# An Introduction to Modern Portfolio Theory

*CQF Lecture 2.1*

# In this lecture, we will see…

- The core concepts of portfolio management and Modern Portfolio Theory:
  - *Risky and risk-free assets;*
  - *Mean-variance analysis;*
  - *Optimal portfolio;*
  - *Diversification;*
  - *Opportunity set and efficient frontier;*
  - *Tangency and market portfolio;*
  - *Sharpe ratio and market price of risk;*
  - *The linear model and the CAPM.*

- The mathematics of optimization, required to solve portfolio selection problem, are treated the companion lecture: "Fundamentals of Optimization and Application to Portfolio Selection."

# Historical note

- Modern Portfolio Theory (MPT) was pioneered by Markowitz in 1952.
- Although “Don’t put all your eggs in the same basket” was a popular say in the investment world even before Markowitz, portfolios tended to be constructed as collections of individual securities selected for their return potential and with little regards for their risks or interactions.
- Markowitz showed that risk and return are equally important.
  - *To produce more returns it is necessary to take more risk (the idea of “**risk-return trade-off**”)*
  - *The only sure way to reduce risk without sacrificing too much return is through **diversification** across enough securities.*
  - *In Markowitz’ framework, diversification benefits depend on the correlation of securities returns.*

*[1] The other breakthrough of the 1950s was the Modigliani-Miller Theorem published in 1958*

# Historical note

- His student William Sharpe then proposed a linear factor model as well as an economic “equilibrium” model called the Capital Asset Pricing Model (CAPM), establishing a clear connection between securities pricing and portfolio selection;
- Other early contributors to the development of MPT include Jack Treynor, who developed the CAPM before Sharpe but never published it, as well as John Lintner and Jan Mossin.
- The development of MPT marked not only the dawn of financial economics[1] but also of quantitative finance as fields of study.
- Sharpe and Markowitz were awarded the Nobel Prize of Economics in 1990 for their contribution to the theory of financial economics.

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# The Setting

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- We are in an economy where $N \geq 2$ assets are traded.
- We start with a wealth of £$W$.
- Our objective is to make the “best” investment of our wealth for a period of $T$ years.
- To achieve this objective, we will constitute a portfolio by buying (or shorting) some or all of the $N$ possible assets.
- We will not revisit our decision up until the end of the period (a one-period or “buy and hold” investment model).

# Assets? Which assets?

- The definition of assets used here is very wide, encompassing all tradable assets on Earth, including:

  – ***Financial assets***: *equity shares, bonds, currencies…*
  – ***Real assets***: *commodities, real estate, collectibles (artwork, fine wine…), manufacturing plants, consumer goods…*
  – ***Intangible assets***: *labour income.*

- Of course, in practice, portfolio managers do not tend to use this definition since they are often limited to a single asset class and country (U.K. equity, U.S. bonds…).

# The core assumption of the MPT

- Our first, and main, assumption is that all the risky assets are fully characterized by:
  - *Their expected return (denoted by $\mu_i$, for Asset i, i = 1,…,N);*
  - *The standard deviation of their returns (denoted by $\sigma_i$, for Asset i, i = 1,…,N);*
  - *The correlation of their return with the return of any other asset (the return correlation of Assets I and j is denoted by $\rho_{ij}$, for i,j = 1,…,N).*
- Note that this assumption is satisfied as long as the distribution of asset returns is Elliptical
  - *Elliptical distribution are an important family of probability distributions;*
  - *Pre-eminent members include the Normal and t distributions.*

# MPT as a Mean-Variance optimization problem

- For Markowitz, the objective of any investor is to achieve either:
  - *The highest return for a given risk budget;*
  - *The lowest level of risk for a given return objective.*

- Under the assumptions made in the previous slide, we could say

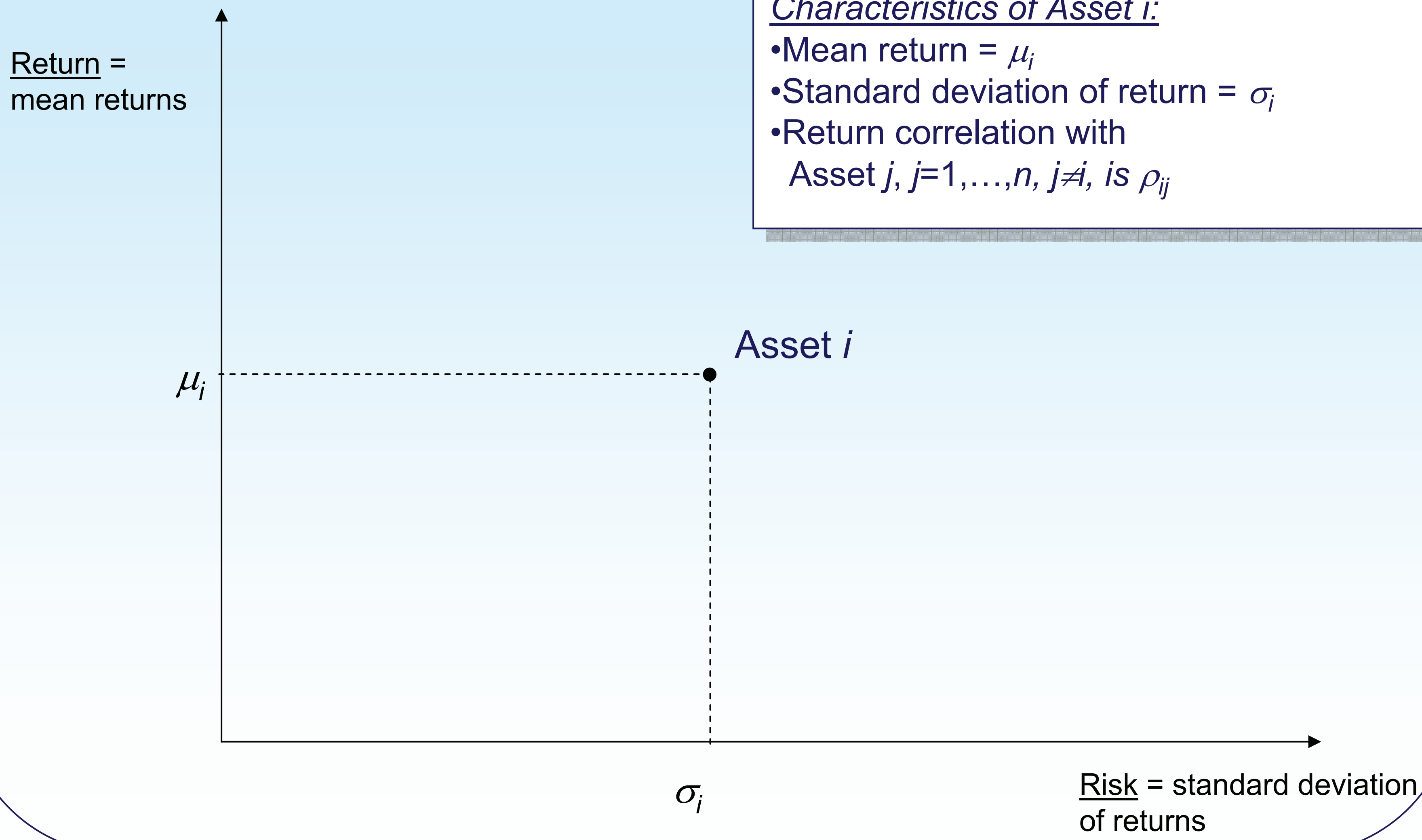
Return := Expected return of the asset / portfolio

Risk := Variance of the returns of the asset / portfolio

- With these definitions of “risk” and “return,” we can express the investor’s objective as a Mean-Variance optimization problem[1].

[1] *More on this in the companion lecture: “Fundamentals of Optimization and Application to Portfolio Selection.”*

# Representing a Risky Asset




*Characteristics of Asset i:*
- Mean return = $\mu_i$
- Standard deviation of return = $\sigma_i$
- Return correlation with Asset $j$, $j$=1,…,$n$, $j \neq i$, is $\rho_{ij}$

# A very special security: the risk-free asset

- What if you do not want to invest in any risky asset? In fact, what if you only want to deposit your money in some bank account at a fixed rate *R*?

- If this "bank" does not have any risk of defaulting, then your deposit does not carry any risk: it is a **risk-free asset** (RFA).

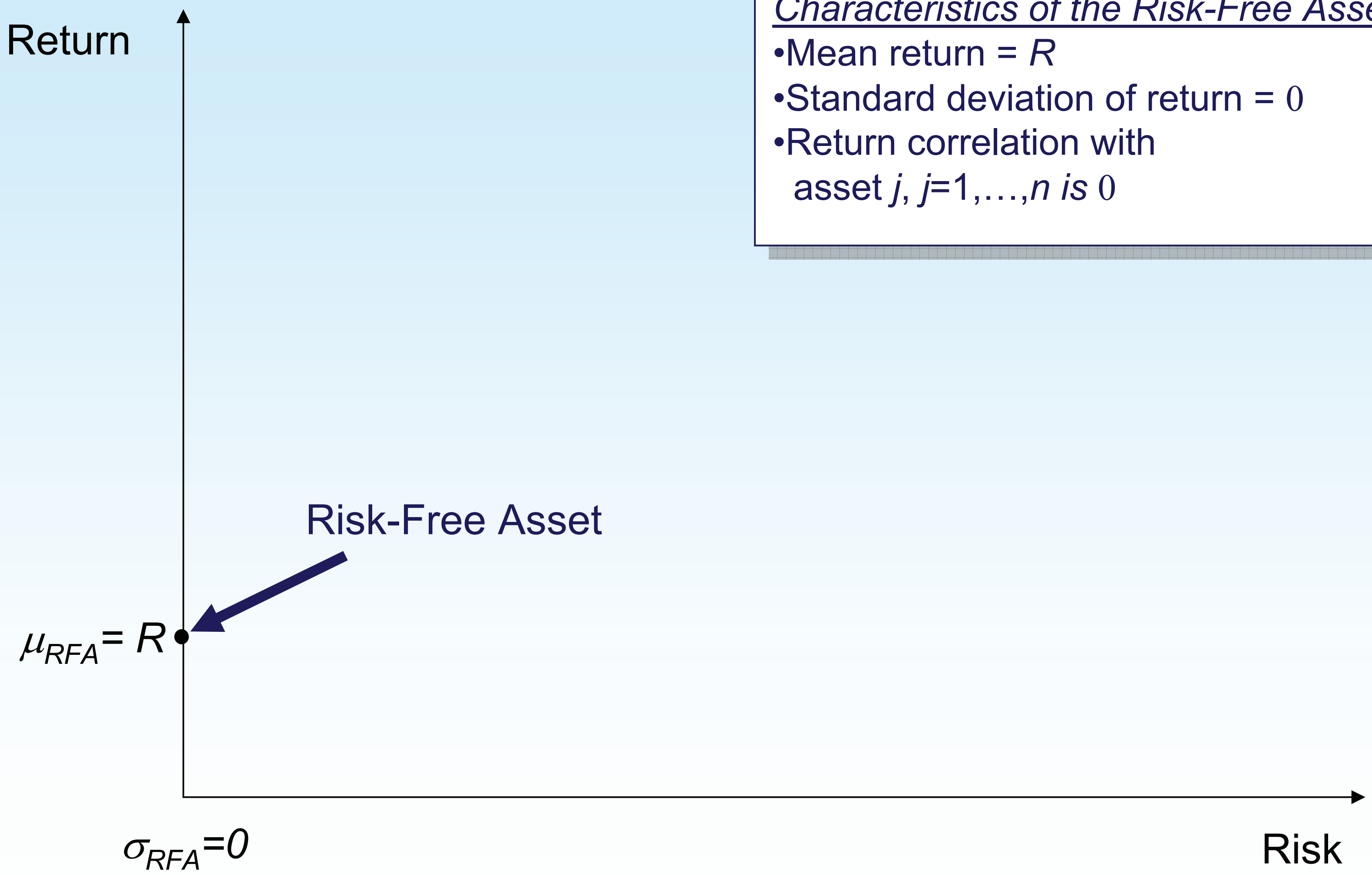
- As a result,
  - *The **expected return** of the RFA, called the risk-free rate, is equal to **R**;*
  - *The **volatility** of the RFR is equal to **0** (since it does not carry any risk!);*
  - *The **correlation** of the RFA with any other asset is also **0**.*

- The concept of risk-free asset is often used in financial economics as a proxy for a secure term deposit.

# Representing the risk-free asset

*Characteristics of the Risk-Free Asset:*

- Mean return = $R$
- Standard deviation of return = $0$
- Return correlation with asset $j$, $j$=1,…,$n$ is $0$

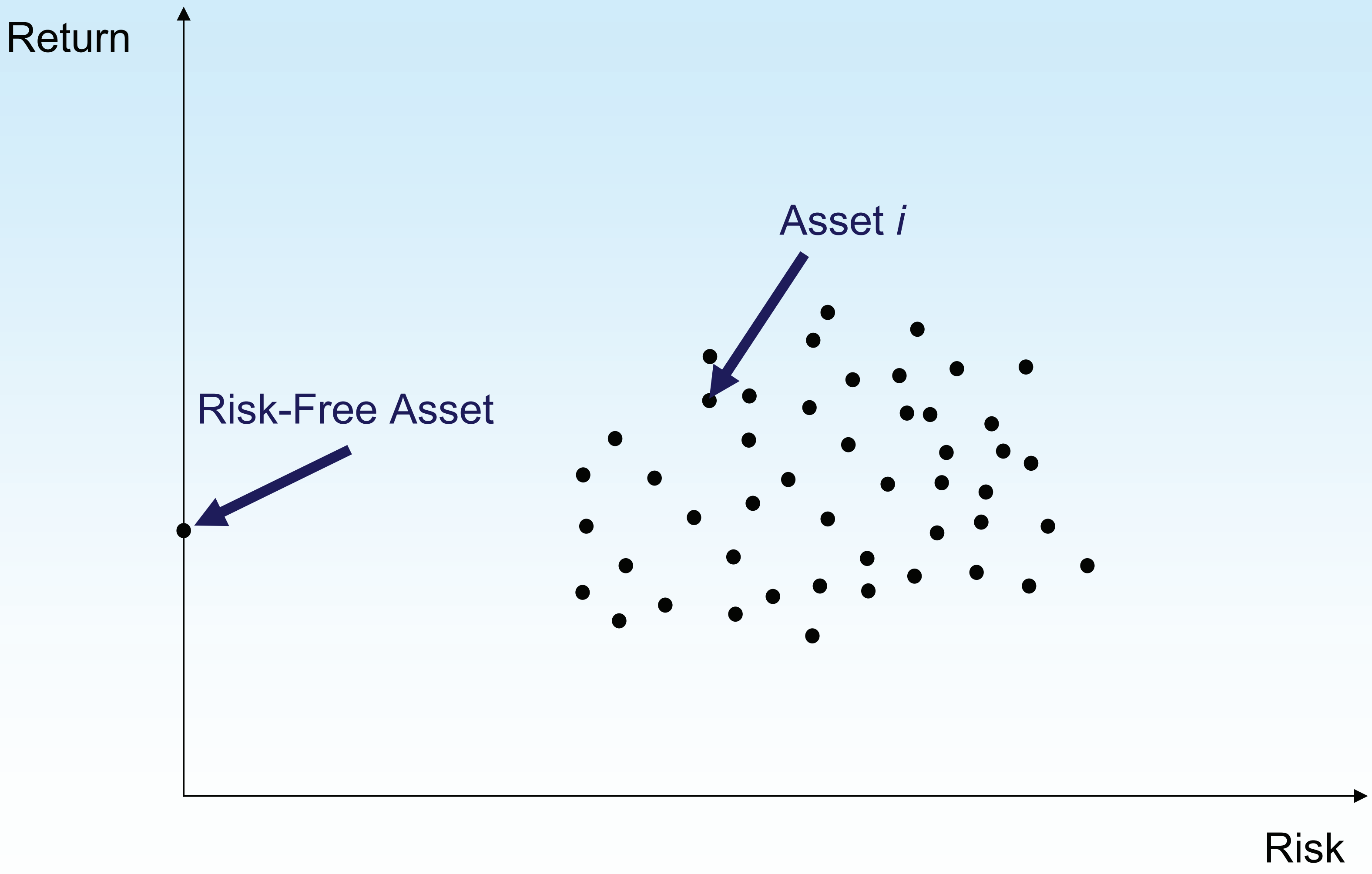


# Additional assumptions

Furthermore, we assume that:

- All statistics are based on total returns, i.e. all dividends and interest paid out are reinvested in the securities.
- Fractional investing is possible;
- Investors can deposit  and borrow freely at the risk-free rate;
- There is no penalty or restriction on short-selling of risky securities;
- The market is “frictionless” in the sense that there is not tax, no transaction fees, and no need for collateral or margins.

# The investment universe



# A Simpler Problem:
# 2 assets and the risk-free asset

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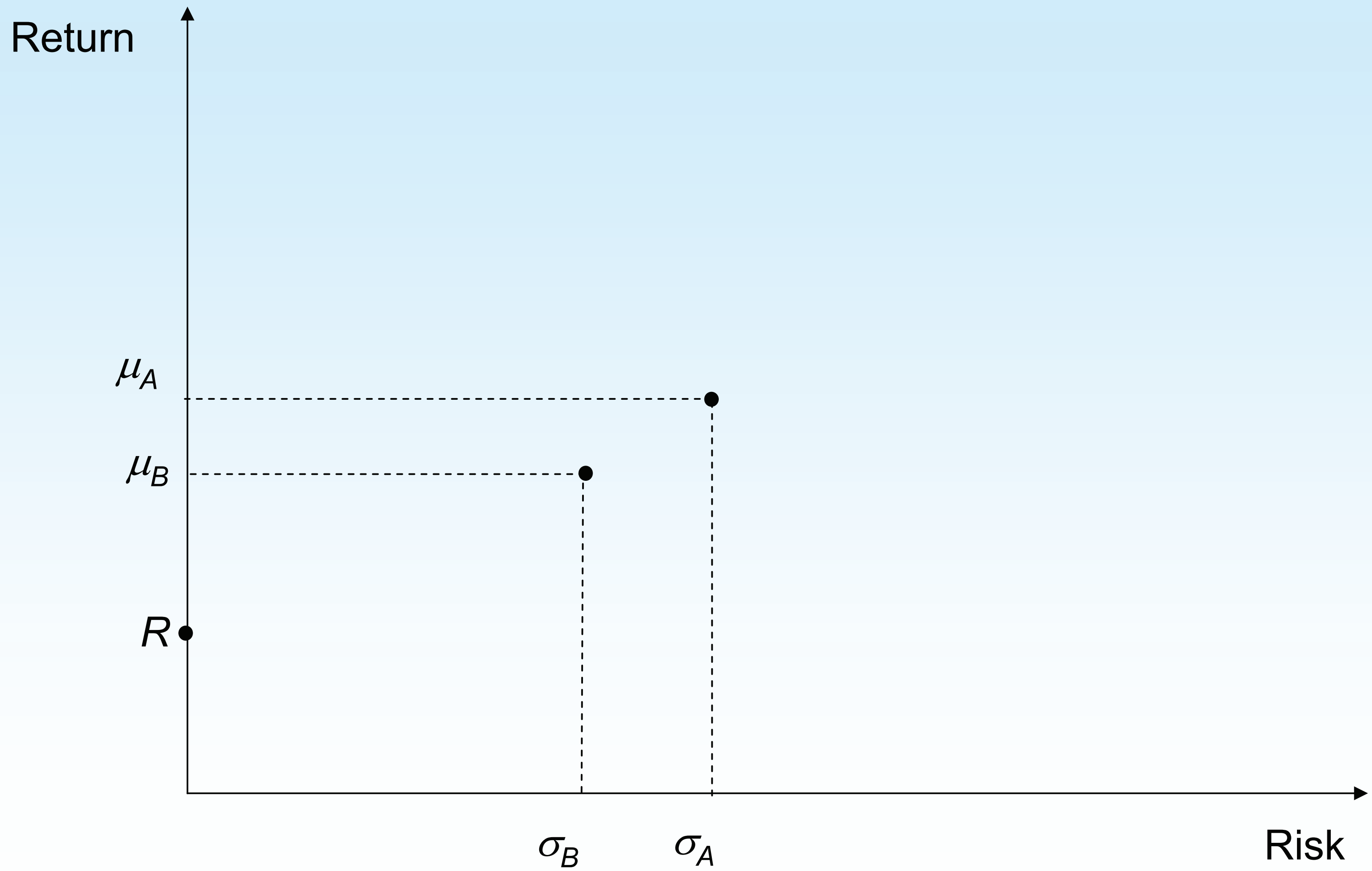
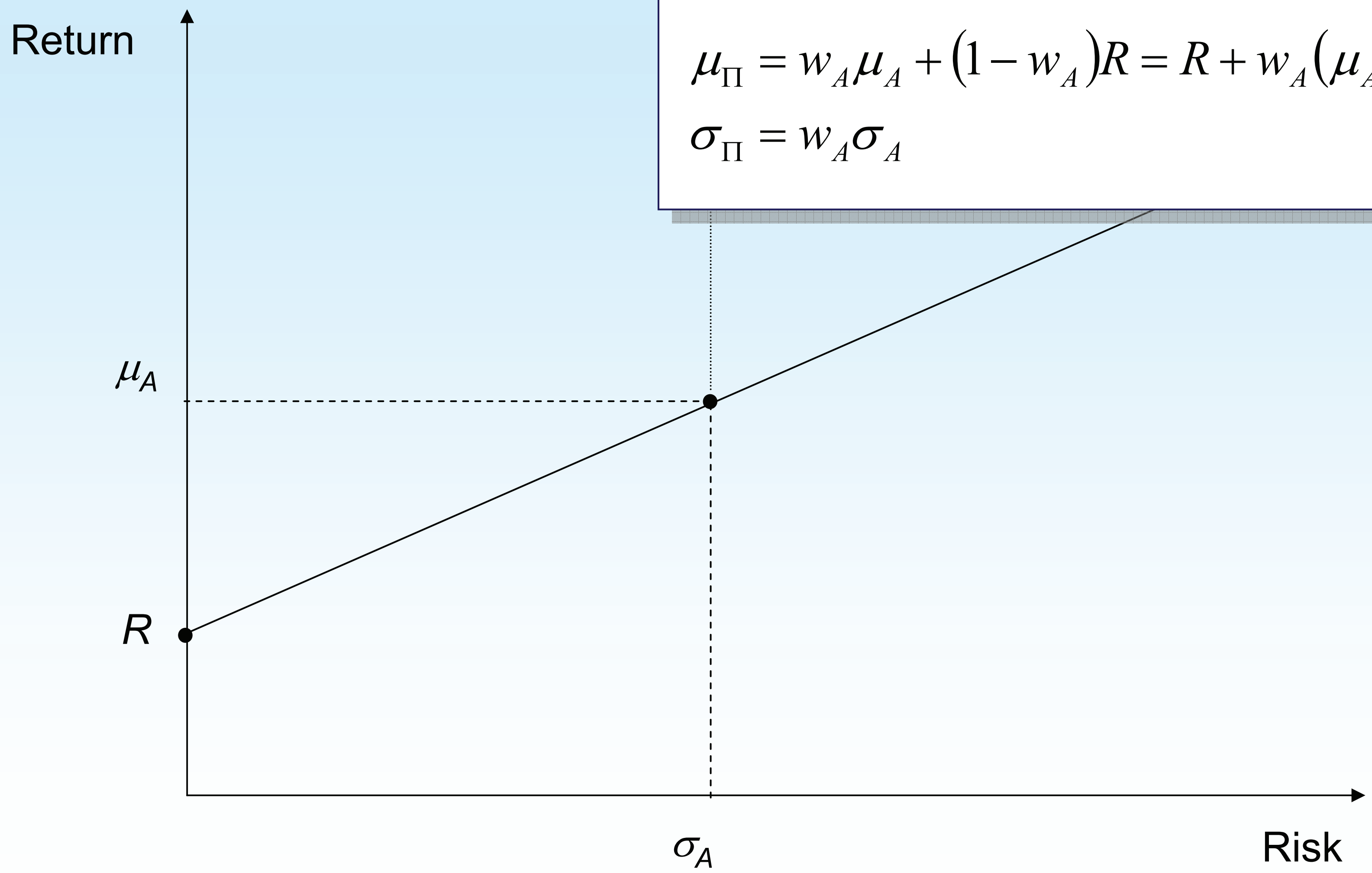


# The Risk-Free Asset and Risky Asset *A*

**Line parametrized by $w_A$:**

$$\mu_\Pi = w_A \mu_A + (1 - w_A) R = R + w_A (\mu_A - R)$$

$$\sigma_\Pi = w_A \sigma_A$$



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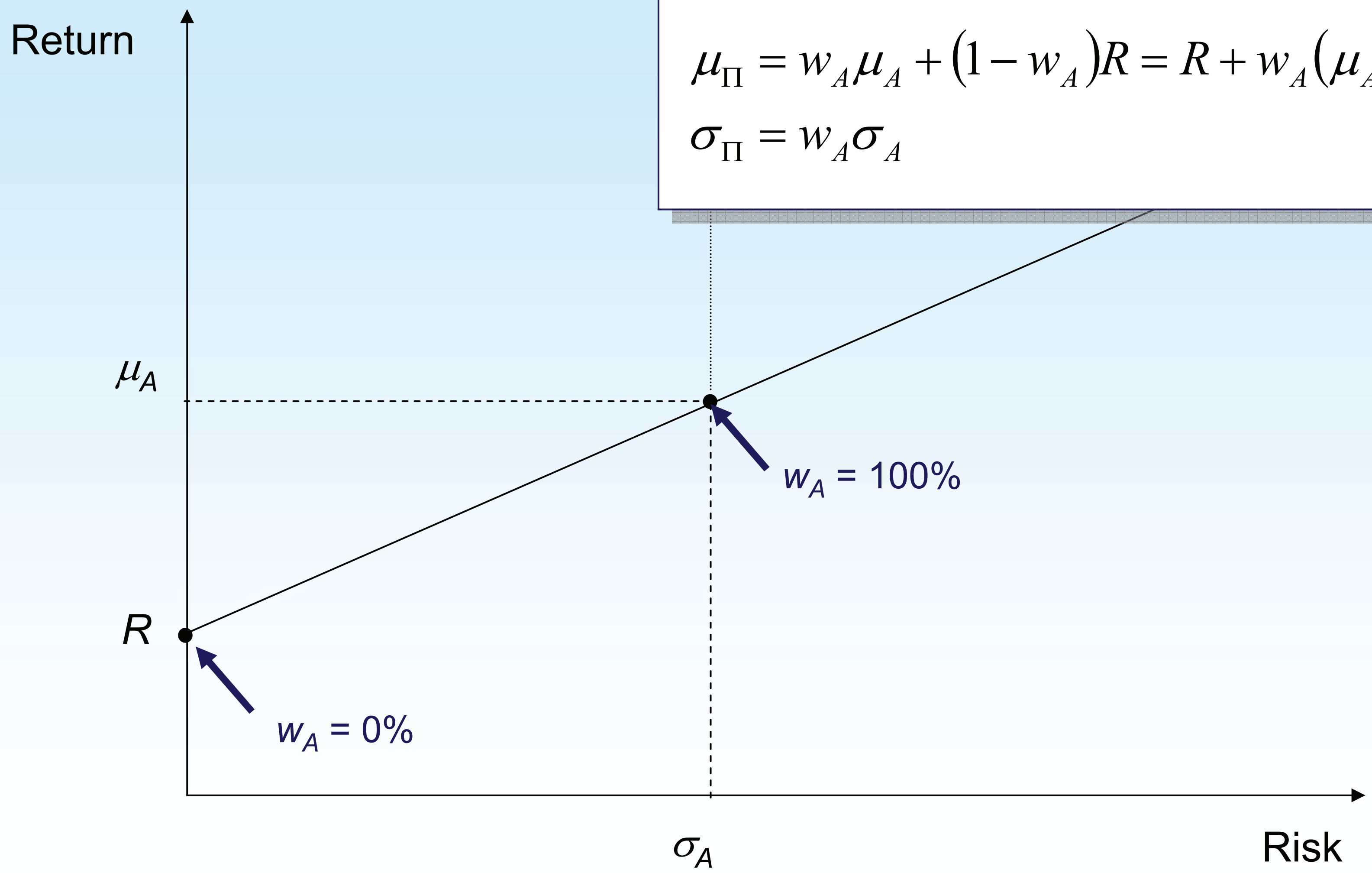

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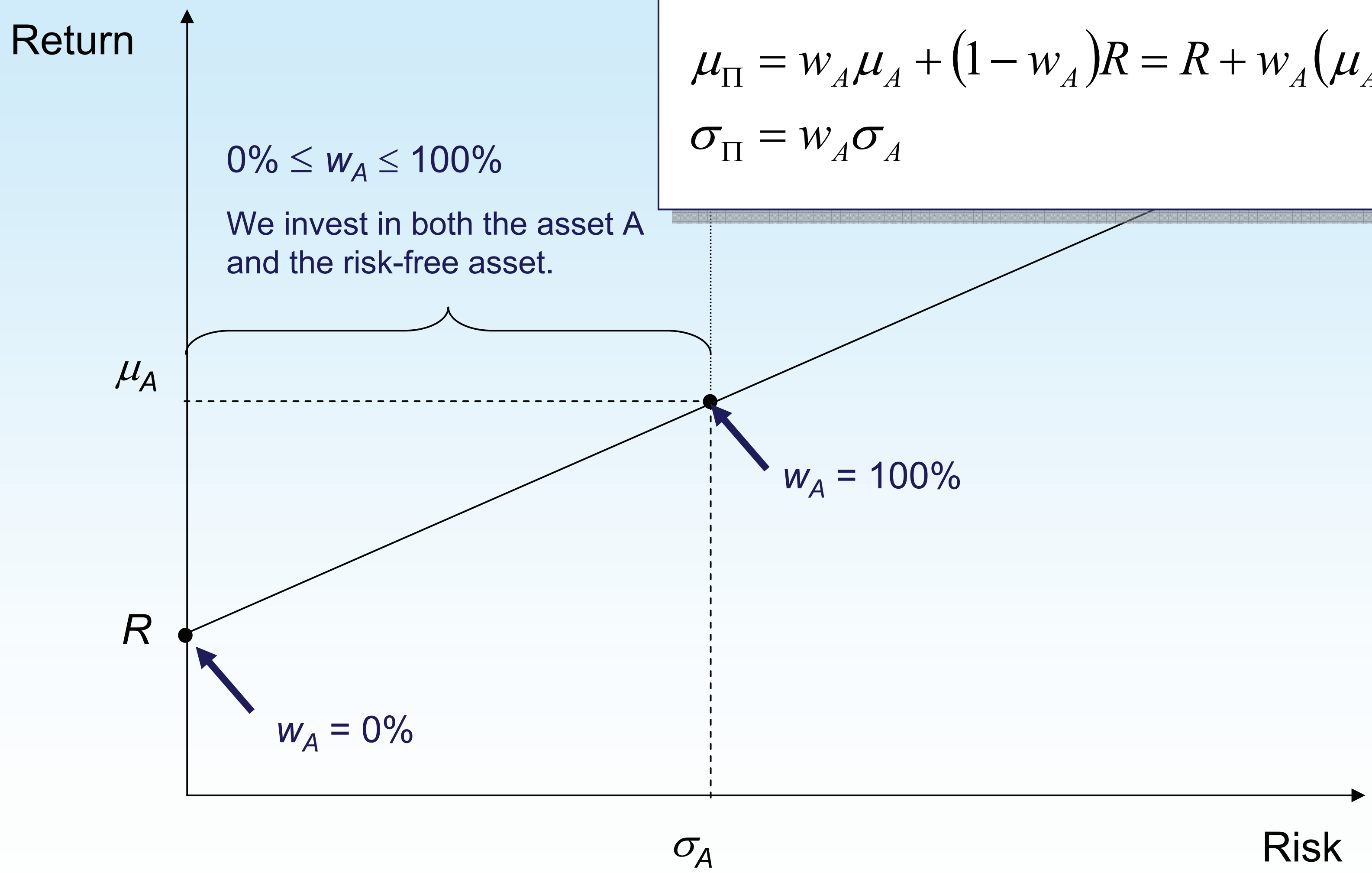

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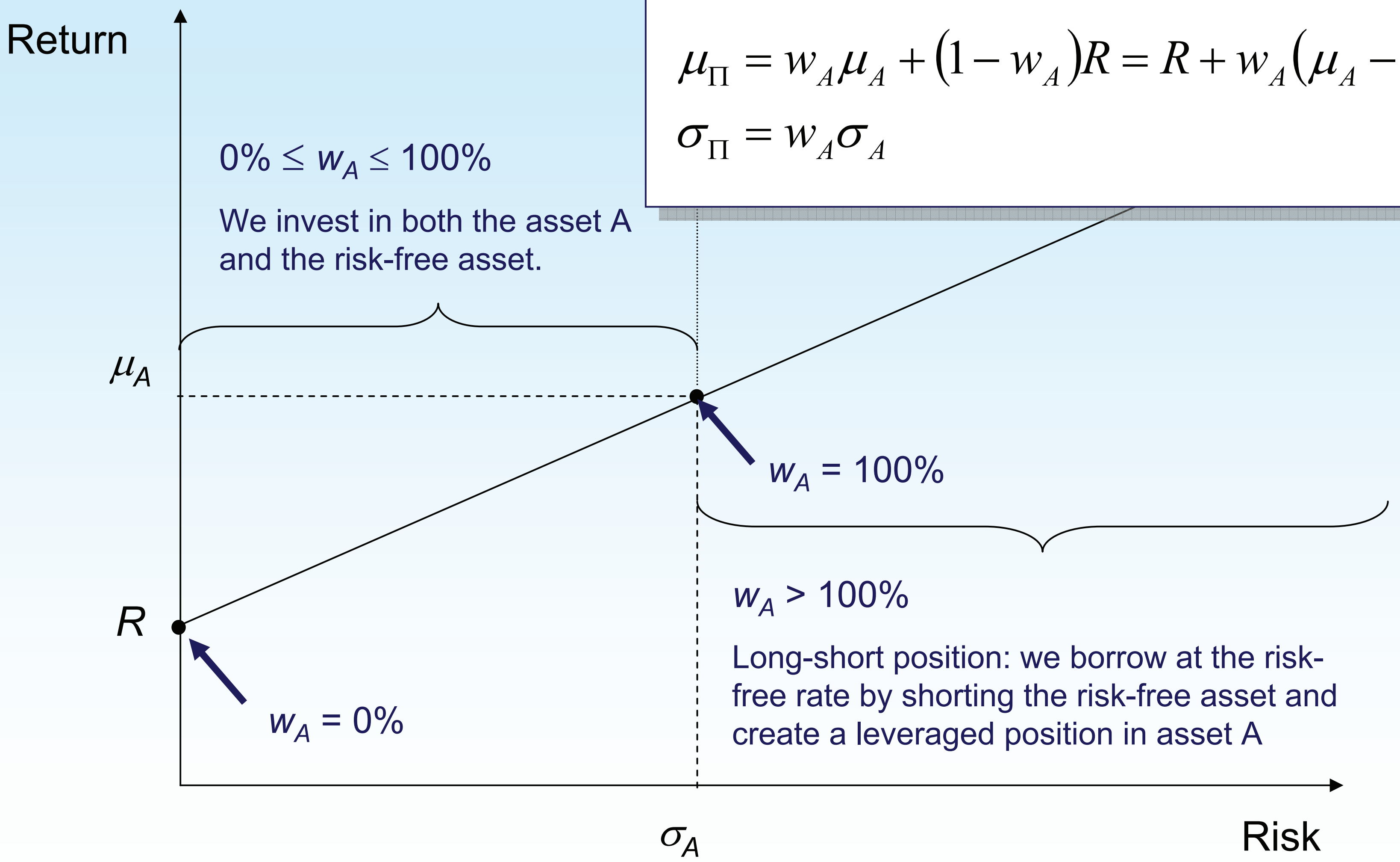
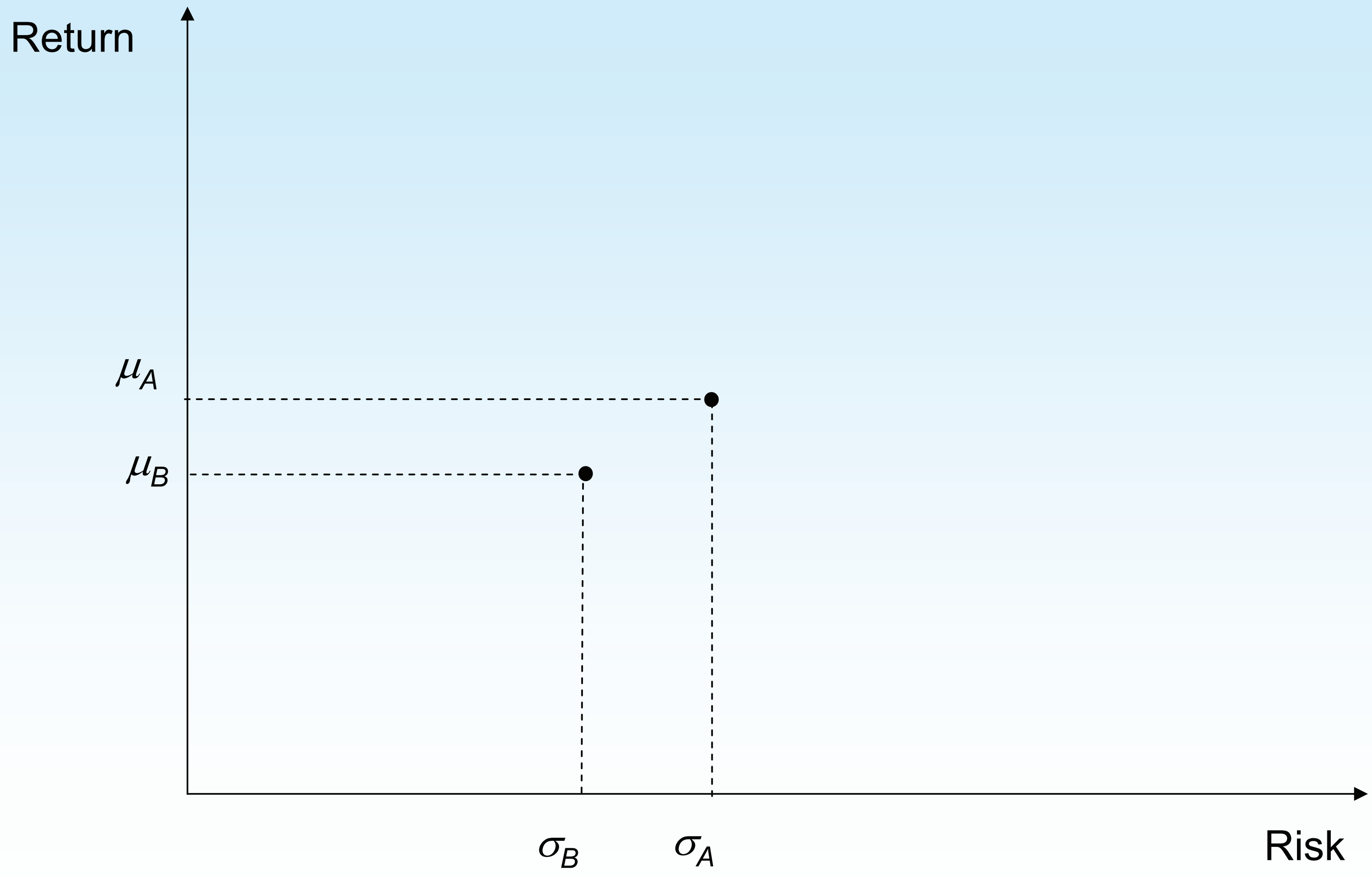


# The case with two risky assets

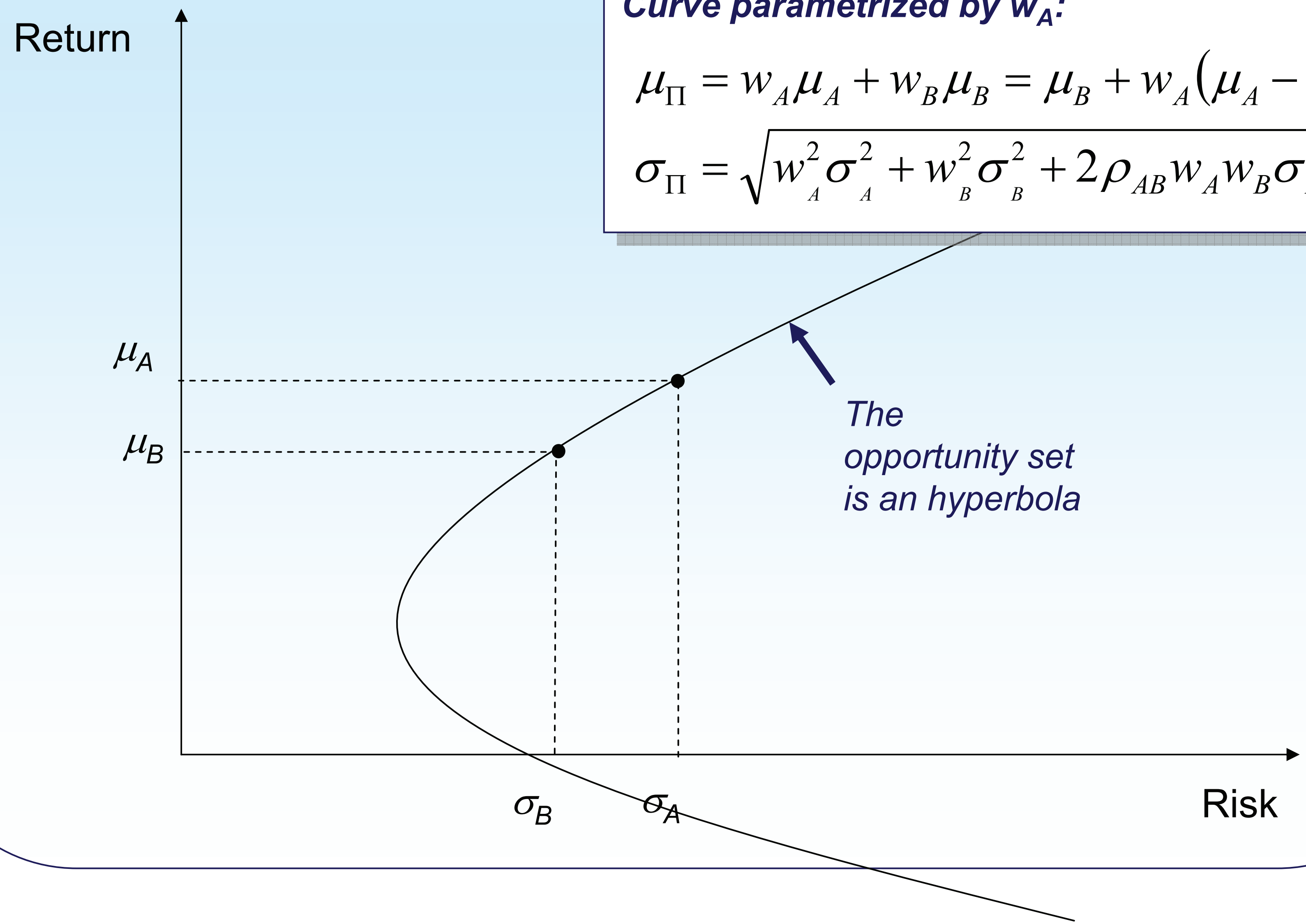


# The opportunity set

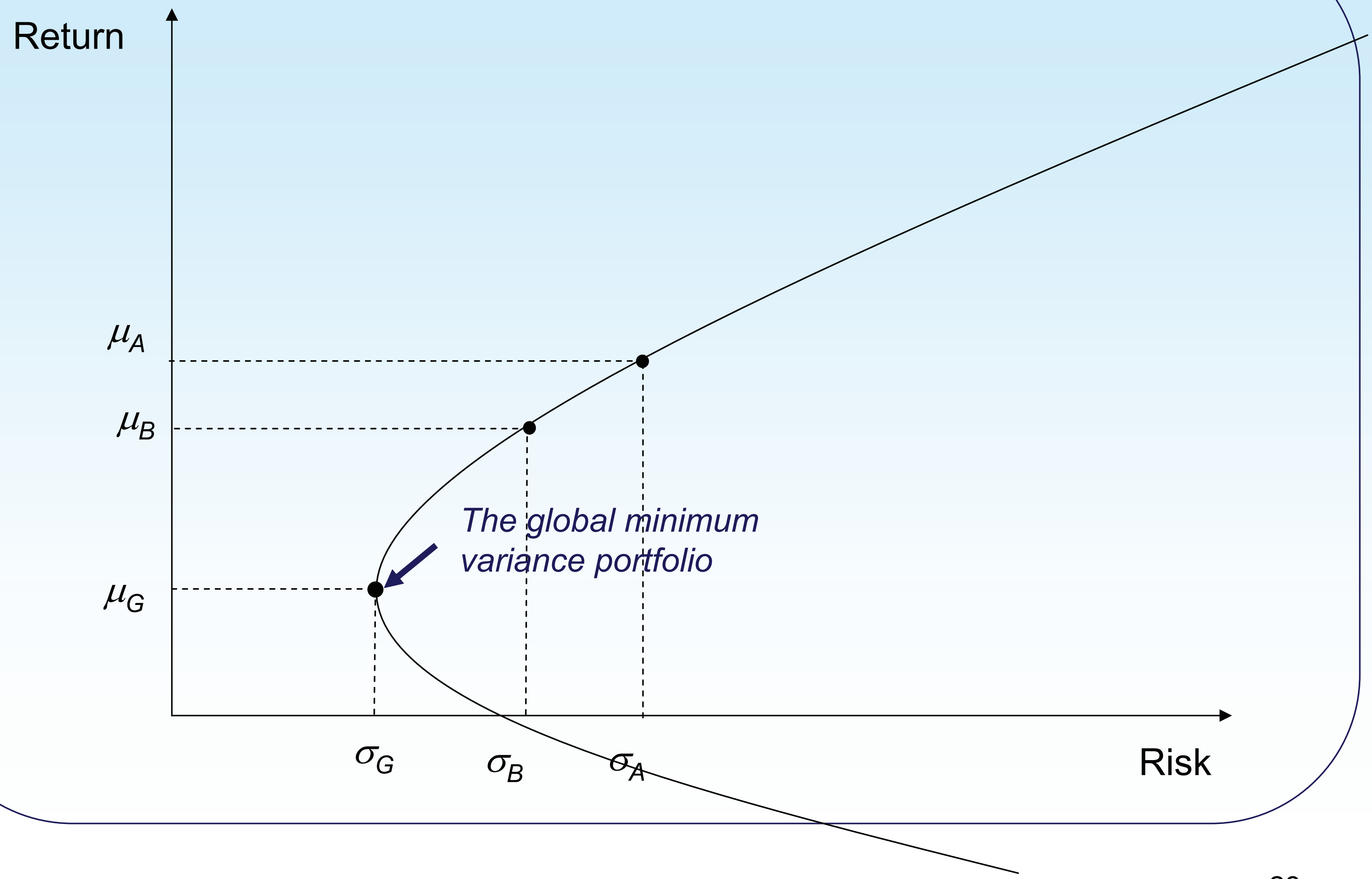
***Curve parametrized by $w_A$:***

$$\mu_\Pi = w_A\mu_A + w_B\mu_B = \mu_B + w_A(\mu_A - \mu_B)$$

$$\sigma_\Pi = \sqrt{w_A^2\sigma_A^2 + w_B^2\sigma_B^2 + 2\rho_{AB}w_A w_B\sigma_A\sigma_B}$$

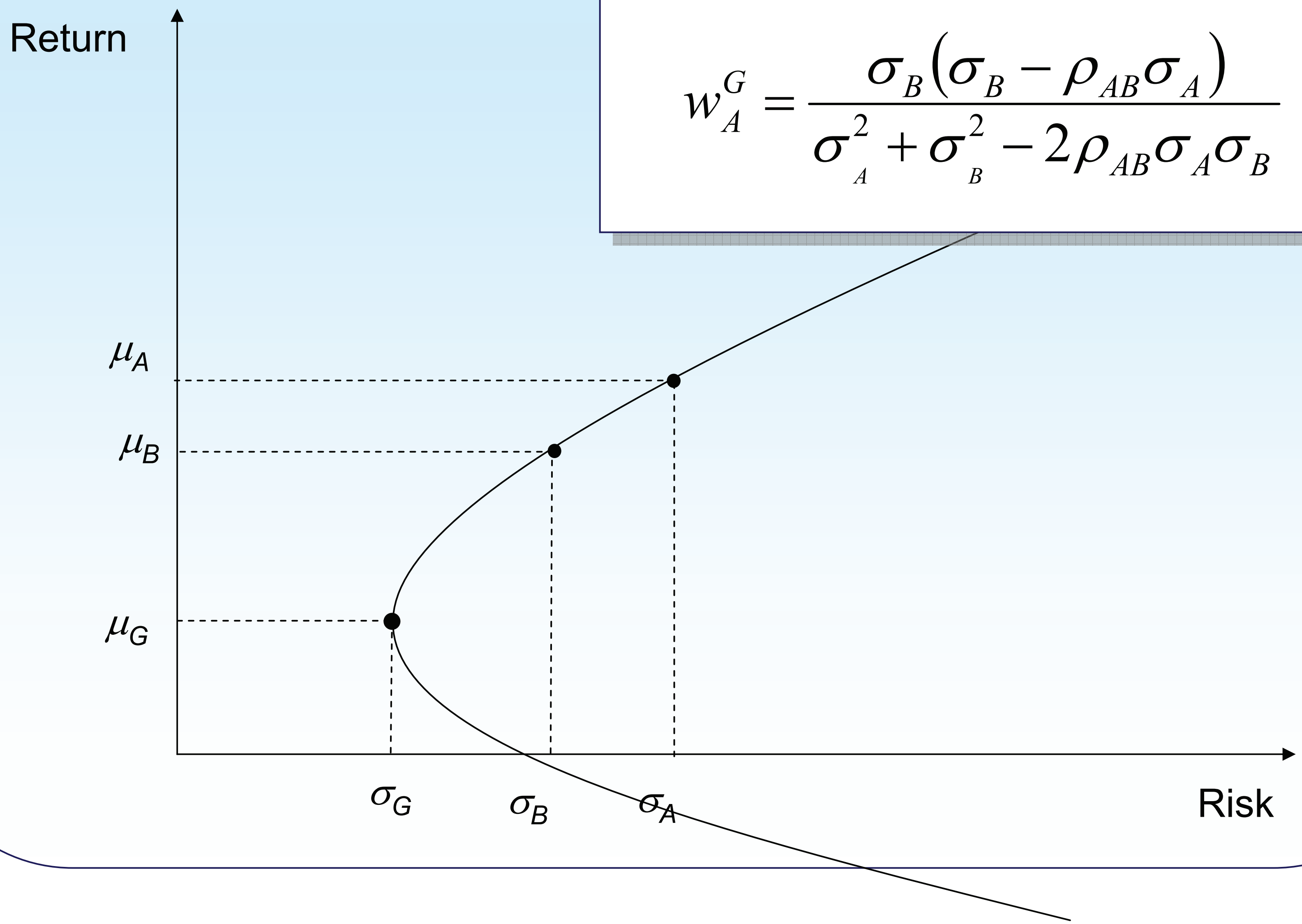


# The global minimum variance portfolio

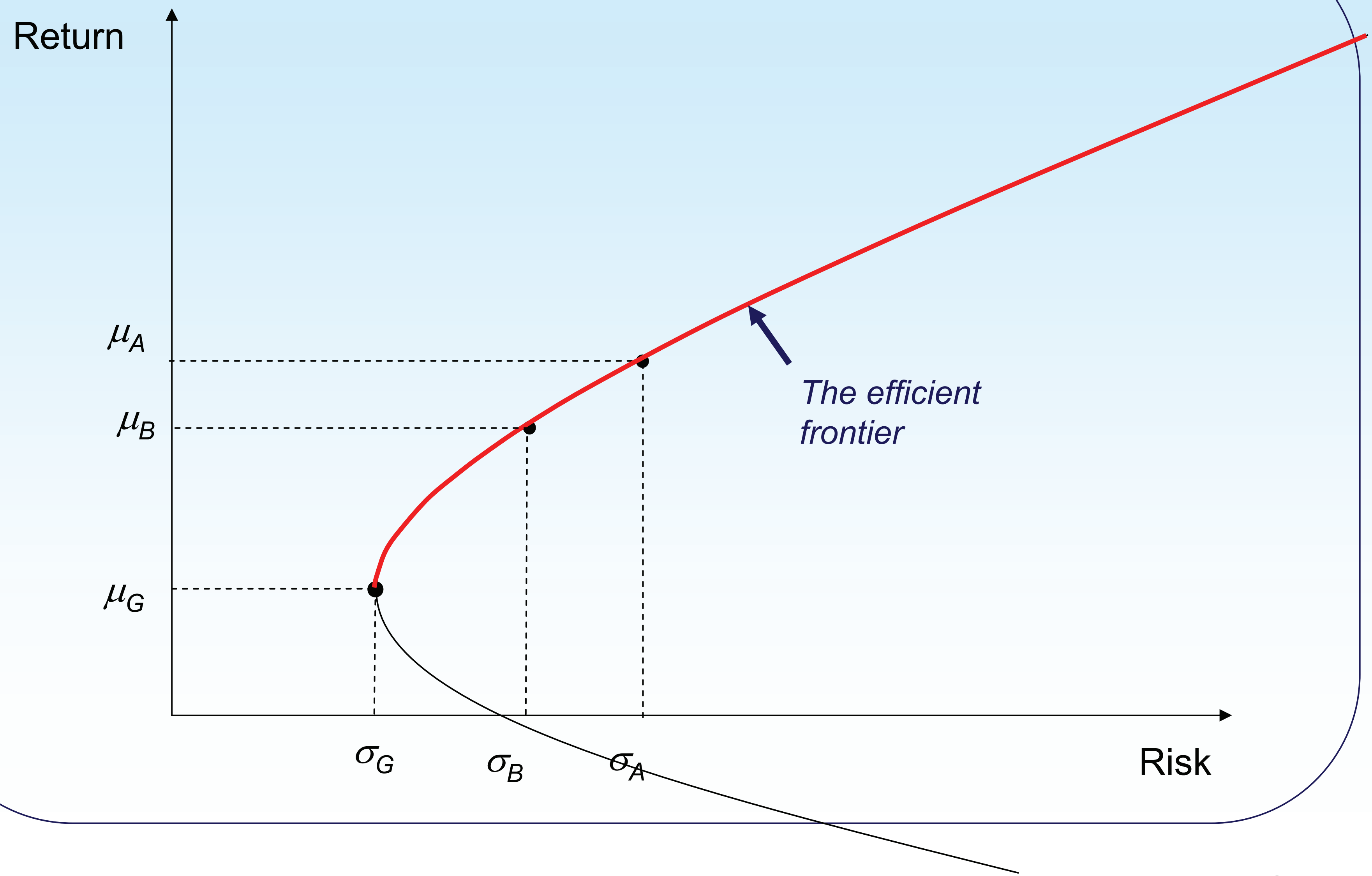


# The global minimum variance portfolio's allocation

$$w_A^G = \frac{\sigma_B(\sigma_B - \rho_{AB}\sigma_A)}{\sigma_A^2 + \sigma_B^2 - 2\rho_{AB}\sigma_A\sigma_B}$$



# The efficient frontier



# Case 1: $\rho_{AB}$ = 1

***Portfolio characteristics:***

$$\mu_{\Pi} = w_A \mu_A + w_B \mu_B = \mu_B + w_A \left( \mu_A - \mu_B \right)$$

$$\sigma_{\Pi} = \left| w_A \sigma_A + w_B \sigma_B \right| = \left| \sigma_B + w_A \left( \sigma_A - \sigma_B \right) \right|$$

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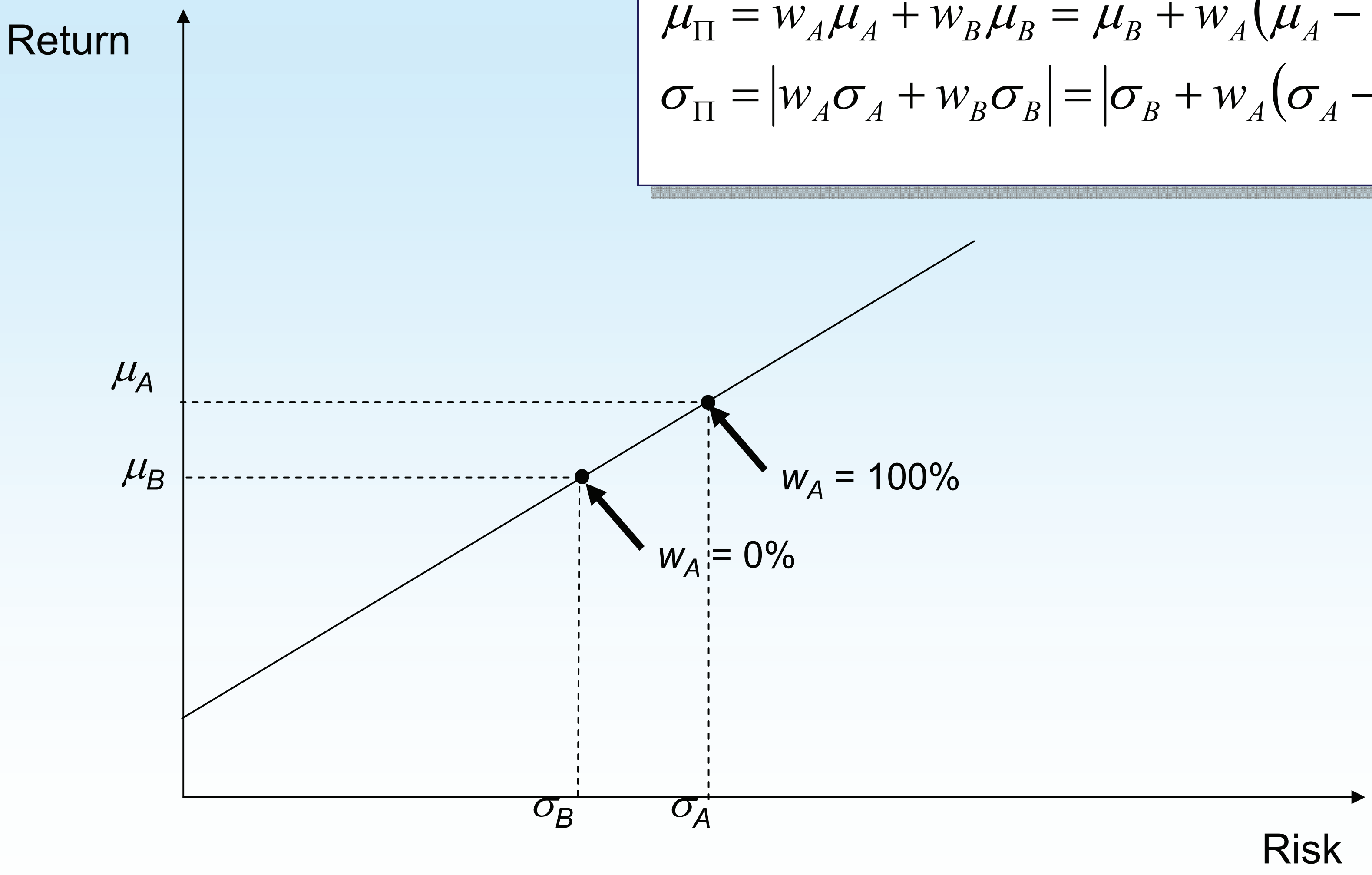

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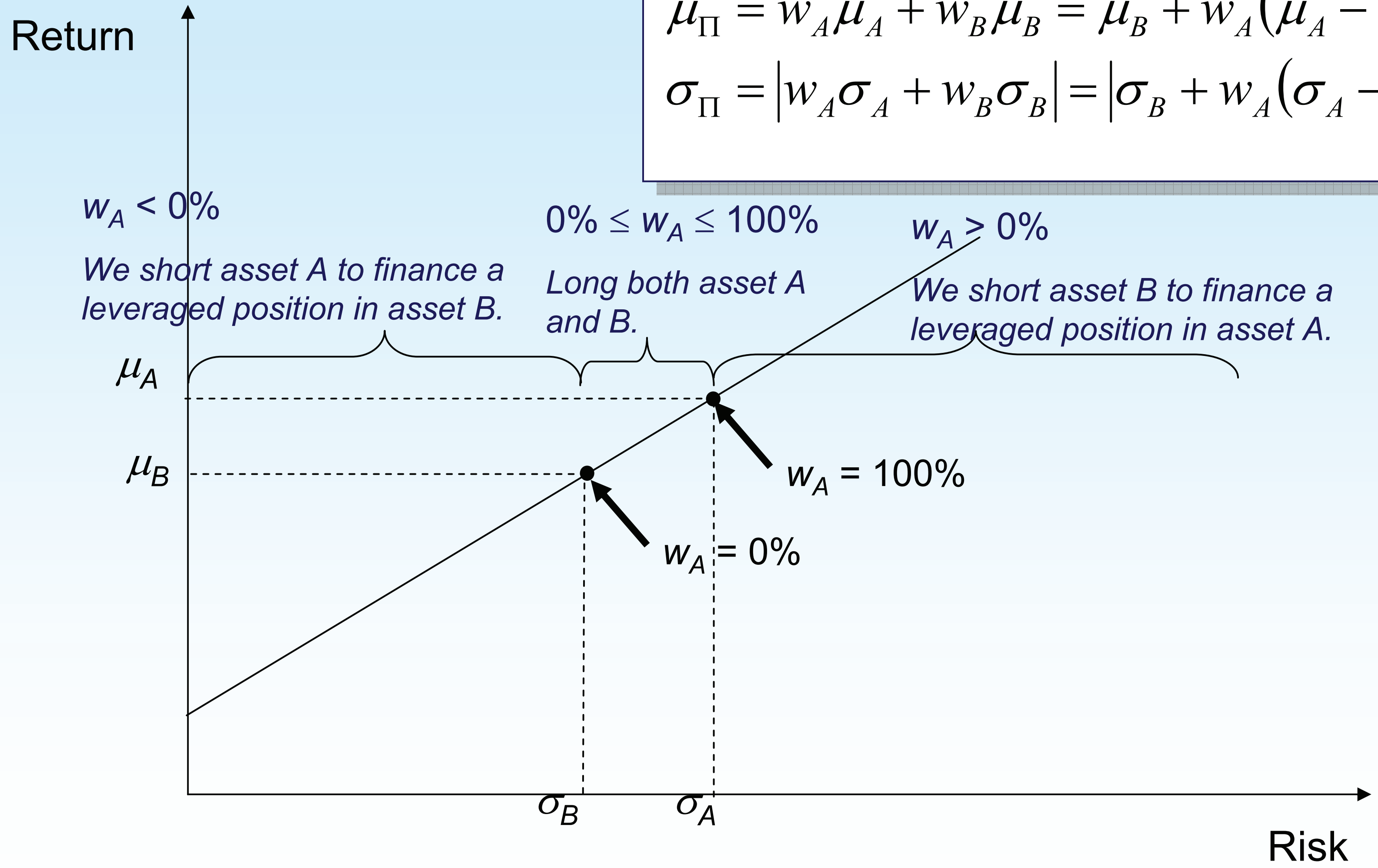


# Case 2: $\rho_{AB}$ = -1

***Portfolio characteristics:***

$$\mu_{\Pi} = w_A \mu_A + w_B \mu_B = \mu_B + w_A(\mu_A - \mu_B)$$

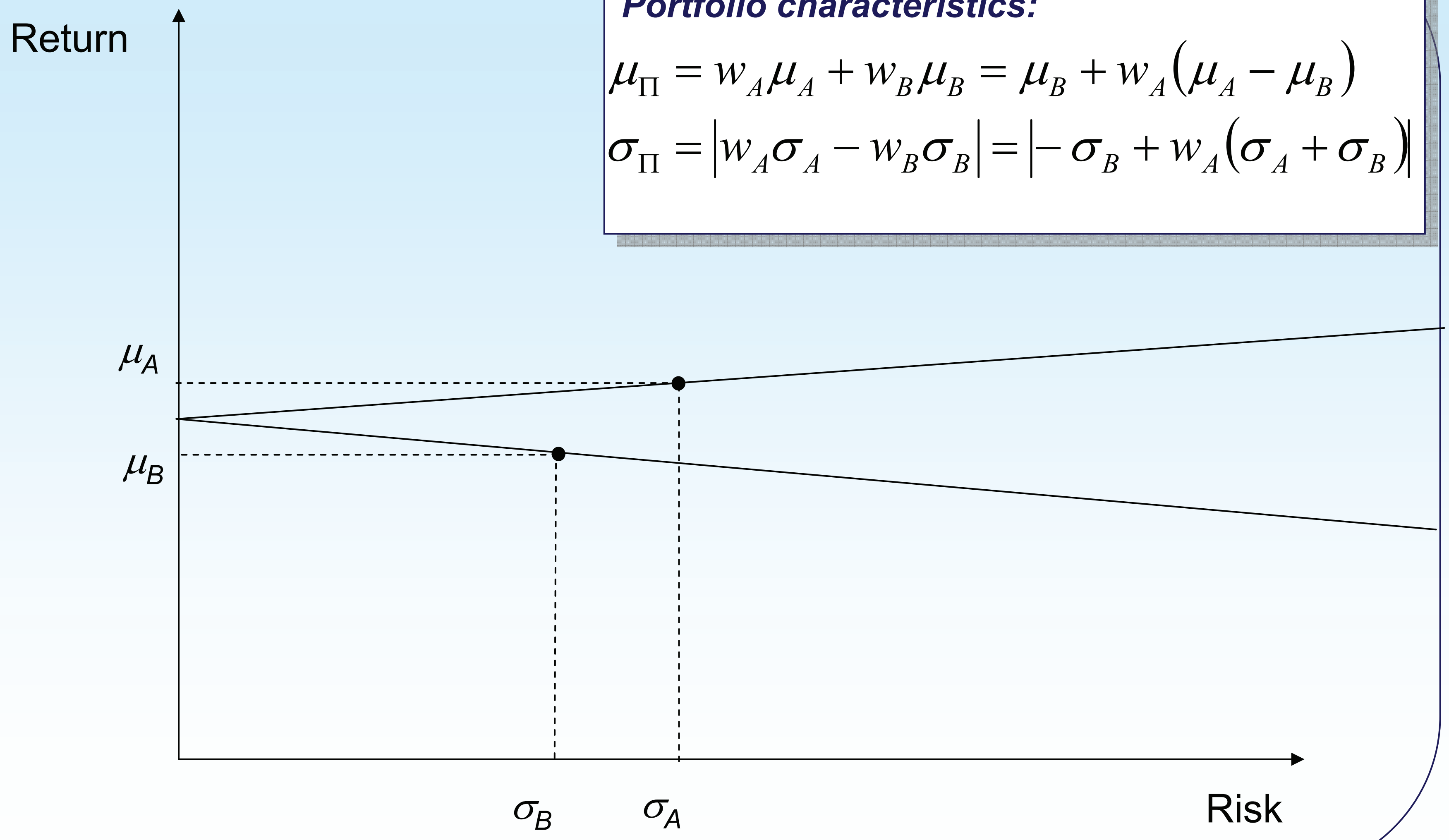
$$\sigma_{\Pi} = |w_A \sigma_A - w_B \sigma_B| = |-\sigma_B + w_A(\sigma_A + \sigma_B)|$$

# Case 2: $\rho_{AB}$ = -1

***Portfolio characteristics:***

$$\mu_\Pi = w_A\mu_A + w_B\mu_B = \mu_B + w_A(\mu_A - \mu_B)$$

$$\sigma_\Pi = \left|w_A\sigma_A - w_B\sigma_B\right| = \left|-\sigma_B + w_A(\sigma_A + \sigma_B)\right|$$

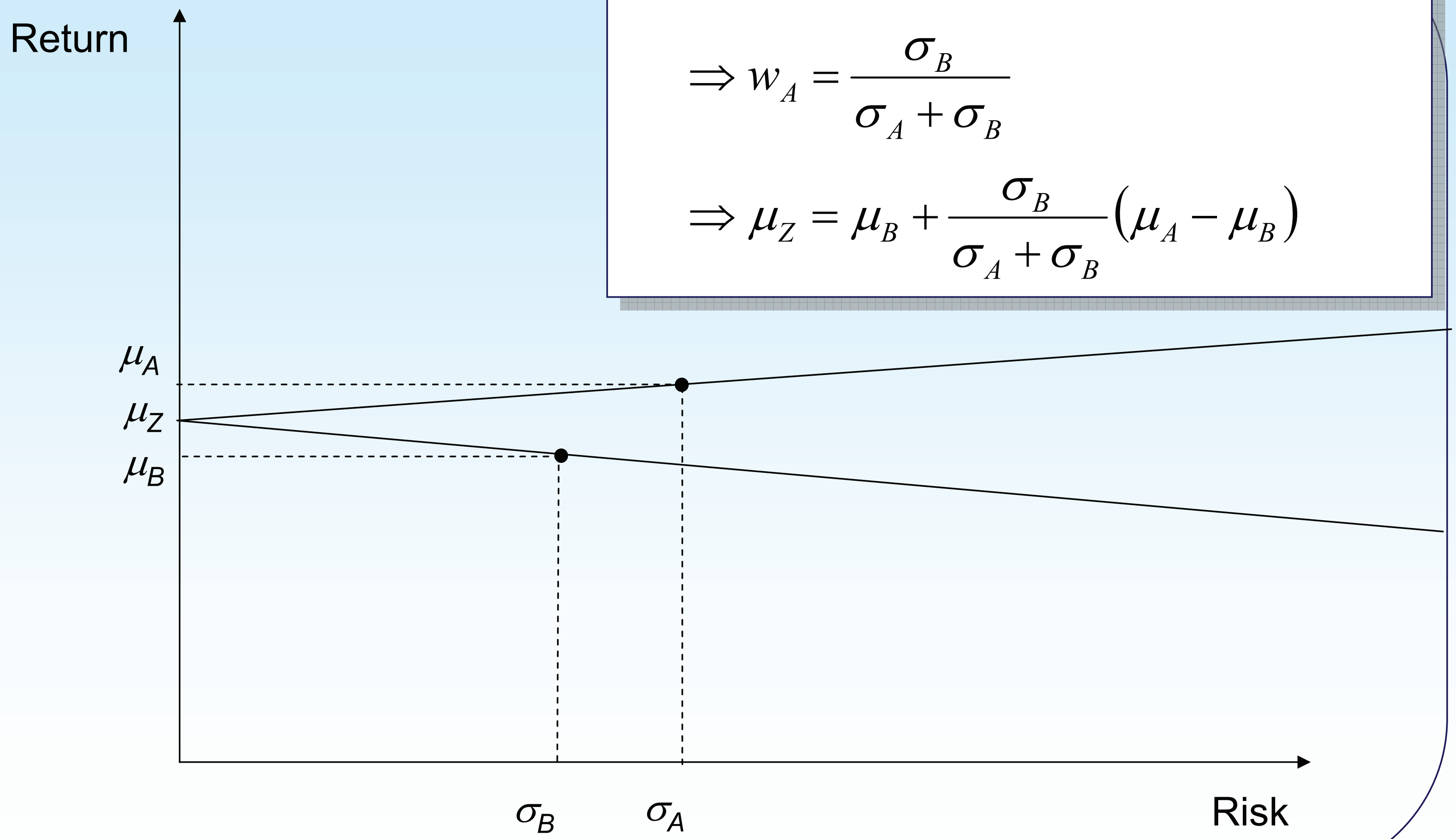


# The zero-variance portfolio

$$\sigma_Z = 0$$

$$\Rightarrow w_A = \frac{\sigma_B}{\sigma_A + \sigma_B}$$

$$\Rightarrow \mu_Z = \mu_B + \frac{\sigma_B}{\sigma_A + \sigma_B}(\mu_A - \mu_B)$$



# Case 3: $\rho_{AB} = 0$

***Portfolio characteristics:***

$$\mu_\Pi = w_A \mu_A + w_B \mu_B = \mu_B + w_A (\mu_A - \mu_B)$$

$$\sigma_\Pi = \sqrt{w_A^2 \sigma_A^2 + w_B^2 \sigma_B^2}$$

# Case 3: $\rho_{AB} = 0$

***Portfolio characteristics:***

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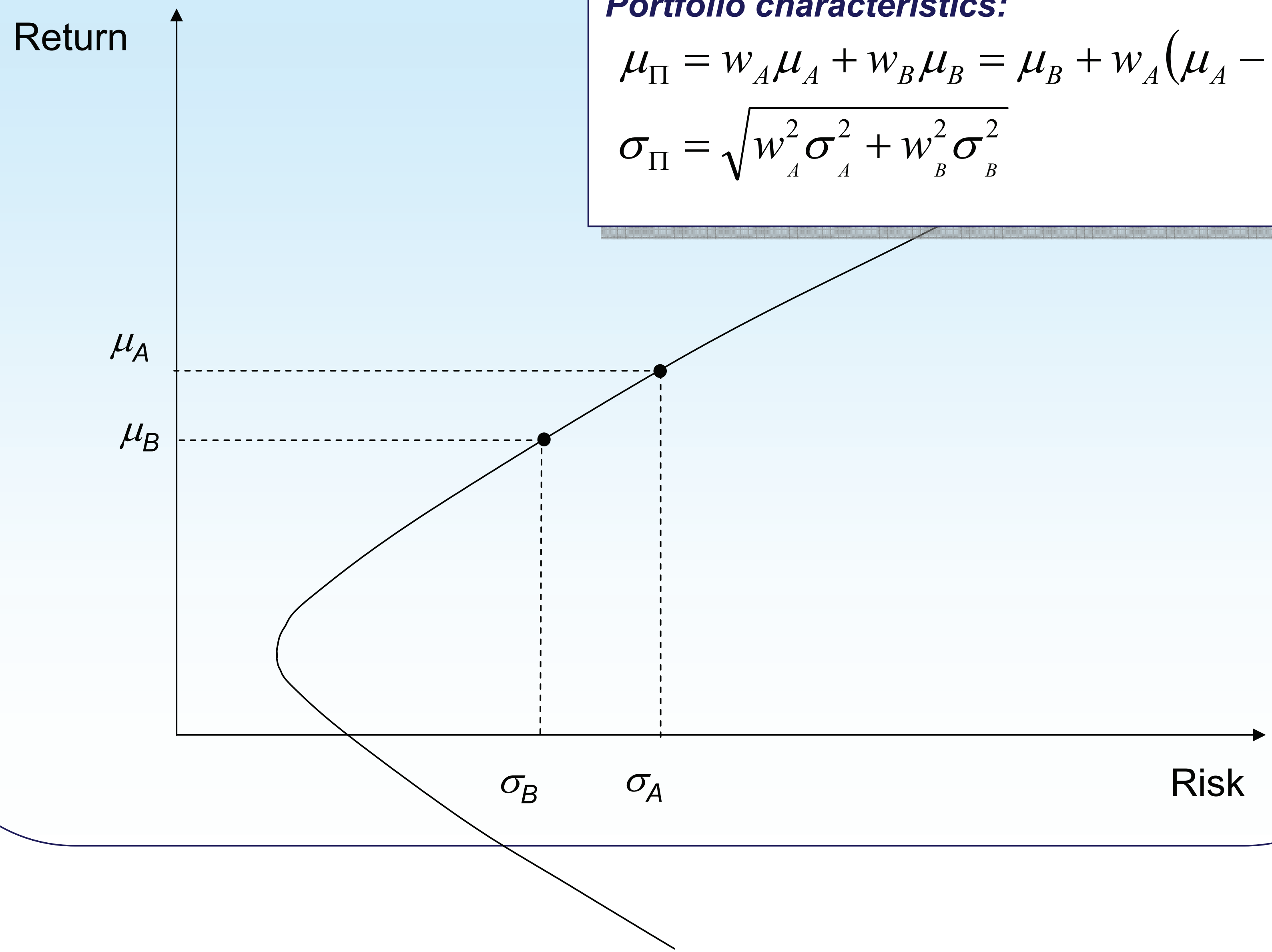
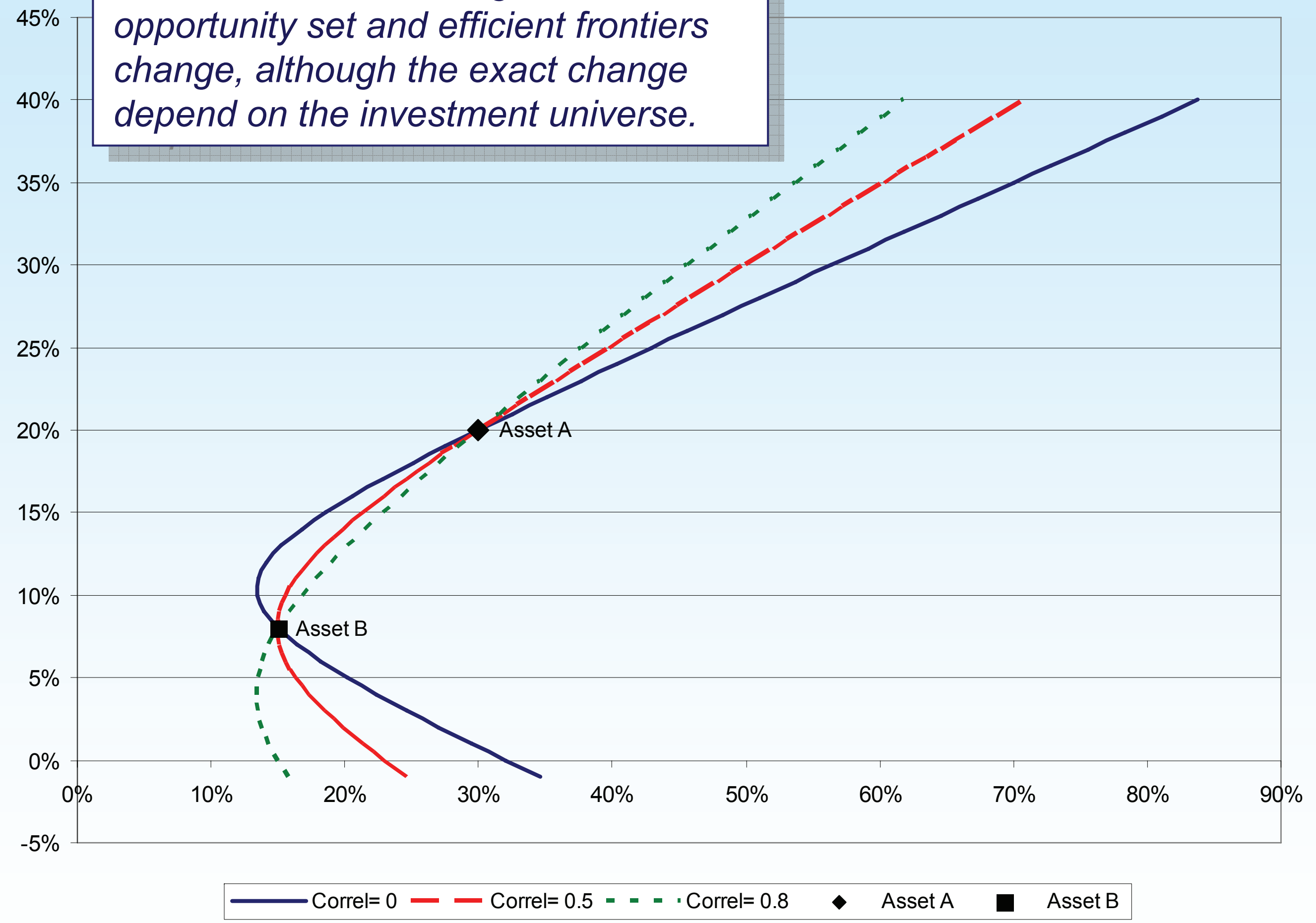


# Conclusion: so what happens as the correlation changes?

*As the correlation changes, the opportunity set and efficient frontiers change, although the exact change depend on the investment universe.*

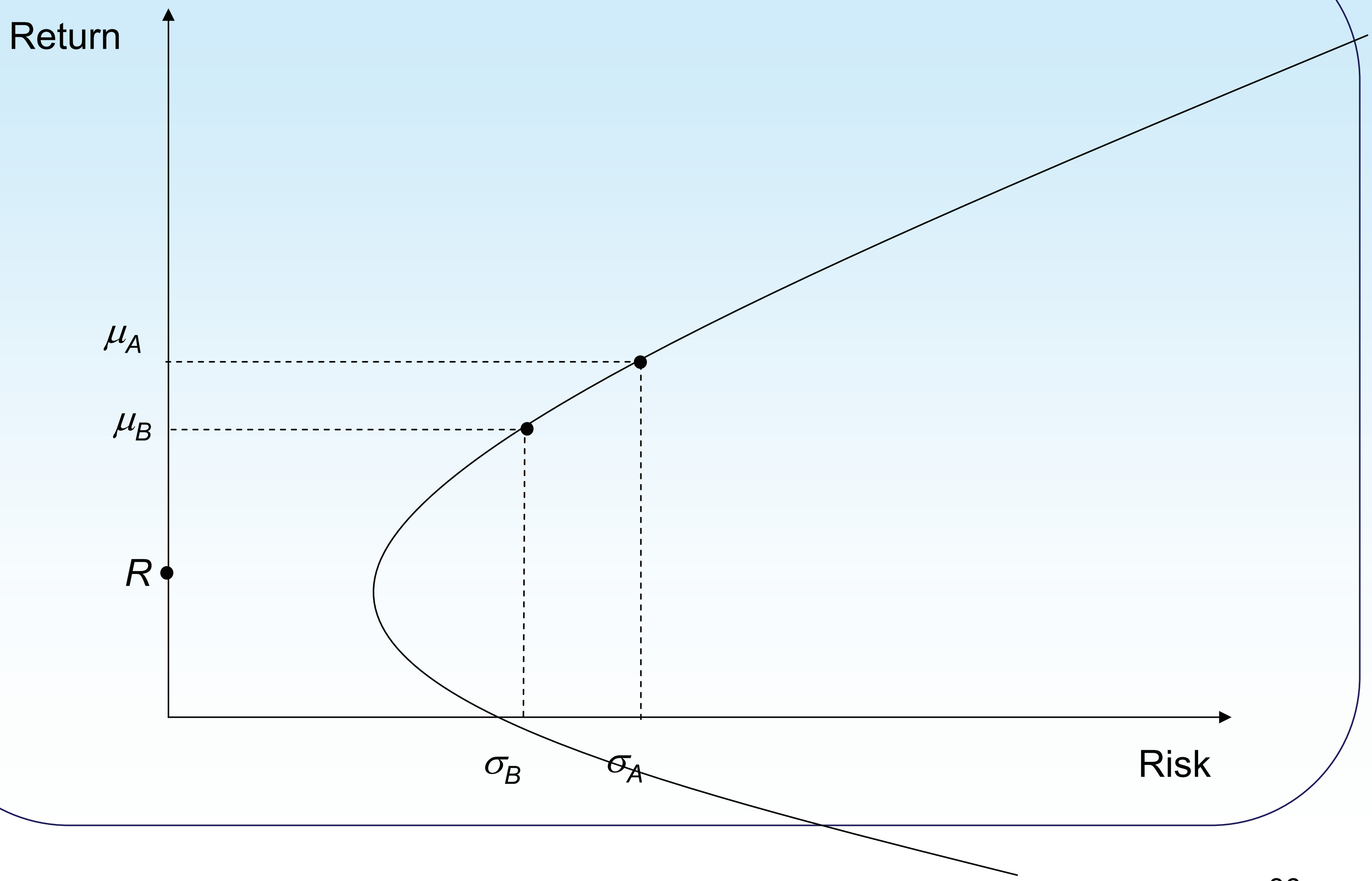


# (Re)introducing the risk-free asset

- What happens if we now consider an allocation between the risk-free asset and the two risky securities?

- Surely, this new problem should be the same thing as:
  - *Selecting a risky portfolio P made of positions in securities A and B, and then;*
  - *Allocating funds between the risk-free asset and the portfolio P.*

# Building the new efficient frontier



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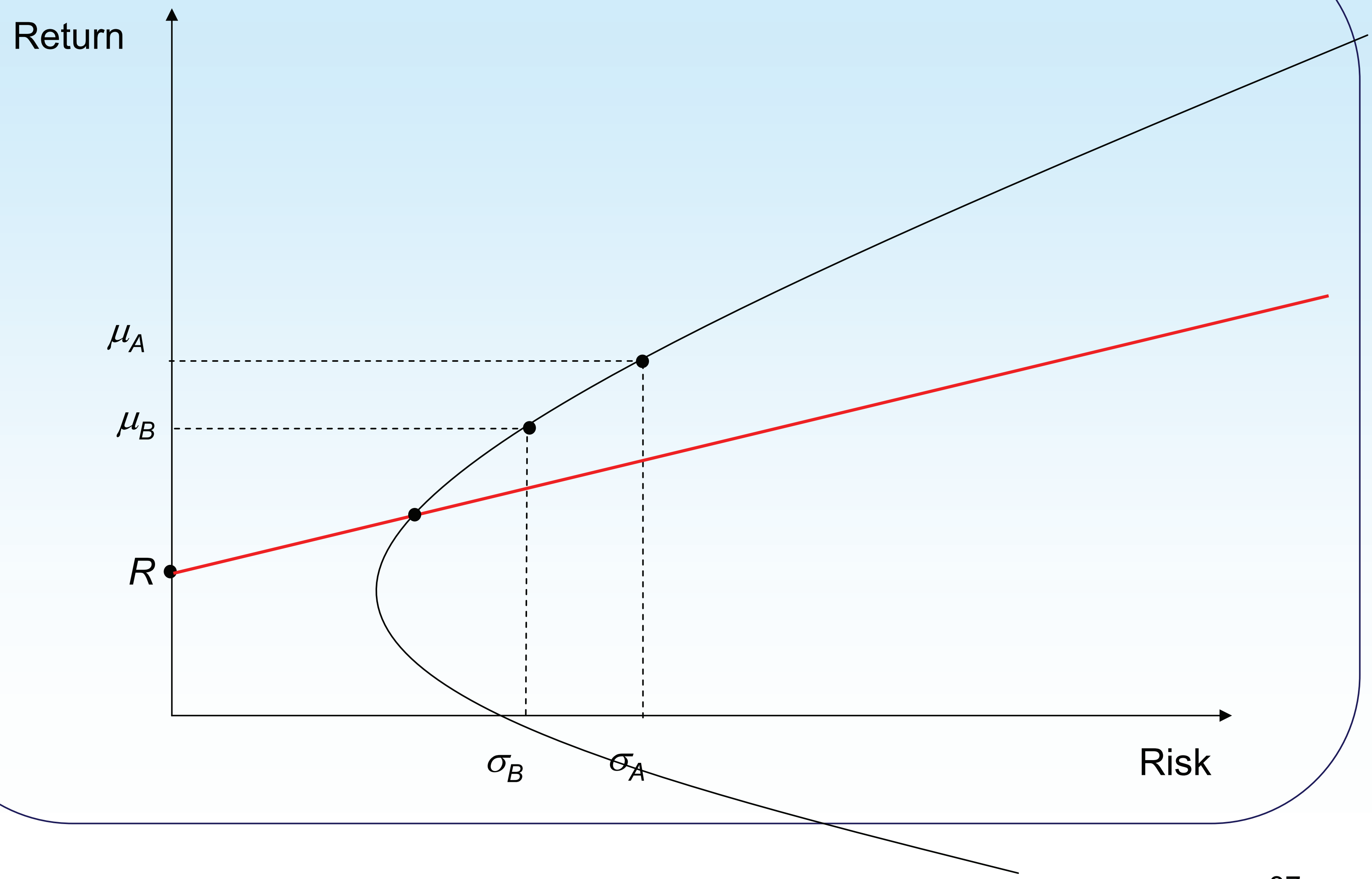

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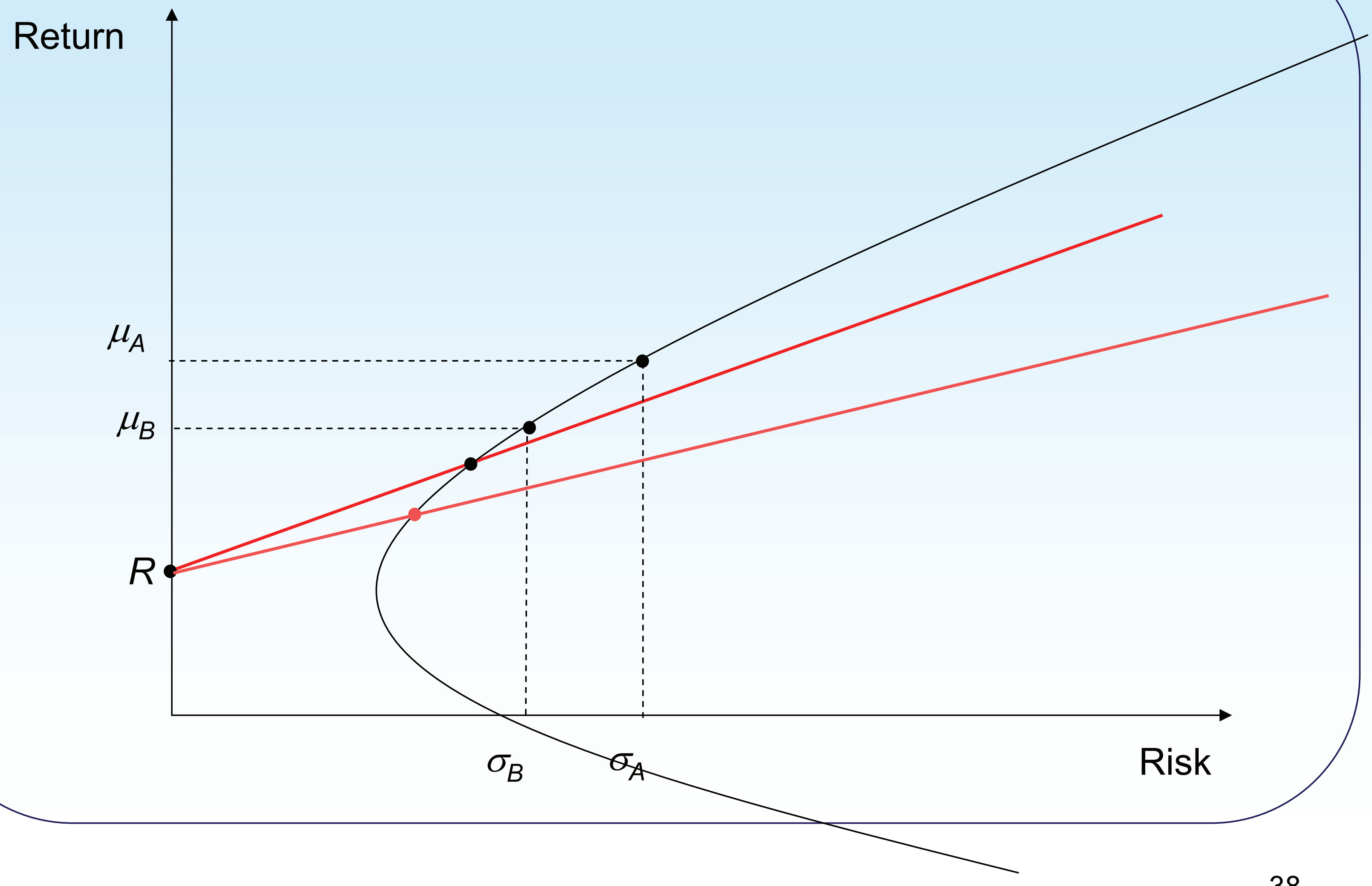

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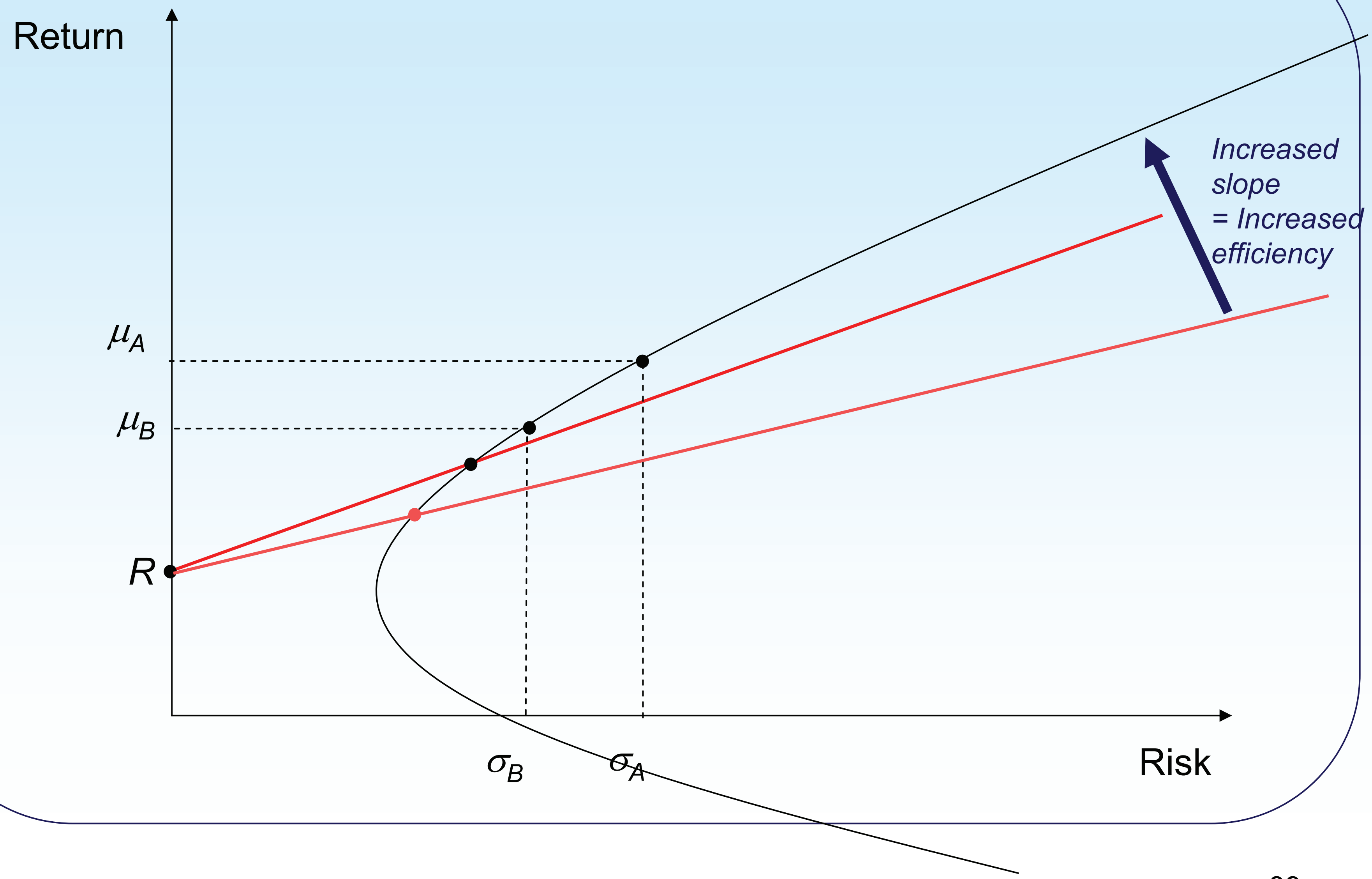

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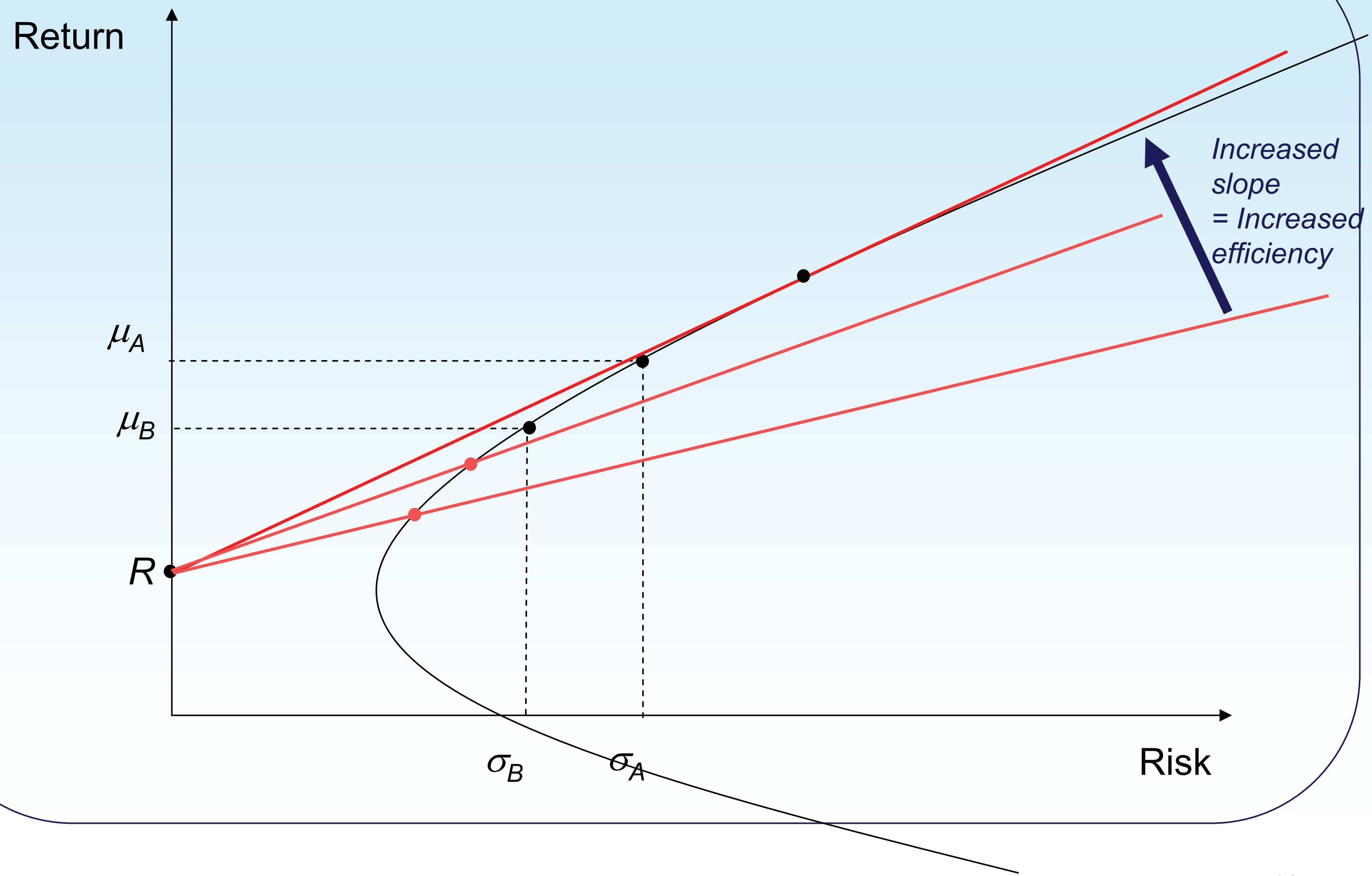
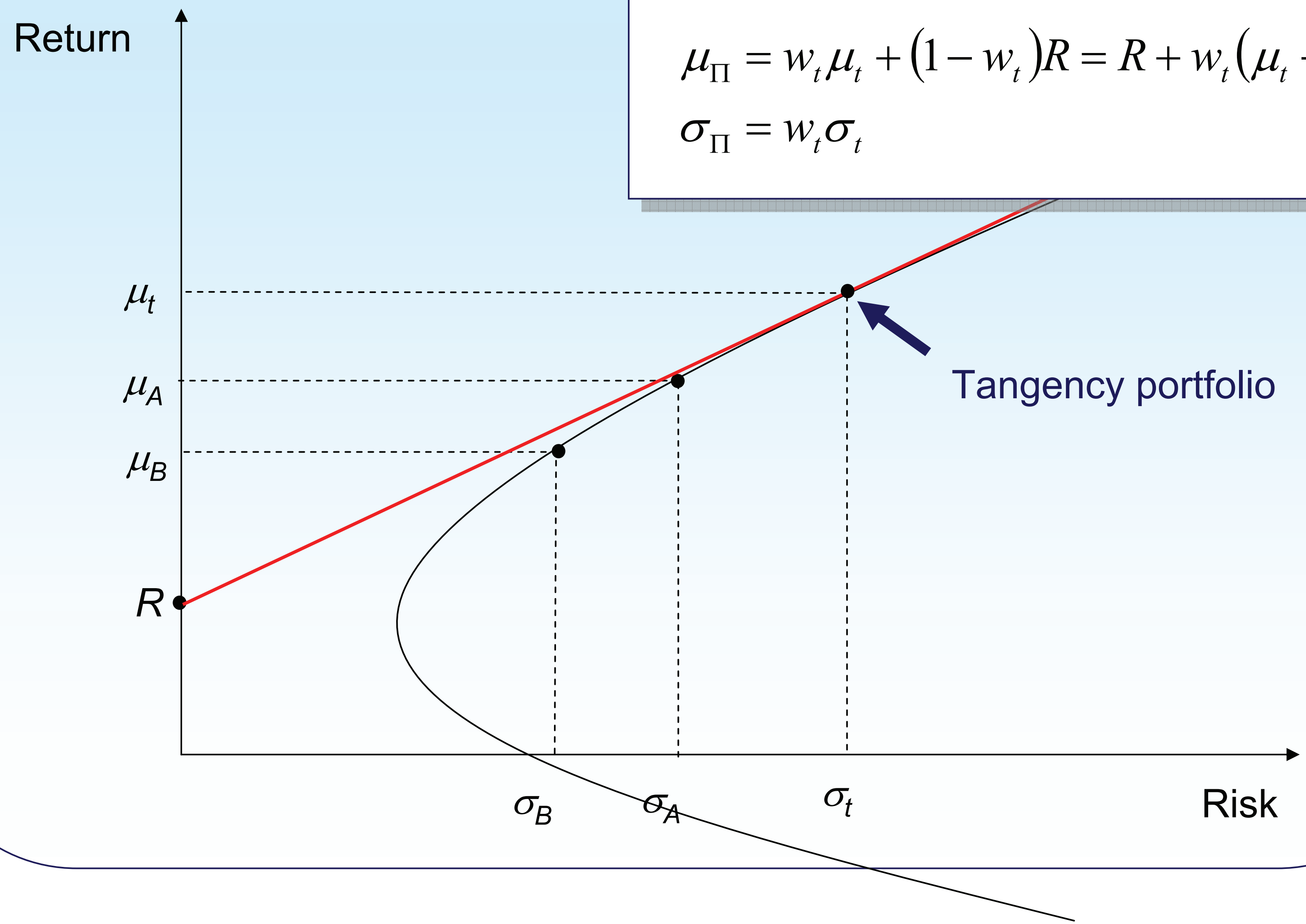


# The tangency portfolio

***Line parametrized by $w_t$:***

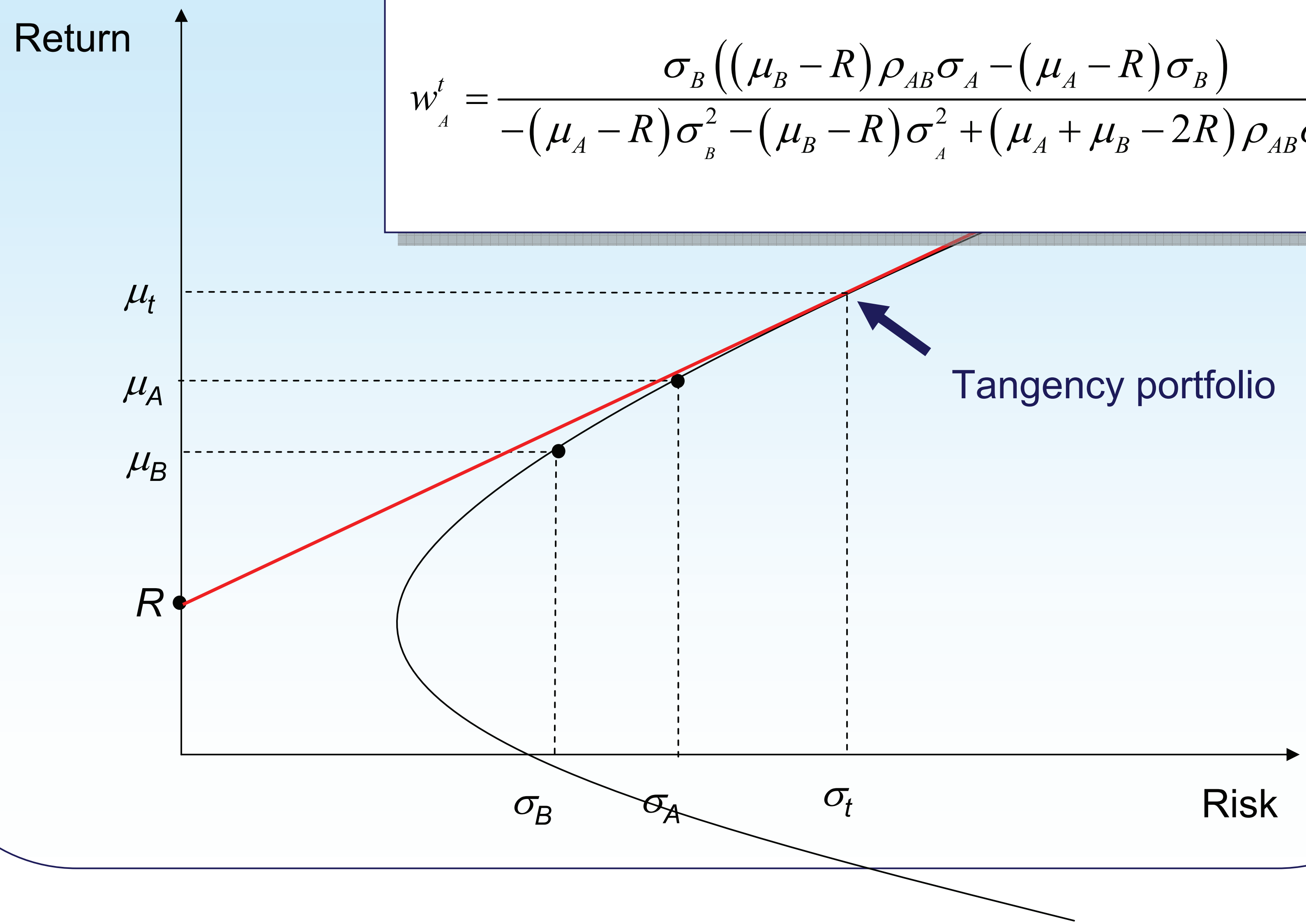
$$\mu_\Pi = w_t \mu_t + (1 - w_t) R = R + w_t (\mu_t - R)$$

$$\sigma_\Pi = w_t \sigma_t$$



# The tangency portfolio's allocation

$$w_A^t = \frac{\sigma_B\left(\left(\mu_B - R\right)\rho_{AB}\sigma_A - \left(\mu_A - R\right)\sigma_B\right)}{-\left(\mu_A - R\right)\sigma_B^2 - \left(\mu_B - R\right)\sigma_A^2 + \left(\mu_A + \mu_B - 2R\right)\rho_{AB}\sigma_A\sigma_B}$$



# Slope of the efficient frontier and tangency portfolio

- We can now express the risk-return relationship more directly.
- By the "risk equation", of the previous slide

$$w_t = \frac{\sigma_\Pi}{\sigma_t}$$

- Substituting in the return "equation" of the previous slide

$$\mu_\Pi = R + \sigma_\Pi \frac{\mu_t - R}{\sigma_t} = R + S_t \sigma_\Pi$$

where

$$S_t = \frac{\mu_t - R}{\sigma_t}$$

- This confirms our insights: the tangency portfolio is the risky portfolio for which the slope $S_t$ is maximized.

# The Sharpe Ratio

- For any investment *C*, one could consider the line of all portfolios made up of *C* and the RFA.

$$\mu_\Pi = R + \sigma_\Pi \frac{\mu_C - R}{\sigma_C} = R + S_C \sigma_\Pi$$

where

$$S_C = \frac{\mu_C - R}{\sigma_C}$$

- The slope $S_c$ is called the **Sharpe ratio** of investment *C*:
  - *It is a key measure of risk-adjusted return representing the excess return (over the risk free rate) per unit of **total** risk taken;*
  - *The higher the Sharpe ratio of a portfolio, the more efficient the portfolio is.*
  - *The risky portfolio with highest Sharpe ratio is the tangency portfolio.*

# Back to the General Problem: *N* risky assets and the risk-free asset

# Back to the general problem: $N$ risky assets

- We now return to the general case in which the market has $N \geq 2$ risky assets and one risk-free asset.

- All the concepts derived in the special case $N = 2$
  - *Opportunity set,*
  - *Efficient frontier,*
  - *Tangency portfolio,*
  - *Sharpe ratio*

  are still valid in the general setting.

- We will consider the following two cases:
  - *Portfolios of risky securities only;*
  - *Portfolios of risk-free and risky securities;*

# Case 1: Risky securities portfolio (Part 1)

- First, consider a portfolio fully invested in risky assets. Denote by $w_i$ the weight invested in asset $i$, $i$ =1,…,$N$.
- Since all of the wealth must be invested in the assets, the proportion of wealth invested or "weights" invested in the various assets must equal 100% of wealth. This leads to the budget equation

$$\sum_{i=1}^{N} w_i = 1$$

- In matrix notation, the budget equation can be expressed as

$$\mathbf{w}^T \mathbf{1}_N = 1$$

where

- $\mathbf{w}$ is the n-element column vector of weights;
- $\mathbf{v}^T$ denotes the transpose of vector $\mathbf{v}$;
- $\mathbf{1}_N$ is the N-element column unit vector, i.e. the vector with all entries set to 1.

# Case 1: Risky securities portfolio (Part 2)

- The expected return of Portfolio $\Pi$ is

$$E[r_\Pi] := \mu_\Pi = \sum_{i=1}^{N} w_i \mu_i$$

- The standard deviation of portfolio returns is

$$\sigma_\Pi = \sqrt{\sum_{i=1}^{N} w_i^2 \sigma_i^2 + 2 \sum_{\substack{i=1 \\ j>1}}^{N} w_i w_j \rho_{ij} \sigma_i \sigma_j}$$

- In matrix notation, we have respectively

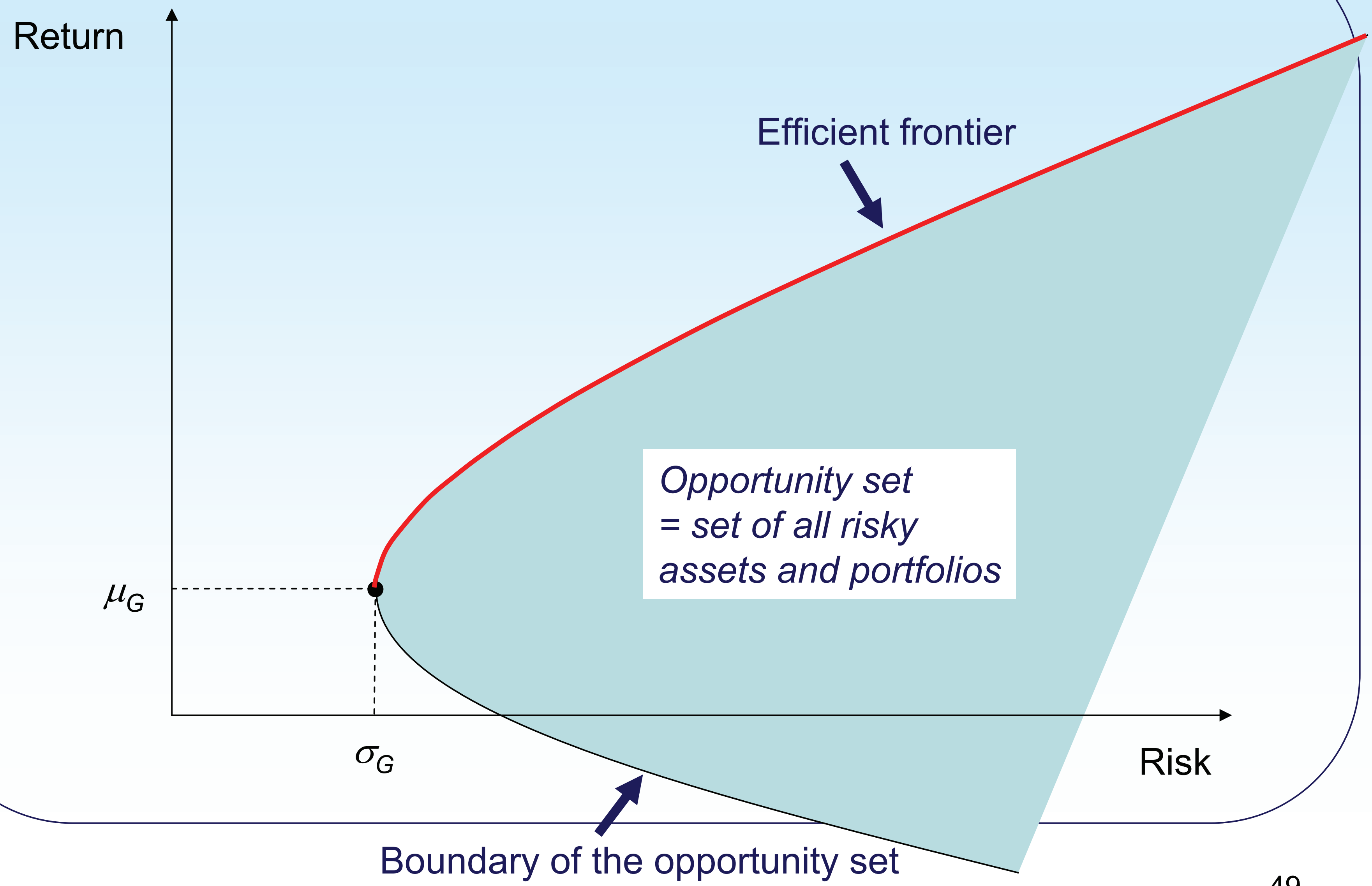
$$\mu_\Pi = \mathbf{w}^T \boldsymbol{\mu}$$

$$\sigma_\Pi = \sqrt{\mathbf{w}^T \boldsymbol{\Sigma} \mathbf{w}}$$

where

- $\boldsymbol{\mu}$ is the n-element column vector of expected returns;
- $\boldsymbol{\Sigma}$ is the covariance matrix.

# Case 1: Risky securities portfolio – efficient frontier



# Case 1: Quantifying diversification (Part 1)

- We will now illustrate how diversification works.

- For convenience, assume that the market is homogeneous
  - *All the securities have the same expected return $\mu_i = \mu$, i=1,…,N;*
  - *All the securities have the same standard deviation of return $\sigma_i = \sigma$, i=1,…,N;*
  - *The securities returns have the same correlation $\rho_{ij} = \rho$, i,j=1,…,N.*

- And we decide to invest equally in all $N$ risky securities so that $w_i = 1/N$.

- What happens to the portfolio return $\mu_i$ and the portfolio risk $\sigma_i$?

# Case 1: Quantifying diversification (Part 2)

- The expected return of the portfolio is

$$\mu_{\Pi} = \sum_{i=1}^{N} w_i \mu_i = N \times \frac{1}{N} \times \mu = \mu$$

The portfolio return stays the same irrespective of the value of *N*: we say that it is **invariant** in *N.*

# Case 1: Quantifying diversification (Part 3)

- The variance of portfolio returns is

$$\sigma_{\Pi}^2 = \sum_{i=1}^{N} w_i^2 \sigma_i^2 + 2\sum_{\substack{i=1 \\ j>1}}^{N} w_i w_j \rho_{ij}\sigma_i\sigma_j$$

$$= \sum_{i=1}^{N} w_i^2 \sigma_i^2 + \sum_{\substack{i=1 \\ j\neq 1}}^{N} w_i w_j \rho_{ij}\sigma_i\sigma_j$$

$$= N \times \frac{1}{N^2} \times \sigma^2 + N \times \left(N-1\right)\frac{1}{N^2} \times \rho \times \sigma^2$$

$$= \frac{N + \rho N\left(N-1\right)}{N^2}\sigma^2$$

$$= \left(\rho + \frac{1-\rho}{N}\right)\sigma^2$$

which shrinks to $\rho\sigma^2$ as N gets large.

# Case 1: Quantifying diversification (Part 4)

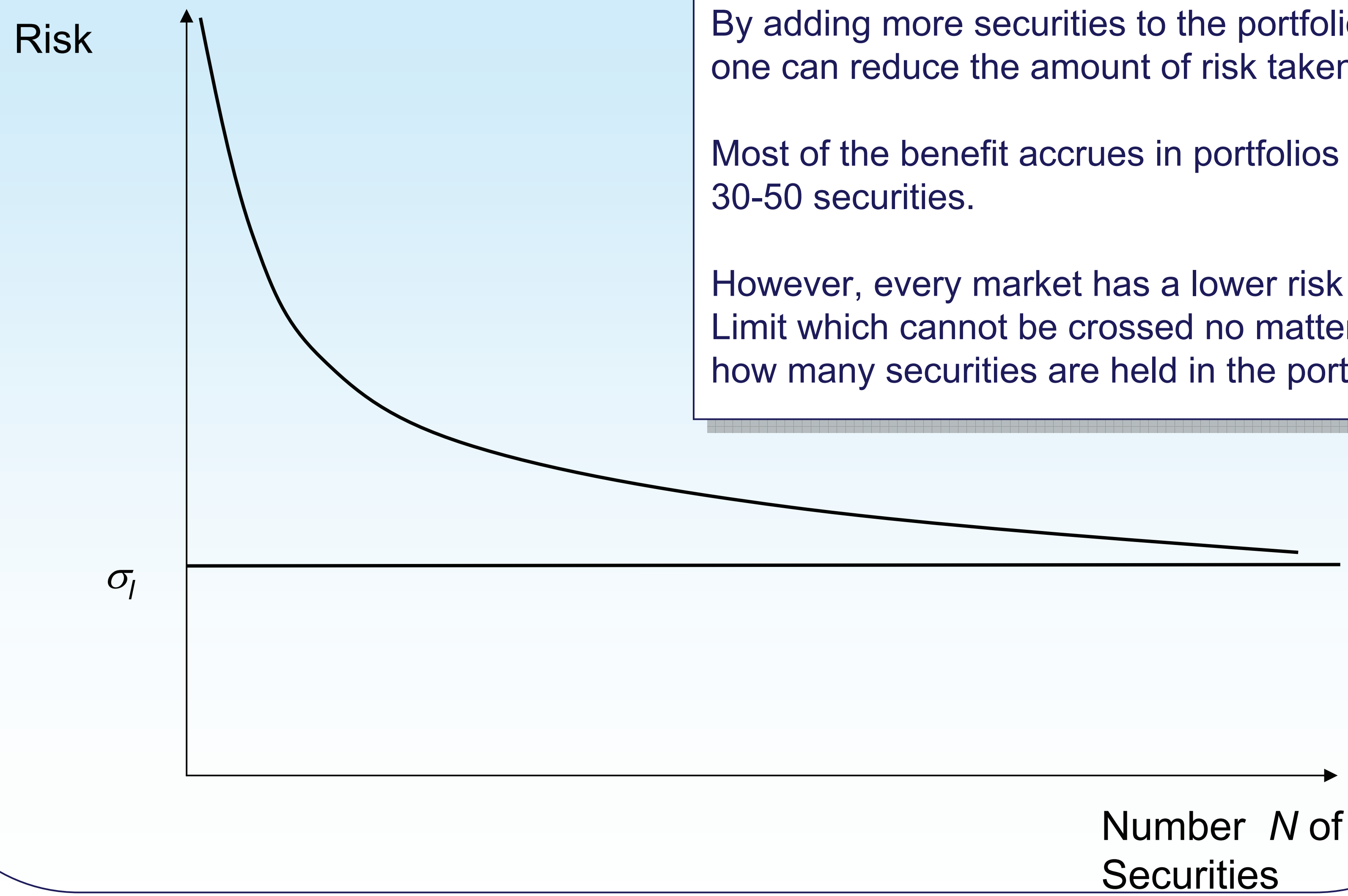
- If $\rho = 0$, then he variance of portfolio returns is

$$\sigma_{\Pi}^2 = \frac{1}{N}\sigma^2$$

which is $O(N^{-1})$.

- When returns are uncorrelated, the standard deviation of portfolio returns actually shrinks like $N^{-1/2}$ as N gets larger.

# Case 1: How far do diversification benefits extend?




By adding more securities to the portfolio, one can reduce the amount of risk taken.

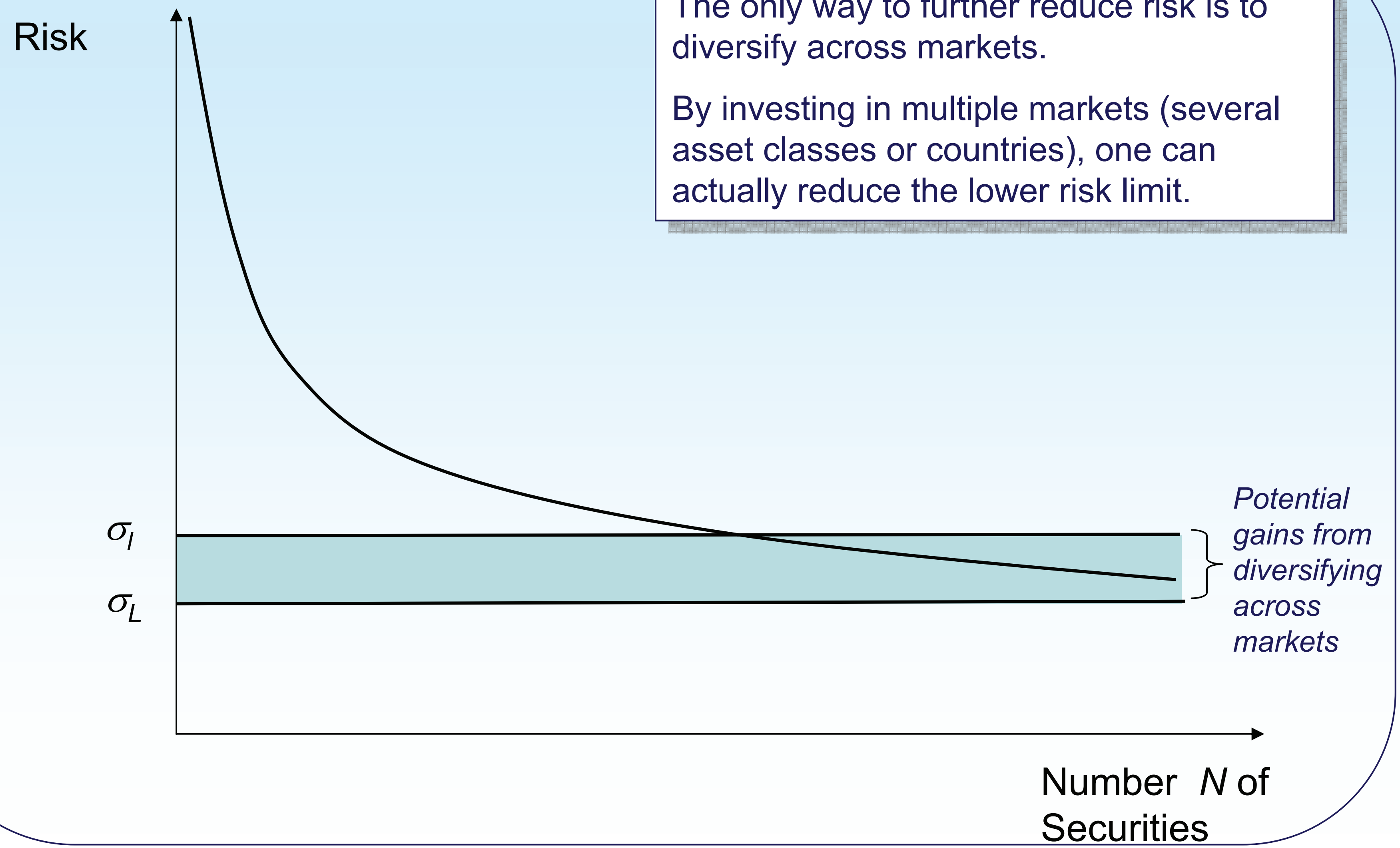
Most of the benefit accrues in portfolios of 30-50 securities.

However, every market has a lower risk Limit which cannot be crossed no matter how many securities are held in the portfolio

# Case 1: Diversifying across markets

The only way to further reduce risk is to diversify across markets.

By investing in multiple markets (several asset classes or countries), one can actually reduce the lower risk limit.



# Case 2: Risk-free and risky portfolio (Part 1)

- Denote by $w_0$ the weight of the risk-free asset in the portfolio. The budget equation in this case is

$$w_0 = 1 - \sum_{i=1}^{N} w_i$$

  and we consider the allocation to the risk-free asset as a residual of the allocation of wealth to the risky assets

- In matrix notation, the budget equation can be expressed as

$$w_0 = 1 - \mathbf{w}^T \mathbf{1}_N$$

# Case 2: Risk-free and risky portfolio (Part 2)

- The expected return of the portfolio is

$$E[r_\Pi] := \mu_\Pi = w_0 R + \sum_{i=1}^{N} w_i \mu_i = R + \sum_{i=1}^{N} w_i (\mu_i - R)$$

- The standard deviation of portfolio returns is still

$$\sigma_\Pi = \sqrt{\sum_{i=1}^{N} w_i^2 \sigma_i^2 + 2 \sum_{\substack{i=1 \\ j>1}}^{N} w_i w_j \rho_{ij} \sigma_i \sigma_j}$$

- In matrix notation,

$$\mu_\Pi = R + \mathbf{w}^T (\boldsymbol{\mu} - \mathbf{1}_N R)$$

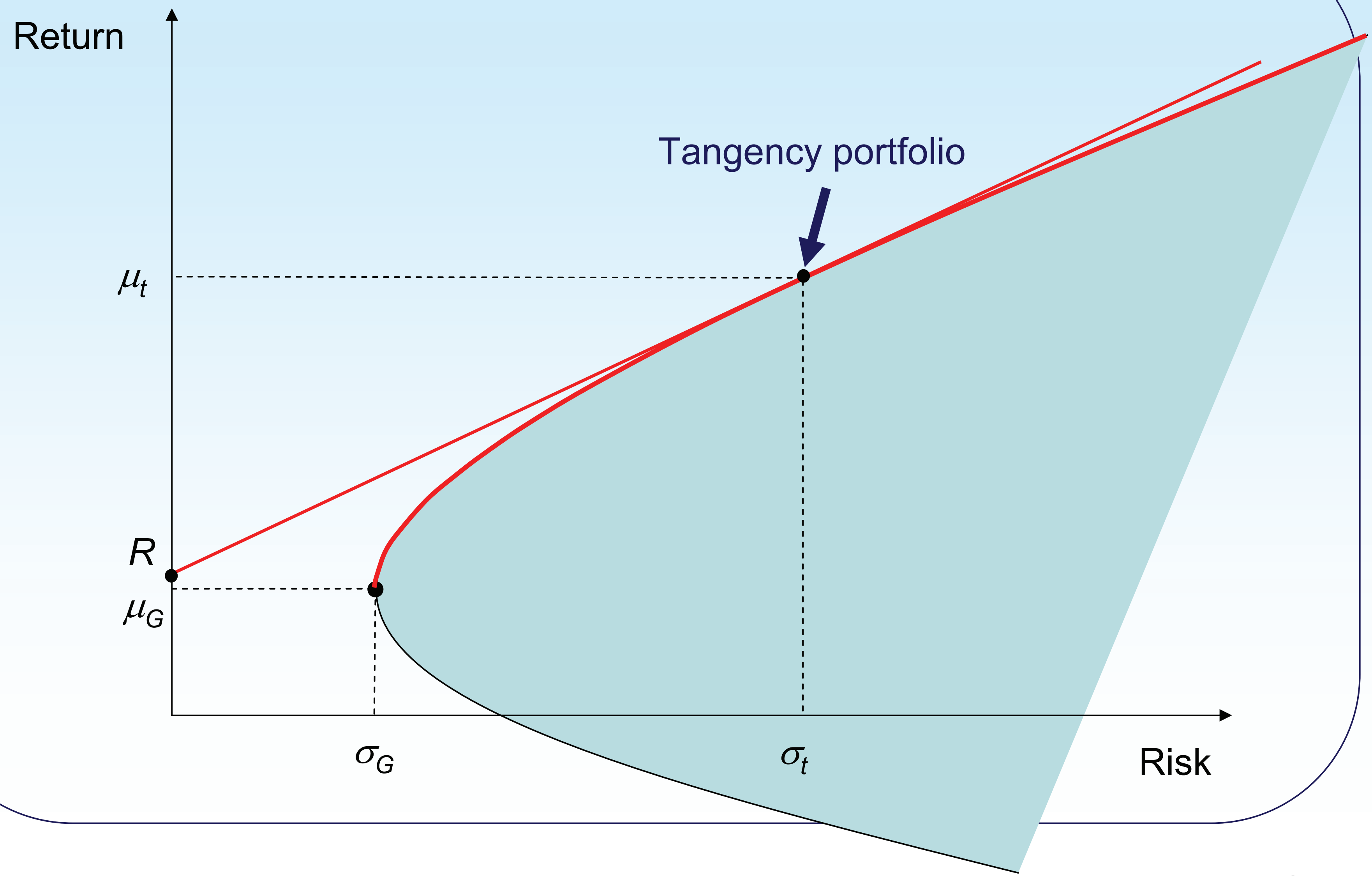
$$\sigma_\Pi = \sqrt{\mathbf{w}^T \boldsymbol{\Sigma} \mathbf{w}}$$

where

– $\boldsymbol{\mu}$ is the n-element column vector of expected returns;

– $\boldsymbol{\Sigma}$ is the covariance matrix.

# Case 2: Risk-free and risky portfolio – efficient frontier



# Further MPT

# Further MPT

- In the last part of this presentation, we introduce some further ideas about the MPT:
  - *The market portfolio and the market price of risk;*
  - *Computational efficiency of the mean-variance analysis;*
  - *The factor model;*
  - *The CAPM;*

# Homogeneity and the market portfolio

- Let's now assume that all the investors on the market share:
  - *The same investment universe of N risky securities and one risk-free asset returning R;*
  - *The same time horizon T;*
  - *The same estimations for the market parameters (expected return, standard deviation and correlation);*

- Then, all the investors will identify (and buy) the same tangency portfolio.
- The relative market values of all the securities on the market will adjust to reflect their allocation within the tangency portfolio.
- Consequently, the tangency portfolio becomes a perfect representation of the underlying asset market.
- In these equilibrium conditions, the tangency portfolio is called the **market portfolio**.

# Sharpe ratio and market price of risk

- The market portfolio is the risky portfolio which maximizes the Sharpe ratio (i.e. the slope of the efficient frontier).

- Since everyone now invests in the market portfolio and the RFA, the Sharpe ratio is interpreted as the ***market price of risk***.

- Indeed, the Sharpe ratio now measures the number of units of extra return generated (above the risk free- rate) per unit of ***market risk*** taken.

- *Note:* You will see more about market price of risk later in the CQF when you learn about interest rate modelling and bond pricing.

# The market in practice

- At the beginning of this presentation, we saw that the investment universe, what we now call “the market”, is comprised of all traded assets.

- However, there currently does not exist any financial index or economic time series capable of tracking the price of all tradable assets.

- The solution adopted in practice is to use a “proxy”: a financial index (such as the S&P 500, or the MSCI World Index) which represents a sizeable share of the assets traded on financial markets.

- The solution is not perfect since it only reflects a small portion of tradable assets, but it is deemed good enough in practice since few portfolio managers venture beyond a few asset classes.

# Computational efficiency of mean-variance analysis

- On of the main problems with what we have done so far is the dimensionality of the problem.

- If we had *N* risky securities (as opposed to two so far), we would need to estimate:
  - *N expected returns;*
  - *N standard deviations;*
  - *N(N-1)/2 correlations.*

- Hence, as *N* gets larger the number of parameters grows at a quadratic rate, i.e. $O(N^2)$.

- This pace is too fast to enable efficient computations.

# The linear factor model: definition

- To reduce the number of parameters, Sharpe postulated a simpler linear model linking portfolio returns to market returns, such as

$$r_i = \alpha_i + \beta_i r_M + \varepsilon_i$$

where

  - *$r_i$ is the actual return of asset i in the period of reference.*
  - *$\beta_i$ represents the exposure of the asset i to the market return and measures the exposure to **systematic risk**;*
  - *$\alpha_i$ represents the base return generated by the asset i;*
  - *$\varepsilon_i$ represents the **idiosyncratic risk** of asset i, a type of residual risk proper to asset i only and unrelated to any other asset or to the market. The assumption is that $\varepsilon_i \sim N(0,e_i^2)$ and $Cov[\varepsilon_i, \varepsilon_j] = 0$ for $i \neq j$.*

- Once the market portfolio (or a proxy) has been identified, the parameters can be estimated through linear regression.

# The factor model: computational efficiency

- With such model, one would only need
  - *N values of* $\alpha$ *;*
  - *N values of* $\beta$ *;*
  - *N values of* $\varepsilon$ *;*

  to parametrize a market with *N* risky assets.

- Hence, as *N* gets larger the number of parameters grows at a linear rate, i.e. *O*(*N*), which enables efficient computations.

# The factor model: some relationships

- Consider investment (i.e portfolio or asset) $C$, then

$$r_C = \alpha_C + \beta_C r_M + \varepsilon_C$$

and

$$E[r_C] := \mu_C = \alpha_C + \beta_C \mu_M$$

- The total risk of $C$, $\sigma_C$, is equal to:

$$\sigma_C = \sqrt{\beta_C^2 \sigma_M^2 + e_C^2}$$

by the properties of the variance.

- Considering in addition an investment $D$, then the covariance of returns between $C$ and $D$ is

$$Cov(C, D) := \sigma_{CD} = \beta_C \beta_D \sigma_M^2$$

by applying the properties of the covariance.

# The factor model: some more relationships

- In particular, if investment *C* is a portfolio of all the risky assets with respective weights $w_i$ in asset i, i=1,…,*N*, we can apply these relationships to deduce that

$$r_C = \sum_{i=1}^{N} w_i \alpha_i + \sum_{i=1}^{N} w_i \beta_i \, r_M + \sum_{i=1}^{N} w_i \varepsilon_i$$

$$E[r_C] := \mu_C = \sum_{i=1}^{N} w_i \, \alpha_i + \sum_{i=1}^{N} w_i \, \beta_i \, \mu_M$$

and

$$\sigma_C = \sqrt{\left( \sum_{i=1}^{N} w_i \, \beta_i \right)^2 \sigma_M^2 + \sum_{i=1}^{N} w_i^2 e_i^2}$$

since by independence of the random variables $\varepsilon_i$ , we have

$$e_C^2 = \sum_{i=1}^{N} w_i^2 e_i^2$$

# Quantifying the diversification benefits (Part 1)

- The factor model sheds a different light on the diversification question.
- For convenience, we will assume that:
  - *all of the idiosyncratic risks are not only independent, but IID, i.e. for all* i, $\varepsilon_i \sim N(0,e^2)$ *for some constant e and* $Cov[\varepsilon_i, \varepsilon_j] = 0$ *for* i ≠ j;
  - *when we invest in a portfolio, we invest an equal proportion in each security, so that* $w_i = w = 1/N$, *and;*
  - *all the securities have the same systematic risk*[1], *so that* $\beta_i = \beta$.

[1] *This last assumption is not necessary, but it makes the argument clearer.*

# Quantifying the diversification benefits (Part 2)

- The formula $\sigma_C = \sqrt{\beta^2 \sigma_M^2 + e^2}$

  shows that the variance of an investment in a security *C* is comprised of both systematic and idiosyncratic risk.

- In the case of an investment in a portfolio of N securities, we have

$$\sigma_\Pi = \sqrt{\left(\sum_{i=1}^{N} w_i \beta_i\right)^2 \sigma_M^2 + \sum_{i=1}^{N} w_i e_i^2}$$

- Taking the limit as N $\rightarrow \infty$, and by the Central Limit Theorem, this last equation becomes

$$\sigma_\Pi = \beta \sigma_M$$

- Idiosyncratic risk has vanished!

# The factor model: an adhoc model

- The linear factor model is an "adhoc" model, i.e.
  - *it is practically convenient,…*
  - *… but it is not theoretically justified.*

- Because it is not theoretically justified, adhoc models do not have any predictive power and should not be used for forecasting purpose[1].

- However, Sharpe also developed a very similar economic model: the Capital Asset Pricing Model.

[1] *Although they are very much used in practice!*

# The CAPM

- The Capital Asset Pricing Model (CAPM)
  - *Is a linear factor model, in which the factor is the market return;*
  - *Is derived directly from the mean-variance analysis (see "Fundamentals of Optimization and Application to Portfolio Selection" for more details);*
  - *Is an equilibrium model: it can be used to predict asset prices;*
  - *Can be applied to any security or portfolio;*
  - *Is expressed in terms of* ***expectations****.*

- For an investment *I*, It takes the form

$$E\left[r_I - R\right] = \beta_I E\left[r_M - R\right]$$

or, alternatively

$$E\left[r_I\right] = R + \beta_I E\left[r_M - R\right]$$

# The CAPM

- The CAPM states that the risk premium on any investment is:
  - *Proportional to the risk premium of the market;*
  - *And the proportionality constant is the* ***degree of systematic risk*** *of the investment.*
- In short, “on average the market is compensating us for taking on systematic risk”.
- Because of the Expectation operator, the CAPM can be used as a predictive model.

# The CAPM

- One variable does not appear in the CAPM: idiosyncratic risk. Where did it go?

- Because we take the expectation, idiosyncratic risk vanishes.

- Read differently, the CAPM implies that only systematic risk should be rewarded, not idiosyncratic risk.

- This is quite logical: since we can diversify away all of our idiosyncratic risk, the market should not compensate us for taking this type of risk.

- This idea is central to financial economics. You will see it again later in the CQF when you learn about the implication of using jump-diffusion processes to price options.

# MPT in Practice

# How is MPT used in practice?

- The impact of MPT on our understanding of financial risks and of the mechanics of portfolio construction cannot be understated.

- In the industry, MPT is routinely used has a frame of reference to
  - *Understand portfolio construction;*
  - *Evaluate financial risks;*
  - *Compute the cost of equity in corporate finance.*

- In a recent survey of trends in quantitative equity management, Fabozzi, Foccardi and Jonas[1] found 30 out of the 36 firms polled (i.e. 83%) actively used Mean-variance optimization.

- However, historically, MPT has suffered two main drawbacks:
  - *Dimensionality;*
  - *Parameter estimation.*

[1] *Fabozzi, F, S. Focardi and C. Jonas. Trends in quantitative equity management: survey Results. Quantitative Finance. 7(2): 115-122. April 2007*

# Dimensionality

- **Dimensionality** was and is still an important concern due to the vast size of financial markets. Although factor models can be used to reduce the dimensionality of the problem, they still do not hold all the answers:
  - *What to do with non-linear assets (bonds, securities with embedded options…)?*
  - *What index/indexes should be used?*
  - *Are the parameters stable over time?*

# Parameter estimation

- Since, optimizers are particularly efficient at taking advantage of the smallest discrepancy in data to reach their objective, **parameter estimation** is critical to get workable investment policies. This phenomenon is often called the “garbage in, garbage out” syndrome.

- The good news is that variance and covariance of returns tend to be quite stable over long periods of time…

- …but the bad news is that it would take hundreds of years of financial data to get a reasonably accurate estimates of expected returns.
  - *few assets have been traded long enough;*
  - *in any case, market conditions change over time which cause “breaks” in the time series of returns.*

# What are the solutions?

- Two school of thoughts developed practical ways of improving the MPT:

  – *The first one, advocates staying in a 1-period framework and improving the optimization process through either*
    - improved parameter estimation techniques (i.e. Bayesian techniques), or;
    - the use of more robust optimization techniques.

  – *The second school promotes the design a multi-period multi-scenario* ***stochastic programming models****. This method has the important advantage of acknowledging*
    - That financial markets are dynamic in nature, and;
    - That it is generally more important to avoid financial disaster in difficult times than generating considerable returns in good times. Thus, scenarios are chosen to model more accurately the left tail of the return distribution.

# Conclusion

# In this lecture, we have seen…

- The key concepts of MPT:
  - *Risky and risk-free assets;*
  - *Mean-variance analysis;*
  - *Optimal portfolio;*
  - *Diversification;*
  - *Opportunity set and efficient frontier;*
  - *Tangency and market portfolio;*
  - *Sharpe ratio and market price of risk;*
  - *The linear model and the CAPM.*

- The drawbacks of MPT: dimensionality and parameter estimation.

- The mathematics of optimization, required to solve portfolio selection problem, are treated the companion lecture: "Fundamentals of Optimization and Application to Portfolio Selection."