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An Exact Bond Option Formula

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ABSTRACT

This paper derives a closed-form solution for European options on pure discount bonds, assuming a mean-reverting Gaussian interest rate model as in Vasicek [8]. The formula is extended to European options on discount bond portfolios.

IN THIS PAPER WE derive a closed-form solution for European options on default-free bonds. We assume that the term structure is completely determined by the value of the instantaneous interest rate $r(t)$ and that $r(t)$ follows a mean-reverting Gaussian (normal) process as in Vasicek [8]. The resulting pricing formula resembles the Black-Scholes formula and has a similar interpretation. Moreover, an option on a portfolio of pure discount bonds (in particular, an option on a coupon bond) decomposes into a portfolio of options on the individual discount bonds in the portfolio.

In the Vasicek model it is assumed that $r(t)$ evolves according to the diffusion process:

$$dr = a(r_0 - r)dt + \sigma dw,$$

where σ, a, and r_0 are positive constants and $w(t)$ is a standard Wiener process. The constant r_0 is interpreted as the historical average instantaneous rate, and a is interpreted as the speed of reversion to this average. It is assumed that prices of bonds and their derivative securities depend on r as the only state variable. Standard arbitrage arguments as in Dothan [3] and Vasicek [8] imply that (i) the price of risk $\lambda(r, t)$ (defined as the expected instantaneous excess return above the riskless rate, divided by the instantaneous standard deviation of return) is the same for all these securities and (ii) the price $U(r, t)$ of a security paying continuously at a rate $h(r, t)$ and yielding a terminal payoff $g(r_T)$ at time T is the solution of

$$U_t + \tfrac{1}{2}\sigma^2 U_{rr} + a(\bar{r} - r)U_r - rU + h = 0, \tag{1}$$

$$U(r, T) = g(r), \tag{2}$$

where $\bar{r} = r_0 + \lambda\sigma/a$. We further assume that λ is a constant.

Let $P(r, t, s)$ denote the price at time t, given that $r(t) = r$, of a pure discount bond maturing at a time s (the solution to (1)–(2) with $T = s$, $g(r) \equiv 1$, and $h \equiv 0$). Let

$$f(r, t, s) = -\partial/\partial s \log P(r, t, s) \tag{3}$$

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denote the forward rate at time t and state r, implied for the instantaneous rate at time s. Finally set

$$v^2(t, s) = \mathrm{var}_{r,t}[r(s)] = \sigma^2(1 - e^{-2a(s-t)})/2a, \tag{4}$$

where the second equality is obtained as in Arnold [1], Section 8.3.

PROPOSITION:

(a) *Under the above assumptions, the solution of* (1)–(2) *is given by*

$$U(r, t) = P(r, t, T)E[g(R_{r,t,T})] + \int_t^T P(r, t, s)E[h(R_{r,t,s}, s)]\, ds, \tag{5}$$

where $R_{r,t,s}$ denotes a normal random variable with mean $f(r, t, s)$ and variance $v^2(t, s)$. Moreover,[1]

$$P(r, t, s) = \exp[\tfrac{1}{2}k^2(t, s) - n(r, t, s)], \tag{6}$$

$$f(r, t, s) = m(r, t, s) - q(t, s), \tag{7}$$

where (denoting $\tau = s - t$)

$$m = m(r, t, s) = e^{-a\tau}r + (1 - e^{-a\tau})\bar{r},$$

$$n = n(r, t, s) = \tau\bar{r} + (r - \bar{r})(1 - e^{-a\tau})/a,$$

$$k^2 = k^2(t, s) = \sigma^2(4e^{-a\tau} - e^{-2a\tau} + 2a\tau - 3)/2a^3,$$

$$q = q(t, s) = \sigma^2(1 - e^{-a\tau})^2/2a^2.$$

(b) *In particular, Equation* (5) *with $g(r) = P(r, T, s)$ and $h \equiv 0$ entails that*

$$E[\tilde{P}] = \textit{forward price} \equiv P(r, t, s)/P(r, t, T),$$

$$\tilde{P} \equiv P(R_{r,t,T}, T, s).$$

Equation (5) *also entails that the price at time t, given that $r(t) = r$, of a call option on the s-maturity pure discount bond with exercise price K and expiration $T < s$ is given by*

$$C(r, t, T, s, K) = P(r, t, T)E[\max\{0, \tilde{P} - K\}]. \tag{8}$$

Moreover, $\tilde{P}$ is lognormal with

$$\mathrm{var}[\log \tilde{P}] = \mathrm{var}_{r,t}[\log P(r(T), T, s)] \equiv \sigma_P{}^2,$$

where

$$\sigma_P = v(t, T)(1 - e^{-a(s-T)})/a.$$

Hence,

$$C(r, t, T, s, K) = P(r, t, s)N(h) - KP(r, t, T)N(h - \sigma_P), \tag{9}$$

[1] Equation (6) can readily be shown to be identical to equation (27) in Vasicek [8]. Also, note that, if g is a constant and h is a deterministic function of time, equation (5) reduces to the obvious present value expression.

where

$$h = \log[P(r, t, s)/P(r, t, T)K]/\sigma_P + \sigma_P/2.$$

(c) *More generally, Equation* (5) *entails that the price C_a at time t and state r of a European call option with exercise price K and expiration T on a portfolio consisting of $a_i > 0$ issues of s_i-maturity discount bonds is given by*

$$C_a = P(r, t, T)E[\max\{0, \tilde{P}_a - K\}],$$

where $\tilde{P}_a = \sum a_j P(R_{r,t,T}, T, s_j)$ and j runs over all indices for which $T < s_j$. Moreover,

$$E[\tilde{P}_a] = \frac{\sum a_j P(r, t, s_j)}{P(r, t, T)} \equiv \textit{forward portfolio price.}$$

One also has the decomposition

$$\max\{0, \tilde{P}_a - K\} = \sum a_j \max\{0, P(R_{r,t,T}, T, s_j) - K_j\}, \tag{10}$$

where $K_j = P(r^*, T, s_j)$ and r^* is the solution to $\sum a_j P(r^*, T, s_j) = K$. Hence,[2]

$$C_a = \sum a_j C(r, t, T, s_j, K_j).$$

The proof is given in the Appendix. The primary conclusion from the proposition is that the price of the European call option equals the discounted expected value of $\max\{0, X - K\}$ for some random variable X (independent of K) with expectation equal to the forward bond (portfolio) price. It can be shown that a similar statement is valid in *all* one-factor term structure models.[3]

The resemblance between the option pricing formula (9) and the Black-Scholes formula is obvious. In both cases, the random variable X above is lognormal, resulting in similar formulas. The discount factor $P(r, t, T)$ plays the role of $e^{-r(T-t)}$ in the Black-Scholes model, and $\sigma_P{}^2$, which is the variance of the logarithm of the price of the underlying security at option expiration, replaces $\sigma^2(T-t)$ of the Black-Scholes model, which has the same meaning. In other words, with these substitutions, the Black-Scholes model and the Vasicek model produce identical option values.

Part (c) of the proposition states that an option on a portfolio is equivalent to a portfolio of options with appropriate strike prices. It is clear from the proof that this decomposition extends to other situations where the prices of the portfolio components are all strictly decreasing (or all strictly increasing) functions of the same state variable.

Equation (9) is also similar to the option pricing formula of Cox, Ingersoll, and Ross [2], equation (32), except for the appearance of the normal distribution instead of the chi-squared distribution. The simpler formula here has the theo-

[2] European put prices follow by put-call parity, namely,

$$\text{call} - \text{put} = \sum a_j P(r, t, s_j) - P(r, t, T)K.$$

[3] A derivation of this more general result and its application to the square root process of Cox, Ingersoll, and Ross [2] and a mean-reverting generalization of the Gaussian continuous-time limit of the Ho and Lee model [5] is given in Jamshidian [6].

retical disadvantage of yielding positive call option prices for arbitrarily large strike prices. However, the magnitude of this deviation is often small, and the computational simplicity of the formula makes it an attractive practical alternative, especially for the evaluation of European options on bond portfolios and coupon bonds.

Appendix

Proof of the Proposition:

(a) Let $\tilde{r}(t)$ be the "risk-neutral interest rate process", defined by $d\tilde{r} = a(\bar{r} - \tilde{r})dt + \sigma dw$. Set $Y(t, s) = \int_t^s \tilde{r}(u)du$. Then it follows from Friedman [4], Theorem 6.5.3, that the solution to (1)–(2) is[4]

$$U(r, t) = E_{r,t}\left[g(\tilde{r}(T))e^{-Y(t,T)} + \int_t^T h(\tilde{r}(s), s)e^{-Y(t,s)}\, ds\right].$$

This is equivalent to

$$U(r, t) = \int_{-\infty}^{\infty} G(r, r', t, T)g(r')\, dr' + \int_t^T \int_{-\infty}^{\infty} G(r, r', t, s)h(r', s)\, dr'\, ds, \tag{11}$$

where

$$G(r, r', t, s) = \int_{-\infty}^{\infty} e^{-y}p(r, t, s, r', y)\, dy \tag{12}$$

and $p(r, t, s, \cdot, \cdot)$ denotes the joint probability density of $\tilde{r}(s)$, $Y(t, s)$ conditional on $\tilde{r}(t) = r$. To calculate p and G, we note that, by Corollary 8.2.4 in Arnold [1], $\tilde{r}(s)$ can be expressed as

$$\begin{aligned}\tilde{r}(s) &= a^{-a(s-t)}\tilde{r}(t) + \int_t^s e^{-a(s-u)}[a\bar{r}\, du + \sigma dw(u)] \\ &= e^{-a(s-t)}(\tilde{r}(t) - \bar{r}) + \bar{r} + \sigma \int_t^s e^{-a(s-u)}\, dw(u)\end{aligned}$$

and, in particular (as in Arnold [1], Section 8.3),

$$E_{r,t}[\tilde{r}(s)] = m(r, t, s), \qquad \mathrm{var}_{r,t}[\tilde{r}(s)] = v^2(t, s),$$

[4] Here $E_{r,t}[\cdot]$ means $E[\cdot \mid \tilde{r}(t) = r]$—similarly for variance.

where m and v^2 are as above.[5] It also follows that $\tilde{r}(s)$ and $Y(t, s)$ are bivariately normally distributed, and $E_{r,t}[Y(t, s)] = n$, $\text{var}_{r,t}[Y(t, s)] = k^2$, and $\text{cov}_{r,t}[\tilde{r}(s), Y(t, s)] = q$.[6] This uniquely determines p, and (12) can now be integrated to yield

$$G(r, r', t, s) = e^{(1/2k^2-n)}(2\pi v^2)^{-1/2}e^{-(r'-(m-q))^2/2v^2}. \tag{13}$$

Setting $h \equiv 0$, $g \equiv 1$, and $T = s$ in (11), (13) implies that $P(r, t, s) = \exp(\frac{1}{2}k^2 - n)$. Taking logarithmic derivative gives $f(r, t, s) = m - q$. In view of (11) and (13), Part (a) is now established.

(b) Applying (5) with $h \equiv 0$ and $g(r) = \max\{0, P(r, T, s) - K\}$ gives (8). The fact that $\tilde{P}$ and $P(r(T), T, s)$ are lognormal follows from (6) and the expression for n, which show that $P(r, T, s)$ is the exponential of a linear function of r. The coefficient of r in this linear term is $(1 - \exp(-a(s - T)))/a$; thus, the expression for σ_P follows. Equation (9) now follows from a well-known calculation involving the lognormal distribution.

(c) The first statement follows as in Part (b). To prove (10), it suffices to show that

$$\max\{0, \textstyle\sum a_j P(r, T, s_j) - K\} = \textstyle\sum a_j \max\{0, P(r, T, s_j) - K_j\}.$$

However, this follows from the fact that all $P(r, T, s_j)$ are decreasing functions of r. Indeed, from the way r^* and K_j are defined, we see that, if $r < r^*$, then $\sum a_j P(r, T, s_j) > K$ and $P(r, T, s_j) > K_j$, with the reverse inequality holding if $r > r^*$.

[5] Note that $\text{var}_{r,t}[\tilde{r}(s)]$ equals $\text{var}_{r,t}[r(s)]$ and does not depend on r. A similar statement holds for the other variance and covariance terms.

[6] These results are obtained by interchanging the order of integration and taking expectation (or covariance). Details of these and other calculations in the paper may be obtained from the author.

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