principal.

Final condition for a cap:

$$V(r, T) = \max(r - r_c, 0) P,$$

where  $r_c$  is the cap rate and  $P$  is the principal.

Final condition for a floor:

$$V(r, T) = \max(r_f - r, 0) P,$$

where  $r_f$  is the floor rate and  $P$  is the principal.

Final condition for a zero-coupon bond:

$$V(r, T) = P,$$

where  $P$  is the principal.

Final condition for a coupon bond:

$$V(r, T) = (1 + c) P,$$

where  $c$  is the (discrete) coupon rate and  $P$  is the principal.

Final condition for a bond option:

$$V(r, T) = \max(Z(r, T) - E, 0),$$

where  $E$  is the exercise price and  $Z(r, t)$  is the value of the underlying bond at time  $t$ .

4. Consider the bond pricing equation

$$\frac{\partial B}{\partial t} + \frac{1}{2}w^2 \frac{\partial^2 B}{\partial r^2} + (u - \lambda w) \frac{\partial B}{\partial r} - rB = 0,$$

where  $dr = (u - \lambda w) dt + w dX$  is the risk-neutral spot rate. Suppose this risk-neutral model is defined by

$$dr = ar^2 dt + br^{3/2} dX.$$

where  $a$  and  $b$  are constants. Suppose we wish to use this to price a new type of interest rate derivative called a "perpetual bond" whose payoff is

$$\max(r - E, 0)$$

this 'perpetual bond' can be exercised at any time, where  $E > 0$  is the exercise price. Find optimal exercise boundary and show that the associated price at this optimal boundary is given by

$$B = \frac{E}{\lambda_1 - 1}$$

where

$$\lambda_1 = \frac{-(a - b^2/2) + \sqrt{(a - b^2/2)^2 + 2b^2}}{b^2}.$$

**Solution** The BPE for this risk-adjusted spot rate is

$$\frac{\partial B}{\partial t} + \frac{1}{2}b^2r^3 \frac{\partial^2 B}{\partial r^2} + ar^2 \frac{\partial B}{\partial r} - rB = 0$$

so perpetual here means independent of time  $t$ , and we have an Euler equation

$$\frac{1}{2}b^2r^3 \frac{d^2 B}{dr^2} + ar^2 \frac{dB}{dr} - rB = 0$$

with solution of form

$$B(r) = r^\lambda$$

so

$$\frac{1}{2}b^2\lambda(\lambda - 1) + a\lambda - 1 = 0 \longrightarrow \frac{1}{2}b^2\lambda^2 + \left(a - \frac{1}{2}b^2\right)\lambda - 1 = 0$$

and

$$\lambda_{1,2} = \frac{-(a - b^2/2) \pm \sqrt{(a - b^2/2)^2 + 2b^2}}{b^2}.$$

so general solution is

$$B(r) = \alpha r^{\lambda_1} + \beta r^{\lambda_2}$$

$\lambda_1 > 0 > \lambda_2$ , and if  $r = 0$  then  $B = 0$  so we must have  $\beta = 0$ . Hence exercise the option when  $r = r^* > E$ , then

$$B(r) = (r^* - E) \left( \frac{r}{r^*} \right)^{\lambda_1}$$

where  $\lambda_1 = \frac{-(a - b^2/2) \pm \sqrt{(a - b^2/2)^2 + 2b^2}}{b^2}$ . Choose  $r^*$  to maximise  $B(r)$ ,  $B$  is max at  $r^*$  for fixed  $r$  :

$$\frac{\partial B}{\partial r^*} = \left( \frac{r}{r^*} \right)^{\lambda_1} - \lambda_1 \frac{(r^* - E)}{r^*} \left( \frac{r}{r^*} \right)^{\lambda_1} = \left( \frac{r}{r^*} \right)^{\lambda_1} \left( 1 - \lambda_1 \frac{(r^* - E)}{r^*} \right) = 0$$

when

$$r^* = \frac{E\lambda_1}{\lambda_1 - 1}$$

then

$$\left. \frac{\partial B}{\partial r} \right|_{r=r^*} = \lambda_1 \frac{(r^* - E)}{r^*} = 1$$

and

$$B = \frac{E}{\lambda_1 - 1}.$$

5. Consider the Vasicek model for the spot rate  $r$  with mean rate  $\bar{r}$  and reversion rate  $\gamma$ . Suppose  $\gamma = 0.1$ ,  $\bar{r} = 0.1$ , and standard deviation  $\sigma = 20\%$ . Price a Zero Coupon Bond that matures in year 10, if the spot rate is 10%. (Very much a spreadsheet based problem).

**Solution:**

Vasicek model is given by:

$$dr = \gamma(\bar{r} - r)dt + \sigma dX$$

Zero coupon bond  $ZCB$  is given by  $Z(r, t; T) = \exp(A - rB)$

where

$$B(t; T) = \frac{1}{\gamma} (1 - \exp(-\gamma(T - t)))$$

$$A(t; T) = \frac{1}{\gamma^2} (B - (T - t)) \left( \bar{r}\gamma^2 - \frac{1}{2}\sigma^2 \right) - \frac{\sigma^2 B^2}{4\gamma}$$

(given in Wilmott). Remember  $\bar{r} = \eta/\gamma$ .

We are given the reversion rate  $\gamma = 0.1$ , mean rate  $\bar{r} = 0.1$  and standard deviation (or diffusion)  $\sigma = 0.2$ . Maturity  $T = 10$ . These quantities substituted in give  $B = 6.321$  and  $A = 2.994$ .

Therefore the bond price is now given as a function of  $r$

$$Z = 19.97 \exp(-6.321r).$$

The spot rate  $r$  is 10%, therefore we have a bond price  $Z = 10.61$ .

6. In class we derived a two factor interest rate model with the BPE given by

$$\frac{\partial V}{\partial t} + \frac{1}{2}w^2 \frac{\partial^2 V}{\partial r^2} + \rho wq \frac{\partial^2 V}{\partial r \partial l} + \frac{1}{2}q^2 \frac{\partial^2 V}{\partial l^2} + (u - \lambda_r w) \frac{\partial V}{\partial r} + (p - \lambda_l q) \frac{\partial V}{\partial l} - rV = 0.$$

where the two state variables evolve according to

$$\begin{aligned} dr &= udt + wdX_1 \\ dl &= pdt + qdX_2. \end{aligned}$$

Given that  $u - \lambda_r w = 0 = p - \lambda_l q$  and  $w = q = \sqrt{a + br + cl}$ , where  $a$ ,  $b$  and  $c$  are constants, derive a set of equations and boundary conditions for  $A$ ,  $B$  and  $C$  such that a bond  $V$  is of the form

$$V = \exp(A(t) + rB(t) + lC(t))$$

is a solution of the BPE with redemption value

$$V(r, l, T; T) = 1.$$

You are not required to solve these equations.

**Solution:** The information given reduces the BPE to

$$\frac{\partial V}{\partial t} + \frac{1}{2}(a + br + cl) \frac{\partial^2 V}{\partial r^2} + \rho(a + br + cl) \frac{\partial^2 V}{\partial r \partial l} + \frac{1}{2}(a + br + cl) \frac{\partial^2 V}{\partial l^2} = rV.$$

We are given  $V = \exp(A(t) + rB(t) + lC(t)) \longrightarrow$

$$\begin{aligned} \dot{V} &= \left( \dot{A}(t) + r\dot{B}(t) + l\dot{C}(t) \right) V \\ V_r &= BV \longrightarrow V_{rr} = B^2V \\ V_l &= CV \longrightarrow V_{ll} = C^2V \\ V_{rl} &= BCV \end{aligned}$$

and substitute in BPE to give

$$\left( \dot{A}(t) + r\dot{B}(t) + l\dot{C}(t) \right) + \frac{1}{2}B^2(a + br + cl) + \rho BC(a + br + cl) + \frac{1}{2}C^2(a + br + cl) = r$$

and now equation coefficients of  $O(1)$ ,  $O(r)$ ,  $O(l)$  to give in turn the following ODE's

$$\begin{aligned}\dot{A}(t) + \frac{1}{2}B^2a + \rho BCa + \frac{1}{2}aC^2 &= 0 \\ \dot{B}(t) + \frac{1}{2}B^2b + \rho BCb + \frac{1}{2}bC^2 &= 1 \\ \dot{C}(t) + \frac{1}{2}B^2c + \rho BCc + \frac{1}{2}cC^2 &= 0\end{aligned}$$

which are solved together with the final condition

$$A(T) = B(T) = C(T) = 0.$$