Εργασία κου. Τσουκαντά

B. Probability

1. (a) It holds that:

$$\int_0^{+\infty} p(x;\lambda)dx=1 \Rightarrow \int_0^{+\infty} A\lambda x e^{-\lambda x^2}dx=1 \Rightarrow -\frac{A}{2}\int_0^{+\infty} e^{-\lambda x^2}(-\lambda x^2)'dx=1 \Rightarrow$$

$$-\frac{A}{2}\cdot e^{-\lambda x^2}\Big|_0^{+\infty}=1 \Rightarrow -\frac{A}{2}(0-1)=1 \Rightarrow \frac{A}{2}=1 \Rightarrow A=2$$ Consequently, $p(x,\lambda)=2\lambda x e^{-\lambda x^2}, x>0, \lambda>0$

(b) It is: $$E(X^{2n})=\int_0^{+\infty} x^{2n}p(x;\lambda)dx=\int_0^{+\infty} x^{2n}2\lambda x e^{-\lambda x^2}dx=-\int_0^{+\infty} x^{2n}\left(e^{-\lambda x^2}\right)'dx=$$

$$=-x^{2n}e^{-\lambda x^2}\Big|_0^{+\infty}+\int_0^{+\infty}2nx^{2n-1}e^{-\lambda x^2}dx=-0+0+\frac{n}{\lambda}\int_0^{+\infty}x^{2(n-1)}2\lambda x e^{-\lambda x^2}dx=\frac{n}{\lambda}E(X^{2(n-1)})$$

It is: $E(X^0)=E(1)=1$

Using a recursive equation rational, it is easy to prove that:

$$E(X^{2n})=\frac{n}{\lambda}E(X^{2(n-1)})=\frac{n(n-1)}{\lambda^2}E(X^{2(n-2)})=\ldots=\frac{n(n-1)\ldots 2}{\lambda^n}E(X^0)=\frac{n!}{\lambda^n}, n=0,1,\ldots$$

2. Usually, it is $RAND(\,)\equiv U(0,1)$

If there is a random variable $X_i \sim U(0,1)$, then $E(X_i)=\frac{1}{2}$, $V(X_i)=\frac{1}{12}$, $i=1,\ldots,12$

Consequently, from the Central Limit Theorem, it is: $\sum_{i=1}^{12}X_i \sim N\left(12\cdot\frac{1}{2}=6, 12\cdot\frac{1}{12}=1\right)$ (approximately)

That means: $$\frac{\sum_{i=1}^{12}RAND(\,)-6}{1}\sim N(0,1)$$

In the same way, $RAND(N)\equiv N\cdot U(0,1)$

If $X_i\sim RAND(N)$, then $E(X_i)=\frac{N}{2}$ and $V(X_i)=\frac{N^2}{12}$, $i=1,\ldots,12$.

Consequently, from the Central Limit Theorem, it is: $\sum_{i=1}^{12}X_i\sim N\left(12\cdot\frac{N}{2}=6N, 12\cdot\frac{N^2}{12}=N^2\right)$

This leads to the conclusion that: $$\frac{\sum_{i=1}^{12}RAND(N)-6N}{N}\sim N(0,1)$$