BSDE's and Feynman-Kac Formula for Lévy Processes with Applications in Finance

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ABSTRACT: In this paper we show the existence and uniqueness of a solution for backward stochastic differential equations driven by a Lévy process with moments of all orders. The results are important both from a mathematical point of view as in finance: An application to Clark-Ocone and Feynman-Kac formulas for Lévy processes is presented. Moreover, the Feynman-Kac formula and the related Partial Differential Integral Equation (PDIE) provide us an analogue of the famous Black-Scholes partial differential equation and is used for the purpose of option pricing in a Lévy market.

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1 Introduction

The first paper concerned with Backward Stochastic Differential Equations (BSDE's) is the paper Bismut (1973), where he introduced a non-linear Ricatti BSDE and showed existence and uniqueness of bounded solutions. Pardoux and Peng (1990) considered general BSDE's and this paper was the starting point for the development of the study of these equations. On the other hand, BSDE's have important applications in the theory of mathematical finance, especially, they play a major role in hedging and non-linear pricing theory for imperfect markets (see El Karoui and Quenez (1997)).

One can consider a BSDE driven by a Brownian motion as a nonlinear generalization of the integral representation theorem for square integrable martingales. Then it is natural to extend these kind of equations to the case of Lévy processes, that is, processes with independent and stationary increments. We recall that a Lévy process consists of three stochastically independent parts: a purely deterministic linear part, a Brownian motion and a pure-jump process. In Situ (1997) BSDE's driven by a Brownian motion and a Poisson point process are studied. Ouknine (1998) considers BSDE's driven by a Poisson random measure. In both papers the main ingredient is the integral representation of square integrable random variables in terms of a Poisson random measure (see Jacod (1979)).

In Nualart and Schoutens (2000) a martingale representation theorem for Lévy processes satisfying some exponential moment condition was proved. The purpose of this paper is to use this martingale representation result to establish the existence and uniqueness of solutions for BSDE's driven by a Lévy process of the kind considered in Nualart and Schoutens (2000). The results are important both from a mathematical point of view as in finance. This is illustratated in the applications. The resulting Clark-Ocone and the Feynman-Kac formulas are fundamental ingredients in the build up of an Malliavin calculus for Lévy process. Moreover, the Feynman-Kac formula and the related Partial Differential Integral Equation (PDIE) also have an important application in finance: they provide us an analogue of the famous Black-Scholes partial differential equation and is used for the purpose of option pricing in a Lévy market.

The paper is organized as follows. Section 2 contains some preliminaries on Lévy processes. Section 3 contains the main result on BSDE's driven by Lévy processes. In Section 4 we have included some applications of BSDE's driven by Lévy processes to the Clark-Ocone, the Feynman-Kac formulas, and option pricing in a Lévy market. Finally, in the appendix one can find detailed proofs of the main results.

2 Preliminaries

Let $X = \{X_t,\ t \geq 0\}$ be a Lévy process defined on a complete probability space $(\Omega, \mathcal{F}, P)$. That is, X is a real-valued process starting from 0 with stationary and independent increments and with càdlàg trajectories. It is known that X_t has a characteristic function of the form

$$E\left(e^{i\theta X_t}\right) = \exp\left[ia\theta t - \frac{1}{2}\sigma^2\theta^2 t + t\int_{\mathbb{R}}\left(e^{i\theta x} - 1 - i\theta x\mathbf{1}_{\{|x|<1\}}\right)\nu(\mathrm{d}x)\right],$$

where $a \in \mathbb{R}$, $\sigma > 0$, and ν is a measure on $\mathbb{R}$ with $\int_{\mathbb{R}}(1 \wedge x^2)\nu(\mathrm{d}x) < \infty$. We will assume that the Lévy measure ν satisfies for some $\lambda > 0$

$$\int_{(-\epsilon,\epsilon)^c} e^{\lambda|x|}\nu(\mathrm{d}x) < \infty,$$

for every $\varepsilon > 0$. This implies that the random variables X_t have moments of all orders. Moreover, it will assure us the existence of the below mentioned Predictable Representation, which we will use in our proofs. We refer to Sato (2000) or Bertoin (1996) for a detailed account on Lévy processes.

For $t \geq 0$, let $\mathcal{F}_t$ denote the σ-algebra generated by the family of random variables $\{X_s, 0 \leq s \leq t\}$ augmented with the P-null sets of $\mathcal{F}$. Fix a time interval $[0,T]$ and set $L^2_T = L^2(\Omega, \mathcal{F}_T, P)$. We will denote by $\mathcal{P}$ the predictable sub-σ-field of $\mathcal{F}_T \otimes \mathcal{B}_{[0,T]}$. First we introduce some notation:

- Let H^2_T denote the space of square integrable and $\mathcal{F}_t-$progressively measurable processes $\phi = \{\phi_t, t \in [0,T]\}$ such that

$$||\phi||^2 = E\left[\int_0^T |\phi_t|^2 \mathrm{d}t\right] < \infty.$$

- M^2_T will denote the subspace of H^2_T formed by predictable processes.

- $H^2_T(l^2)$ and $M^2_T(l^2)$ are the corresponding spaces of l^2-valued processes equipped with the norm

$$||\phi||^2 = E\left[\int_0^T \sum_{i=1}^{\infty} |\phi_t^{(i)}|^2 \mathrm{d}t\right].$$

- Set $\mathcal{H}^2_T = H^2_T \times M^2_T(l^2)$.

Following Nualart and Schoutens (2000) we define for every $i = 1, 2, ...$ the so-called power-jump processes $\{X_t^{(i)}, t \geq 0\}$ and their compensated version $\{Y_t^{(i)} = X_t^{(i)} - E[X_t^{(i)}], t \geq 0\}$, also called the Teugels martingales, as follows

$$\begin{aligned} X_t^{(1)} &= X_t \text{ and } X_t^{(i)} = \sum_{0<s\leq t} (\Delta X_s)^i \text{ for } i = 2, 3, ... \\ Y_t^{(i)} &= X_t^{(i)} - \ E[X_t^{(i)}] = X_t^{(i)} - t\ E[X_1^{(i)}] \text{ for } i \geq 1. \end{aligned}$$

An orthonormalization procedure can be applied to the martingales $Y^{(i)}$ in order to obtain a set of pairwise strongly orthonormal martingales $\{H^{(i)}\}_{i=1}^{\infty}$ such that each $H^{(i)}$ is a linear combination of the $Y^{(j)}$, $j = 1, ..., i$:

$$H^{(i)} = c_{i,i} Y^{(i)} + c_{i,i-1} Y^{(i-1)} + ... + c_{i,1} Y^{(1)}.$$

It was shown in Nualart and Schoutens (2000) that the coefficients $c_{i,k}$ correspond to the orthonormalization of the polynomials $1, x, x^2, ...$ with respect to the measure $\mu(\mathrm{d}x) = x^2 \nu(\mathrm{d}x) + \sigma^2 \delta_0(\mathrm{d}x)$:

$$q_{i-1}(x) = c_{i,i} x^{i-1} + c_{i,i-1} x^{i-2} + ... + c_{i,1}.$$

Set

$$\begin{aligned} p_i(x) &= x q_{i-1}(x) = c_{i,i} x^i + c_{i,i-1} x^{i-1} + ... + c_{i,1} x \\ \tilde{p}_i(x) &= x(q_{i-1}(x) - q_{i-1}(0)) = c_{i,i} x^i + c_{i,i-1} x^{i-1} + ... + c_{i,2} x^2. \end{aligned}$$

Then

$$\begin{aligned} H_t^{(i)} &= \sum_{0<s\leq t} \left(c_{i,i} \left(\Delta X_s \right)^i \ + ... + c_{i,2} \left(\Delta X_s \right)^2 \right) + c_{i,1} X_t \\ &\quad -tE \left[c_{i,i} X_1^{(i)} + ... + c_{i,2} X_1^{(2)} \right] - t c_{i,1} E\left[X_1 \right] \\ &= q_{i-1}(0) X_t + \sum_{0<s\leq t} \tilde{p}_i(\Delta X_s) - tE \left[\sum_{0<s\leq 1} \tilde{p}_i(\Delta X_s) \right] - t q_{i-1}(0) E\left[X_1 \right]. \end{aligned}$$

As a consequence, $\Delta H_t^{(i)} = p_i(\Delta X_t)$ for each $i \geq 1$. In the particular case $i = 1$ we obtain

$$H_t^{(1)} = c_{1,1}(X_t - tE[X_1]),$$

where

$$c_{1,1} = \left[\int_{\mathbb{R}} y^2 \nu(\mathrm{d}y) + \sigma^2 \right]^{-1/2}$$

and

$$E[X_1] = a + \int_{\{|z|\geq 1\}} z\nu(\mathrm{d}z).$$

In the case $\int_{\mathbb{R}} |z| v(\mathrm{d}z) < \infty$, assuming $a = \int_{\{|z|<1\}} z\nu(\mathrm{d}z)$, we obtain $E[X_1] = \int_{\mathbb{R}} z\nu(\mathrm{d}z)$.

The main results in Nualart and Schoutens (2000) is the Predictable Representation Property (PRP): Every square integrable random variable $F \in L_T^2$ has a representation of the form

$$F = E[F] + \sum_{i=1}^{\infty} \int_0^T Z_s^{(i)} \mathrm{d}H_s^{(i)},$$

where Z_t is a predictable process in the space $M_T^2(l^2)$.

Remark: If $\nu = 0$, we are in the classical Brownian case and all non-zero degree polynomials $q_i(x)$ will vanish, giving $H_t^{(i)} = 0$, $i = 2, 3, \ldots$. If μ has only mass in 1, we are in the Poisson case; and also here $H_t^{(i)} = 0$, $i = 2, 3, \ldots$. Both case are degenerate cases in this Lévy framework.

¿From these observations, it is not so hard to see that the PRP property shows that financial markets based on a non-Brownian or non-Poissonian Lévy process, i.e. with a stock price behaviour $S_t = \exp(X_t)$, are so called incomplete, meaning that perfectly replicating or hedging strategies do not exists for all relevant contingent claims.

3 BSDE for Lévy Processes

Taking into account the results and notation presented in the previous section, it seems natural to consider the BSDE

$$-\mathrm{d}Y_t = f(t, Y_{t-}, Z_t)\mathrm{d}t - \sum_{i=1}^{\infty} Z_t^{(i)} \mathrm{d}H_t^{(i)}, \qquad Y_T = \xi, \tag{1}$$

where:

- $H_t^{(i)}$ is the orthonormalized Teugels martingale of order i associated with the Lévy process X.
- $f : \Omega \times [0, T] \times \mathbb{R} \times M_T^2(l^2) \to \mathbb{R}$ is a measurable function such that $f(., 0, 0) \in H_T^2$.

- f is uniformly Lipschitz in the first two components, i.e., there exists $C > 0$ such that $\mathrm{d}t \otimes \mathrm{d}P$ a.s., for all (y_1, z_1) and (y_2, z_2) in $\mathbb{R} \times l^2$

$$|f(t, y_1, z_1) - f(t, y_2, z_2)| \leq C\left(|y_1 - y_2| + \|z_1 - z_2\|_{l^2}\right).$$

- $\xi \in L_T^2$.

If (f, ξ) satisfies the above assumptions, the pair (f, ξ) is said to be **standard data** for the BSDE. A solution of the BSDE is a pair of processes, $\{(Y_t, Z_t), 0 \leq t \leq T\} \in H_T^2 \times M_T^2(l^2)$ such that the following relation holds for all $t \in [0, T]$:

$$Y_t = \xi + \int_t^T f(s, Y_{s-}, Z_s)\mathrm{d}s - \sum_{i=1}^{\infty} \int_t^T Z_s^{(i)} \mathrm{d}H_s^{(i)}. \tag{2}$$

Note that the progressive measurability of $\{(Y_t, Z_t), 0 \leq t \leq T\}$ implies that (Y_0, Z_0) is deterministic.

A first key-result concerns the existence and uniqueness of solution of BSDE:

Theorem 1 *Given standard data (f, ξ), there exists a unique solution (Y, Z) which solves the BSDE (2).*

The proof can be found in the Appendix, as the proof of the continuous dependency of the solution on the final data ξ and the function f.

Theorem 2 *Given standard data (f, ξ) and (f', ξ'), let (Y, Z) and $(Y'Z')$ be the unique adapted solutions of the BSDE (2) corresponding to (f, ξ) and (f', ξ'). Then*

$$\begin{aligned} & E\left[\int_0^T \left(|Y_{s-} - Y'_{s-}|^2 + \sum_{i=1}^{\infty} |Z_s^{(i)} - Z_s'^{(i)}|^2\right) ds\right] \\ \leq \; & C\left(E[|\xi - \xi'|^2] + E\left[\int_0^T |f(s, Y_{s-}, Z_s) - f'(s, Y_{s-}, Z_s|^2 ds\right]\right). \end{aligned}$$

4 Applications

Suppose our Lévy process X_t has no Brownian part, i.e. $X_t = at + L_t$, where L_t is pure jump process with Lévy measure $\nu(\mathrm{d}x)$.

4.1 Clark-Ocone Formula and Feynman-Kac Formula

Let us consider the simple case of a BSDE where $f = 0$, and the terminal random variable ξ is a function of X_T, that is,

$$-\mathrm{d}Y_t = -\sum_{i=1}^{\infty} Z_t^{(i)} \mathrm{d}H_t^{(i)}; \qquad\qquad Y_T = g(X_T)$$

or equivalently

$$Y_t = g(X_T) - \sum_{i=1}^{\infty} \int_t^T Z_s^{(i)} \mathrm{d}H_s^{(i)}, \tag{3}$$

where $E(g(X_T)^2) < \infty$. Let $\theta = \theta(t,x)$ be the solution of the following PDIE (Partial Differential Integral Equation) with terminal value g:

$$\frac{\partial \theta}{\partial t}(t,x) + \int_{\mathbb{R}} (\theta(t,x+y) - \theta(t,x) - \frac{\partial \theta}{\partial x}(t,x)y)\ \nu(\mathrm{d}y) + a' \frac{\partial \theta}{\partial x}(t,x) = 0,$$
$$\theta(T,x) = g(x), \tag{4}$$

where $a' = a + \int_{\{|y| \geq 1\}} y\nu(\mathrm{d}y)$. Set

$$\theta^{(1)}(t,x,y) = \theta(t,x+y) - \theta(t,x) - \frac{\partial \theta}{\partial x}(t,x)y. \tag{5}$$

The following result is a version of the Clark-Ocone formula for functions of a Lévy process. Again the proof can be found in the Appendix.

Proposition 3 *Suppose that θ is a $C^{1,2}$ function such that $\frac{\partial \theta}{\partial x}$ and $\frac{\partial^2 \theta}{\partial x^2}$ are bounded by a polynomial function of x, uniformly in t, then the unique adapted solution of (3) is given by*

$$\begin{aligned}
Y_t &= \theta(t, X_t) \\
Z_t^{(i)} &= \int_{\mathbb{R}} \theta^{(1)}(t, X_{t-}, y) p_i(y) \nu(dy) \quad \text{for } i \geq 2, \\
Z_t^{(1)} &= \int_{\mathbb{R}} \theta^{(1)}(t, X_{t-}, y) p_1(y) \nu(dy) + \frac{\partial \theta}{\partial x}(t, X_{t-}) (\int_{\mathbb{R}} y^2 \nu(dy))^{1/2},
\end{aligned}$$

where $\theta = \theta(t,x)$ is the solution of the PDIE (4) and $\theta^{(1)}(t,x,y)$ is given by (5).

Now by taking expectations we derive that the solution $\theta(t,x)$ to our PDIE (4) equation has the stochastic representation

$$\theta(t,x) = E[g(X_T) | X_t = x].$$

This is an extension of the classical Feynman-Kac Formula.

If $\int_{\mathbb{R}} |y|\nu(\mathrm{d}y) < \infty$, and we take $a = \int_{\{|y|<1\}} y\nu(\mathrm{d}y)$, then the equation (4) reduces to

$$\frac{\partial \theta}{\partial t}(t,x) + \int_{\mathbb{R}} (\theta(t,x+y) - \theta(t,x))\nu(\mathrm{d}y) = 0,$$
$$\theta(T,x) = g(x),$$

and taking into account that $p_1(y) = y(\int_{\mathbb{R}} y^2\nu(\mathrm{d}y))^{-1/2}$ in Proposition 3 we have

$$Z_t^{(1)} = \int_{\mathbb{R}} [\theta(t, X_{t-} + y) - \theta(t, X_{t-})\ ]\, p_1(y)\nu(\mathrm{d}y).$$

Example: Consider the very special case where we have a compensated Poisson process $X_t = N_t - \lambda t$. Then

$$H_t^{(1)} = \frac{1}{\sqrt{\lambda}}(N_t - \lambda t) = \frac{X_t}{\sqrt{\lambda}} \text{ and } H_t^{(i)} = 0 \text{ for } i = 2, 3, \ldots.$$

Note that $p_1(x) = \frac{x}{\sqrt{\lambda}}$ and $p_i(x) = 0$, $i = 2, 3, \ldots.$ Moreover the PDIE (4) reduces to

$$(\theta(t,x+1) - \theta(t,x)) - \lambda\frac{\partial \theta}{\partial x}(t,x) + \frac{\partial \theta}{\partial t}(t,x) = 0,$$
$$\theta(T,x) = g(x).$$

The Clark-Ocone Formula is now given by

$$g(X_T) = E[g(X_T)] + \int_t^T \theta(s, X_{s-} + 1) - \theta(s, X_{s-})\mathrm{d}X_s.$$

4.2 Nonlinear Clark-Haussman-Ocone Formula and Feynman-Kac Formula

Let us consider the BSDE

$$-\mathrm{d}Y_t = f(t, Y_t, Z_t)\mathrm{d}t - \sum_{i=1}^{\infty} Z_t^{(i)}\mathrm{d}H_t^{(i)}; \qquad Y_T = g(X_T) \tag{6}$$

or equivalently

$$Y_t = g(X_T) + \int_t^T f(s, Y_{s-}, Z_s)\mathrm{d}s - \sum_{i=1}^{\infty}\int_t^T Z_s^{(i)}\mathrm{d}H_s^{(i)}.$$

Suppose that $\theta = \theta(t,x)$ satisfies the following PDIE:

$$\begin{gathered}\frac{\partial \theta}{\partial t}(t,x) + \int_{\mathbb{R}} \theta^{(1)}(t,x,y)\nu(\mathrm{d}y) + a' \frac{\partial \theta}{\partial x}(t,x) \\ +f\left(t, \theta(t,x)\ , \left\{\theta^{(i)}(t,x)\right\}_{i=1}^{\infty}\right) = 0, \\ \theta(T,x) = g(x).\end{gathered} \tag{7}$$

where as in the previous section, we define $\theta^{(1)}(t,x,y)$ by (5),

$$\theta^{(1)}(t,x) = \int_{\mathbb{R}} \theta^{(1)}(t,x,y)p_1(y)\nu(\mathrm{d}y) + \frac{\partial \theta}{\partial x}(t,x)(\int_{\mathbb{R}} y^2 \nu(\mathrm{d}y))^{1/2}, \tag{8}$$

and for $i \geq 2$

$$\theta^{(i)}(t,x) = \int_{\mathbb{R}} \theta^{(1)}(t,x,y)p_i(y)\nu(\mathrm{d}y). \tag{9}$$

Proposition 4 *Suppose that θ is a $C^{1,2}$ function such that $\frac{\partial \theta}{\partial x}$ and $\frac{\partial^2 \theta}{\partial x^2}$ are bounded by a polynomial function of x, uniformly in t. Then the (unique) adapted solution of (6) is given by*

$$\begin{aligned} Y_t &= \theta(t, X_t) \\ Z_t^{(i)} &= \int_{\mathbb{R}} \theta^{(1)}(t, X_{t-}, y)p_i(y)\nu(dy) \quad \text{for } i \geq 2, \\ Z_t^{(1)} &= \int_{\mathbb{R}} \theta^{(1)}(t, X_{t-}, y)p_1(y)\nu(dy) + \frac{\partial \theta}{\partial x}(t, X_{t-})(\int_{\mathbb{R}} y^2 \nu(dy))^{1/2}. \end{aligned}$$

where $\theta = \theta(t,x)$ is the solution of the PDIE (7) and $\theta^{(1)}(t,x,y)$ is given by (5).

Notice that taking expectations we get

$$\begin{gathered}\theta(t,x) = E[g(X_T)|X_t = x] + \\ E\left[\int_t^T f\left(s, \theta(s, X_{s-})\ , \left\{\ \theta^{(i)}(s, X_{s-})\right\}_{i=1}^{\infty}\right) \mathrm{d}s | X_t = x\right].\end{gathered}$$

Example: Consider again the very special case where we have a Poisson process N_t with $E[N_t] = \lambda t$. Set $X_t = N_t - \lambda t$. Then the PDIE (7) reduces to

$$\begin{gathered}(\theta(t,x+1) - \theta(t,x)) - \lambda \frac{\partial \theta}{\partial x}(t,x) + \frac{\partial \theta}{\partial t}(t,x) + \\ f\left(t, \theta(t,x)\ , \theta(t,x+1) - \theta(t,x)\right) = 0, \\ \theta(T,x) = g(x).\end{gathered} \tag{10}$$

And we derive the nonlinear Feynman-Kac Formula:

$$
\begin{aligned}
\theta(t,x) &= E[g(X_T)|X_t = x] \\
&\quad + E\left[\int_t^T f\left(s, \theta(s, X_{s-}), \theta(s, X_{s-}+1) - \theta(s, X_{s-})\right) \mathrm{d}s | X_t = x\right].
\end{aligned}
$$

4.3 Option Pricing

Assume a market consisting of one riskless asset (the bond) with price process given by $B_t = e^{rt}$, where r is compound interest rate, and one risky asset (the stock), with price process:

$$S_t = S_0 \exp(X_t),$$

where X_t is a Lévy process. Denote by $P(\mathrm{d}x)$ the probability measure of X_1.

In the last two decades several particular choices for non-Brownian Lévy processes where proposed. Madan and Seneta [16] have proposed a Lévy process with variance gamma distributed increments. We mention also the Hyperbolic Model proposed by Eberlein and Keller (1995). In the same year Barndorff-Nielsen (1995) proposed the normal inverse Gaussian Lévy process. Recently the CMGY model was introduced in Carr et al. (2000). Finally, we mention the Meixner model (see Grigelionis (1999) and Schoutens (2001)). All models give a much better fit to the data and lead to an improvement with respect to the Black-Scholes model.

We recall the density f, the cumulant generating function K, the drift a, and the Lévy measure ν, for the Meixner Process $\{M_t, t \geq 0\}$, for which we will illustrated the method.

$$
\begin{aligned}
\frac{P_{\mathrm{Meix}}(\mathrm{d}x)}{\mathrm{d}x} = f_{\mathrm{Meixner}}(x;\alpha,\beta,\delta,\mu) &= \frac{(2\cos\frac{\beta}{2})^{2\delta} e^{\frac{\beta(x-\mu)}{\alpha}} \left(\left|\Gamma(\delta + \frac{i(x-\mu)}{\alpha})\right|\right)^2}{\pi\alpha\Gamma(2\delta)}, \\
K_{\mathrm{Meixner}}(\theta;\alpha,\beta,\delta,\mu) &= \mu\,\theta + 2\delta\left(\log\cos\frac{\beta}{2} - \log\cos\frac{\alpha\,\theta+\beta}{2}\right), \\
a_{\mathrm{Meixner}}(\alpha,\beta,\delta,\mu) &= \mu + \alpha\,\delta\,\tan\frac{\beta}{2} - 2\,\delta\int_1^\infty \frac{\sinh\frac{\beta\,x}{\alpha}}{\sinh\frac{\pi\,x}{\alpha}}\,\mathrm{d}x, \\
\nu_{\mathrm{Meixner}}(\mathrm{d}x;\alpha,\beta,\delta,\mu) &= \frac{\delta\,e^{\frac{\beta\,x}{\alpha}}}{x\sinh\frac{\pi\,x}{\alpha}}\mathrm{d}x,
\end{aligned}
$$

where $\alpha > 0$, $-\pi < \beta < \pi$, $\mu \in \mathbb{R}$, and $\delta > 0$.

¿From the form of the cumulant generating function one easily deduces that the density at any time t can be calculated by multiplying the parameters δ and μ by t for both cases.

Given our market model, let $G(S_T) = F(X_T)$ denote the payoff of the derivative at its time of expiry T. In case of the European call with strike price K, we have $G(S_T) = (S_T - K)^+$ or equivalently $F(X_T) = (S_0 \exp(X_T) - K)^+$. According to the fundamental theorem of asset pricing (see Delbaen and Schachermayer 1994) the arbitrage free price V_t of the derivative at time $t \in [0, T]$ is given by

$$V_t = E_Q[e^{-r(T-t)} G(S_T) | \mathcal{F}_t],$$

where the expectation is taken with respect to an equivalent martingale measure $Q(\mathrm{d}x)$ and $\mathcal{F} = \{\mathcal{F}_t, 0 \leq t \leq T\}$ is the natural filtration of $X = \{X_t, 0 \leq t \leq T\}$. An equivalent martingale measure is a probability measure which is equivalent (it has the same null-sets) to the given (historical) probability measure and under which the discounted process $\{e^{-rt} S_t\}$ is a martingale. Unfortunately for most models, in particular the more realistic ones, the class of equivalent measures is rather large and often covers the full no-arbitrage interval. In this perspective the Black-Scholes model, where there is an unique equivalent martingale measure, is very exceptional. Models with more than one equivalent measures are called incomplete.

Our Lévy model is such an incomplete model. Following Gerber and Shiu (1994) and Gerber and Shiu (1996), we can, by using the so-called Esscher transform, easily find at least one equivalent martingale measure, which we will use in the sequel for the valuation of derivative securities. The choice of the Esscher measure may be justified by a utility maximizing argument (see Gerber and Shiu (1996)).

Let K be the cumulant generating function of X under the measure $P(\mathrm{d}x)$, and let θ be the solution of $K(\theta + 1) - K(\theta) = r$. Then, we define the risk-neutral measure $Q(\mathrm{d}x)$ as the probability measure with the Radon-Nykodym derivative with respect to $P(\mathrm{d}x)$ given by $Q(\mathrm{d}x)/P(\mathrm{d}x) = \exp(\theta x - K(\theta))$.

For our Meixner-example, the parameters for the Esscher transforms are easily found; explicit values for θ can be found in Schoutens (2001) or Grigelionis (1999). In the Meixner case one only has to shift β to $\beta + \alpha\theta$ to get the density under the measure $Q(\mathrm{d}x)$. This means that under the risk-neutral measure $Q(\mathrm{d}x)$ our process M_t is again a Meixner process. In all such cases where the underlying process is a Lévy process in the risk-neutral world and the price $V_t = V(t, M_t)$ at time t of a given derivative satisfies some regularity conditions (i.e. $V(t,x) \in C^{(1,2)}$), the function $V(t,x)$ can also be obtained by solving a partial differential integral equation (PDIE) with a boundary condition:

$$\begin{aligned} a\frac{\partial}{\partial x}V(t,x) + \frac{\partial}{\partial t}V(t,x) + \int_{-\infty}^{+\infty}\left(V(t,x+y) - V(t,x) - y\frac{\partial}{\partial x}V(t,x)\right)\nu^Q(\mathrm{d}y) \\ = rV(t,x) \\ V(T,x) = F(x) \end{aligned} \quad ,$$

where $\nu^Q(\mathrm{d}y)$ is the Lévy measure of the risk-neutral distribution $Q(\mathrm{d}x)$. This PDIE is the analogue of the famous Black-Scholes partial differential equation and follows from the above Feynman-Kac formula for Lévy Processes. In the Meixner case, it is clear that:

$$\nu^Q(\mathrm{d}x) = d\frac{\exp((a\theta + b)x/a)}{x\sinh(\pi x/a)}\mathrm{d}x.$$

and $a = a_{\mathrm{Meixner}}(\alpha, \beta + \alpha\theta, \delta, \mu)$.

Appendix: Proofs of the Results

Proof of Theorem 1:

We define a mapping Φ from $\mathcal{H}_T^2$ into itself such that $(Y,Z) \in \mathcal{H}_T^2$ is a solution of the BSDE if and only if it is a fixed point of Φ. Given $(U,V) \in \mathcal{H}_T^2$, we define $(Y,Z) = \Phi(U,V)$ as follows:

$$Y_t = E\left[\xi + \int_t^T f(s, U_{s-}, V_s)ds|\mathcal{F}_t\right],\ 0 \le t \le T,$$

and $\{Z_t, 0 \le t \le T\}$ is given by the martingale representation of Nualart and Schoutens (2000) applied to the square integrable random variable

$$\xi + \int_0^T f(s, U_{s-}, V_s)ds,$$

i.e.,

$$\begin{aligned} \xi + \int_0^T f(s, U_{s-}, V_s)ds &= E\left[\xi + \int_0^T f(s, U_{s-}, V_s)ds\right] \\ &\quad + \sum_{i=1}^{\infty}\int_0^T Z_s^{(i)}dH_s^{(i)}. \end{aligned}$$

Taking the conditional expectation with respect to $\mathcal{F}_t$ in the last identity yields

$$Y_t + \int_0^t f(s, U_{s-}, V_s) ds = Y_0 + \sum_{i=1}^{\infty} \int_0^t Z_s^{(i)} dH_s^{(i)},$$

from which we deduce that

$$Y_t = \xi + \int_t^T f(s, U_{s-}, V_s) ds - \sum_{i=1}^{\infty} \int_t^T Z_s^{(i)} dH_s^{(i)}$$

and we have shown that $(Y, Z) \in \mathcal{H}_T^2$ solves our BSDE if and only if it is a fixed point of Φ.

Next we prove that Φ is a strict contraction on $\mathcal{H}_T^2$ equipped with the norm

$$\|(Y, Z)\|_\beta = \left(E \left[\int_0^T e^{\beta s} \left(|Y_{s-}|^2 + \sum_{i=1}^{\infty} |Z_s^{(i)}|^2 \right) ds \right] \right)^{1/2},$$

for a suitable $\beta > 0$. Let (U, V) and (U', V') be two elements of $\mathcal{H}_T^2$ and set $\Phi(U, V) = (Y, Z)$ and $\Phi(U', V') = (Y', Z')$. Denote $(\overline{U}, \overline{V}) = (U - U', V - V')$ and $(\overline{Y}, \overline{Z}) = (Y - Y', Z - Z')$.

Applying Itô's formula from $s = t$ to $s = T$, to $e^{\beta s} (Y_s - Y_s')^2$, it follows that

$$\begin{aligned} e^{\beta t} \left(Y_t - Y_t' \right)^2 &= -\beta \int_t^T e^{\beta s} \left(Y_{s-} - Y_{s-}' \right)^2 ds \\ &\quad -2 \int_t^T e^{\beta s} \left(Y_{s-} - Y_{s-}' \right) d(Y_s - Y_s') \\ &\quad - \int_t^T e^{\beta s} d[Y - Y', Y - Y']_s. \end{aligned} \tag{11}$$

We have

$$\begin{aligned} -d(Y_t - Y_t') &= (f(t, U_{t-}, V_t) - f(t, U_{t-}', V_t'))dt \\ &\quad - \sum_{i=1}^{\infty} \left(Z_t^{(i)} - Z_t'^{(i)} \right) dH_t^{(i)}, \\ d \left[Y - Y', Y - Y' \right]_t &= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left(Z_t^{(i)} - Z_t'^{(i)} \right) \left(Z_t^{(j)} - Z_t'^{(j)} \right) d[H^{(i)}, H^{(j)}]_t, \\ \left\langle H^{(i)}, H^{(j)} \right\rangle_t &= \delta_{ij} t. \end{aligned}$$

Hence, taking expectations in (11), we have

$$E\left[e^{\beta t}\left(Y_t - Y'_t\right)^2\right] + \sum_{i=1}^{\infty} E\left[\int_t^T e^{\beta s}\left(Z_s^{(i)} - Z_s'^{(i)}\right)^2 ds\right]$$
$$= -\beta E\left[\int_t^T e^{\beta s}\left(Y_{s-} - Y'_{s-}\right)^2 ds\right]$$
$$+2E\left[\int_t^T e^{\beta s}\left(Y_{s-} - Y'_{s-}\right)\left(f(s, U_{s-}, V_s) - f(s, U'_{s-}, V'_s)\right) ds\right].$$

Using the fact that f is Lipschitz with constant C yields

$$E\left[e^{\beta t}\left(Y_t - Y'_t\right)^2\right] + \sum_{i=1}^{\infty} E\left[\int_t^T e^{\beta s}(\left(Z_s^{(i)} - Z_s'^{(i)}\right)^2 ds\right]$$
$$\leq -\beta E\left[\int_t^T e^{\beta s}(Y_{s-} - Y'_{s-})^2 ds\right]$$
$$+2CE\left[\int_t^T e^{\beta s}\left|Y_{s-} - Y'_{s-}\right|\left(|U_{s-} - U'_{s-}| + \sqrt{\sum_{i=1}^{\infty}|V_s^{(i)} - V_s'^{(i)}|^2}\right) ds\right].$$

If we now use the fact that for every $c > 0$ and $a, b \in \mathbb{R}$ we have that $2ab \leq ca^2 + \frac{1}{c}b^2$ and $(a+b)^2 \leq 2a^2 + 2b^2$, we obtain

$$E\left[e^{\beta t}\left|Y_t - Y'_t\right|^2\right] + \sum_{i=1}^{\infty} E\left[\int_t^T e^{\beta s}\left(Z_s^{(i)} - Z_s'^{(i)}\right)^2 ds\right]$$
$$\leq (4C^2 - \beta)E\left[\int_t^T e^{\beta s}\left|Y_s - Y'_s\right|^2 ds\right]$$
$$+\frac{1}{2}E\left[\int_t^T e^{\beta s}\left(|U_{s-} - U'_{s-}|^2 + \sum_{i=1}^{\infty}|V_s^{(i)} - V_s'^{(i)}|^2\right) ds\right].$$

Taking now $\beta = 4C^2 + 1$, and noting that $e^{\beta t}E[(Y_t - Y'_t)^2] \geq 0$, we finally derive

$$E\left[\int_t^T e^{\beta s}\left|Y_s - Y'_s\right|^2 ds\right] + \sum_{i=1}^{\infty} E\left[\int_t^T e^{\beta s}(Z_s^{(i)} - Z_s'^{(i)})^2 ds\right]$$
$$\leq \frac{1}{2}E\left[\int_t^T e^{\beta s}\left(|U_{s-} - U'_{s-}|^2 + \sum_{i=1}^{\infty}|V_s^{(i)} - V_s'^{(i)}|^2\right) ds\right],$$

that is,

$$\|(Y,Z)\|_\beta^2 \leq \frac{1}{2}\|(U,V)\|_\beta^2,$$

from which it follows that Φ is a strict contraction on $\mathcal{H}_T^2$ equipped with the norm $\|\cdot\|_\beta$ if $\beta = 4C^2+1$. Then Φ has a unique fixed point and the theorem is proved. $\diamond$

Proof of Theorem 2:

Applying Itô's formula from $s=t$ to $s=T$, to $(Y_s - Y'_s)^2$, it follows that

$$\begin{aligned}
\left(Y_T - Y'_T\right)^2 - \left(Y_t - Y'_t\right)^2 &= 2\int_t^T \left(Y_{s-} - Y'_{s-}\right) d(Y_s - Y'_s) \\
&\quad + \int_t^T d[Y-Y', Y-Y']_s.
\end{aligned}$$

Taking expectations and using the relations

$$\begin{aligned}
-d(Y_t - Y'_t) &= f(t, Y_{t-}, Z_t) - f'(t, Y'_{t-}, Z'_t)dt \\
&\quad - \sum_{i=1}^{\infty}\left(Z_t^{(i)} - Z_t^{\prime(i)}\right) dH_t^{(i)} \\
d[Y-Y', Y-Y']_t &= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left(Z_t^{(i)} - Z_t^{\prime(i)}\right)\left(Z_t^{(j)} - Z_t^{\prime(j)}\right) d[H^{(i)}, H^{(j)}]_t, \\
\left\langle H^{(i)}, H^{(j)}\right\rangle_t &= \delta_{ij} t,
\end{aligned}$$

we have

$$\begin{aligned}
& E[\left(Y_t - Y'_t\right)^2] + \sum_{i=1}^{\infty} E\left[\int_t^T \left|Z_s^{(i)} - Z_s^{\prime(i)}\right|^2 ds\right] \\
= \;& E[\left(\xi - \xi'\right)^2] \\
& + 2E\left[\int_t^T \left(Y_{s-} - Y'_{s-}\right)\left(f(s, Y_{s-}, Z_s) - f'(s, Y'_{s-}, Z'_s)\right) ds\right].
\end{aligned}$$

Using the Lipschitz property of f', and computations similar to those of the proof of Theorem 1 we obtain

$$
\begin{aligned}
& E[|Y_t - Y'_t|^2] + \frac{1}{2} E\left[\int_t^T \sum_{i=1}^{\infty} |Z_s^{(i)} - Z_s'^{(i)}|^2 ds \right] \\
\leq \quad & E[|\xi - \xi'|^2] + (1 + 2C' + 2C'^2) E\left[\int_t^T |Y_{s-} - Y'_{s-}|^2 ds \right] \\
& + E\left[\int_t^T |f(s, Y_{s-}, Z_s) - f'(s, Y_{s-}, Z_s|^2 ds \right].
\end{aligned}
$$

Then by Gronwall's inequality the result follows. $\diamond$

Lemma 5 *Let $h : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ be a random function measurable with respect to $\mathcal{P} \otimes \mathcal{B}_{\mathbb{R}}$ such that*

$$
|h(s, y)| \leq a_s(y^2 \wedge |y|) \quad a.s., \tag{12}
$$

where $\{a_s, 0 \leq s \leq T\}$ is a nonnegative predictable process such that $E[\int_0^T a_s^2 ds] < \infty$. Then for each $t \in [0,T]$ we have

$$
\sum_{t < s \leq T} h(s, \Delta X_s) = \sum_{i=1}^{\infty} \int_t^T \langle h(s, \cdot), p_i \rangle_{L^2(\nu)} \, dH_s^{(i)} + \int_t^T \int_{\mathbb{R}} h(s, y) \nu(dy) ds.
$$

Proof of Lemma 5: Because (12) implies that $E[\int_0^t \int_{\mathbb{R}} |h(s,y)|^2 \nu(dy) ds] < \infty$, we have that

$$
M_t = \sum_{0 < s \leq t} h(s, \Delta X_s) - \int_0^t \int_{\mathbb{R}} h(s, y) \nu(dy) ds.
$$

is a square integrable martingale. By the Predictable Representation Theorem, there exists a process ϕ in the space $M_T^2(l^2)$ such that

$$
M_t = \sum_{i=1}^{\infty} \int_0^t \phi_s^{(i)} dH_s^{(i)}.
$$

Taking into account that $\langle H^{(i)}, H^{(j)} \rangle_t = t\delta_{ji}$, we have

$$
\left\langle M, H^{(i)} \right\rangle_t = \int_0^t \phi_s^{(i)} ds. \tag{13}
$$

On the other hand, using that $\Delta M_s \Delta H^{(i)}_s = h(s, \Delta X_s) p_i(\Delta X_s)$ we obtain

$$\left\langle M, H^{(i)} \right\rangle_t = \int_0^t \int_{\mathbb{R}} h(s,y) p_i(y) \nu(dy) ds. \tag{14}$$

Consequently, (13) and (14) imply

$$\phi^{(i)}_s = \int_{\mathbb{R}} h(s,y) p_i(y) \nu(dy),$$

and the result follows. $\diamond$

Proof of Proposition 3:

Under the hypotheses of Proposition 3 the function $\theta^{(1)}(t,x,y)$ given by 5) satisfies the hypotheses in Lemma 5 imposed on h due to the mean value theorem, when we take $x = X_{t-}$.

Apply Itô's lemma to $\theta(s, X_s)$ from $s = t$ to $s = T$:

$$\begin{aligned} \theta(T, X_T) - \theta(t, X_t) &= \int_t^T \frac{\partial \theta}{\partial t}(s, X_{s-}) ds + \int_t^T \frac{\partial \theta}{\partial x}(s, X_{s-}) dX_s \qquad (15) \\ &\quad + \sum_{t<s\leq T} \left[\theta(s, X_s) - \theta(s, X_{s-}) - \frac{\partial \theta}{\partial x}(s, X_{s-}) \Delta X_s \right]. \end{aligned}$$

If we apply Lemma 5 to $h(s,y) = \theta(s, X_{s-} + y) - \theta(s, X_{s-}) - \frac{\partial \theta}{\partial x}(s, X_{s-}) y$, we obtain

$$\begin{aligned} &\sum_{t<s\leq T} \left[\theta(s, X_s) - \theta(s, X_{s-}) - \frac{\partial \theta}{\partial x}(s, X_{s-}) \Delta X_s \right] \\ &= \sum_{i=1}^{\infty} \int_t^T \left(\int_{\mathbb{R}} \theta^{(1)}(s, X_{s-}, y) p_i(y) \nu(dy) \right) dH^{(i)}_s \\ &\quad + \int_t^T \int_{\mathbb{R}} \theta^{(1)}(s, X_{s-}, y) \ \nu(dy) ds. \qquad (16) \end{aligned}$$

Hence, substituting (16) into (15) yields

$$
\begin{aligned}
& g(X_T) - \theta(t, X_t) \\
= \quad & \int_t^T \frac{\partial \theta}{\partial t}(s, X_{s-}) ds + \int_t^T \frac{\partial \theta}{\partial x}(s, X_{s-}) dX_s \\
& + \sum_{i=1}^{\infty} \int_t^T \left(\int_{\mathbb{R}} \theta^{(1)}(s, X_{s-}, y) p_i(y) \nu(dy) \right) dH_s^{(i)} \\
& + \int_t^T \int_{\mathbb{R}} \theta^{(1)}(s, X_{s-}, y) \ \nu(dy) ds.
\end{aligned} \tag{17}
$$

Notice that

$$
X_t = Y_t^{(1)} + tE(X_1) = (\int_{\mathbb{R}} y^2 \nu(dy))^{1/2} H_t^{(1)} + tE(X_1),
$$

and

$$
E(X_1) = a + \int_{\{|y| \geq 1\}} y \nu(dy).
$$

We also have $Y_0 = E[Y_0] = E[g(\mathrm{X}_T)]$ so we can rewrite (17) as

$$
\begin{aligned}
g(X_T) \quad = \quad & E[g(X_T)] + \int_t^T \frac{\partial \theta}{\partial x}(s, X_{s-}) (\int_{\mathbb{R}} y^2 \nu(dy))^{1/2} dH_s^{(1)} \\
& + \sum_{i=1}^{\infty} \int_t^T \left(\int_{\mathbb{R}} \theta^{(1)}(s, X_{s-}, y) p_i(y) \nu(dy) \right) dH_s^{(i)},
\end{aligned}
$$

which completes the proof of the Proposition. $\diamond$

Proof of Proposition 4:

Apply Itô's lemma to $\theta(s, X_s)$ from $s = t$ to $s = T$. By using Lemma 5, we obtain the equality (17). Now, using (7) we get

$$
\begin{aligned}
g(X_T) - \theta(t, X_t) = - \int_t^T f\left(s, \theta(s, X_{s-}) \ , \left\{ \ \theta^{(k)}(s, X_{s-}) \right\}_{k=1}^{\infty} \right) ds \\
+ \int_t^T \left[\int_{\mathbb{R}} \theta^{(1)}(s, X_{s-}, y) p_1(y) \nu(dy) + \frac{\partial \theta}{\partial x}(s, X_{s-}) (\int_{\mathbb{R}} y^2 \nu(dy))^{1/2} \right] dH_s^{(1)} \\
+ \sum_{i=2}^{\infty} \int_t^T \left[\int_{\mathbb{R}} \theta^{(1)}(s, X_{s-}, y) p_i(y) \nu(dy) \right] dH_s^{(i)},
\end{aligned}
$$

which completes the proof of the Proposition. $\diamond$

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