Value at Risk and Volatility

In this lecture. . .

- The meaning of Value at Risk (VaR)
- How VaR is calculated in practice
- Simulations and bootstrapping
- Simple volatility estimates
- The exponentially weighted moving average

By the end of this lecture you will be able to

- calculate the risk in a portfolio of assets

- use simulation methods for calculating VaR

- estimate volatility in two different ways

Introduction

Randomness plays a very important role in the financial world.

The assumption of a probabilistic description for assets is crucial to the development of models for investment, option pricing and the understanding of risk.

In this lecture we look at the theory of the risk in portfolios of stocks and the practice of measuring risk in portfolios of options.

We then consider simple models for estimating volatility.

Risk Measurement

We have seen how risk can be interpreted as a standard deviation of returns.

It is common practice now for banks and hedge funds to quote a measure of this risk in terms of how much money is at stake in their investments.

Such measures are known as **Value at Risk** (**VaR**).

Definition of Value at Risk

One of the definitions of Value at Risk, and the definition now commonly intended, is the following.

> Value at Risk is an estimate,
> with a given degree of confidence,
> of how much one can lose from one's
> portfolio over a given time horizon.

Watch out for...

- Size of possible loss
- Time horizon
- Degree of confidence

Examples:

1. The one-week VaR for trader X is \$112,000 at the 95% confidence level

2. The daily VaR for hedge fund Y is \$357,000 at the 99% level

3. The daily VaR for bank Z is \$87million at the 95% level

Key points

- VaR is an executive summary

- The portfolio can be that of a single trader, with VaR measuring the risk that he is taking with the firm's money, or that of a desk, or it can be the portfolio of the entire firm

- The time horizon of interest may be one day, say, for trading activities or months for portfolio management. It is supposed to be the timescale associated with the orderly liquidation of the portfolio, meaning the sale of assets at a sufficiently low rate for the sale to have little effect on the market. Thus the VaR is an estimate of a loss that can be realized, not just a 'paper' loss.

- The degree of confidence is typically set at 95%, 97.5%, 99% etc. How should this be chosen?

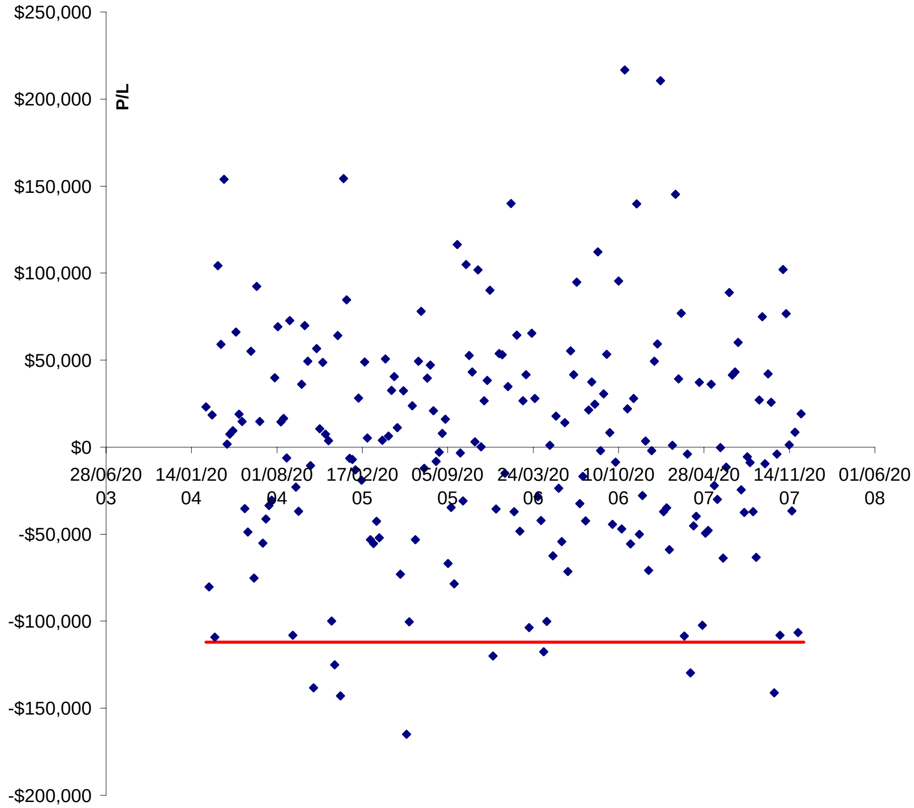
- VaR is calculated assuming normal market circumstances, meaning that extreme market conditions such as crashes are not considered, or are examined separately.

- Thus, effectively, VaR measures what can be expected to happen during the day-to-day operation of an institution.

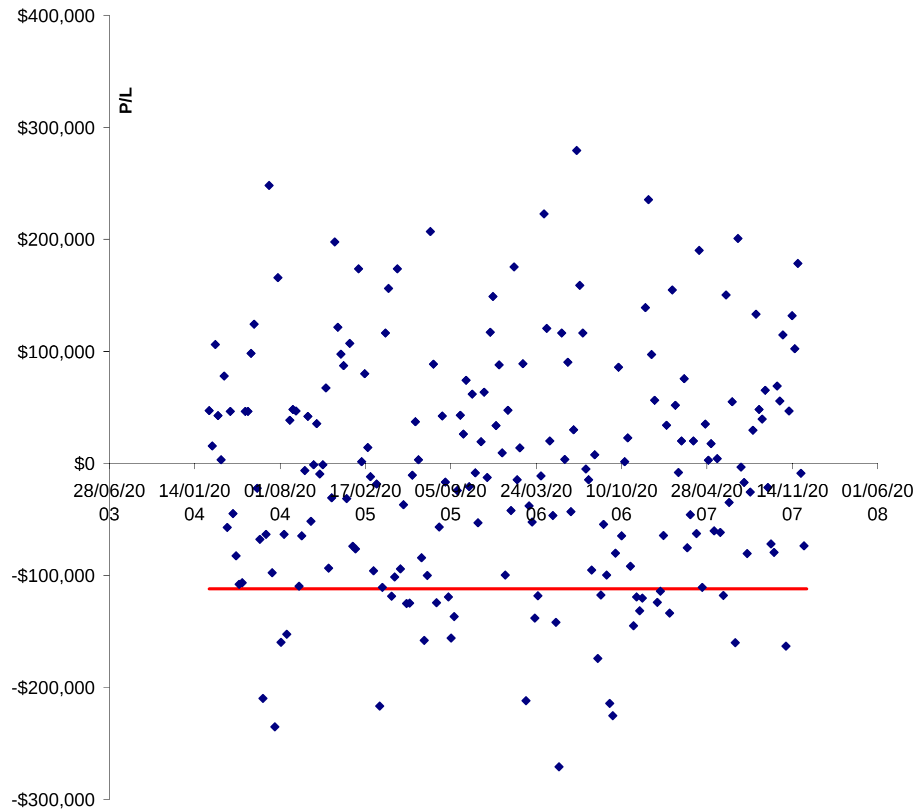
Let's see one of those examples again:

1. One-week VaR for trader X is \$112,000 at the 95% level

The 95% level means that in this example we would lose no more than \$112,000 19 weeks out of 20. A time series of P&L might look like this:



But what if you calculate the theoretical one-week VaR to be $112,000 at the 95% level and we get the following?



Some mathematics for a simple equity. . .

We hold a quantity Δ of a stock with price S and volatility σ. We want to know with 95% certainty what is the maximum we can lose over a time horizon δt.

First, we'll assume that the distribution of possible returns is Normal.

Second, let's calculate the standard deviation. The standard deviation of the stock price over this time horizon is

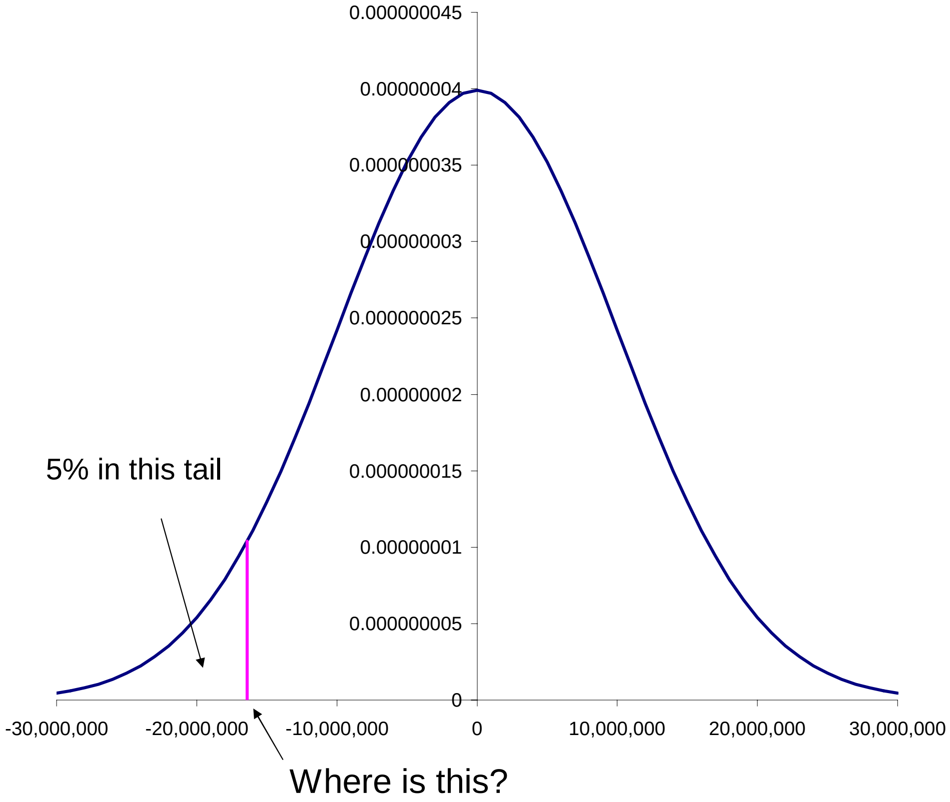
$$\sigma S\ \delta t^{1/2}.$$

We hold a number Δ of the stock, therefore the standard deviation for Δ shares is given by

$$\Delta \sigma S\ \delta t^{1/2}.$$

Third, we look at the left-hand 5% tail of the distribution.

Since the time horizon is so small, we can reasonably assume that the mean is zero.



The beauty of Normal distributions is that they are all the same!

The 5% (or any %) tail is always the same number of standard deviations away from the mean.

Here is the table of factors for converting from a confidence level and a standard deviation to a VaR number.

Degree of confidence	**Number of standard deviations from the mean**
99%	2.326342
98%	2.053748
97%	1.88079
96%	1.750686
95%	1.644853
90%	1.281551

We only need the table for the standardized Normal distribution (mean zero, standard deviation one) because we can get to any other Normal distribution by translation and scaling.

So, if we are interested in 95% confidence, or the 5% tail, then we need to multiply the standard deviation by a factor of 1.64. (Remember that we are assuming that the mean is zero.)

Answer:

$$\text{VaR} = 1.64\ \Delta\sigma S\ \delta t^{1/2}.$$

Generally, if we have a required degree of confidence, then

$$\text{VaR} = \text{Factor} \;\; \Delta\sigma S\,\delta t^{1/2},$$

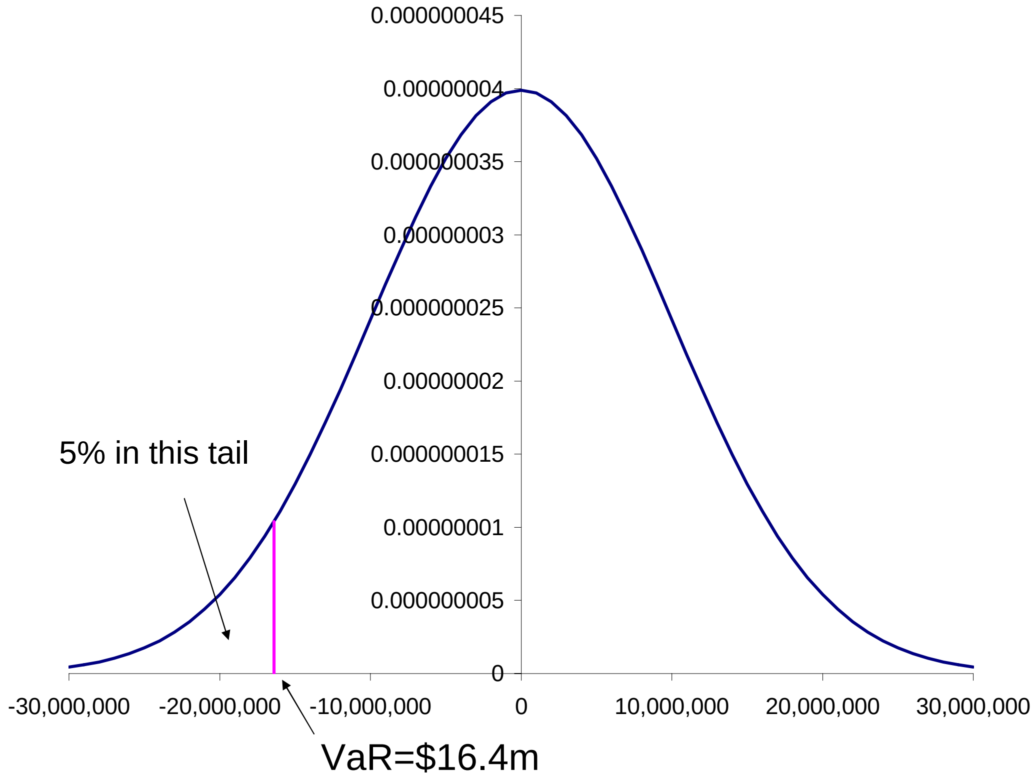
where 'Factor' comes from the table.

Example: We hold 1000 of a stock with market price \$67. The stock has a 23% volatility. What is the one-day VaR at the 99% confidence level?

$$2.33 \times 1000 \times 0.23 \times \$67 \times \frac{1}{\sqrt{252}} = \$2,262.$$

The stock holding itself is worth \$67,000.

In pictures: An investment has a standard deviation of $10m over the time horizon of interest. What is the Value at Risk at the 95% confidence level?

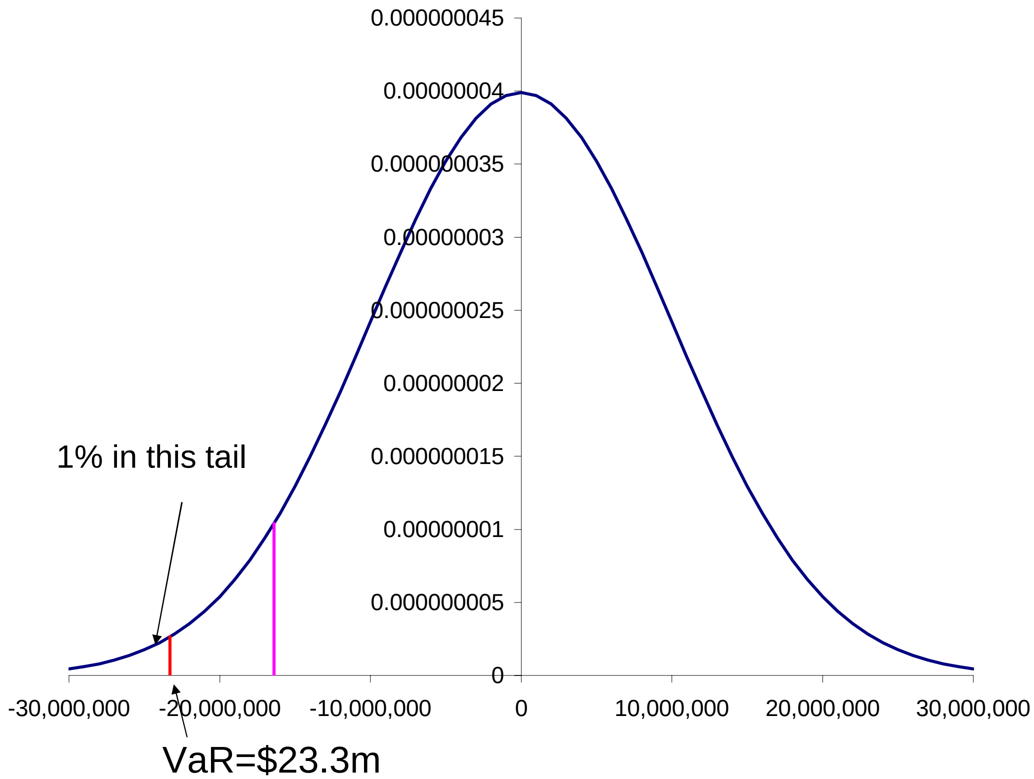


With the assumption of Normally distributed returns (with mean of zero) the answer would be $16.4m.

The 95% level means that in this example we would lose no more than \$16.4m 19 weeks out of 20.

If we want to play it safer, and be 99% certain what our loss would be, what would the answer be? Larger or smaller?

With the same assumptions the answer is \$23.3m.



The factor of 2.33 comes from the previous table.

With the assumption of Normally distributed changes in the value of a portfolio we can say that 95% confidence is equivalent to 1.64 standard deviations away from the mean and that 99% is equivalent to 2.33 standard deviations away from the mean.

Longer time horizons and the role of the drift

We have assumed that the return on the asset is Normally distributed *with a mean of zero*. The assumption of zero mean is valid for short time horizons: the standard deviation of the return scales with the square root of time but the mean scales with time itself.

- For longer time horizons, the return is shifted by an amount proportional to the time horizon.

Thus for longer timescales the VaR expression should be modified to account for the drift of the asset. If the rate of this drift is μ then

$$\text{VaR} = \Delta S \left(\mu \, \delta t - \text{Factor} \; \sigma \, \delta t^{1/2} \right).$$

VaR for a portfolio of equities

If we know the volatilities of all the assets in our portfolio and the correlations between them then we can calculate the VaR for the whole portfolio.

If the volatility of the of the ith asset is σ_i and the correlation between the ith and jth assets is ρ_{ij} (with $\rho_{ii} = 1$), then the VaR for the portfolio consisting of M assets with a holding of Δ_i of the ith asset is

$$\text{Factor } \delta t^{1/2} \sqrt{\sum_{j=1}^{M} \sum_{i=1}^{M} \Delta_i \Delta_j \sigma_i \sigma_j \rho_{ij} S_i S_j}.$$

(Think back to MPT.)

VaR for derivatives

The key point about estimating VaR for a portfolio containing derivatives is that, even if the change in the underlying *is* Normal, the essential nonlinearity in derivatives means that the change in the derivative can be far from Normal.

- Nevertheless, if we are concerned with very small movements in the underlying, for example over a very short time horizon, we may be able to approximate for the sensitivity of the portfolio to changes in the underlying by the option's delta.

- For larger movements we may need to take a higher order approximation.

We see these approaches and pitfalls next.

The delta approximation

Consider a portfolio of derivatives with a single underlying, S. The sensitivity of an option, or a portfolio of options, to the underlying is the delta, Δ.

If the standard deviation of the distribution of the underlying is $\sigma S\ \delta t^{1/2}$ then the standard deviation of the distribution of the option position is

$$\sigma S\ \delta t^{1/2}\Delta.$$

Δ must here be the delta of the whole position.

It is a small step to the following estimate for the VaR of a portfolio containing options:

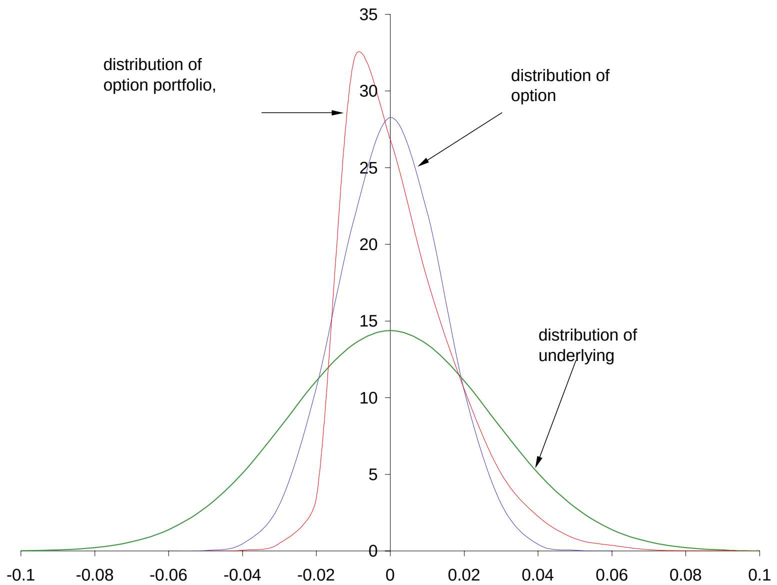
$$\text{Factor}\ \delta t^{1/2} \sqrt{\sum_{j=1}^{M} \sum_{i=1}^{M} \Delta_i \Delta_j \sigma_i \sigma_j \rho_{ij} S_i S_j}.$$

Here Δ_i is the rate of change of the *portfolio* with respect to the ith asset.

The role of curvature (gamma)

The delta approximation is satisfactory for small movements in the underlying.

- The effect of gamma is to introduce an asymmetry.



Distributions of stock and option changes.

And this completely messes up all the nice properties of the combination of linear instruments (equities) and Normal distributions.

There are no longer any nice formulæ for standard deviations when you have portfolios of non-linear instruments.

What can be done?

Use of valuation models

The obvious way around the problems associated with non-linear instruments is to use a simulation for the random behavior of the underlyings and then use valuation formulæ or algorithms to deduce the distribution of the changes in the whole portfolio.

This is the ultimate solution to the problem but has the disadvantage that it can be very slow.

After all, we may want to run tens of thousands of simulations.

Simulations

Sometimes we might want to estimate value at risk by simulating future asset behavior rather than use the previous formula.

This will often be the case if we have options in our portfolio.

- **Monte Carlo**, based on the generation of Normally distributed random numbers

- **Bootstrapping** using actual asset price movements taken from historical data

Monte Carlo

Monte Carlo simulation is the generation of a distribution of returns and/or asset price paths by the use of random numbers.

The technique can be applied to VaR: using numbers drawn from a Normal distribution, to build up a distribution of future scenarios.

For each of these scenarios use some pricing methodology to calculate the value of a portfolio (of the underlying asset and its options) and thus directly estimate its VaR.

Bootstrapping

Another method for generating a series of random movements in assets is to use historical data.

- The data that we use will consist of daily returns, say, for all underlying assets going back several years. The data for each day is recorded as a vector, with one entry per asset.

Suppose, for example, that we have real time-series data for N assets and that our data is daily data stretching back four years, resulting in approximately 1000 daily *returns* for each asset.

We are going to use these returns for simulation purposes. This is done as follows.

Spreadsheet showing bootstrap data.

	A	B	C	D	E	F	G	H	I	J	K	L
1	**Index**	**Prob.**	**TELEBRAS**	**ELETROBRAS**	**PETROBRAS**	**CVDR**	**USIMINAS**	**YPF**	**TAR**	**TEO**	**TGS**	**PEREZ**
2	1	0.001	-0.0152210	0.0180185	0.0118345	-0.0240975	-0.0111733	0.0355909	0.0185764	0.0071943	0.0121214	-0.0182190
3	2	0.001	0.0091604	0.0072214	-0.0046130	0.0001039	-0.0226244	0.0207620	0.0121953	0.0106953	0.0000000	-0.0026393
4	3	0.001	-0.0498546	-0.0397883	-0.0324353	-0.0246926	-0.0115608	0.0068260	-0.0121953	-0.0106953	0.0119762	-0.0067506
5	4	0.001	-0.0357762	-0.0494712	-0.0358569	-0.1001323	-0.0602100	-0.0068260	-0.0123458	-0.0254097	0.0000000	-0.0144520
6	5	0.001	0.0033058	-0.0204013	0.0112413	0.0264184	0.0114388	-0.0068729	0.0061920	0.0036697	-0.0240976	0.0096238
7	6	0.001	0.0424306	0.0260111	-0.0255043	-0.0022909	0.0004155	0.0067340	0.0452054	0.0438519	0.0115608	0.0140863
8	7	0.001	0.0279317	0.0006216	0.0006216	0.0006216	-0.0234759	0.0040080	-0.0208341	0.0000000	-0.0114287	0.0000000
9	8	0.001	-0.0139801	-0.0383626	-0.0346393	0.0161040	0.0122250	0.0000000	0.0259755	0.0176996	0.0114287	-0.0020299
10	9	0.001	-0.0052219	-0.0036563	0.0403377	0.0142520	0.0229896	0.0063898	0.0101524	0.0085349	0.0114287	-0.0019638
11	10	0.001	-0.0130693	0.0000000	-0.0116961	-0.0047282	0.0112996	0.0000000	0.0100503	0.0112677	0.0112996	0.0081304
12	11	0.001	-0.0106656	-0.0148639	0.0069306	-0.0048541	-0.0114030	0.0000000	-0.0100503	-0.0169496	-0.0112996	-0.0166294
13	12	0.001	0.0079035	0.0184167	-0.0023392	-0.0240975	0.0112996	0.0000000	-0.0050633	-0.0085837	0.0335227	-0.0170073
14	13	0.001	0.0000000	-0.0072225	-0.0284021	0.0001035	-0.0226248	0.0000000	-0.0257084	-0.0380719	-0.0110498	-0.0063762
15	14	0.001	0.0000000	-0.0042230	0.0242792	-0.0004134	-0.0126835	-0.0061920	0.0100001	0.0057471	0.0109291	0.0144183
16	15	0.001	-0.0475377	-0.0138620	-0.0251059	-0.0444106	-0.0254146	-0.0062305	-0.0150379	-0.0057471	0.0000000	0.0300138
17	16	0.001	0.0297428	0.0102652	0.0114445	0.0714950	-0.0008256	0.0415490	0.0277796	0.0309303	0.0408220	-0.0035537
18	17	0.001	-0.0037045	-0.0192060	-0.0248187	0.0078541	-0.0129914	-0.0116280	-0.0045977	0.0052771	0.0000000	0.0007187
19	18	0.001	-0.0208341	-0.0205823	-0.0264421	-0.0021947	-0.0267507	0.0231224	0.0091744	-0.0052771	-0.0304592	0.0152095
20	19	0.001	0.0568874	0.0828276	0.1052577	0.0477150	0.0612661	0.0271019	0.0753494	0.0746435	0.0388398	0.0599762
21	20	0.001	0.0655911	0.0209154	0.0209922	0.0420456	0.0224756	-0.0173917	0.0165293	0.0223058	0.0000000	0.0232360
22	21	0.001	-0.0091109	0.0309987	0.0615672	-0.0428678	0.0221646	0.0058309	0.0122201	0.0145988	0.0392207	0.0139353
23	22	0.001	0.0231470	-0.0036810	-0.0003084	0.0158210	0.0099482	0.0342891	0.0160646	0.0264121	0.0204089	0.0208262
24	23	0.001	0.0157998	-0.0014713	-0.0122804	-0.0094697	-0.0014375	0.0277026	0.0504962	0.0440396	0.0492710	0.0285456
25	24	0.001	-0.0045147	-0.0054003	-0.0008614	-0.0020500	0.0379553	-0.0054795	-0.0191210	-0.0136988	0.0000000	0.0101176
26	25	0.001	0.0270287	0.0275012	0.0583711	0.0001022	0.0361421	0.0280130	-0.0038536	0.0069849	0.0000000	0.0034104
27	26	0.001	-0.0089286	0.0068362	0.0038686	0.0248968	-0.0087868	0.0000000	-0.0116506	-0.0093241	0.0000000	0.0110554
28	27	0.001	0.0133632	0.0232183	0.0512933	0.0202027	0.0436750	-0.0167135	-0.0117880	-0.0285055	-0.0099503	-0.0080555
29	28	0.001	-0.0110341	0.0417395	0.0176995	0.0506932	0.0085108	0.0000000	-0.0240012	-0.0268636	0.0000000	0.0005324
30	29	0.001	-0.0044544	-0.0289144	0.0069770	-0.0002423	-0.0002043	-0.0056023	0.0000000	0.0024907	0.0000000	-0.0037901
31	30	0.001	0.0044544	0.0001022	-0.0344781	-0.0075315	0.0086859	0.0056023	0.0202027	0.0049628	0.0204089	0.0147792
32	31	0.001	-0.0044544	-0.0065956	-0.0225751	0.0151071	-0.0173439	0.0111112	-0.0040080	0.0024722	0.0396091	0.0090663

- Assign an ‘index’ to each daily change. That is, we assign 1000 numbers, one for each vector of returns. To visualize this, imagine writing the returns for all of the N assets on the blank pages of a notebook. On page 1 we write the changes in asset values that occurred from 8th July 1998 to 9th July 1998. On page 2 we do the same, but for the changes from 9th July to 10th July 1998. On page 3. . . from 10th to 11th July etc. We will fill 1000 pages if we have 1000 data sets.

- Now, draw a number from 1 to 1000, uniformly distributed; it is 534. Go to page 534 in the notebook.

- Change all assets from today’s value by the vector of returns given on the page.

- Now draw another number between 1 and 1000 at random and repeat the process. Increment this new value again using one of the vectors.

- Continue this process until the required time horizon has been reached. This is one realization of the path of the assets.

- Repeat this simulation to generate many, many possible realizations to get an accurate distribution of all future prices.

By this method we generate a distribution of possible future scenarios based on historical data.

Note how we keep together all asset changes that happen on a certain date. By doing this we ensure that we capture any correlation that there may be between assets.

A problem with classical VaR. . .

A common criticism of traditional VaR has been that it does not satisfy all of certain commonsense criteria.

Artzner et al. (1997) defined the following set of sensible criteria that a measure of risk, $\rho(X)$ where X is a set of outcomes, should satisfy.

These are as follows. . .

1. Sub-additivity: $\rho(X+Y) \leq \rho(X)+\rho(Y)$. This just says that if you add two portfolios together the total risk can't get any worse than adding the two risks separately. Indeed, there may be cancellation effects or economies of scale that will make the risk better.

2. Monotonicity: If $X \leq Y$ for each scenario then $\rho(Y) \leq \rho(X)$. If one portfolio has better values than another under all scenarios then its risk will be better.

3. Positive homogeneity: For all $\lambda > 0$, $\rho(\lambda X) = \lambda\rho(X)$. Double your portfolio then you double your risk.

4. Translation invariance: For all constant c, $\rho(X+c) = \rho(X)-c$. Think of just adding cash to a portfolio, this would come off your risk.

A risk measure that satisfies all of these is called **coherent**.

The traditional, simple VaR measure is not coherent since it does not satisfy the sub-additivity condition.

Sub-additivity is an obvious requirement for a risk measure, otherwise there would be no risk benefit to adding uncorrelated new trades into a book.

If you have two portfolios X and Y then this benefit can be defined as

$$\rho(X) + \rho(Y) - \rho(X + Y).$$

Sub-additivity says that this can only be non negative.

Lack of sub-additivity is a risk measure can be exploited in a form of regulatory arbitrage. All a bank has to do is create subsidiary firms, in a reverse form of the above example, to save regulatory capital.

With a coherent measure of risk, specifically because of its sub-additivity, one can simply add together risks of individual portfolios to get a conservative estimate of the total risk.

Coherent measures:

An example of a measure that is coherent is **Expected Shortfall**. This is calculated as the average of all the P&Ls making up the tail percentile of interest. Suppose we are working with the 5% percentile, rather than quoting this number (this would be traditional VaR) instead calculate the average of all the P&Ls in this 5% tail.

How do we estimate σ for all of these formulæ?

Is this just a statistical exercise or is some ‘modeling’ required?

This depends on whether we believe volatility is a constant parameter or is changing.

The simplest volatility estimate: constant volatility/moving window

If we believe that volatility is constant and we have a total of N days' data, we can estimate volatility σ using

$$\sigma^2 = \frac{1}{N\ \delta t} \sum_{i=1}^{N} R_i^2$$

where

$$R_i = \frac{S_i - S_{i-1}}{S_{i-1}}$$

and is the return on the ith day. As usual, δt is the time step.

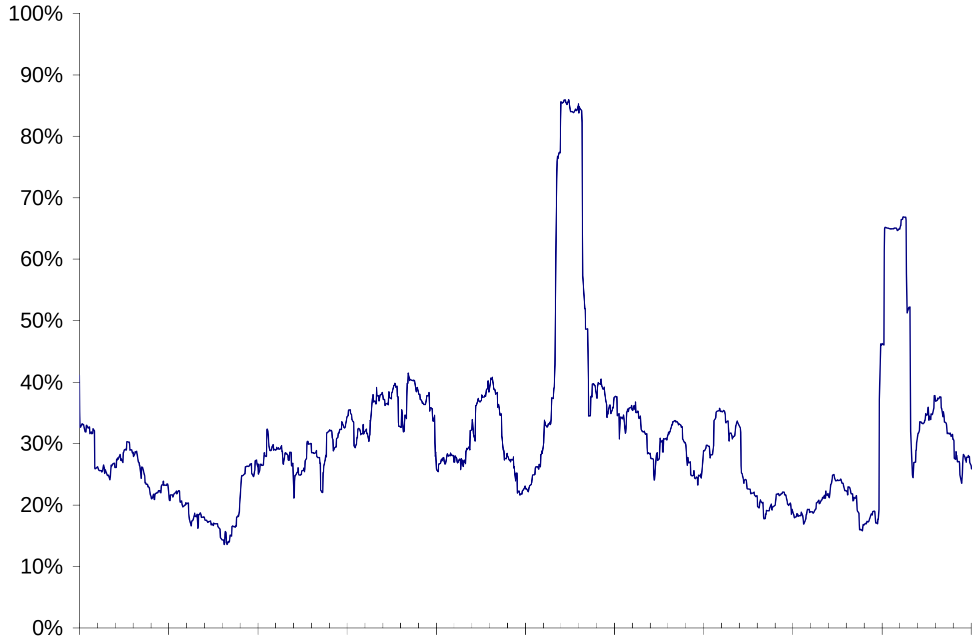
If volatility is thought to be slowly varying in time then it would be natural to take a smaller subset of all our data for estimating volatility.

We might use the previous 60 daily returns for calculating volatility, if we think that 60 days is a timescale over which volatility is roughly unchanged.

This is the 'default' estimate of volatility if, for example, we want to value an option with 60 days left until expiration.

Bloomberg calculates such simple volatility numbers.

Here is an example of a volatility calculated in such a way.



There are major problems associated with this volatility measure. The result shows spurious ‘plateauing’ after big moves in the stock.

Exponentially weighted moving average

But why should all of the last 60, say, days be equally important?

Surely, if the volatility is not constant, is changing, then there will be more information in the more recent stock price moves than in data from a long time ago.

This leads us to looking at volatility estimates in which more recent returns are weighted more heavily than returns from the past.

Consider this estimate for the volatility σ_n on the nth day:

$$\sigma_n^2 = \frac{1}{\delta t}(1-\lambda)\left(R_n^2 + \lambda R_{n-1}^2 + \lambda^2 R_{n-2}^2 + \lambda^3 R_{n-3}^2 + \cdots\right).$$

This can be written as

$$\sigma_n^2 = \frac{(1-\lambda)}{\delta t}\sum_{i=1}^{\infty} \lambda^{i-1} R_{n-i+1}^2.$$

The parameter λ must be greater than zero and less than one.

This is an example of an **exponentially-weighted moving average estimate**. The more recent the return, the more weight is attached. The sum extends back to the beginning of time.

The coefficient $1-\lambda$ ensures that the weights all add to one.

The above expression looks like it requires data going back to the dawn of time!

However, the expression can also be written in a simplified form.

$$\sigma_n^2 = \frac{1}{\delta t}(1-\lambda)\left(R_n^2 + \lambda R_{n-1}^2 + \lambda^2 R_{n-2}^2 + \lambda^3 R_{n-3}^2 + \cdots\right)$$

$$= \frac{1}{\delta t}(1-\lambda)R_n^2 + \frac{\lambda}{\delta t}(1-\lambda)\left(R_{n-1}^2 + \lambda R_{n-2}^2 + \lambda^2 R_{n-3}^2 + \cdots\right).$$

Therefore

$$\sigma_n^2 = \lambda\sigma_{n-1}^2 + \frac{(1-\lambda)}{\delta t}R_n^2.$$

Note that this uses the most recent return *and* the previous estimate of the volatility.

How do we choose λ?

There is a timescale associated with the choice of λ, and it is related to the idea of half life in radioactive decay!

After how many days does the impact of a return decrease to one half of the impact it originally had?

If the number of days is m then

$$\lambda^m = \frac{1}{2}.$$

So

$$m \ln(\lambda) = \ln(0.5) = -\ln(2).$$

Therefore

$$m = -\frac{\ln(2)}{\ln(\lambda)}.$$

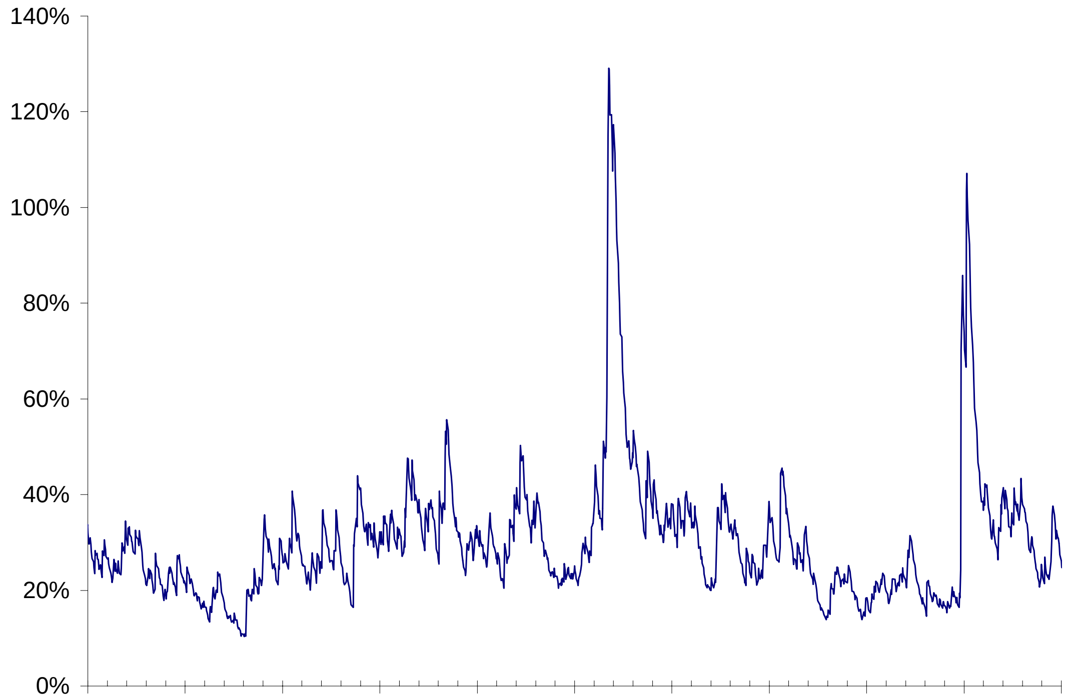
Or

$$\lambda = e^{-\ln(2)/m}.$$

If we know λ then we know how many days are relevant.

Alternatively, if we know how many days are relevant in the calculation then we know what λ to use.

With the same returns data as before. . .



Summary

Please take away the following important ideas

- It is common practice for banks and hedge funds to estimate how much money they could, this is called Value at Risk

- Value at risk is quoted over a given time horizon and with a specified degree of confidence

- Volatility is probably not constant and therefore we may want to use more advanced methods for its estimation than a simple standard deviation