CQF Exercise 2.6 Solutions - Binomial Method

(1) Binomial tree for share price is

$$\begin{array}{ccc} & & 84 \\ 80 & & \\ & & 76 \end{array}$$

Binomial tree for option price V is

$$\begin{array}{ccc} & & 5 \;\; (= \max(84-79,0)) \\ V & & \\ & & 0 \;\; (= \max(76-79,0) \end{array}$$

Now set up a Black-Scholes hedged portfolio, $V - \Delta S$, then binomial tree for its value is

$$\begin{array}{ccc} & & 5-84\Delta \\ V-80\Delta & & \\ & & -76\Delta \end{array}$$

For risk-free portfolio choose Δ such that $5 - 84\Delta = -76\Delta \Rightarrow \Delta = \frac{5}{8}$. So in absence of arbitrage, $V - 80\Delta = -76\Delta$, and $V = 2.5$.

(2) Binomial tree for share price is

$$\begin{array}{ccc} & & 98 \\ 92 & & \\ & & 86 \end{array}$$

Binomial tree for option price V is

$$\begin{array}{ccc} & & 8 \;\; (= \max(98-90,0)) \\ V & & \\ & & 0 \;\; (= \max(86-90,0) \end{array}$$

Now set up a Black-Scholes hedged portfolio, $V - \Delta S$, then binomial tree for its value is

$$\begin{array}{ccc} & & 8-98\Delta \\ V-92\Delta & & \\ & & -86\Delta \end{array}$$

For risk-free portfolio choose Δ such that $8 - 98\Delta = -86\Delta \Rightarrow \Delta = \frac{2}{3}$. So in absence of arbitrage, since portfolio is riskless, it must earn risk-free rate $r = 2\%$ and $V - 92\Delta = \exp(-0.02)(-86\Delta)$, then $V = \frac{2}{3}(92 - 86\exp(-0.02)) = 5.14$.

(3) Binomial tree for share price is

$$\begin{array}{ccc} & & 17 \\ 15 & & \\ & & 13 \end{array}$$

Binomial tree for option price V is

$$
\begin{array}{ccl}
 & & 130 \;\; \big(= \max(17^2 - 159, 0)\big) \\
V & & \\
 & & 10 \;\; \big(= \max(13^2 - 159, 0\big)
\end{array}
$$

Now set up a Black-Scholes hedged portfolio, $V - \Delta S$, then binomial tree for its value is

$$
\begin{array}{ccc}
 & & 130 - 17\Delta \\
V - 15\Delta & & \\
 & & 10 - 13\Delta
\end{array}
$$

For risk-free portfolio choose Δ such that $130 - 17\Delta = 10 - 13\Delta \Rightarrow \Delta = 30$. So in absence of arbitrage, $V - 15\Delta = 10 - 13\Delta$, and $V = 70$.

(**4**) Binomial tree for share price is

$$
\begin{array}{ccc}
 & & 92 \\
75 & & \\
 & & 59
\end{array}
$$

Time is 3 months - i.e. $\frac{1}{4}$ year, interest rate $r = 0$, so risk neutral probability that share price increases satisfies

$$92p + 59\,(1 - p) = 75$$

Re-arranging gives

$$p = \frac{75 - 59}{33} = 0.485$$

So probability of a fall would be given by $1 - p = 0.515$.

(5) Using data in question 4) now. We know from above risk-neutral probability is $p = 0.485$. The binomial tree for option price V is

$$
\begin{array}{ccl}
 & & 7 \;\; (= \max(92 - 85, 0)) = V^+ \\
V & & \\
 & & 0 \;\; (= \max(59 - 85, 0) = V^-
\end{array}
$$

Then value of option $V = pV^+ + (1 - p)\,V^- \Rightarrow V = 0.485\,(V^+) = 3.395$

(6) Binomial tree for share price is

$$
\begin{array}{ccc}
 & & 92 \\
75 & & \\
 & & 59
\end{array}
$$

Time is 3 months - i.e. $\frac{1}{4}$ year, interest rate $r = 4\%$, so risk neutral probability that share price increases satisfies

$$92p + 59(1-p) = 75\exp\left(\frac{0.04}{4}\right)$$

Re-arranging gives

$$p = \frac{75e^{0.01} - 59}{33} = 0.5077$$

So probability of a fall would be given by $1 - p = 0.4923$.

(7)

$$pu + (1-p)v = e^{\mu\ \delta t} \tag{a}$$

$$pu^{\ 2} + (1-p)v^{\ 2} = e^{\left(2\mu+\sigma^2\right)\ \delta t} \tag{b}$$

$u.(a) + v(a)$ gives

$$(u+v)e^{\mu\ \delta t} = pu^2 + uv - puv + pvu + v^{\ 2} - pv^{\ 2}.$$

Rearrange to get

$$(u+v)e^{\mu\ \delta t} = pu^2 + + (1-p)v^{\ 2} + uv$$

and we know from (b) that $pu^{\ 2} + (1-p)v^{\ 2} = e^{\left(2\mu+\sigma^2\right)\ \delta t}$ and $uv = 1$. Hence we have

$$\begin{aligned}(u+v)e^{\mu\ \delta t} &= e^{\left(2\mu+\sigma^2\right)\ \delta t} + 1 \Rightarrow \\ (u+v) &= e^{-\mu\ \delta t} + e^{\left(\mu+\sigma^2\right)\ \delta t}.\end{aligned}$$

Now recall that the quadratic equation $ax^2 + bx + c = 0$ with roots α and β has

$$\alpha + \beta = -\frac{b}{a}; \quad \alpha\beta = \frac{c}{a}.$$

We have

$$\begin{aligned}(u+v) &= e^{-\mu\ \delta t} + e^{\left(\mu+\sigma^2\right)\ \delta t} \equiv -\frac{b}{a} \\ uv &= 1 \equiv \frac{c}{a}\end{aligned}$$

hence u and v satisfy

$$(x-u)(x-v)=0$$

to give the quadratic

$$x^2-(u+v)x+uv=0 \Rightarrow$$

$$x=\frac{(u+v)\pm\sqrt{(u+v)^2-4uv}}{2}$$

so with $u>1$

$$u=\frac{1}{2}\left(e^{-\mu\ \delta t}+e^{\left(\mu+\sigma^2\right)\ \delta t}\right)+\frac{1}{2}\sqrt{\left(e^{-\mu\ \delta t}+e^{\left(\mu+\sigma^2\right)\ \delta t}\right)^2-4}$$

(8)

share price		Option price	
	103		$\max[103-100,0]=3$
100		V	
	98		$\max[98-100,0]=0$

Δ hedged portfolio: $V-\Delta S \Rightarrow$

	$3-103\Delta$
$V-100$	
	-98Δ

risk-free $\Rightarrow$ $3-103\Delta=-98\Delta \Rightarrow \Delta=0.6$
$V-100=-98\Delta \rightarrow V=1.2$
$\Pi=3-103\Delta=\boxed{-58.8}$

$103p+98(1-p)=100 \rightarrow p=0.4 \qquad \therefore 1-p=0.6$

If interest rates are non zero, and there is a discount factor of 0.99, how does this affect the results? Fill in the blanks in the following diagram.

As in the first part $\Delta=0.6$
Earns at risk-free rate, $\therefore V-100=0.99(-98\Delta) \rightarrow V=1.788$
$\Pi=V-100\Delta=-58.212$
$S=\left[pS^{+}+(1-p)S^{-}\right]0.99 \Rightarrow 100\div 0.99=103p+(1-p)98 \Rightarrow 0.602$
$\therefore 1-p=0.398$