1) PLAMEN STILIANOV

We consider a portfolio fully invested in four risky assets (A, B, C, D). Denoted by w_i the weight invested in asset i, i = A, B, C, D.

Since all of the weight must be invested in the assets, the proportion of wealth invested in the various assets must equal 100% of wealth. This leads to the budget equation expresses in matrix notation $\mathbf{w}^T\mathbf{1}_N = 1$.

a)

Correlation Matrix

	A	B	C	D
A	1	0.2	0.5	0.3
B	0.2	1	0.7	0.4
C	0.5	0.7	1	0.9
D	0.3	0.4	0.9	1

σ	μ
0.07	0.04
0.12	0.08
0.18	0.12
0.26	0.15

Covariance Matrix ∑

	A	B	C	D
A	0.0049	0.0017	0.0063	0.0055
B	0.0017	0.0144	0.0151	0.0125
C	0.0063	0.0151	0.0324	0.0421
D	0.0055	0.0125	0.0421	0.0676

Inverse Cov Matrix Σ^{-1}

A	B	C	D
1301.39012	1086.95652	-1909.36278	883.89871
1086.95652	1207.72947	-1952.49597	905.79710
-1909.36278	-1952.49597	3390.58866	-1597.91899
883.89871	905.79710	-1597.91899	771.80345

b)

Our objective function is the portfolio variance, and we will minimize it with respect to the portfolio weights. Actually, instead of using the portfolio variance, we will use a factor of ½ to ease our calculations. Since the factor is positive, it does not affect the value of the optimal vector of the weights w*.

$$\min_{w} \frac{1}{2} w^T \Sigma w^*$$

We have two constraint:

First constraint is: portfolio return must be equal to prespectified level m= 0.1 as;

$$\mu_n = \mu^T w^* = m = 0.1$$

The second constraint on the weights called 'budget equation'. The sum of all the weights must necessary equal 1.. Since there is no risk – free assets, our welth must be entirely invested in a combination of the four assets.

$$w^T\mathbf{1} = 1$$

The problem is an optimization with equality constraints. Therefore it can be solved using the method of Langrange.

$$L(w,\lambda,\gamma) = \frac{1}{2} w^T \Sigma w^* + \lambda(m - \mu^T w) + \gamma(1 - \mathbf{1}^T w)$$

After solving first order condition and second order condition, we reached the optimal weight vector w*.

$$w^* = \Sigma^{-1}(\lambda\mu + \gamma\mathbf{1})$$

$A=1^T \times \Sigma^{-1} \times 1$	1505.26	λ	0.25426
$B=\mu^T \times \Sigma^{-1} \times 1$	50.59	γ	-0.00788
$C=\mu^T \times \Sigma^{-1} \times \mu$	1.96	m	10%

	Inverse Cov Matrix Σ^{-1}				
	A	B	C	D	$\lambda\mu + \gamma 1$
A	1301.39012	1086.95652	-1909.36278	883.89871	0.00229
B	1086.95652	1207.72947	-1952.49597	905.79710	0.01246
C	-1909.36278	-1952.49597	3390.58866	-1597.91899	0.02263
D	883.89871	905.79710	-1597.91899	771.80345	0.030258

The optimal asset allocation to obtain a return m = 10% is given by;

	w_{ex}^*
A	5.87%
B	75.90%
C	-31.95%
D	50.18%
	100.00%

Threfore, the portfolio return μ_π can be written as

$$\mu_\pi = \mu^T w^*$$

and the portfolio variance and portfolio standard deviation are given by,

$$\sigma_\pi^2 = w^T \Sigma w, \quad \sigma_\pi = \sqrt{\sigma_\pi^2}$$

μ_π	0.1
σ_π^2	0.0175
σ_π	0.1325

We have used the explicit resolution of optimazation problem to compute the vector of weights of an efficent portfolio with the following constraints;

$w_{ex}^{*T}\mu=0.1$	0.1
$w_{ex}^{*T}1=1$	1

Inverse Cov Matrix Σ^{-1}

A	B	C	D	1 Vec		
1301.39012	1086.95652	-1909.36278	883.89871	1		
1086.95652	1207.72947	-1952.49597	905.79710	1	$A=1^T \times \Sigma^{-1} x$	1505.26
-1909.36278	-1952.49597	3390.58866	-1597.91899	1	$B=\mu^T \times \Sigma^{-1} x$	50.59
883.89871	905.79710	-1597.91899	771.80345	1	$C=\mu^T \times \Sigma^{-1} x$	1.96

The global minimum varience portfolio's assets allocation is given by

$$w_g = \frac{\Sigma^{-1}}{A}\mathbf{1}$$

	w_g
A	**90.54%**
B	**82.91%**
C	**-137.46%**
D	**64.01%**
	100.00%

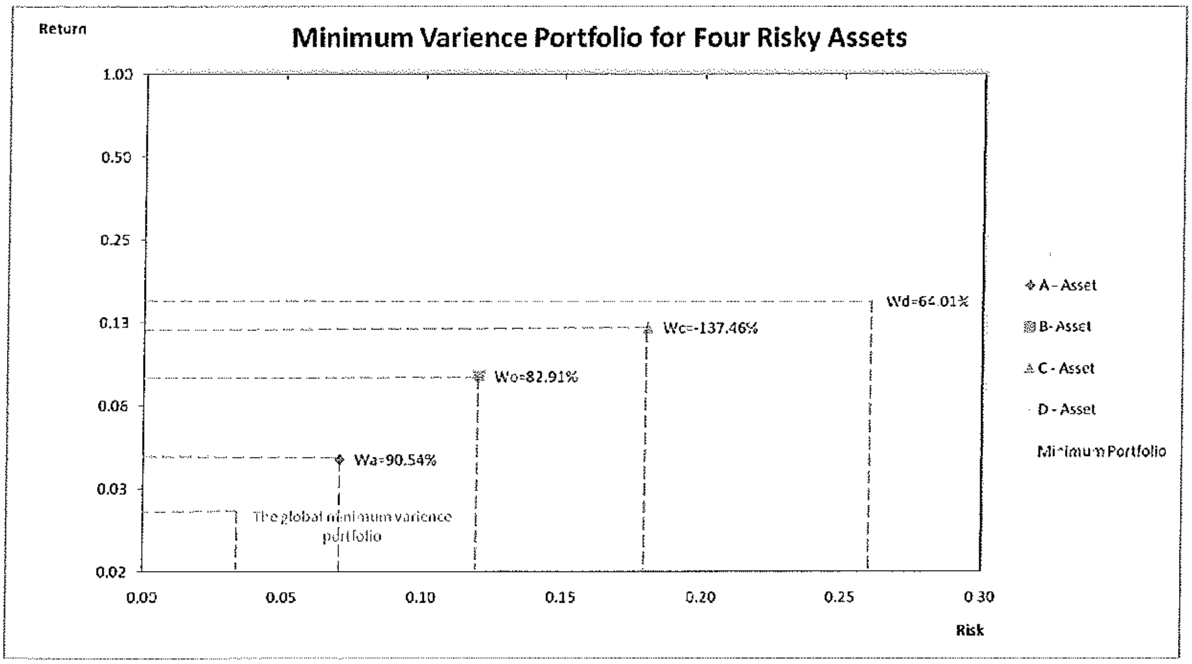
$w_g^T\Sigma$	0.000664	0.000664	0.000664	0.000664

Its return is

$$m_g = \frac{B}{A}$$ m_g **0.0336**

and its standard deviation is equal to

$$\sigma_g = \sqrt{wg^T \Sigma wg} = \sqrt{\frac{1}{A}}$$ σ_g **0.0258**



2)

		Payoff		
Stock	Profit	X_{call}=£15	X_{call}=£17.5	X_{call}=£20
£0.00	-0.5	-4	4	-0.5
£14.00	-0.5	-4	4	-0.5
£14.50	-0.5	-4	4	-0.5
£15.00	-0.5	-4	4	-0.5
£15.50	0	-3.5	4	-0.5
£16.00	0.5	-3	4	-0.5
£16.50	1	-2.5	4	-0.5
£17.00	1.5	-2	4	-0.5
£17.50	2	-1.5	4	-0.5
£18.00	1.5	-1	3	-0.5
£18.50	1	-0.5	2	-0.5
£19.00	0.5	0	1	-0.5
£19.50	0	0.5	0	-0.5
£20.00	-0.5	1	-1	-0.5
£20.50	-0.5	1.5	-2	0
£21.00	-0.5	2	-3	0.5
£21.50	-0.5	2.5	-4	1
£22.00	-0.5	3	-5	1.5
£22.50	-0.5	3.5	-6	2
£23.00	-0.5	4	-7	2.5
£23.50	-0.5	4.5	-8	3
£24.00	-0.5	5	-9	3.5
£24.50	-0.5	5.5	-10	4
£25.00	-0.5	6	-11	4.5
£25.50	-0.5	6.5	-12	5
£26.00	-0.5	7	-13	5.5

Butterfly spread	Premium	Strike Price
Buy Call Option x 1	£4.00	£15.00
Sell Call Option x 2	£2.00	£17.50
Buy Call Option x 1	£0.50	£20.00

Components

Short two ATM call options, long one ITM call option and long one OTM call option.

Risk / Reward

Maximum Loss: Limited to the ATM strike less the ITM strike less the net premium paid for the spread.

Maximum Gain: Limited to the net premium received from the spread.

Characteristics

When to use: When you are neutral on market direction and bearish on volatility.

With a long butterfly your losses are limited. This means that you make money when the market remains flat over the life of the options.

CALL $V(S,T) = \max(S-\alpha-5, 0)$

15
20

5
10

payoff 0 0
OPTION 5 0

0
0

0
0

TIME 0 T_1 T_2

V_2

V_1

V V_0

V_{-1}

V_{-2}

0 T_1 T_2

II) $V(S,T) = \max(\alpha-5-S, 0)$ PUT

$V_2 = \alpha - \alpha - 20 = 0$

$V_0 = \alpha - \alpha = 0$

$V_{-2} = \alpha - \alpha + 20 = 20$

$V_1 = pV_2 + (1-p)V_0 = 0 + 0 = 0$

$V_{-1} = pV_0 + (1-p)V_{-2} = 0 + \frac{1}{2}\cdot 20 = 10$

$V = pV_1 + pV_{-1} = 0 + \frac{1}{2}\cdot 10 = 5$

$V_2(S,T) = \max(\alpha-5-\alpha-20, 0) = 0$

$V_0(S,T) = \max(\alpha-5-\alpha, 0) = 0$

$V_{-2}(S,T) = \max(\alpha-5-\alpha+20, 0) = 15$

$V_1(S,T) = \max(\alpha-5-\alpha-10, 0) = 0$

$V_{-1}(S,T) = \max(\alpha-5-\alpha+10, 0) = 5$

$V(S,T) = \max(\alpha-5-\alpha, 0) = 0$

0
0

0
0

payoff 0 0
OPTION 5 0

5
10

15
20

TIME 0 T_1 T_2

V_2

V_1

V V_0

V_{-1}

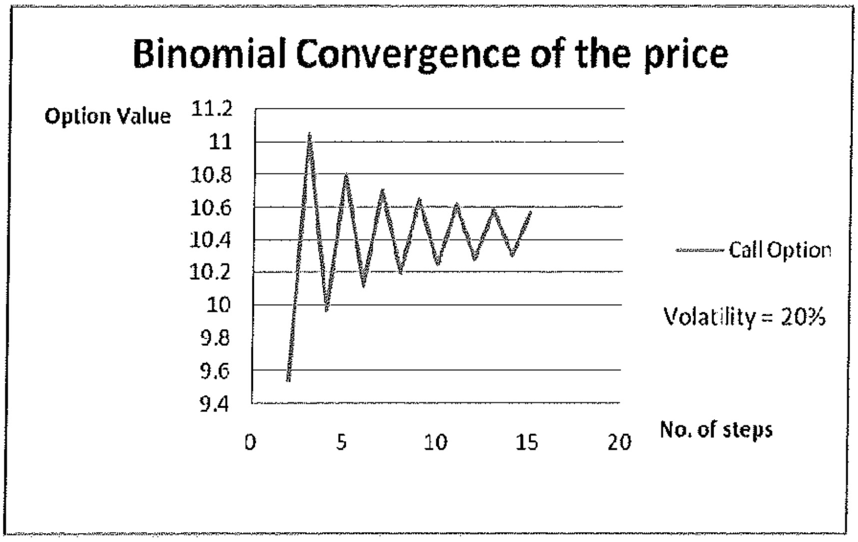
V_{-2}

0 T_1 T_2

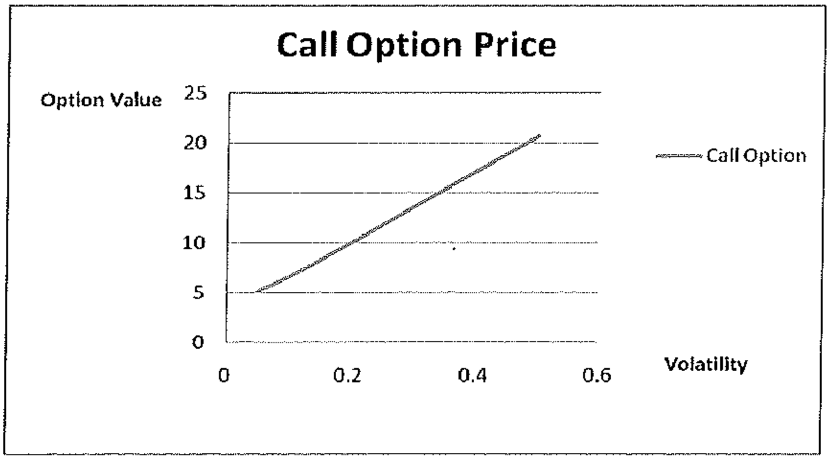
5)

Calculation	Binomial Method
Stock	100
Strike Price	100
Int. Rates	5%
Expiration	1
Volatility	20%
Time steps	4

Time step	Call Option
2	9.540501
3	11.04387
4	9.970523
5	10.80593
6	10.12557
7	10.70379
8	10.2051
9	10.64716
10	10.25341
11	10.6112
12	10.28585
13	10.58635
14	10.30913
15	10.56815



Volatility	Call Option
0.05	5.14369127
0.1	6.54752816
0.15	8.22390147
0.2	9.97052292
0.25	11.7425546
0.3	13.5240018
0.35	15.307326
0.4	17.0882073
0.45	18.8637695
0.5	20.6318765



6

a)

Exponentially weighted moving average models

The exponentially weighted moving average (EWMA) is essentially a simple extension of the historical average volatility measure, which allows more recent observations to have a stronger impact on the forecast of volatility than older data points. Under an EWMA specification, the latest observation carries the largest weight, and weights associated with previous observations decline exponentially over time. This approach has two advantages over the simple historical model. First, volatility is in practice likely to be affected more by recent events, which carry more weight, than events further in the past. Second, the effect on volatility of a single given observation declines at an exponential rate as weights attached to recent events fall. On the other hand, the simple historical approach could lead to an abrupt change in volatility once the shock falls out of the measurement sample. And if the shock is still included in a relatively long measurement sample period, then an abnormally large observation will imply that the forecast will remain at an artificially high level even if the market is subsequently tranquil. The exponentially weighted moving average model can be expressed in several ways, e.g.

$$\sigma_t^2 = (1 - \lambda)\sum_{i=0}^{\infty} \lambda^{f-1}(r_{t-i} - \bar{r}) \quad \text{with } \lambda \ (0 < \lambda < 1)$$

where σ_t^2 is the estimate of the variance for period t, which also becomes the forecast of future volatility for all periods, $\bar{r}$ is the average return estimated over the observations and λ is the 'decay factor', which determines how much weight is given to recent versus older observations.

b)

The decay factor could be estimated, but in many studies is set at 0.94 as recommended by RiskMetrics, producers of popular risk measurement software.

$$m = \frac{\ln(2)}{\ln(\lambda)} \quad \text{or} \quad \lambda = e^{\frac{-\ln(2)}{m}}$$

If we change λ from 95% to 85% with will effect m, how many days are relevant. For e.g. λ = 95% then m = 13 days, and λ = 85% then m = 4 days.

7

$$e^{i\epsilon\omega} = \cos\epsilon\omega + i\sin\epsilon$$
$$e^{-i\epsilon\omega} = \cos\epsilon\omega - i\sin\epsilon$$

$$f(x) = \frac{1}{2\epsilon} \qquad \epsilon < |x| \le \epsilon$$

$$\int_{-\epsilon}^{\epsilon} f(x)\, e^{ix\omega}\, dx = \hat{f}(\omega) = \int_{-\epsilon}^{\epsilon} \frac{1}{2\epsilon} e^{ix\omega}\, dx =$$

$$= \frac{1}{2}\int_{-\epsilon}^{\epsilon} \frac{1}{\epsilon} e^{ix\omega}\, dx = \frac{1}{2}\int_{-\epsilon}^{0} \frac{1}{\epsilon} e^{ix\omega}\, dx + \frac{1}{2}\int_{0}^{\epsilon} \frac{1}{\epsilon} e^{ix\omega}\, dx =$$

$$= \left[\frac{\ln|\epsilon|}{2} \cdot \frac{e^{ix\omega}}{i\omega}\right]_{-\epsilon}^{0} + \left[\frac{\ln|\epsilon|}{2} \cdot \frac{e^{ix\omega}}{i\omega}\right]_{0}^{\epsilon} =$$

$$= \left(\frac{1}{2} \cdot \frac{1}{i\omega} + \frac{\ln|\epsilon|}{2} \frac{e^{-i\epsilon\omega}}{i\omega}\right) + \left[\frac{\ln|\epsilon|}{2} \cdot \frac{e^{i\epsilon\omega}}{i\omega} - \frac{1}{2} \cdot \frac{1}{i\omega}\right] =$$

$$= \left(\cancel{\frac{1}{2i\omega}} + \frac{\ln|\epsilon|}{2i\omega} \cdot \cos\epsilon\omega - i\sin\epsilon\right) + \frac{\ln|\epsilon| \cdot (\cos\epsilon\omega + i\sin\epsilon)}{2i\omega}$$

$$- \cancel{\frac{1}{2i\omega}} = \frac{\ln|\epsilon|}{2i\omega}\left(\cos\epsilon\omega - \cancel{i\sin\epsilon} + \cos\epsilon\omega + \cancel{i\sin\epsilon}\right)$$

$$= \frac{\ln|\epsilon|}{\cancel{2}\, i\cancel{\omega}} \cdot \cancel{2}\cos\epsilon\cancel{\omega} = \frac{\ln|\epsilon|}{i} \cos\epsilon$$

$$\Rightarrow \lim_{\epsilon \to 0^+} \hat{f}(\omega) = \frac{1}{\cancel{4}\sqrt{2\pi}} \cdot \frac{\ln|0|}{i} = \cos 0 =$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{i}$$

8

$$\int_1^4 (x-1)^2(4-x)^3\,dx$$

$$u = (x-1) \quad du = dx$$

$$4-x = -(x-4) = -[(x-1)+3] = u-3$$

$$\int_0^3 u^2(u-3)^3\,du \qquad B(m,n) = \int_0^1 x^{m}(1-x)^{n}\,dx$$

$m-1 = 2 \rightarrow m = 3$

$n-1 = 3 \rightarrow n = 4$

$$B(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)} = \frac{\Gamma(3)\,\Gamma(4)}{\Gamma(7)}$$

9) 1

$$x^2 y'' + 2xy' + y = 0 \qquad 1 : x^2$$

$$y^2 + \overset{= p(x)}{\frac{2x}{x^2}} y' + \overset{q(x)}{\frac{1}{x^2}} y = 0$$

$$y^2 + \frac{2}{x} y' + \frac{1}{x^2} y$$

$p(x)$, $q(x)$ do not have Taylor exp. for $x=0$

$$p(x) = \frac{2}{x} \Rightarrow x\,p(x) = 2$$

$$q(x) = \frac{1}{x^2} \Rightarrow x^2 q(x) = 1$$

However $x\,p(x)$ and $x^2 q(x)$ have Taylor expansion, so

$x=0$ is a regular singular point of D.E

9

11) $x^2y'' + xy' + (x^2-4)y = 0 \mid \div x^2$

$P(x)$ $Q(x)$

$y'' + \frac{\cancel{x}}{\cancel{x^2}_{x}} y' + \frac{x^2-4}{x^2} y = 0$

$P(x) = \frac{1}{x}$; $Q(x) = \frac{x^2-4}{x^2}$

$P(x)$ and $Q(x)$ do no have Taylor Expansion
so $x=0$ is
$xP(x) = 1$ $x^2Q(x) = x^2-4$

However $xP(x)$ and $x^2Q(x)$ have

Taylor expansion, therefore

$x=0$ is a regular singular
point for D.E.

9

III

$$\underset{P(x)}{x\,y''} + x^2 y' + \underset{Q(x)}{y} = 0 \quad \big/ : x$$

$$y'' + \frac{x^2}{x} + \frac{1}{x}\,y = 0$$

$P(x) = x \qquad Q(x) = \frac{1}{x} \qquad x\,P(x) = x^2 \quad x^2 Q(x) = x$

$q(x)$ does not have Taylor expansion, so $x\,P(x)$ and $x^2 Q(x)$ have Taylor expansion at $x = 0$ is a regular singular point for D.E

IV)

$$(1 - x^2)^2 y'' + x y' + y = 0 \quad \big/ : (1 - x^2)^2$$

$$y'' + \frac{x}{(1-x^2)^2}\,y'' + \frac{1}{(1-x^2)^2} = 0$$

$$P(x) = \frac{x}{(1-x^2)^2} = \frac{x}{((1-x)(1+x))^2} \qquad Q(x) = \frac{1}{(1-x^2)^2} = \frac{1}{(1+x)^2(1-x)^2}$$

$p(x)$ and $q(x)$ are analytical at $x = 0$, which is an ordinary point, but both diverged at ± 1, so $x = 1$ and $x = -1$ are regular points

$$(1-x)\,P(x) = \frac{x}{(1+x)^2(1-x)}; \quad (1-x^2)^2 q(x) = \frac{1}{(1+x^2)^2}$$

$$(1+x)\,P(x) = \frac{x}{(1-x)^2(1+x)}; \quad (1+x^2)^2 q(x) = \frac{1}{(1-x^2)^2}$$

For $x = -1$ at $(1+x)P(x)$ and $(1+x^2)Q(x)$ are analytic and hence $x = -1$ is a regular singular point, the same for $x = 1$ at $(1-x)P(x)$ and $(1-x^2)Q(x)$ is regular singular point v)

$$(x+5)\,y'' + xy = 0 \quad \Big|\, \div\, x+5$$

$$y'' + \frac{x^4}{x+5}\,y = 0 \qquad Q(x)$$

$$P(x) = 0 \qquad Q(x) = \frac{x^4}{(x+5)}$$

$$(x+5)^2 Q(x) = x^4 (x+5)$$

$x=-5$, $x=0$ has an irregular singular point for D.E

$$(x+5)^2 Q(x) = x^4 (x+5)$$

10

1) $$I = \int_1^2 \int_{x^2}^{4x^2} (x+y)\, dy\, dx$$

$$I = \int_{x^2}^{4x^2} (x+y)\, dy = \left(xy + \frac{y^2}{2} \right)\Big|_{x^2}^{4x^2} =$$

$$\cancel{= 4x^2 y + \frac{16x^4}{2} - }$$

$$= \left(x \cdot 4x^2 + \frac{16x^4}{2} \right) - \left(x \cdot x^2 + \frac{x^4}{2} \right) =$$

$$= 4x^3 + \frac{16x^4}{2} - x^3 - \frac{x^4}{2} = \frac{15}{2}x^4 + 3x^3$$

$$I = \int_\phi^2 \left(\frac{15}{2}x^4 + 3x^3 \right) dx = \left(\frac{\cancel{15}^{3}}{2} \cdot \frac{1}{\cancel{5}} x^5 + 3 \cdot \frac{1}{4} x^4 \right)\Big|_\phi^2 =$$

$$= \left(\frac{3}{2}x^5 + \frac{3}{4}x^4 \right)\Big|_1^2 = \left(\frac{3}{\cancel{2}} \cdot 2^{\cancel{5}\,4} + \frac{3}{4} 2^4 \right) - \left(\frac{3}{2} \cdot 1^5 + \frac{3}{4} \cdot 1^4 \right)$$

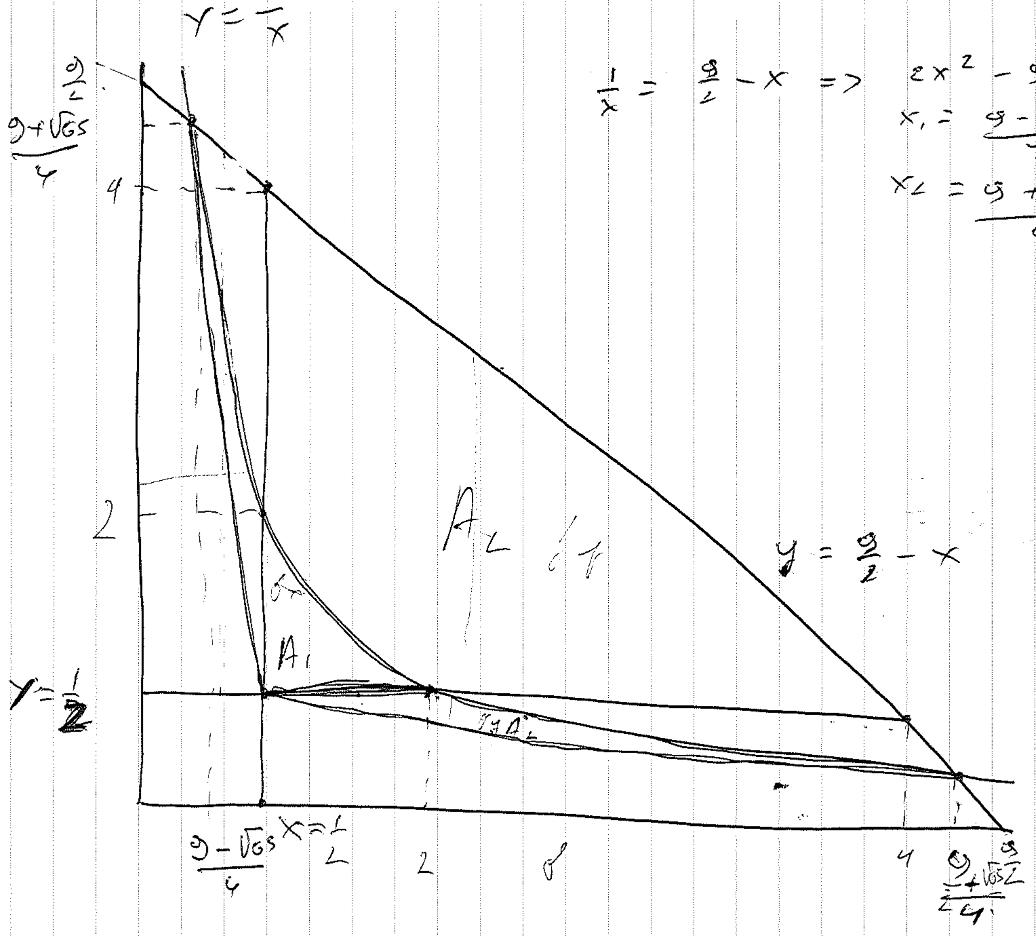
$$= \left(3 \cdot 2^4 + \frac{3 \cdot \cancel{16}^{4}}{\cancel{4}} \right) - \left(\frac{\overset{2}{3}}{\underset{4}{2}} + \frac{3}{4} \right) = (\overset{48}{3 \cdot 16} + 12) - \left(\frac{6+3}{4} \right)$$

$$= \overset{4}{60} - \frac{9}{4} = \frac{240 - 9}{4} = \frac{231}{4}$$

$$\frac{1}{x} = \frac{9}{2} - x \Rightarrow 2x^2 - 9x + 2$$

$$x_1 = \frac{9-\sqrt{65}}{4}$$

$$x_2 = \frac{9+\sqrt{65}}{4}$$



4.25

$\frac{9}{2} = 4.5$

$4.5 - \frac{1}{2} =$

$\frac{9}{2} \cdot x$

$$\begin{array}{r} 4.5 \\ -\ 0.5 \\ \hline 4.0 \end{array}$$

$$A_1 = \int_{\frac{9-\sqrt{65}}{4}}^{\frac{1}{2}} \int_{\frac{1}{x}}^{\frac{9}{2}-x} dy\,dx$$

$$I = \int_{\frac{1}{x}}^{\frac{9}{2}-x} dy\,\cancel{dx} = \left[y\right]_{\frac{1}{x}}^{\frac{9}{2}-x} = \frac{9}{2} - x - \frac{1}{x}$$

$$A_1 = \int_{\frac{9-\sqrt{65}}{4}}^{\frac{1}{2}} \frac{9}{2} - x - \frac{1}{x}\,dx = \left|\frac{9}{2}x - \frac{x^2}{2} - \ln|x|\right|_{\frac{9-\sqrt{65}}{4}}^{\frac{1}{2}}$$

$$0$$

$$//$$

$$= \frac{9}{2}\cdot\frac{1}{2} - \frac{1}{2}\cdot\frac{1}{4} - (\ln 1 - \ln 2)$$

$$-\left(\frac{9}{2}\frac{(9-\sqrt{65})}{4} - \frac{1}{2}\cdot\frac{(9-\sqrt{65})^2}{4^2} - \ln\frac{9-\sqrt{65}}{4}\right.$$

$$= \frac{9}{4} - \frac{1}{8} + \ln 2 - \frac{9\cdot(9-\sqrt{65})}{8} - \frac{1}{2}\cdot\frac{(9-\sqrt{65})^2}{16} -$$

$$\ln(9-\sqrt{65}) \cancel{+} 2\ln 2 =$$

$$= \frac{17}{8} - \frac{9(9-\sqrt{65})}{8} - \frac{(9-\sqrt{65})^2}{32} + 3\ln 2 - \ln(9-\sqrt{65})$$

$$= \frac{17}{8} - \frac{9(9-\sqrt{65})}{8} - \frac{(9-\sqrt{65})^2}{32} + 3\ln 2 - \ln(9-\sqrt{65})$$

$$A_3 = \int_2^{\frac{9+\sqrt{65}}{4}} \int_{\frac{1}{x}}^{\frac{9}{2}-x} dy\, dx =$$

$$I = \int_{\frac{1}{x}}^{\frac{9}{2}-x} dy = |y|_{\frac{1}{x}}^{\frac{9}{2}-x} = \frac{9}{2} - x - \frac{1}{x}$$

$$\int_2^{\frac{9+\sqrt{65}}{4}} \left(\frac{9}{2} - x - \frac{1}{x}\right) dx = \left|\frac{9}{2}x - \frac{x^2}{2} - \ln\right|_2^{\frac{9+\sqrt{65}}{4}}$$

$$\frac{9}{2} \cdot \frac{9+\sqrt{65}}{4} - \frac{1}{2}\frac{(9+\sqrt{65})^2}{4^2} - \ln \frac{9+\sqrt{65}}{4} + 2\ln 2$$

$$-\left(\frac{9}{4} \cdot 4 - \frac{4^2}{4} - \ln 2\right) =$$

$$\frac{9}{2} \cdot \frac{9+\sqrt{65}}{4} - \frac{(9+\sqrt{65})^2}{32} - \ln \frac{9+\sqrt{65}}{\cancel{4}} + 2\ln 2$$

$$-7 + \ln 2 =$$

$$\frac{9(9+\sqrt{65})}{8} - 7 - \frac{(9+\sqrt{65})^2}{32} + 3\ln 2 - \ln 9 + \sqrt{65}$$

$\cancel{+ \ln 4}$

$$A_2 = \int_{\frac{1}{2}}^{4}\int_{\frac{1}{2}}^{\frac{9}{2}-x} dy\,dx$$

$$I = \int_{\frac{1}{2}}^{\frac{9}{2}-x} 1\,dy = \left| y \right|_{\frac{1}{2}}^{\frac{9}{2}-x} = \frac{9}{2} - x - \frac{1}{2} = 4 - x$$

$$\int_{\frac{1}{2}}^{4} (4-x)\,dx = \left| 4x - x^2 \right|_{\frac{1}{2}}^{4} = (4\cdot 4 - 4^2) -$$

$$4\cdot\frac{1}{2} - \left(\frac{\cancel{x}}{2}\right)^2 = (16 - 16) = 2 - \frac{1}{4}$$

10/111

$$x^2+y^2 = 4y \Rightarrow x^2+y^2-4y=0$$

$$x^2+y^2-4y+4=4$$

$y = r\sin\theta$

$$\boxed{\begin{array}{l} r = 4\sin\theta \\ r^2 = 4r\sin\theta \end{array}}$$

$$\int_{\pi/4}^{\pi/3}\int_0^{4\sin\theta} r\,dr\,d\theta = \left[\frac{r^2}{2}\right]_0^{4\sin\theta} =$$

$$\int_{\pi/4}^{\pi/3} \cancel{16}^{\,8} \frac{\sin^2\theta}{\cancel{2}}\,d\theta = 8\left[\frac{\theta}{2}-\frac{\sin 2\theta}{4}\right)_{\pi/4}^{\pi/3} =$$

$$\frac{\cancel{8}}{\cancel{4}}\left[2\theta - \sin 2\theta\right]_{\pi/4}^{\pi/3} = 2\left[2\theta - \sin 2\theta\right]_{\pi/4}^{\pi/3} =$$

$$= \left[\frac{4\pi}{3} - 2\sin\frac{2\pi}{3}\right] - \left[\frac{4\pi}{4} - 2\sin\frac{2\pi}{4}\right]$$

$$= \frac{4\pi}{3} - 2\sin\frac{2\pi}{3} - \pi + 2\sin\frac{\pi}{2}$$

$$= 2 + \frac{4\pi}{3} - \frac{3\pi}{3} - 2\sin\frac{2\pi}{3}$$

$$= 2 + \frac{\pi}{3} - \cancel{2}\cdot\frac{\sqrt{3}}{\cancel{2}} = 2 + \frac{\pi}{3} - \sqrt{3}$$

$\frac{\pi}{4} < \theta < \frac{\pi}{3}$ $\qquad$ $r = 2\sqrt{y}$

$r^2 = 4y =$

$0 < r < 2\sqrt{y}$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \int_0^{2\sqrt{y}} r\,dr\,d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \Big[r \Big]_0^{2\sqrt{y}} =$$

$\cancel{x^2 + y^2 = 4y}$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} 2\sqrt{y}\, d\theta =$$

$$\Big[2\sqrt{y}\,\theta \Big]_{\frac{\pi}{4}}^{\frac{\pi}{3}} = 2\sqrt{y}\left[\frac{\pi}{3} - \frac{\pi}{4}\right] =$$

$$= \cancel{2}\sqrt{y}\,\frac{4\pi - 3\pi}{\cancel{12}_{6}} = \sqrt{y}\,\frac{\pi}{6}$$

$x^2 + y^2 = 4y$

$x^2 = -y^2 + 4y$

$x^2 = -(y^2 - 4y)$

(11)

$$dY(t) = f(t)\,dt + g(t)\,dX$$

$$Y(0) = 0$$

$$Z(t) = e^{Y_t} \qquad \frac{\partial z(t,S_t)}{\partial t} = f'(t)\,z(t,S_t)$$

$$M(t) = z(t, S_t) \qquad \frac{\partial z}{\partial X}(t,S_t) = z(t,S_t)$$

$$\frac{\partial^2 z}{\partial X^2}(t,S_t) = z(t,S_t)$$

THEN BY ITÔ'S LEMMA

$$dM(t) = \frac{\partial z}{\partial t}(t,S_t)\,dt + \frac{\partial z}{\partial X}(t,S_t)\,dS + \frac{1}{2}\sigma^2\frac{\partial^2 z}{\partial X^2}(t,S_t)\,dt =$$

$$= \left(f'(t) + \mu + \frac{1}{2}\sigma^2\right) z(t,S_t)\,dt + \sigma\, z(t,S_t)\,dX =$$

$$= \left(f'(t) + \mu + \frac{1}{2}\sigma^2\right) M(t)\,dt + \sigma M(t)\,dX$$

FOR $M(t)$ TO BE A MARTINGALE, WE NEED IT TO BE DRIFTLESS, HENCE ITS DRIFT MUST BE EQUAL TO 0 AND THUS

$$f'(t) + \mu + \frac{1}{2}\sigma^2 = 0 \qquad \text{THEN}$$

$$\boxed{f(t) = -\left(\mu + \frac{1}{2}\sigma^2\right)t}$$

$$Y_t = f(x)\,dt = -\left(\mu + \frac{1}{2}\sigma^2\right)t$$

$$Z(t) = e^{-\left(\mu + \frac{1}{2}\sigma^2\right)t}$$

12

$$m_n(t) = E\left(X^n(t)\right) \quad n = 1, 2, 3 \ldots$$

$$\frac{\partial m_n(t)}{\partial t} = 0 \qquad X_0 = 0$$

$$\frac{\partial m_n(t)}{\partial X} = n X_{(t)}^{n-1}$$

$$\frac{\partial^2 m_n(t)}{\partial X^2} = n(n-1) X_t^{n-2}$$

ITÔ's LEMMA

$$m_n(t) = X_0 + \frac{\partial m_n(t)}{\partial X_t} + \frac{1}{2} \frac{\partial^2 m_n(t)}{\partial X_t^2}$$

$$dm_n = \frac{1}{2} n(n-1) X^{n-2} dt + n X^{n-1} dX$$

Since X_t is B.M. then $X_0 = 0$ and therefore $Y_0 = 0$, thus integrating SDE over $[0, T]$

we get $m_n(t) = \frac{1}{2} n(n-1) \int_0^T X_t^{n-2} dt + n \int_0^t X^{n-1} dX$

TAKING EXPECTATIONS AND LINEARITY OF THE EXPECTATION OPERATOR

$$E(m_n(t)) = \frac{1}{2} n(n-1) E\left[\int_0^T X_t^{n-2} dt\right] + nE\left[\int_0^T X_t^{n-1} dX\right]$$

NOW THE ITÔ's INTEGRAL $\int_0^t X_t^{n-1} dX$ IS MARTINGALE AND $E\left[\int_0^T X_t^{n-1} dX\right] = 0$

By Fubini's Theorem we can change the order of integration

$$E[m_n(t)] = \frac{1}{2} n(n-1) \int_0^t E[X_t^{n-2}]\, dt =$$

$$= E[(X_t - X_0)^{n-2}] = t - 0 = t$$

$$E[m_n(t)] = \frac{1}{2} n(n-1) \int_0^t t\, dt$$

test: $m_4 = 3t^2$

$$E[m_4(t)] = \frac{1}{\cancel{2}} \overset{2\cdot 3}{\cancel{4}(4-1)} \left[\frac{t^2}{2}\right]_0^t = \cancel{2} \cdot 3 \cdot \frac{1}{\cancel{2}} t^2 = 3t^2$$

$$m_n(t) = \frac{1}{2} n(n-1) \int_0^t X_t^{n-2}\, dt$$

if $m_n = X^n \Rightarrow X^{n-2} = m_{n-2}$

$$\boxed{m_n(t) = \frac{1}{2} n(n-1) \int_0^t m_{n-2}\, dt}$$

$m_4 = 3t^2$

$$m_6 = \frac{1}{\cancel{2}} \overset{3}{\cancel{6}}(6-1) \int_0^t \underbrace{m_4}_{=3t^2}\, dt = 15 \int_0^t 3t^2\, dt =$$

$$= 15\left[\frac{\cancel{3}}{\cancel{3}} t^3\right]_0^t = 15t^3 \Rightarrow \boxed{m_6 = 15t^3}$$