# CQF Examination Module 1

January 2009

**Instructions**

All questions must be attempted. Books and lecture notes may be referred to.

**Help from others is not permitted**. Your answers, must be handed in to the examiner.

You may use the result of the error function, $\int_{-\infty}^{\infty} \exp\left(-x^2\right) dx = \sqrt{\pi}$ without proof.

**Note:** There will be an exam workshop on Saturday 31 January, conducted in New York (starting 9am local time). All delegates will be able to log in, the recorded session will then be available on the internet immediately.

## A. Basic Mathematics

1. Given that the Taylor series for the function

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + ...,$$

use this to show the following

$$\begin{aligned} \frac{x}{\left(1+x^2\right)^2} &= x - 2x^3 + 3x^5 - .. \\ \log\left(1-x\right) &= -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - .. \end{aligned}$$

2. Consider complex number division $\dfrac{a+ib}{c+id}$ which we express as $re^{i\alpha}$. Using Euler's identity **only**, work out the precise form for the quotient's modulus $r$ and argument $\alpha$. Your calculations should not involve division of complex numbers.

3. Using row operations (only) evaluate the following determinant $\begin{vmatrix} y-z & z-x & x-y \\ z-x & x-y & y-z \\ x-y & y-z & z-x \end{vmatrix}$ and give your solution in the simplest form.

4. Find the eigenvalues and eigenvectors of the following matrix

$$\begin{pmatrix} 3 & 3 & 3 \\ 3 & -1 & 1 \\ 3 & 1 & -1 \end{pmatrix}.$$

Verify that the eigenvectors are mutually orthogonal and hence diagonalize the matrix. Show all working.

## B. Probability

**N.B**. $\mathbb{E}[f(x)]$ is the expectation of the function $f(x)$, given some probability density function $p(x)$.

1. (a) Consider the probability density function $p(x;\ \lambda)$

$$p(x;\ \lambda) = \begin{cases} A\lambda x \exp\left(-\lambda x^2\right) & x \geq 0 \\ 0 & x < 0 \end{cases}$$

where $\lambda\,(> 0)$ and $A$ are both constants. Calculate the value of $A$.

(b) Show that the even moments of $p(x;\ \lambda)$ are given by ,

$$E\left[x^{2n}\right] = \frac{n!}{\lambda^n}, \qquad n = 0,\ 1,\ 2, ....$$

This can be done by recursion.

2. In class we saw that summing up the RAND() function and subtracting off 6 gives a standard normal, i.e.

$$\sum_{1}^{12} \mathrm{RAND}\,() - 6 \longrightarrow \phi \sim N\,(0,1)\,.$$

Obtain a similar expression for using a number $N$ of the RAND() function and verify that this also gives $\phi \sim N\,(0,1)$. Further, show that your formula is consistent with the Central Limit Theorem.

# C. Stochastic Calculus

**N.B.** $X$ is standard Brownian motion.

1. Find the stochastic differential equation (sde) $df$ for the function $f$ in each of the following cases.

**a)** $f(X) = \ln(X^n)$

**b)** $f(X) = \exp(nX)$

**c)** $f(X) = a^X$ where $a > 1$

Show that in **b)** and **c)**, the SDE can be also written $\frac{df}{f} = A\ dt + BdX$ and give the form of the constants $A$ and $B$.

2. Consider the diffusion process for the spot rate $r$ which evolves according to the stochastic differential equation

$$dr = -ardt + bdX.$$

Both $a$ and $b$ are constants. Write down the forward Fokker-Planck equation for the transition probability density function $p(r', t')$ for this process, where a primed variable refers to a future state/time.

By solving the Fokker-Planck equation which you have obtained, obtain the **steady state** probability distribution $p_\infty(r')$, which is given by

$$p_\infty = \sqrt{\frac{a}{b^2\pi}} \exp\left(-\tfrac{a}{b^2}r'^2\right).$$

3. **(a)** Show that

$$G = \exp(t + a\exp(X(t)))$$

is a solution of the stochastic differential equation

$$dG(t) = G\left(1 + \tfrac{1}{2}(\ln G - t) + \tfrac{1}{2}(\ln G - t)^2\right)dt + G(\ln G - t)\,dX.$$

**(b)** By considering the form $S(t) = (A + X/\alpha)^\alpha$ show that

$$dS = \tfrac{1}{3}S^{1/3}dt + S^{2/3}dX,$$

where $\alpha$ should be determined.

## D. Further Mathematical Methods

1. Consider the time independent Black-Scholes equation

$$\frac{1}{2}\sigma^2 S^2 \frac{d^2V}{dS^2} + rS\frac{dV}{dS} - rV = 0.$$

for the unknown function $V(S)$, where the volatility $\sigma$ and interest rate $r$ are constant. Show that the general solution is

$$V(S) = AS + BS^{-2r/\sigma^2}.$$

If

$$\begin{aligned} \lim_{S\longrightarrow\infty} V(S) &\longrightarrow S \\ V(S^*) &= S^* - E \end{aligned}$$

obtain the constants $A$ and $B$ to present a particular solution.

2. Find the general solutions of the following:

   (i) $xy' = y + \sqrt{x^2 + y^2}$

   (ii) $y' = \dfrac{2x + 9y - 20}{6x + 2y - 10}$

   (iii) $y' = \dfrac{3x - 4y - 2}{3x - 4y - 3}$

   (iv) $2y' + y = (x - 1)\, y^3$

   (v) $(x + 3y - 1)\, dx + (3x - 2y + 4)\, dy = 0$

3. (i) Calculate $\left(\sqrt{3} + i\right)^{25}$

   (ii) Find $\sin 5\theta$ and $\cos 5\theta$ in terms of powers of $\sin\theta$ and $\cos\theta$

   (iii) Find $\sin^5\theta$ and $\cos^5\theta$ in terms of $\sin n\theta$ and $\cos n\theta,\ n \in \mathbb{N}$

   (iv) Find <u>all</u> the roots of $x^6 - 1$.

4. If $z = x + iy$, solve the following equation

$$\cos z = 4$$