Incomplete Markets: Jump Diffusion and Stochastic Volatility

In this lecture. . .

- the Poisson process for modeling jumps
- hedging in the presence of jumps
- how to price derivatives when the path of the underlying can be discontinuous
- modeling volatility as a stochastic variable
- how to price contracts when volatility is stochastic
- the market price of volatility risk

By the end of this lecture you will be able to

- model discontinuities using Poisson processes

- understand the difficulties associated with hedging when the underlying asset path is possibly discontinuous

- model volatility as a stochastic variable

- price options when volatility is stochastic

Introduction

There is plenty of evidence that financial quantities, be they equities, currencies or interest rates, for example, do not follow the lognormal random walk that has been the foundation of almost everything we have seen so far.

Several theories have been put forward for the non-Normality of the empirical distribution.

Two of these are

- Assets can jump in value

- Volatility is stochastic

Evidence for non Normality

Let's look at some data to see just how far from Normal the returns really are.

There are several ways to visualize the difference between two distributions, in our case the difference between the empirical distribution and the Normal distribution.

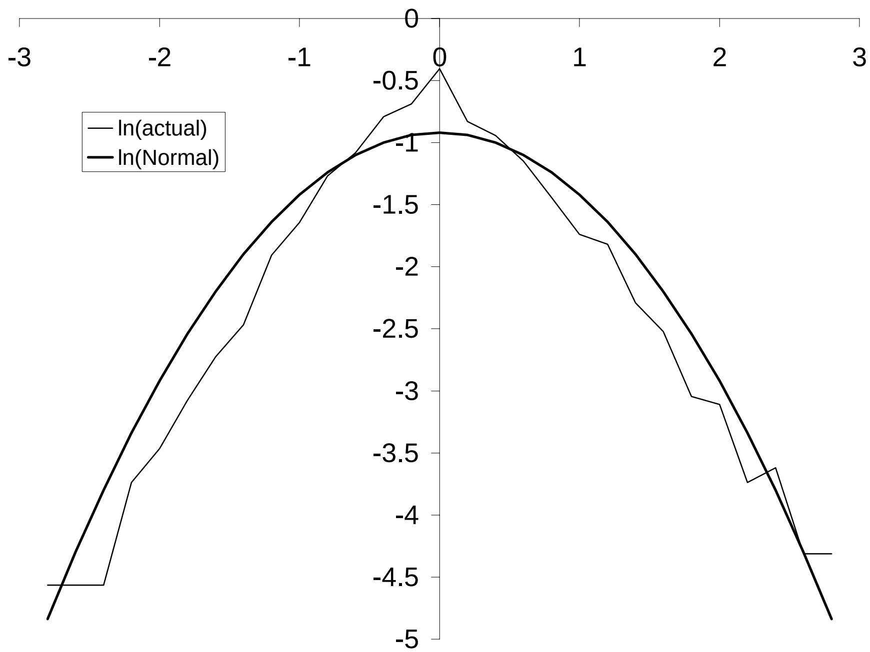
The obvious ways are to plot

1. the two probability density functions

2. the difference between the two cumulative distribution functions

Another useful plot is the logarithm of the pdf for the actual distribution and the Normal distribution.

The latter is simply a parabola, but what is the former? Such a plot is shown below. The log of the pdf of the actual distribution looks more linear than quadratic in the tails.



The final picture is called a **Quantile-Quantile** or **Q-Q plot**.

This is a common way of visualizing the difference between two distributions when you are particularly interested in the tails of the distribution.

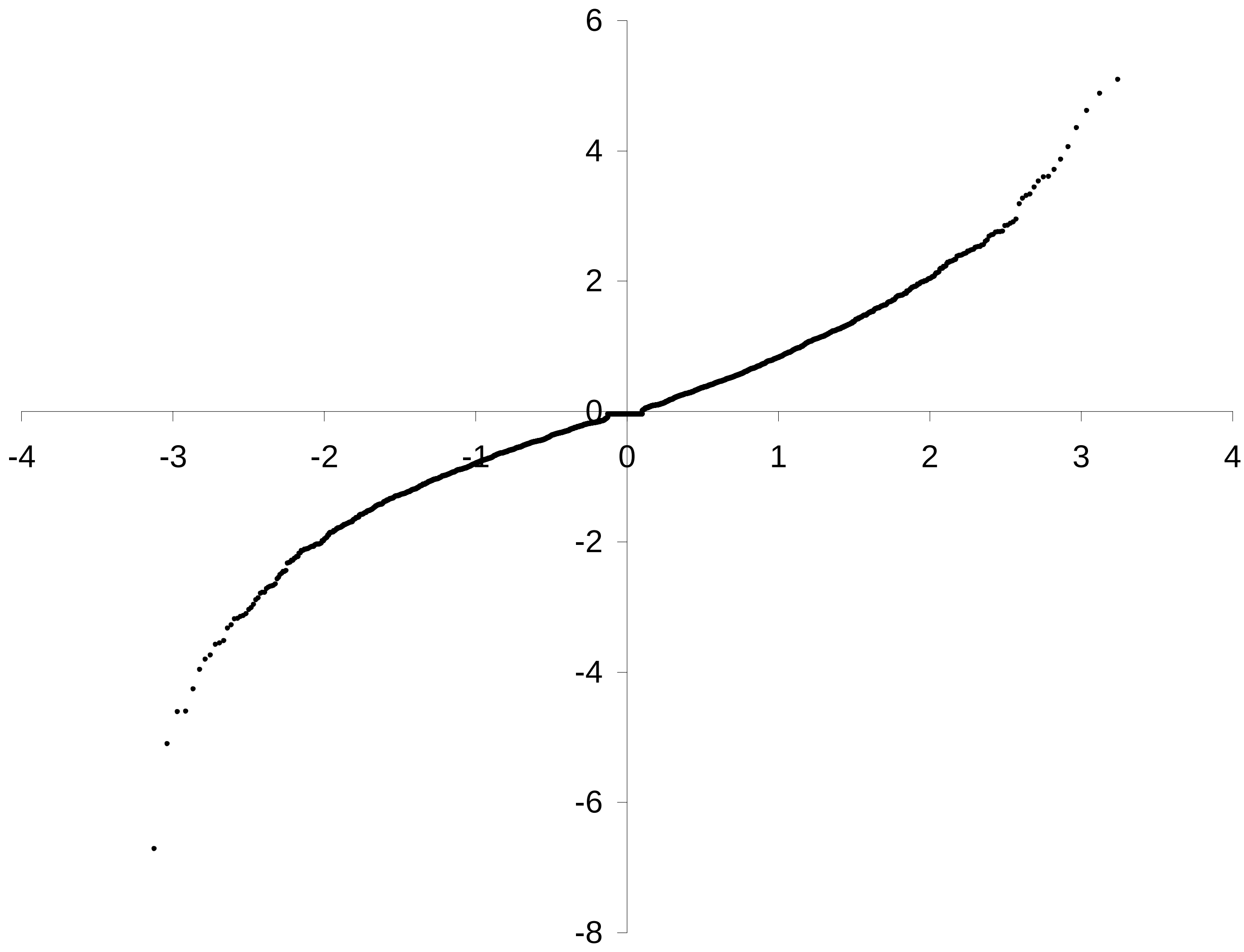
This plot is made up as follows.

1. Rank the empirical returns in order from smallest to largest, call these y_i with an index i going from 1 to n.

2. For the Normal distribution find the returns x_i such that the cumulative distribution function at x_i has value i/n.

3. Now plot each point (x_i, y_i).

The better the fit between the two distributions, the closer the line is to straight. In the present case the line is far from straight, due to the extra weight in the tails.



Q-Q plot for Xerox daily returns and the standardized Normal distribution.

Discontinuous price paths

One of the striking features of real financial markets is that every now and then there is a sudden unexpected fall or crash (or rally).

- These sudden movements occur far more frequently than would be expected from a Normally distributed return with a reasonable volatility.

On all but the shortest timescales the move looks discontinuous, the prices of assets have jumped.

This is important for the theory and practice of derivatives because *it is usually not possible to hedge through the crash*.

Poisson processes

The basic building block for the random walks we have considered so far is continuous Brownian motion based on the Normally-distributed increment.

We can think of this as adding to the return from one day to the next a Normally distributed random variable with variance proportional to time step.

The extra building block we need for the **jump-diffusion model** for an asset price, is the **Poisson process**.

A Poisson process dq is defined by

$$dq = \begin{cases} 0 & \text{with probability } 1 - \lambda\, dt \\ 1 & \text{with probability } \lambda\, dt. \end{cases}$$

There is a probability $\lambda\, dt$ of a jump in q in the time step dt.

The parameter λ is called the **intensity** of the Poisson process.

The scaling of the probability of a jump with the size of the time step is important in making the resulting process 'sensible,' i.e. there being a finite chance of a jump occuring in a finite time, with q not becoming infinite.

This Poisson process can be incorporated into a model for an asset in the following way:

$$dS = \mu S \, dt + \sigma S \, dX + (J-1) S \, dq. \tag{1}$$

We assume that there is no correlation between the Brownian motion and the Poisson process.

If there is a jump ($dq = 1$) then S immediately goes to the value JS. We can model a sudden 10% fall in the asset price by $J = 0.9$.

We can generalize further by allowing J to also be a random quantity.

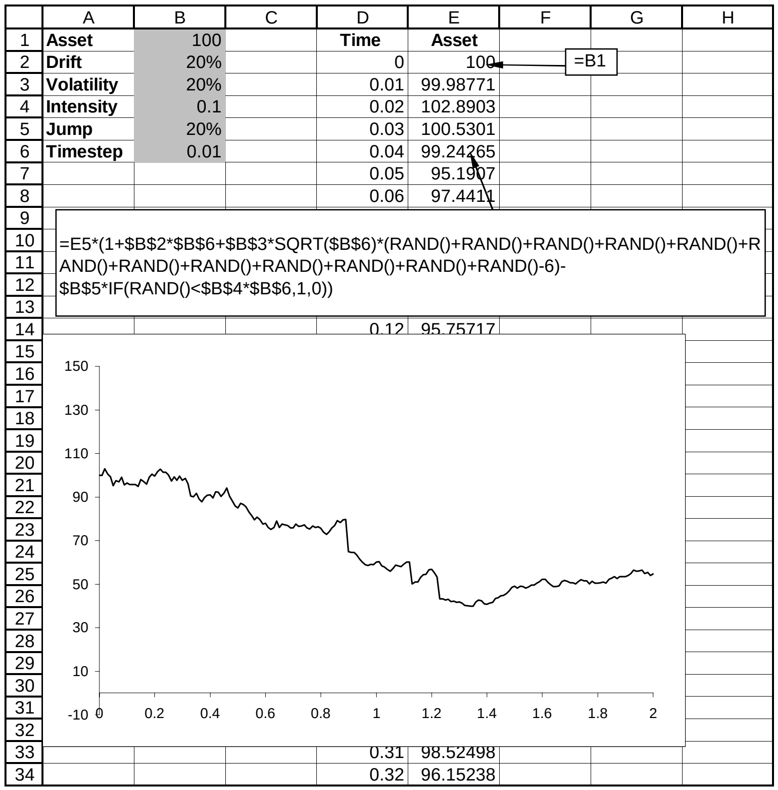
We assume that it is drawn from a distribution with probability density function $P(J)$, again independent of the Brownian motion and Poisson process.

The random walk in $\log S$ follows from (1):

$$d(\log S) = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma\, dX + (\log J) dq.$$

This is just a jump-diffusion version of Itô.

Below is a spreadsheet showing how to simulate the random walk for S. In this simple example the stock jumps by 20% at random times given by a Poisson process.



Spreadsheet simulation of a jump-diffusion process.

Hedging options when there are jumps

Now let us build up a theory of derivatives in the presence of jumps.

Begin by holding a portfolio of the option and $-\Delta$ of the asset:

$$\Pi = V(S,t) - \Delta S.$$

The change in the value of this portfolio is

$$d\Pi = \left(\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt + \left(\frac{\partial V}{\partial S} - \Delta\right)(\mu S\, dt + \sigma S\, dX)$$
$$+ \left(V(JS,t) - V(S,t) - \Delta(J-1)S\right) dq.$$

Again, this is a jump-diffusion version of Itô.

If there is no jump at time t so that $dq = 0$, then we could have chosen $\Delta = \partial V/\partial S$ to eliminate the risk.

If there is a jump and $dq = 1$ then the portfolio changes in value by an $O(1)$ amount, *that cannot be hedged away*.

In that case perhaps we should choose Δ to minimize the variance of $d\Pi$.

- This presents us with a dilemma. We don't know whether to hedge the small(ish) diffusive changes in the underlying which are always present, or the large moves which happen rarely.

Let us pursue both of these possibilities.

Hedging the diffusion

If we choose

$$\Delta = \frac{\partial V}{\partial S}$$

we are following a Black–Scholes type of strategy, hedging the diffusive movements.

The change in the portfolio value is then

$$d\Pi = \left(\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \left(V(JS,t) - V(S,t) - (J-1)S\frac{\partial V}{\partial S} \right) dq.$$

The portfolio now evolves in a deterministic fashion, except that every so often there is a non-deterministic jump in its value.

Robert Merton argued that if the jump component of the asset price process is uncorrelated with the market as a whole, then the risk in the discontinuity should not be priced into the option.

- Diversifiable risk should not be rewarded.

In other words, we can take expectations of this expression and set that value equal to the risk-free return from the portfolio.

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV +$$

$$\lambda E\left[V(JS,t) - V(S,t)\right] - \lambda\frac{\partial V}{\partial S} SE\left[(J-1)\right] = 0.$$

Here $E[\cdot]$ is the expectation taken over the jump size J, which can also be written

$$E[x] = \int xP(J)dJ,$$

where $P(J)$ is the probability density function for the jump size.

If J is known then just drop the $E[\cdot]$.

Aside: Are we working with real or risk-neutral expectations? At the moment real (Merton's argument), but later we'll look at the concept of risk neutrality when there are jumps.

This is a pricing equation for an option when there are jumps in the underlying.

The important point to note about this equation that makes it different from others we have derived is its non-local nature.

- That is, the equation links together option values at distant S values, instead of just containing local derivatives.

Naturally, the value of an option here and now depends on the prices to which it can instantaneously jump.

There is a simple closed-form solution of this equation in a special case.

That special case if when J is lognormally distributed. I.e. the logarithm of J is Normally distributed.

Why might there be a closed-form solution in this case?

If the logarithm of J is Normally distributed with standard deviation σ' and 'mean' $k = E[J-1]$ then the price of a European non-path-dependent option can be written as

$$\sum_{n=0}^{\infty} \frac{1}{n!} e^{-\lambda'(T-t)} (\lambda'(T-t))^n V_{BS}(S, t; \sigma_n, r_n).$$

In the above

$$\lambda' = \lambda(1+k), \quad \sigma_n^2 = \sigma^2 + \frac{n\sigma'^2}{T-t} \quad \text{and} \quad r_n = r - \lambda k + \frac{n \log(1+k)}{T-t},$$

and V_{BS} is the Black–Scholes formula for the option value in the absence of jumps.

- This formula can be interpreted as the sum of individual Black–Scholes values each of which assumes that there have been n jumps, and they are weighted according to the probability that there will have been n jumps before expiry.

Hedging the jumps

In the above we hedged the diffusive element of the random walk for the underlying.

Another possibility is to hedge both the diffusion and jumps 'together.'

For example, we could choose Δ to minimize the variance of the hedged portfolio, after all, this is ultimately what hedging is about.

The change in the value of the portfolio with an arbitrary Δ is, to leading order,

$$d\Pi = \left(\frac{\partial V}{\partial S} - \Delta\right)(\mu S\, dt + \sigma S\, dX)$$

$$+\,(-\Delta(J-1)S + V(JS,t) - V(S,t))\, dq + \cdots.$$

The variance in this change, which is a measure of the risk in the portfolio, is

$$\text{var}[d\Pi] = \left(\frac{\partial V}{\partial S} - \Delta\right)^2 \sigma^2 S^2 dt +$$

$$\lambda E\left[(-\Delta(J-1)S + V(JS,t) - V(S,t))^2\right] dt + \cdots. \qquad (2)$$

This is minimized by the choice

$$\Delta = \frac{\lambda E\left[(J-1)(V(JS,t)-V(S,t))\right] + \sigma^2 S\frac{\partial V}{\partial S}}{\lambda S E\left[(J-1)^2\right] + \sigma^2 S}.$$

(To see this, differentiate (2) with respect to Δ and set the resulting expression equal to zero.)

If we value the options as a pure discounted real expectation under this best-hedge strategy then we find that

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial s^2} + S\frac{\partial V}{\partial S}\left(\mu - \frac{\sigma^2}{d}(\mu + \lambda k - r)\right) - rV$$

$$+\lambda E\left[(V(JS,t) - V(S,t))\left(1 - \frac{J-1}{d}(\mu + \lambda k - r)\right)\right] = 0,$$

where

$$d = \lambda E\left[(J-1)^2\right] + \sigma^2.$$

What about risk neutrality?

Does the concept of risk neutrality have any role when there are jumps?

N.B. The above uses 'real' expectations.

Let's see a special case, known jump *size*, J, but unknown timing.

Start with

$$dS = \mu S \, dt + \sigma S \, dX + (J - 1)S \, dq$$

but with J given.

There are now *two* sources of risk (there were three before), dX and dq.

Let's see if we can eliminate risk by having *two* hedging instruments, the stock and another option.

(You will recall this from the stochastic interest rate lecture and will see it again in stochastic volatility modelling.)

Construct a portfolio of the option and $-\Delta$ of the asset, and $-\Delta_1$ of another option, V_1:

$$\Pi = V(S,t) - \Delta S - \Delta_1 V_1.$$

The change in the value of this portfolio is

$$d\Pi = \left(\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - \Delta_1 \left(\frac{\partial V_1}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_1}{\partial S^2} \right) \right) dt$$

$$+ \left(\frac{\partial V}{\partial S} - \Delta - \Delta_1 \frac{\partial V_1}{\partial S} \right) (\mu S \, dt + \sigma S \, dX)$$

$$+ \left(V(JS, t) - V(S, t) - \Delta(J-1)S - \Delta_1 \left(V_1(JS, t) - V_1(S, t) \right) \right) dq.$$

To eliminate dX terms choose

$$\frac{\partial V}{\partial S} - \Delta - \Delta_1 \frac{\partial V_1}{\partial S} = 0,$$

and to eliminate dq terms choose

$$V(JS,t) - V(S,t) - \Delta(J-1)S - \Delta_1 \left(V_1(JS,t) - V_1(S,t)\right) = 0.$$

So

$$\Delta_1 = \frac{(J-1)S\frac{\partial V}{\partial S} - V(JS,t) + V(S,t)}{(J-1)S\frac{\partial V_1}{\partial S} - V_1(JS,t) + V_1(S,t)}$$

and

$$\Delta = \frac{\partial V}{\partial S} - \frac{\partial V_1}{\partial S} \times \frac{(J-1)S\frac{\partial V}{\partial S} - V(JS,t) + V(S,t)}{(J-1)S\frac{\partial V_1}{\partial S} - V_1(JS,t) + V_1(S,t)}.$$

All risk now eliminated, so set return on portfolio equal to risk-free rate.

End result:

$$\frac{\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+rS\frac{\partial V}{\partial S}-rV}{(J-1)S\frac{\partial V}{\partial S}-V(JS,t)+V(S,t)}$$

$$=\frac{\frac{\partial V_1}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V_1}{\partial S^2}+rS\frac{\partial V_1}{\partial S}-rV_1}{(J-1)S\frac{\partial V_1}{\partial S}-V_1(JS,t)+V_1(S,t)}.$$

Same 'functional' on each side.

So

$$\frac{\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 V}{\partial S^2}+rS\frac{\partial V}{\partial S}-rV}{(J-1)S\frac{\partial V}{\partial S}-V(JS,t)+V(S,t)}$$

$=$ universal quantity, independent of type of option

$$=-\lambda'.$$

Final equation is

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 V}{\partial S^2}+rS\frac{\partial V}{\partial S}-rV$$

$$+\lambda'\left(V(JS,t)-V(S,t)-(J-1)S\frac{\partial V}{\partial S}\right)=0.$$

This is the same equation as before but with risk-neutral λ' instead of real λ.

How do you solve these equations?

- **Monte Carlo:** The solutions of the partial integro-differential equations you get with jump-diffusion models can still be interpreted as 'the present value of the expected payoff.' So all you have to do is to simulate the relevant random walk for the underlying (risk neutral) many times, calculate the average payoff and then present value it. As always!

- **Finite differences:** The partial integro-differential equations can still be solved by finite differences but the method will no longer be 'local' since the governing equation contains integrations over all asset prices.

Pros and cons of jump-diffusion models

Pros:

- Evidence (and common sense) suggests that assets can jump in value
- Jump models can capture extreme implied volatility skews (such as seen close to expiration)
- More parameters means that calibration can be 'better'

Cons:

- The foundations are a bit shaky (can't hedge, hedge diffusion or minimize risk, real versus risk neutral)

Stochastic volatility

Volatility does not behave how the Black–Scholes equation would like it to behave; it is not constant, it is not predictable, it's not even directly observable.

This makes it a prime candidate for modeling as a random variable.

There is plenty of evidence that returns on equities, currencies and commodities are not Normally distributed, they have higher peaks and fatter tails than predicted by a Normal distribution. This has been cited as evidence for non-constant volatility.

A stochastic differential equation for volatility

We continue to assume that S satisfies

$$dS = \mu S\, dt + \sigma S\, dX_1,$$

but we further assume that volatility satisfies

$$d\sigma = p(S, \sigma, t)\, dt + q(S, \sigma, t)\, dX_2. \tag{3}$$

The two increments dX_1 and dX_2 have a correlation of ρ. The choice of functions $p(S, \sigma, t)$ and $q(S, \sigma, t)$ is crucial to the evolution of the volatility, and thus to the pricing of derivatives.

The value of an option with stochastic volatility is a function of three variables, $V(S, \sigma, t)$.

The pricing equation

The new stochastic quantity that we are modeling, the volatility, is not a traded asset.

Thus, when volatility is stochastic we are faced with the problem of having a source of randomness that cannot be easily hedged away.

Because we have two sources of randomness we must hedge our option with two other contracts, one being the underlying asset as usual, but now we also need another option to hedge the volatility risk.

We therefore must set up a portfolio containing one option, with value denoted by $V(S, \sigma, t)$, a quantity $-\Delta$ of the asset and a quantity $-\Delta_1$ of another option with value $V_1(S, \sigma, t)$.

We have

$$\Pi = V - \Delta S - \Delta_1 V_1. \tag{4}$$

The change in this portfolio in a time dt is given by

$$d\Pi = \left(\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \rho\sigma q S \frac{\partial^2 V}{\partial S \partial \sigma} + \tfrac{1}{2} q^2 \frac{\partial^2 V}{\partial \sigma^2} \right) dt$$

$$-\Delta_1 \left(\frac{\partial V_1}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_1}{\partial S^2} + \rho\sigma q S \frac{\partial^2 V_1}{\partial S \partial \sigma} + \tfrac{1}{2} q^2 \frac{\partial^2 V_1}{\partial \sigma^2} \right) dt$$

$$+ \left(\frac{\partial V}{\partial S} - \Delta_1 \frac{\partial V_1}{\partial S} - \Delta \right) dS$$

$$+ \left(\frac{\partial V}{\partial \sigma} - \Delta_1 \frac{\partial V_1}{\partial \sigma} \right) d\sigma.$$

where we have used Itô's lemma on functions of S, σ and t.

To eliminate all randomness from the portfolio we must choose

$$\frac{\partial V}{\partial S} - \Delta - \Delta_1 \frac{\partial V_1}{\partial S} = 0,$$

to eliminate dS terms, and

$$\frac{\partial V}{\partial \sigma} - \Delta_1 \frac{\partial V_1}{\partial \sigma} = 0,$$

to eliminate $d\sigma$ terms.

Therefore we choose

$$\Delta_1 = \frac{\frac{\partial V}{\partial \sigma}}{\frac{\partial V_1}{\partial \sigma}}$$

and

$$\Delta = \frac{\partial V}{\partial S} - \frac{\frac{\partial V}{\partial \sigma}}{\frac{\partial V_1}{\partial \sigma}} \frac{\partial V_1}{\partial S}.$$

This leaves us with

$$d\Pi = \frac{\partial V}{\partial t}dt + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}dt + \rho\sigma S q \frac{\partial^2 V}{\partial S\,\partial\sigma}dt + \tfrac{1}{2}q^2\frac{\partial^2 V}{\partial\sigma^2}dt$$

$$-\Delta_1\left(\frac{\partial V_1}{\partial t}dt + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_1}{\partial S^2}dt + \rho\sigma S q \frac{\partial^2 V_1}{\partial S\,\partial\sigma}dt + \tfrac{1}{2}q^2\frac{\partial^2 V_1}{\partial\sigma^2}dt\right)$$

$$= r\Pi\,dt = r(V - \Delta S - \Delta_1 V_1)dt,$$

where we have used arbitrage arguments to set the return on the portfolio equal to the risk-free rate.

As it stands, this is *one* equation in the *two* unknowns, V and V_1.

This contrasts with the earlier Black–Scholes case with one equation in the one unknown.

Collecting all V terms on the left-hand side and all V_1 terms on the right-hand side we find that

$$\frac{\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+\rho\sigma S q\frac{\partial^2 V}{\partial S\,\partial\sigma}+\frac{1}{2}q^2\frac{\partial^2 V}{\partial\sigma^2}+rS\frac{\partial V}{\partial S}-rV}{\frac{\partial V}{\partial\sigma}}$$

$$=\frac{\frac{\partial V_1}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V_1}{\partial S^2}+\rho\sigma S q\frac{\partial^2 V_1}{\partial S\,\partial\sigma}+\frac{1}{2}q^2\frac{\partial^2 V_1}{\partial\sigma^2}+rS\frac{\partial V_1}{\partial S}-rV_1}{\frac{\partial V_1}{\partial\sigma}}.$$

We are lucky that the left-hand side is a functional of V but not V_1 and the right-hand side is a function of V_1 but not V.

Therefore both sides can only be functions of the *independent* variables, S, σ and t.

Thus we have

$$\frac{\partial V}{\partial t}+\tfrac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+\rho\sigma S q\frac{\partial^2 V}{\partial S\,\partial\sigma}+\tfrac{1}{2}q^2\frac{\partial^2 V}{\partial\sigma^2}+rS\frac{\partial V}{\partial S}-rV=-(p-\lambda q)\frac{\partial V}{\partial\sigma},$$

for some function $\lambda(S,\sigma,t)$.

Reordering this equation, we usually write

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+\rho\sigma S q\frac{\partial^2 V}{\partial S\,\partial\sigma}+\frac{1}{2}q^2\frac{\partial^2 V}{\partial\sigma^2}+rS\frac{\partial V}{\partial S}+(p-\lambda q)\frac{\partial V}{\partial\sigma}-rV=0 \tag{5}$$

The function $\lambda(S,\sigma,t)$ is called the **market price of (volatility) risk**.

The market price of volatility risk

If we can solve Equation (5) then we have found the value of the option, and the hedge ratios.

But note that we find *two* hedge ratios, $\frac{\partial V}{\partial S}$ and $\frac{\partial V}{\partial \sigma}$.

- We have two hedge ratios because we have two sources of randomness that we must hedge away.

Because one of the modeled quantities, the volatility, is not traded we find that the pricing equation contains a market price of risk term.

What does this term mean?

Let's see what happens if we only hedge to remove the stock risk.

Suppose we hold one of the option with value V, and satisfying the pricing equation (5), delta hedged with the underlying asset only i.e. we have

$$\Pi = V - \Delta S.$$

The change in this portfolio value is

$$d\Pi = \left(\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \rho\sigma q S \frac{\partial^2 V}{\partial S \partial \sigma} + \tfrac{1}{2} q^2 \frac{\partial^2 V}{\partial \sigma^2} \right) dt$$

$$+ \left(\frac{\partial V}{\partial S} - \Delta \right) dS + \frac{\partial V}{\partial \sigma} d\sigma.$$

Because we are delta hedging the coefficient of dS is zero. We find that

$$d\Pi - r\Pi\, dt =$$

$$\left(\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \rho\sigma q S \frac{\partial^2 V}{\partial S \partial \sigma} + \tfrac{1}{2} q^2 \frac{\partial^2 V}{\partial \sigma^2} + rS \frac{\partial V}{\partial S} - rV \right) dt + \frac{\partial V}{\partial \sigma} d\sigma$$

$$= q \frac{\partial V}{\partial \sigma} (\lambda\, dt + dX_2).$$

The return on this partially hedged portfolio in excess of the risk-free return is

$$q\frac{\partial V}{\partial \sigma}(\lambda\, dt + dX_2).$$

Observe that for every unit of volatility risk, represented by dX_2, there are λ units of extra return, represented by dt. Hence the name 'market price of risk.'

Going back to the pricing equation ...

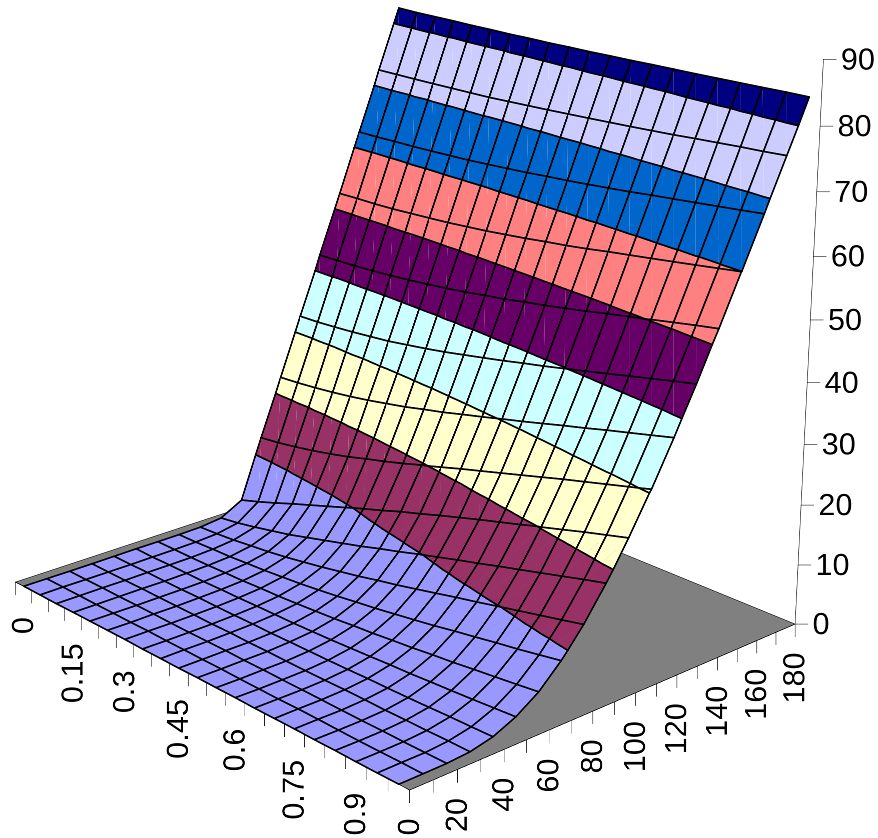
$$\frac{\partial V}{\partial t}+\tfrac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+\rho\sigma S q\frac{\partial^2 V}{\partial S\,\partial\sigma}+\tfrac{1}{2}q^2\frac{\partial^2 V}{\partial\sigma^2}+rS\frac{\partial V}{\partial S}+(p-\lambda q)\frac{\partial V}{\partial\sigma}-rV=0.$$

The quantity $p-\lambda q$ is called the **risk-neutral drift rate** of the volatility.

Recall that the risk-neutral drift of the underlying asset is r and not μ.

When it comes to pricing derivatives, it is the risk-neutral drift that matters and not the real drift, whether it is the drift of the asset or of the volatility.

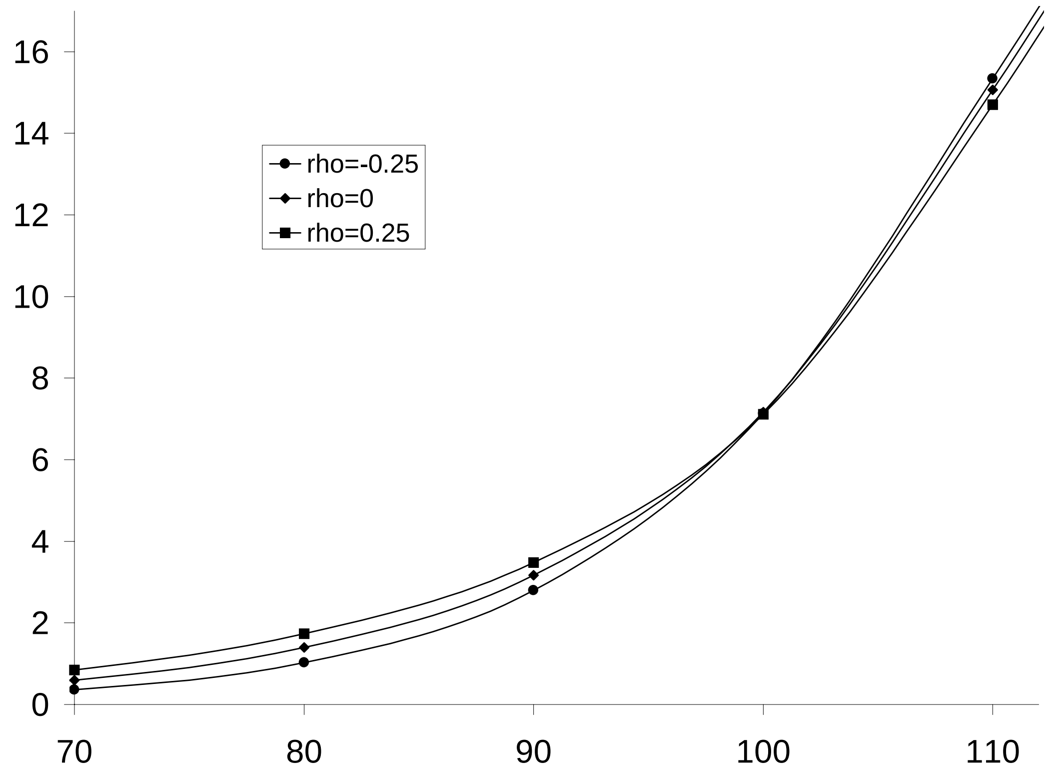
Stochastic volatility: an example



The next figure shows plots of call values for three different values of correlation, for the same volatility.

In this example the correlation effect is quite large for slightly out-of-the-money options.

For in-the-money options the effect on the option's time value is also significant.



Call value when volatility is stochastic; three different correlations.

Popular Models

In all of these we have $dS = \mu S\, dt + \sigma S\, dX_1$ and write $v = \sigma^2$.

GARCH-diffusion:

$$dv = (a - bv)dt + cv\, dX_2.$$

Popular because it is the continuous-time limit of a GARCH process. (Therefore relatively easy to get parameter estimates.)

Square-root model (Heston):

$$dv = (a - bv)dt + c\sqrt{v}\, dX_2.$$

Popular because there are closed-form solutions for European options.

3/2 model:

$$dv = (av - bv^2)dt + cv^{3/2}\,dX_2.$$

Popular because it has a closed-form solution.

Ornstein–Uhlenbeck process:

With $y = \ln v$,

$$dy = (a - by)dt + c\,dX_2.$$

Popular because it matches data well.

The Heston model

$$dv = (a - bv)dt + c\sqrt{v}\, dX_2.$$

There are ‘closed-form solutions’ for simple options in this model.

How to find the formulae:

1. Change variables, $x = \ln S$

2. Take Fourier transform in x

3. Solve transformed equation (now a diffusion equation in just time and v)

4. Invert the transform (numerical integration)

How to use Heston

There are four parameters in the model, speed of mean reversion, level of mean reversion, volatility of volatility, correlation. And also potentially a market price of volatility risk parameter.

The main four parameters can be chosen by matching data or by calibration.

Experience suggests that calibrated parameters are very unstable, and often unreasonable. (For example, the best fit to market prices might result in a correlation of exactly -1.)

The Heston model with jumps

Increasingly popular are **stochastic volatility with jumps models** (SVJ).

Jump models require a parameter to measure probability of a jump (a Poisson process) and a distribution for the jumps.

Pros: More parameters allow better fitting. The jump component of the model has most impact over short time scales. Therefore use longer-dated options to fit the stochastic volatility parameters and the shorter-dated options to fit the jump component.

Cons: Mathematics slightly more complicated (and again we must work in the transform domain). Hedging is even harder when the underlying stock process is potentially discontinuous.

Case Study: The REGARCH model and its diffusion limit

REGARCH = Range-based Exponential GARCH

'Range-based' refers to the use of the daily range, defined as the difference between the highest and lowest log asset price recorded throughout the day.

'Exponential' refers to modeling the logarithm of the variance.

Diffusion limits exist for all GARCH-type of processes. That is, they can be expressed in continuous time using stochastic differential equations.

(This is achieved via 'moment matching.' The statistical properties of the discrete-time GARCH processes are recreated with the continuous-time sdes.)

The REGARCH model becomes the following three-factor model:

$$dS = \mu S\, dt + \sigma_1 S\, dX_0$$

$$d(\ln \sigma_1) = a_1(\ln \sigma_2 - \ln \sigma_1)dt + b_1 dX_1$$

$$d(\ln \sigma_2) = a_2(c_2 - \ln \sigma_2)dt + b_2 dX_2.$$

This is a three-factor model, with two volatilities. σ_1 represents the actual volatility of the asset returns, which is stochastic.

The σ_2 represents the level to which σ_1 reverts, and is itself stochastic.

For pricing options we must replace these sdes with the risk-neutral versions:

$$dS = rS\,dt + \sigma_1 S\,dX_0$$

$$d(\ln \sigma_1) = a_1(\ln \sigma_2 - \ln \sigma_1 - \lambda_1 b_1/a_1)dt + b_1 dX_1$$

$$d(\ln \sigma_2) = a_2(c_2 - \ln \sigma_2 - \lambda_2 b_2/a_2)dt + b_2 dX_2.$$

The λs represent the market prices of risk.

The a and b coefficients and the correlations between the three sources of randomness give this system seven parameters.

These parameters are related to the parameters of the original REGARCH model and can be estimated from asset data.

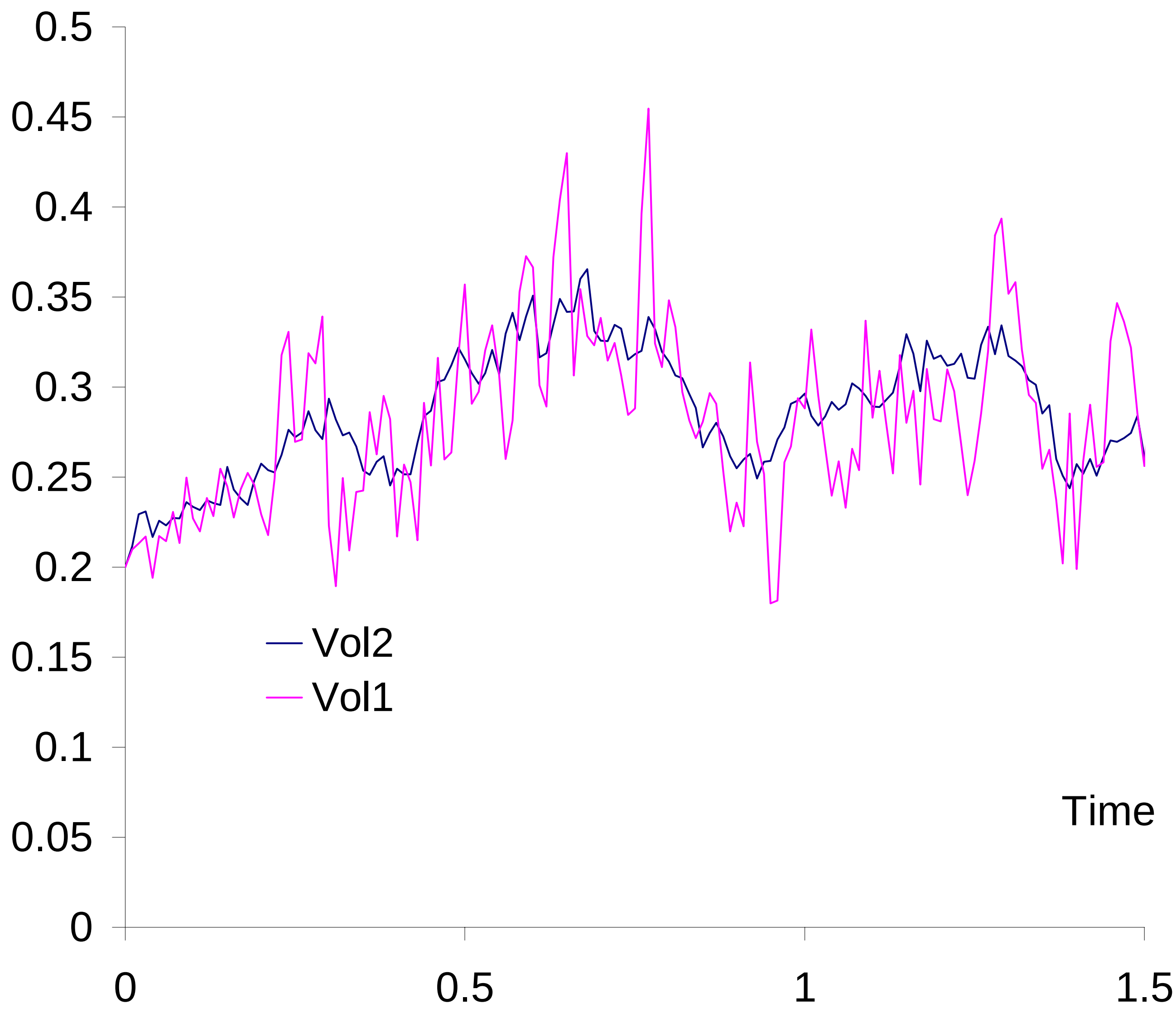
Example:

$$a_1 = 56.6, \quad b_1 = 1.138, \quad a_2 = 2.82, \quad b_2 = 0.388, \quad c_2 = -1.25.$$

($\lambda_1 = \lambda_2 = 0$.)

σ_1 is very rapidly mean reverting to the level of σ_2. This is a 'short-term' volatility. The time scale for mean reversion is about one week.

σ_2, the 'long-term volatility, reverts more slowly, over a period of about six months.

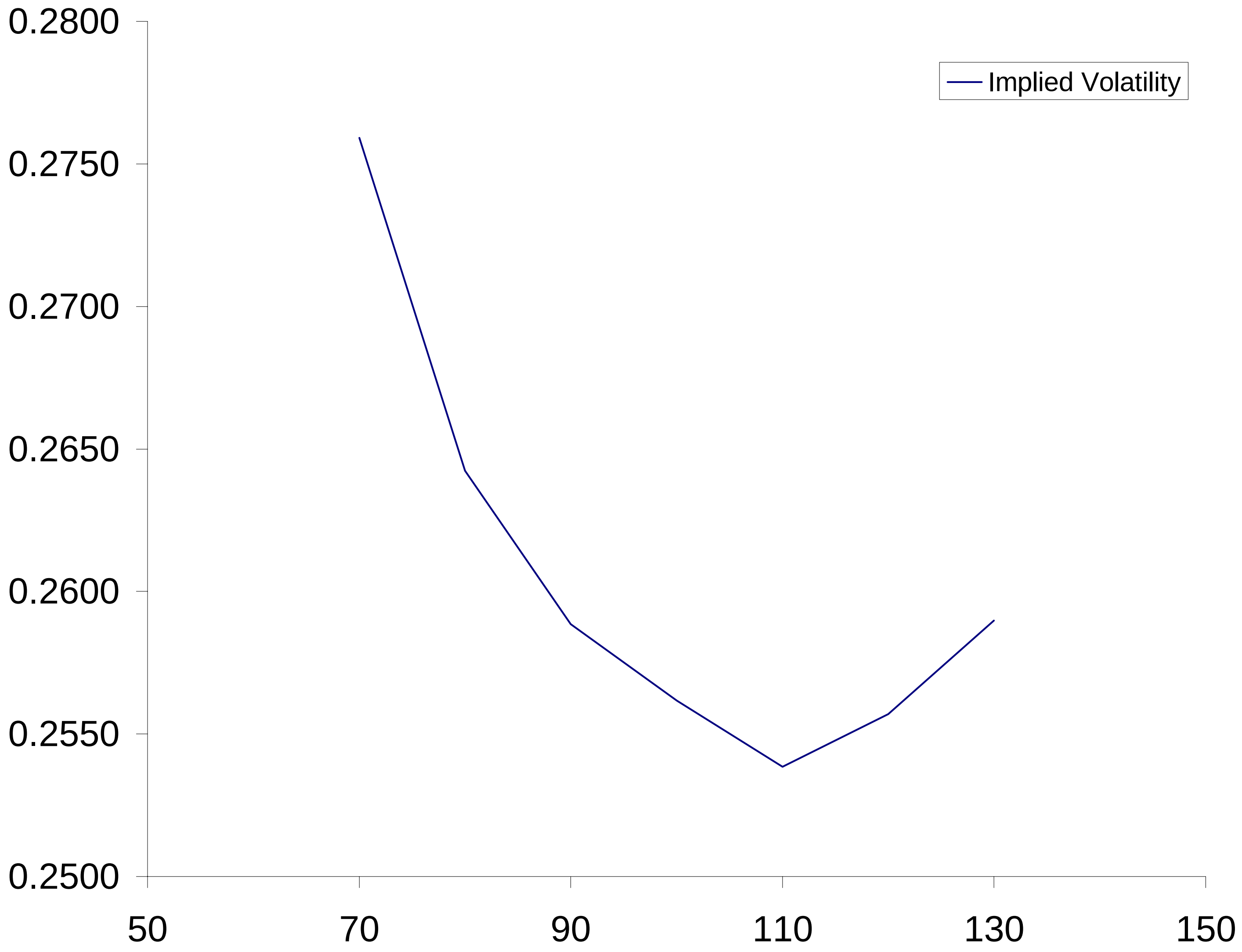


0.5
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
Vol2
Vol1
Time
0
0.5
1
1.5

Results from Monte Carlo simulation:

Type	C	C	C	C	C	C	C
Strike	70	80	90	100	110	120	130
Expiration	1	1	1	1	1	1	1
Option Value	34.11	25.69	18.41	12.57	8.18	5.24	3.32
Delta	0.94	0.88	0.77	0.63	0.47	0.33	0.23
Gamma	0.00	0.01	0.02	0.02	0.02	0.02	0.01
Theta	-4.98	-7.66	-7.69	-6.78	-2.78	-1.16	-0.94
Vega 1	-0.04	0.10	0.66	0.97	0.63	0.63	0.56
Vega 2	4.04	7.19	11.34	13.89	14.72	14.14	11.29

Implied Vol.	0.28	0.26	0.26	0.26	0.25	0.26	0.26
BS Delta	0.95	0.88	0.77	0.63	0.48	0.35	0.24
BS Gamma	0.00	0.01	0.01	0.01	0.02	0.01	0.01
BS Theta	-4.53	-5.78	-6.87	-7.36	-7.05	-6.21	-5.13
BS Vega	10.88	20.22	30.57	37.86	39.84	36.98	31.43



0.2800
0.2750
0.2700
0.2650
0.2600
0.2550
0.2500
50
70
90
110
130
150
Implied Volatility

How do you solve these equations?

- **Monte Carlo:** The solutions of the two-factor partial differential equations you get with stochastic volatility models can still be interpreted as 'the present value of the expected payoff.' So all you have to do is to simulate the relevant random walks for the underlying and volatility (risk neutral) many times, calculate the average payoff and then present value it. As always!

- **Finite differences:** The partial differential equations can still be solved by finite differences but you will need to work with a three-dimensional grid.

Pros and cons of stochastic volatility models

Pros:

- Evidence (and common sense) suggests that volatility changes, possibly randomly

- More parameters means that calibration can be 'better'

Cons:

- As with any incomplete-market model hedging is only possible if you believe in the market price of (volatility) risk

Summary

Please take away the following important ideas

- There is a simple model for jumps, using the Poisson process
- It is theoretically much harder to hedge options when there are jumps in the stock price
- It is possible to model volatility as a stochastic process, but there are very few clues as to which is the best model
- Whenever you have a variable that is not traded you will get a market price of risk term in the equation