Exercise 5.5
Collateralized Debt Obligations

1. **Synthetic CDO**. A balance sheet synthetic CDO is comprised of the following underlying portfolio:

Assets:	125 single-name CDS
Principal:	0.8 million
Maturity:	5 years
CDS spread:	200 bps
Payments:	Act/360 quarterly in arrears

The CDO is structured with the following capital structure:

Tranche	Attachment point	Expected Loss	Fair Spread	Rating
Senior	7%-10%	0.002%	L+45	AAA
Class A	5%-7%	0.1%	L+70	AA-
Class B	2%-5%	2.3%	L+120	BBB-
Class C	0%-2%	26.27%	Excess spread	NR

(a) which noteholders are long correlation? Which tranche is the most sensitive to changes in default correlation? Why is this?

(b) how concerned are mezzanine noteholders with changes in the level of default correlation?

(c) how many defaults must there be before the Senior note experiences capital loss? Assume 0% recovery. If we assume 40%

recovery how much more protection does this afford the Senior noteholder?

(d) How many defaults must there be before the implied rating of the note is downgraded assuming no recovery and downgrade occurs when entire equity tranche is lost?

2. (a) Consider a random default time X that, given default intensity parameter θ, can be modeled as an exponential distribution, i.e.,

$$\text{Prob}(X \leq x|\theta) = 1 - e^{-\theta x}.$$

Now assume θ is Gamma distribution, i.e.,

$$\theta \sim \Gamma(\alpha, \beta),$$

so that the PDF of θ is $g(\theta)$, where

$$g(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta\theta}.$$

Show that marginal distribution of X is

$$F(x) = \text{Prob}(X \leq x) = 1 - \left(1 + \frac{x}{\beta}\right)^{-\alpha}.$$

Hint: Integrate conditional marginal distribution $F(x|\theta)$ w.r.t. θ to find unconditional marginal $F(x)$.

(b) Suppose conditional on θ, there exists two independent and identically distributed default times X_1 and X_2, such that their joint distribution function is $F(X_1, X_2)$, by finding F show that the associated copula function is

$$C(u_1, u_2) = u_1 + u_2 - 1 + \left((1 - u_1)^{-\frac{1}{\alpha}} + (1 - u_2)^{-\frac{1}{\alpha}} - 1\right)^{-\alpha}.$$

Hint: To find joint distribution you can use the result $F(x_1, x_2) = 1 - \Pr(X_1 > x_1) - \Pr(X_2 > x_2) + \Pr(X_1 > x_1, X_2 > x_2)$, then identify marginal distributions hidden in $F(x_1, x_2)$ hence express it in terms of uniforms. This question actually shows that joint distribution function can be expressed as copula function.