Know Your Weapon

Dr. Espen Gaarder Haug

• Bronzin 1908 1900

Market Formula (Bachelier-Thorp)

1904

Nelson

$$c = Se^{(b-r)T}N(d_1) - Xe^{-rT}N(d_2)$$
$$p = Xe^{-rT}N(-d_2) - Se^{(b-r)T}N(-d_1)$$

Where:
$$d_1 = \frac{\ln(S/X) + (b + \sigma_{X,T}{}^2/2)T}{\sigma_{X,T}\sqrt{T}}$$
$$d_2 = d_1 - \sigma_{X,T}\sqrt{T}$$

1973

$b = r$ $b = 0$

$b = r - r_f$

S = Asset price
X = Strike
T = Years to maturity
r = risk - free - rate
b = cost - of - carry
$\sigma_{X,T}$ = volatility that can be different for each strike and maturity

See Haug 2007 "Derivatives Models on Models" chapter 2

Black-Scholes-Merton

$$c = Se^{(b-r)T}N(d_1) - Xe^{-rT}N(d_2)$$
$$p = Xe^{-rT}N(-d_2) - Se^{(b-r)T}N(-d_1)$$

Where:

$$d_1 = \frac{\ln(S/X) + (b + \sigma^2/2)T}{\sigma\sqrt{T}}$$
$$d_2 = d_1 - \sigma\sqrt{T}$$

S = Asset price
X = Strike
T = Years to maturity
r = risk - free - rate
b = cost - of - carry
σ = volatility

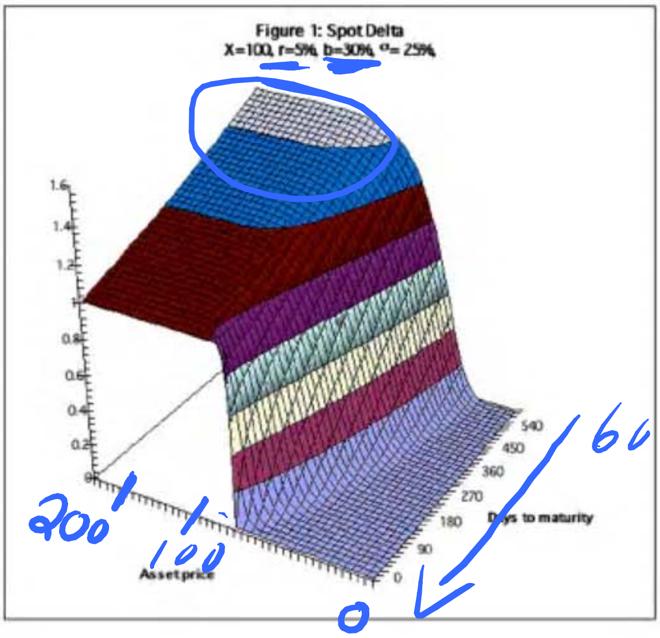
Delta Greeks

- Delta
- Delta mirror strikes
- Strike from delta
- Elasticity

Delta higher than one

$$\Delta_{call} = \frac{\partial c}{\partial S} = e^{(b-r)T} N(d_1)$$

$$\Delta_{put} = \frac{\partial p}{\partial S} = -e^{(b-r)T} N(-d_1)$$



6TC

Delta Mirror Strikes

-40 40

$$X_P = \frac{S^2}{X_C} e^{(2b+\sigma^2)T}, \quad X_C = \frac{S^2}{X_P} e^{(2b+\sigma^2)T}$$

ln(S/X)

Special case delta symmetric straddle (Wystrup(1999)):

$$X_C = X_P = Se^{(b+\sigma^2/2)T}$$

X=S

Delta symmetric asset: $S = Xe^{(-b-\sigma^2/2)T}$

At this strike the delta is $\Delta_C = \frac{e^{(b-r)T}}{2}, \quad \Delta_P = -\frac{e^{(b-r)T}}{2}$

=Δ=50

$$c = \frac{Se^{(b-r)T}}{2} - X^{-rT} N(-\sigma\sqrt{T}), \quad p = X^{-rT} N(\sigma\sqrt{T}) - \frac{Se^{(b-r)T}}{2}$$

T=3/12

12% - 12.50%

12.96%

$\Delta = 25\%$

$T = 3/12$

$\sigma = 12\%$

$r =$

$r_f =$

Strikes from delta

Wystrup(1999):

$$X_C = S\exp[N^{-1}(\Delta_C e^{(r-b)T})\sigma\sqrt{T} + (b+\sigma^2/2)T]$$

$$X_P = S\exp[N^{-1}(-\Delta_P e^{(r-b)T})\sigma\sqrt{T} + (b+\sigma^2/2)T]$$

Robust and accurate approximation of inverse cumulative normal distribution needed, Moro(1995).

$\boxed{\text{AUD}}$ FX V T

1W m T

1M

2M

3M

6M

9M

12M

11%

DdeltaDvol

$$\frac{\partial c}{\partial S \partial \sigma} = \frac{\partial p}{\partial S \partial \sigma} = \frac{-e^{(b-r)T} d_2}{\sigma} n(d_1)$$

0.30

Maximal value at

$$S_L = X e^{-bT - \sigma\sqrt{T}\sqrt{4+T\sigma^2}/2}$$

Minimal value at

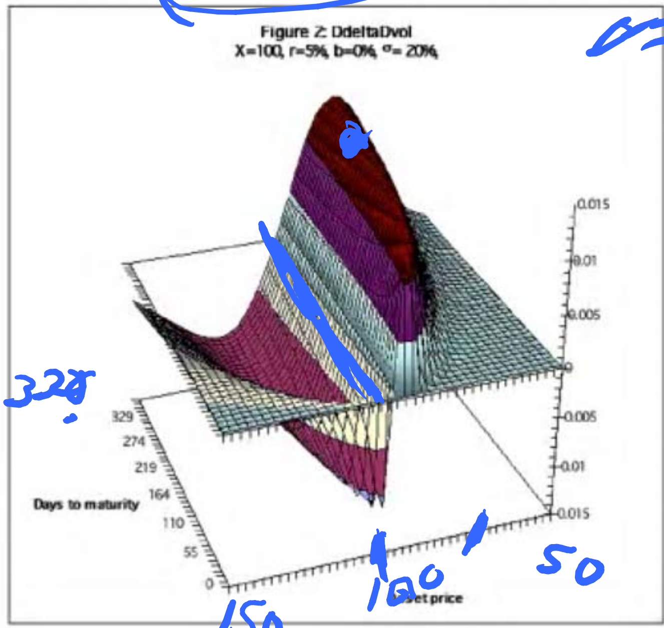
$$S_U = X e^{-bT + \sigma\sqrt{T}\sqrt{4+T\sigma^2}/2}$$

Minimal value at

$$X_L = S e^{bT - \sigma\sqrt{T}\sqrt{4+T\sigma^2}/2}$$

Maximal value at

$$X_U = S e^{bT + \sigma\sqrt{T}\sqrt{4+T\sigma^2}/2}$$



Useful Tools

- A library
- Paper and pencil
- Mathematica
- Maple
- Matlab (?)
- Others ?

Implementation:

VBA, VB, C/C++, Java….

you name it

Elasticity

$$\Lambda_{call} = \Delta_{call}\frac{S}{call}, \quad \Lambda_{put} = \Delta_{put}\frac{S}{put}$$

Option volatility: $\sigma_O \approx \sigma\,|\,\Lambda\,|$ Compound options

Option Beta, expected return satisfy the CAPM equation (Merton-71):

$$E[return] = r + E[r_m - r]\beta_i$$

$$\beta_C = \frac{S}{call}\Delta_C\beta_S = \Lambda_C\beta_S, \quad \beta_P = \frac{S}{put}\Delta_P\beta_S = \Lambda_P\beta_S$$

Option Sharp ratios

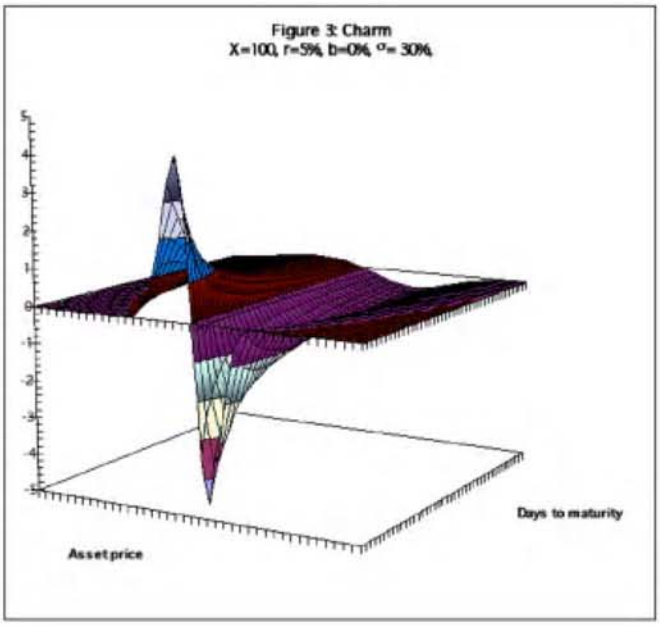
$$\frac{\mu_O - r}{\sigma_O} = \frac{\mu_S - r}{\sigma}$$

Smile?

Charm

$$\frac{\partial\Delta_C}{\partial T} = -e^{(b-r)T}\left[n(d_1)\left(\frac{b}{\sigma\sqrt{T}} - \frac{d_2}{2T}\right) + (b-r)N(d_1)\right]$$

$$\frac{\partial\Delta_P}{\partial T} = -e^{(b-r)T}\left[n(d_1)\left(\frac{b}{\sigma\sqrt{T}} - \frac{d_2}{2T}\right) - (b-r)N(-d_1)\right]$$



F

V

O

F C

M O

Gamma Greeks

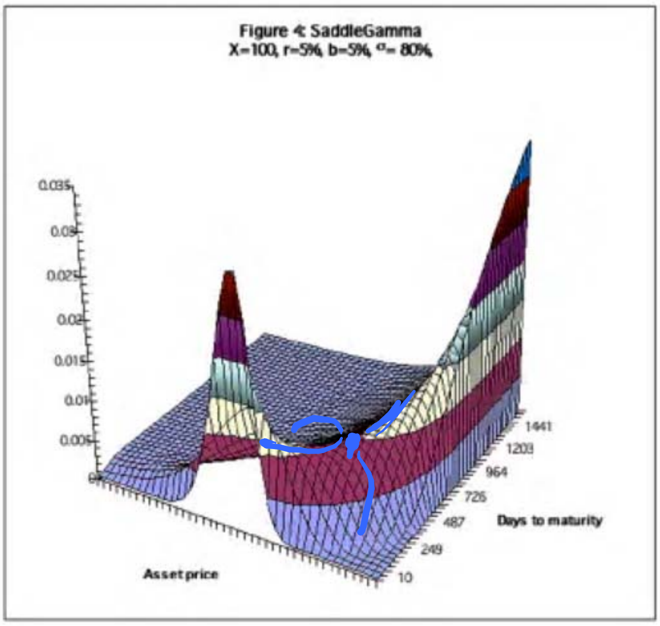
- Gamma
- Saddle gamma
- GammaP
- Gamma symmetry
- DGammaDVol
- DGammaDspot
- DGammaDTime

Saddle Gamma
Alexander Adamchuk www.wilmott.com

$$T_{\Gamma} = \frac{1}{2(\sigma^2 + b)}$$

$$S_{\Gamma} = X e^{(-b-3\sigma^2/2)T_S}$$

$$\Gamma_S = \frac{e^{(b-r)T}\sqrt{\frac{e}{\pi}}\sqrt{\frac{b}{\sigma^2}+1}}{X}$$



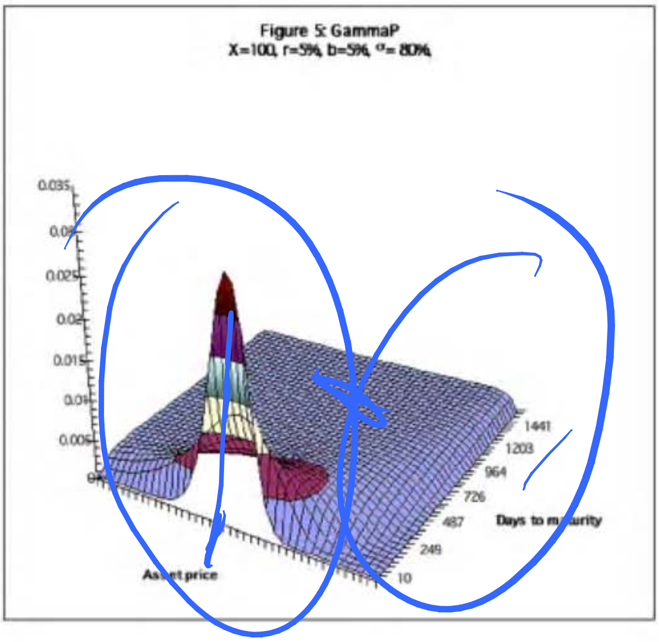
GammaP

$$\Gamma_P = \Gamma \frac{S}{100}$$

Max GammaP at

$$S = Xe^{(-b-\sigma^2/2)T}$$

$$X = Se^{(b+\sigma^2/2)T}$$



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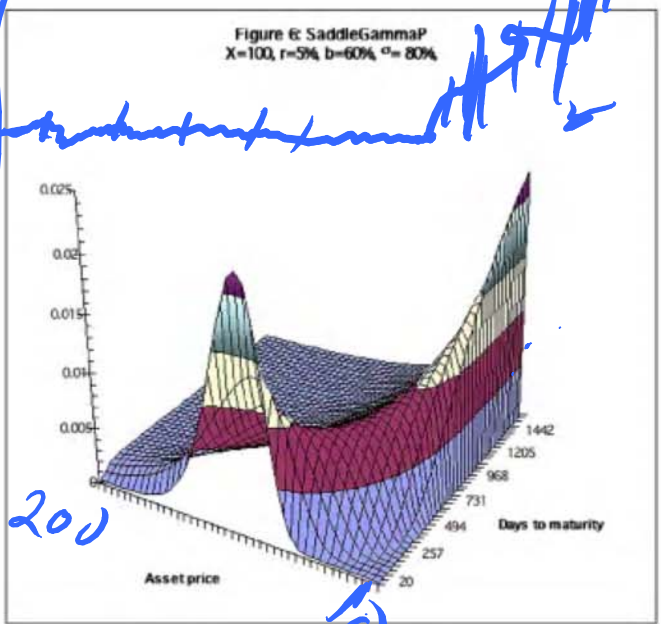
Saddle GammaP

•Spot gamma

•Forward gamma



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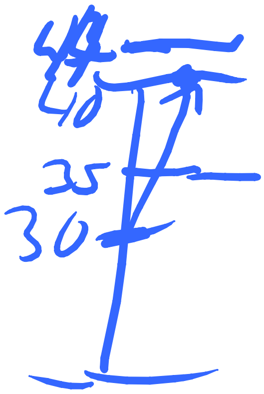
$F = Se^{bT}$

Gamma-symmetry

Put-call symmetry Bates(1991) and Carr and Bowie (1994):

$$c(S,X,T,r,b,\sigma)=\frac{X}{Se^{bT}}p\left(S,\frac{(Se^{bT})^2}{X},T,r,b,\sigma\right)$$

100



Gamma-symmetry

$$\Gamma(S,X,T,r,b,\sigma)=\frac{X}{Se^{bT}}\Gamma\left(S,\frac{(Se^{bT})^2}{X},T,r,b,\sigma\right)$$

Also gives vega and cost-of-carry symmetry

DgammaDvol $\quad \frac{\partial\Gamma}{\partial\sigma}=\Gamma\left(\frac{d_1d_2-1}{\sigma}\right) \qquad \frac{\partial\Gamma_P}{\partial\sigma}=\Gamma_P\left(\frac{d_1d_2-1}{\sigma}\right)$

Positive outside interval

$$S_L=Xe^{-bT-\sigma\sqrt{T}\sqrt{4+T\sigma^2}/2}$$

$$S_U=Xe^{-bT+\sigma\sqrt{T}\sqrt{4+T\sigma^2}/2}$$

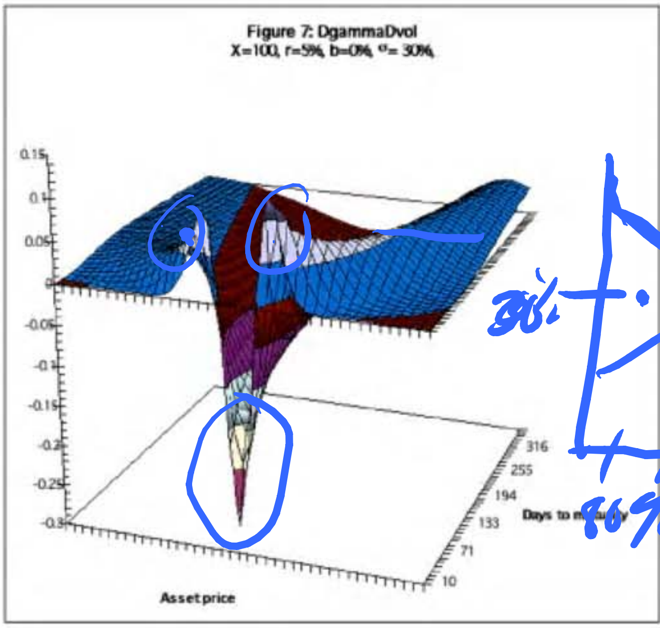
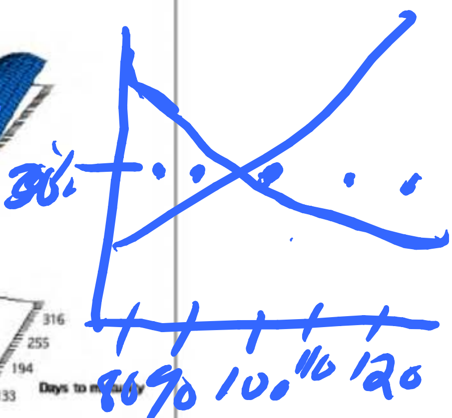


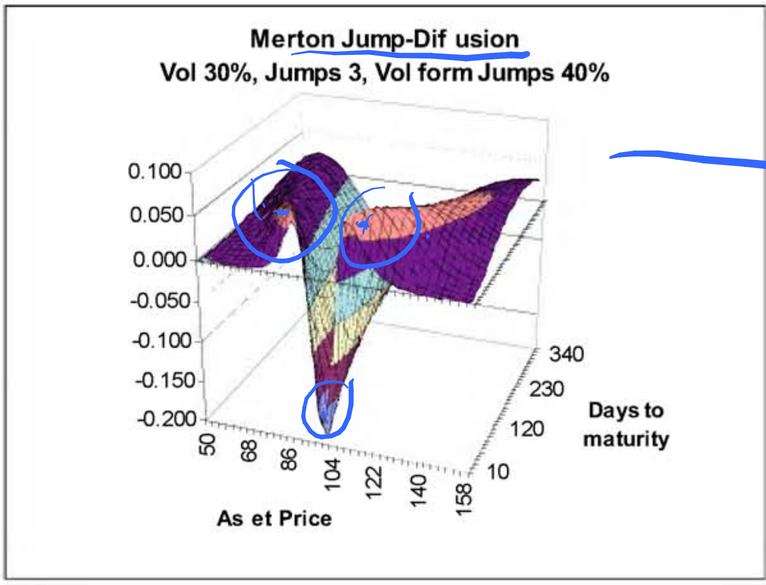
Figure 7: DgammaDvol
X=100, r=5%, b=0%, σ=30%



$\sigma=30\%$

$\sigma_R=40\%$

$\sigma_R=20\%$



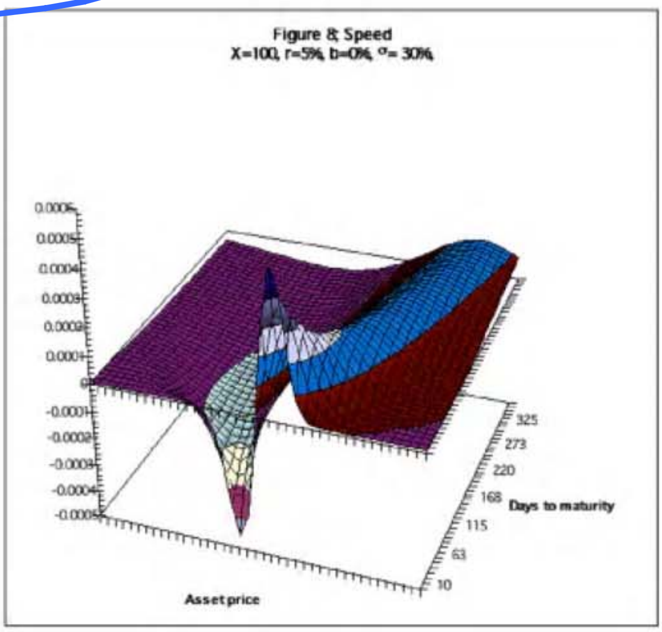
Call

Asset price (S)	80.00
Strike price (X)	100.00
Time to maturity (T)	0.25
Risk-free rate (r)	5.00%
Volatility (σ)	30.00%
Jumps per year (λ)	3.00
Percent of total volatility (γ)	40.00%
Value	0.5255

Speed (DgammaDspot)

$$\frac{\partial^3 c}{\partial S^3} = -\frac{\Gamma\left(1+\frac{d_1}{\sigma\sqrt{T}}\right)}{S}$$

$$SpeedP = -\Gamma\frac{d_1}{S}$$



Speed is used by Fouque, Papanicolaou, and Sircar (2000) as part of stochastic vol model

DgammaDtime

$$\frac{\partial \Gamma}{\partial T} = \Gamma\left(r - b + \frac{bd_1}{\sigma\sqrt{T}} + \frac{1-d_1 d_2}{2T}\right)$$

$$\frac{\partial \Gamma_P}{\partial T} = \Gamma_P\left(r - b + \frac{bd_1}{\sigma\sqrt{T}} + \frac{1-d_1 d_2}{2T}\right)$$

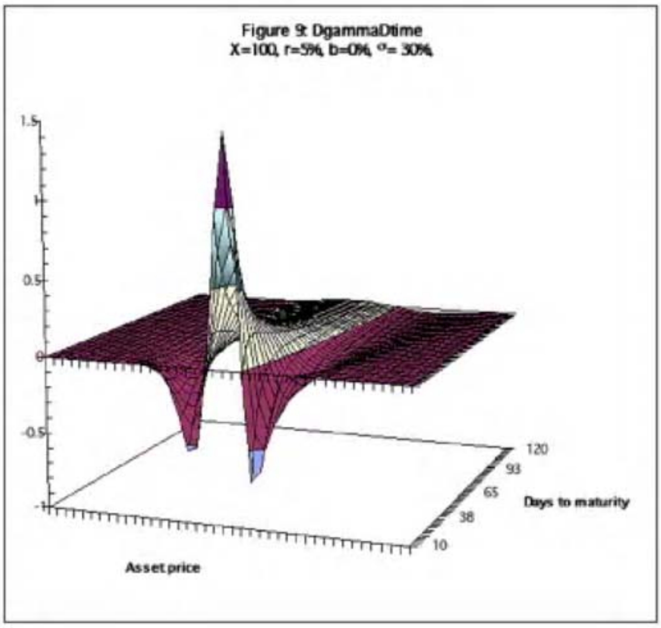


Figure 9: DgammaDtime
X=100, r=5%, b=0%, σ= 30%

Numerical Greeks

S=100$

•More robust (?)
•Model independent
•Faster to implement (?)

c(100+0.1)
c(100-0.1)

Two-sided finite difference

$$\Delta_C \approx \frac{c(S+\Delta S, X, T, r, b, \sigma) - c(S-\Delta S, X, T, r, b, \sigma)}{2\Delta S}$$

Backward derivative:

4 2/365

$$\Theta \approx \frac{c(S, X, T, r, b, \sigma) - c(S, X, T-\Delta T, r, b, \sigma)}{\Delta T}$$

Numerical Greeks

20%

21% 19% ΔS

$$\Delta_C \approx \frac{c(S+\Delta S, X, T, r, b, \sigma_1) - c(S-\Delta S, X, T, r, b, \sigma_2)}{2\Delta S}$$

$$\Theta \approx \frac{c(S, X, T, r, b, \sigma_1) - c(S, X, T-\Delta T, r, b, \sigma_2)}{\Delta T}$$

Gamma and other second derivatives, central finite difference

$$\Gamma \approx \frac{c(S+\Delta S, \ldots) - 2c(S, \ldots) + c(S-\Delta S, \ldots)}{\Delta S^2}$$

Speed and other third order derivatives, central finite difference

$$Speed \approx \frac{1}{\Delta S^3}[c(S+2\Delta S, \ldots) - 3c(S+\Delta S, \ldots) + 3c(S, \ldots) - c(S-\Delta S, \ldots)]$$

Know Your Weapon
Part 2

Numerical Greeks

What about mixed derivatives? For example DdeltaDvol and Charm

$$DdeltaDvol \approx \frac{1}{4\Delta S\Delta\sigma}[c(S+\Delta S,...,\sigma+\Delta\sigma)-c(S+\Delta S,...,\sigma-\Delta\sigma)$$
$$-c(S-\Delta S,...,\sigma+\Delta\sigma)+c(S-\Delta S,...,\sigma-\Delta\sigma)]$$

Vega “Greeks”

- ~~Vega~~
- Vega maximum
- VegaP
- Vega symmetry
- Vega Leverage
- DVegaDvol
- DVegaDtime

X=100

Vega $\frac{\partial c}{\partial \sigma} = Se^{(b-r)T} n(d_1)\sqrt{T}$

$p = c \approx 0.4 \cdot S \cdot \sigma\sqrt{T} \cdot e^{-rT}$

S=200

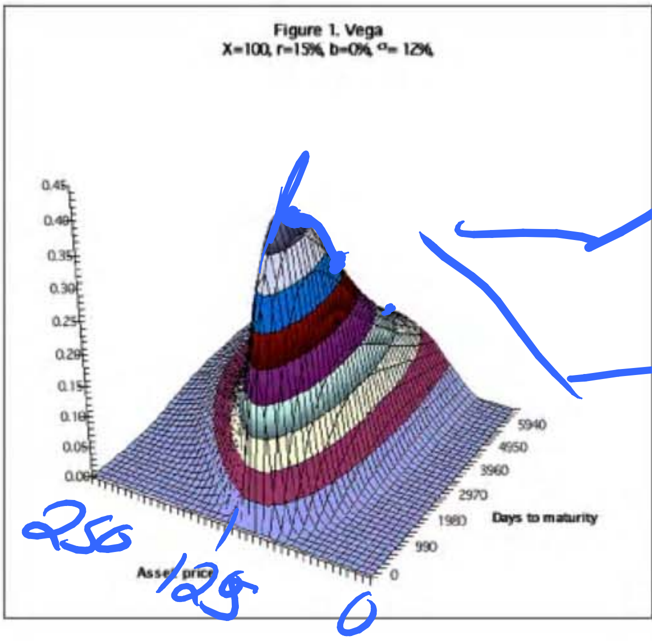
Vega local max

$$S = Xe^{(-b+\sigma^2/2)T}$$
$$X = Se^{(b+\sigma^2/2)T}$$

Global maximum

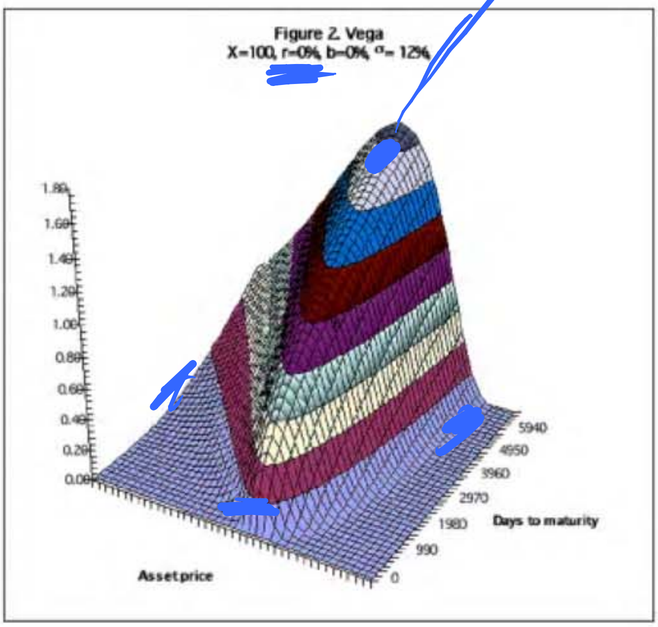
$$T_V = \frac{1}{2r}$$
$$S_V = Xe^{(-b+\sigma^2/2)T_V}$$
$$= Xe^{\frac{-b+\sigma^2/2}{2r}}$$

$$Vega(S_V, T_V) = \frac{X}{2\sqrt{re\pi}}$$



Why the Vega top?

Discounting at some point will dominate over volatility effect (Vega).



Vega-symmetry

Put-call symmetry Bates(1991) and Carr and Bowie (1994):

$$c(S, X, T, r, b, \sigma) = \frac{X}{Se^{bT}} p(S, \frac{(Se^{bT})^2}{X}, T, r, b, \sigma)$$

Vega-symmetry

$$Vega(S, X, T, r, b, \sigma) = \frac{X}{Se^{bT}} Vega(S, \frac{(Se^{bT})^2}{X}, T, r, b, \sigma)$$

Also gives gamma and cost-of-carry symmetry

Vega-gamma relationship

Taleb(1997):

$$Vega = \Gamma \sigma S^2 T$$

Vega from delta

$$Vega = Se^{(b-r)T} \sqrt{T} n[N^{-1}(e^{(r-b)T} |\Delta|)]$$

Gamma from delta

$$\Gamma = \frac{e^{(b-r)T} n[N^{-1}(e^{(r-b)T} |\Delta|)]}{S\sigma\sqrt{T}}$$

VegaP

Vega gives dollar change in option value for one percent point change in implied volatility. VegaP gives dollar change in option value for percentage move in volatility.

$$VegaP = \frac{\sigma}{10} Se^{(b-r)T} n(d_1)\sqrt{T}$$

VegaP makes much more sense when comparing sensitivity to changes in Implied volatility.

If you want to speculate on an increase in implied volatility what type of options offers the most bang for the bucks?

10"

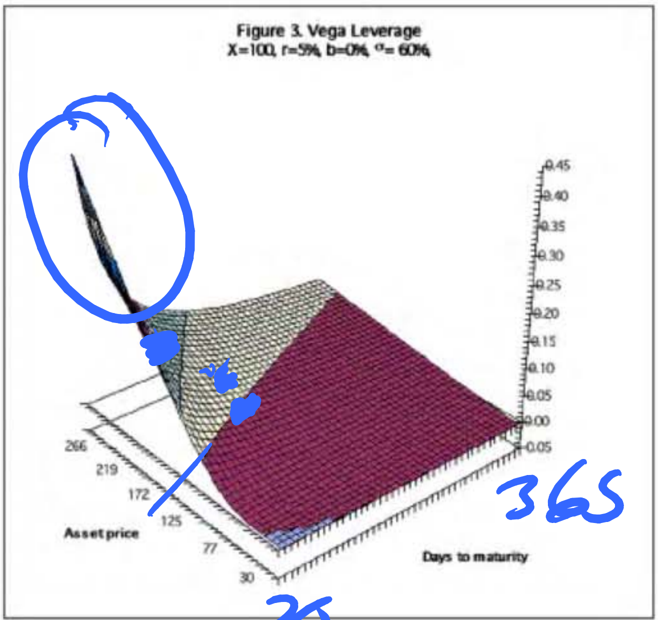
IBM T = →

X = ATM OTM

Vega leverage

Percent change in option value for percent point change in implied volatility.

$$Vega\frac{\sigma}{call}, \quad Vega\frac{\sigma}{put}$$



365

30

DvegaDvol Vomma/Volga

$$\frac{\partial^2 c}{\partial \sigma^2} = Vega\left(\frac{d_1 d_2}{\sigma}\right)$$

Positive outside

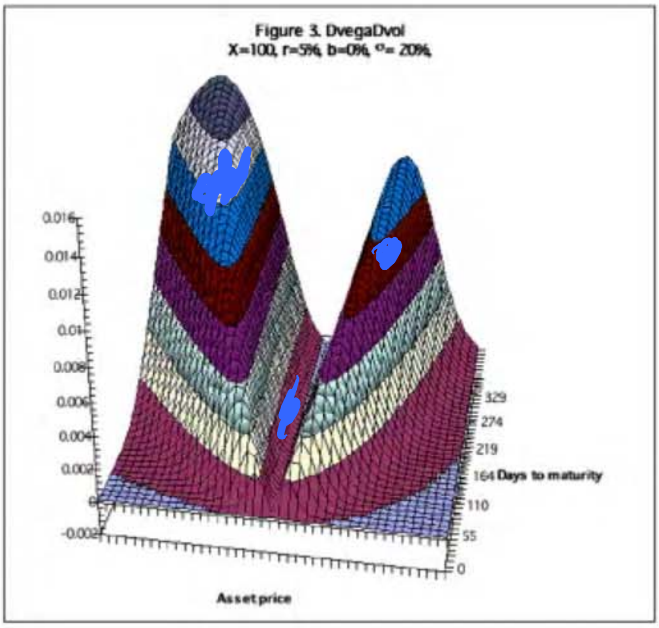
$$S_L = Xe^{(-b-\sigma^2/2)T}$$

$$S_U = Xe^{(-b+\sigma^2/2)T}$$

Positive outside

$$X_L = Se^{(b-\sigma^2/2)T}$$

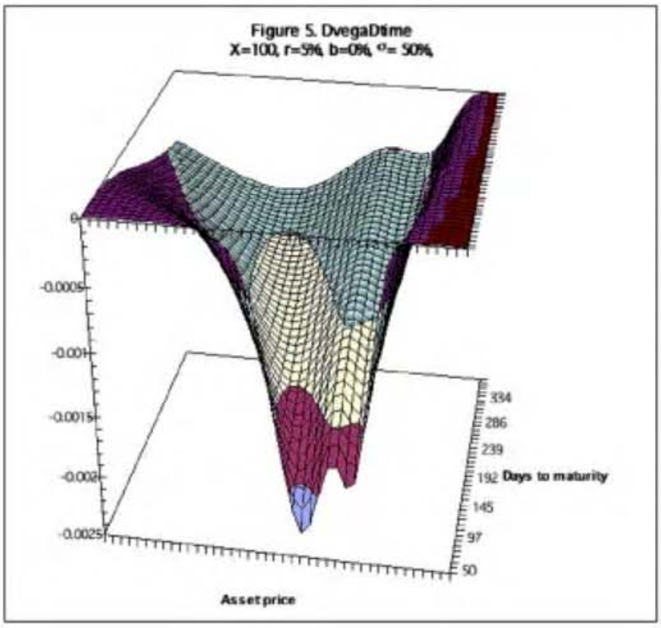
$$S_U = Se^{(b+\sigma^2/2)T}$$



S VoVVol

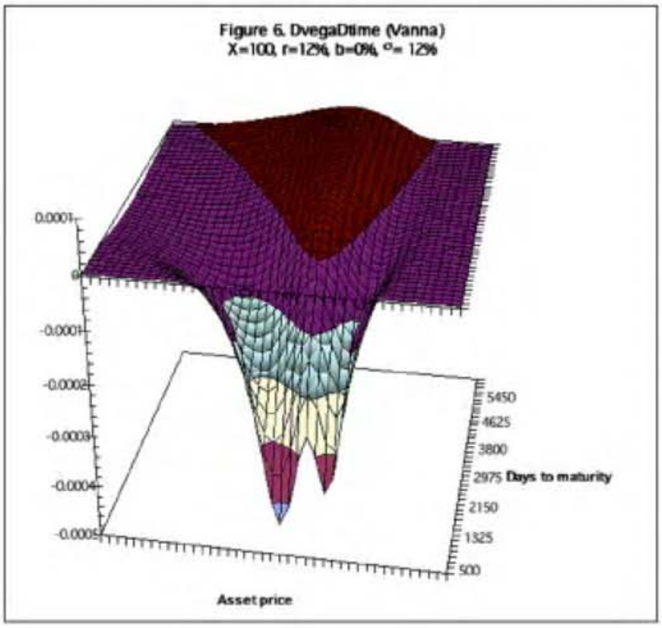
DvegaDtime

$$\frac{\partial^2 c}{\partial \sigma \partial T} = Vega\left(r - b + \frac{bd_1}{\sigma\sqrt{T}} - \frac{1 + d_1 d_2}{2T}\right)$$



DvegaDtime

$$\frac{\partial^2 c}{\partial \sigma \partial T} = Vega\left(r - b + \frac{bd_1}{\sigma\sqrt{T}} - \frac{1 + d_1 d_2}{2T}\right)$$



Theta

$$\Theta_C = -\frac{\partial c}{\partial T} = -\frac{Se^{(b-r)T} n(d_1)}{2\sqrt{T}} - (b-r)Se^{(b-r)T} N(d_1) - rXe^{-rT} N(d_2)$$

$$\Theta_C = -\frac{\partial c}{\partial T} = -\frac{Se^{(b-r)T} n(d_1)\sigma}{2\sqrt{T}} + (b-r)Se^{(b-r)T} N(-d_1) + rXe^{-rT} N(-d_2)$$

Drift-less theta

$$\theta_C = \theta_P = -\frac{Sn(d_1)}{2\sqrt{T}}$$

Theta symmetry

$$\theta(S, X, T, 0, 0, \sigma) = \frac{X}{S}\theta(S, \frac{S^2}{X}, T, 0, 0, \sigma)$$

Bleed-offset volatility

$$\frac{\Theta}{Vega}$$

Rho

$$\rho_C = \frac{\partial c}{\partial r} = TXe^{-rT} N(d_2), \quad \rho_P = \frac{\partial p}{\partial r} = -TXe^{-rT} N(-d_2)$$

In case of options on futures (b=0)

$$\rho_C = \frac{\partial c}{\partial r} = -Tc, \quad \rho_P = \frac{\partial p}{\partial r} = -Tp$$

Probability "Greeks"

Black Scholes

Risk neutral probability of ending up in-the-money

$$\zeta_C = N(d_2) > 0, \quad \zeta_P = N(-d_2) > 0$$

Strike-delta

$$\frac{\partial c}{\partial X} = -e^{-rT} N(d_2), \quad \frac{\partial p}{\partial X} = e^{-rT} N(-d_2)$$

Probability mirror strikes

$$X_P = \frac{S^2}{X_C} e^{(2b-\sigma^2)T}, \quad X_C = \frac{S^2}{X_P} e^{(2b-\sigma^2)T}$$

Probability neutral straddle

$$X_C = X_P = Se^{(b-\sigma^2/2)T}$$

Probability "Greeks"

Strikes from probability

$$X_C = S\exp[-N^{-1}(p_i)\sigma\sqrt{T} + (b - \sigma^2/2)T]$$
$$X_P = S\exp[N^{-1}(-p_i)\sigma\sqrt{T} + (b - \sigma^2/2)T]$$

Risk neutral probability density

$$RND = \frac{\partial^2 c}{\partial X^2} = \frac{\partial^2 p}{\partial X^2} = \frac{n(d_2)e^{-rT}}{X\sigma\sqrt{T}}$$

Probability neutral straddle

$$X_C = X_P = Se^{(b-\sigma^2/2)T}$$

Risk neutral probability density $RND = \frac{\partial^2 c}{\partial X^2} = \frac{\partial^2 p}{\partial X^2} = \frac{n(d_2)e^{-rT}}{X\sigma\sqrt{T}}$

Breeden and Litzenberger (1978)

1994=

1994 04-

1994 -

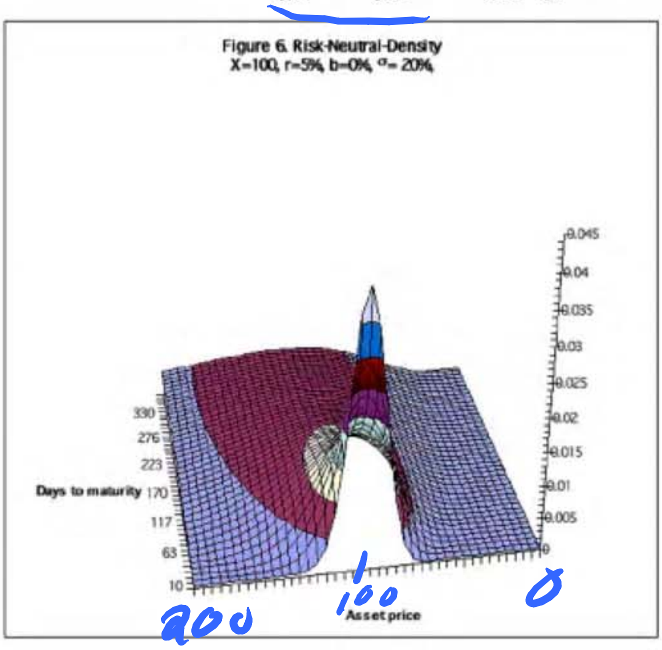


Figure 6. Risk-Neutral-Density
X=100, r=5%, b=0%, σ= 20%

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Probability "Greeks"

Risk neutral probability of ever being in-the-money

$$p_C = (X/S)^{\mu+\lambda} N(-z) + (X/S)^{\mu-\lambda} N(-z + 2\lambda\sigma\sqrt{T})$$
$$p_p = (X/S)^{\mu+\lambda} N(z) + (X/S)^{\mu-\lambda} N(z - 2\lambda\sigma\sqrt{T})$$

where

$$z = \frac{\ln(X/S)}{\sigma\sqrt{T}} + \lambda\sigma\sqrt{T}, \quad \mu = \frac{b - \sigma^2/2}{\sigma^2}, \quad \lambda = \sqrt{\mu^2 + \frac{2r}{\sigma^2}}$$