A. Basic Mathematics

1. Given that the Taylor series for the function

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + ...,$$

use this to show the following

$$\begin{aligned} \frac{x}{(1+x^2)^2} &= x - 2x^3 + 3x^5 - .. \\ \log(1-x) &= -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - .. \end{aligned}$$

2. Consider complex number division $\dfrac{a+ib}{c+id}$ which we express as $re^{i\alpha}$. Using Euler's identity **only**, work out the precise form for the quotient's modulus r and argument α. Your calculations should not involve division of complex numbers.

3. Using row operations (only) evaluate the following determinant $\begin{vmatrix} y-z & z-x & x-y \\ z-x & x-y & y-z \\ x-y & y-z & z-x \end{vmatrix}$ and give your solution in the simplest form.

4. Find the eigenvalues and eigenvectors of the following matrix

$$\begin{pmatrix} 3 & 3 & 3 \\ 3 & -1 & 1 \\ 3 & 1 & -1 \end{pmatrix}.$$

Verify that the eigenvectors are mutually orthogonal and hence diagonalize the matrix. Show all working.

B. Probability

N.B. $\mathbb{E}[f(x)]$ is the expectation of the function $f(x)$, given some probability density function $p(x)$.

1. (a) Consider the probability density function $p(x;\ \lambda)$

$$p(x;\ \lambda) = \begin{cases} A\lambda x \exp\left(-\lambda x^2\right) & x \geq 0 \\ 0 & x < 0 \end{cases}$$

where $\lambda\,(>0)$ and A are both constants. Calculate the value of A.

(b) Show that the even moments of $p(x;\ \lambda)$ are given by ,

$$E\left[x^{2n}\right] = \frac{n!}{\lambda^n}, \qquad n = 0,\ 1,\ 2, ...$$

This can be done by recursion.

2. In class we saw that summing up the RAND() function and subtracting off 6 gives a standard normal, i.e.

$$\sum_1^{12} \mathrm{RAND}\,() - 6 \longrightarrow \phi \sim N\,(0,1)\,.$$

Obtain a similar expression for using a number N of the RAND() function and verify that this also gives $\phi \sim N\,(0,1)$. Further, show that your formula is consistent with the Central Limit Theorem.

C. Stochastic Calculus

N.B. X is standard Brownian motion.

1. Find the stochastic differential equation (sde) df for the function f in each of the following cases.

a) $f(X) = \ln(X^n)$

b) $f(X) = \exp(nX)$

c) $f(X) = a^X$ where $a > 1$

Show that in **b)** and **c)**, the SDE can be also written $\dfrac{df}{f} = A\ dt + BdX$ and give the form of the constants A and B.

2. Consider the diffusion process for the spot rate r which evolves according to the stochastic differential equation

$$dr = -ar dt + bdX.$$

Both a and b are constants. Write down the forward Fokker-Planck equation for the transition probability density function $p(r', t')$ for this process, where a primed variable refers to a future state/time.

By solving the Fokker-Planck equation which you have obtained, obtain the **steady state** probability distribution $p_\infty(r')$, which is given by

$$p_\infty = \sqrt{\frac{a}{b^2\pi}} \exp\left(-\tfrac{a}{b^2} r'^2\right).$$

3. **(a)** Show that

$$G = \exp(t + a\exp(X(t)))$$

is a solution of the stochastic differential equation

$$dG(t) = G\left(1 + \tfrac{1}{2}(\ln G - t) + \tfrac{1}{2}(\ln G - t)^2\right) dt + G(\ln G - t)\, dX.$$

(b) By considering the form $S(t) = (A + X/\alpha)^\alpha$ show that

$$dS = \tfrac{1}{3} S^{1/3} dt + S^{2/3} dX,$$

where α should be determined.

D. Further Mathematical Methods

1. Consider the time independent Black-Scholes equation

$$\frac{1}{2}\sigma^2 S^2 \frac{d^2V}{dS^2} + rS\frac{dV}{dS} - rV = 0.$$

for the unknown function $V(S)$, where the volatility σ and interest rate r are constant. Show that the general solution is

$$V(S) = AS + BS^{-2r/\sigma^2}.$$

If

$$\begin{aligned} \lim_{S\longrightarrow\infty} V(S) &\longrightarrow S \\ V(S^*) &= S^* - E \end{aligned}$$

obtain the constants A and B to present a particular solution.

2. Find the general solutions of the following:

(i) $xy' = y + \sqrt{x^2 + y^2}$

(ii) $y' = \dfrac{2x + 9y - 20}{6x + 2y - 10}$

(iii) $y' = \dfrac{3x - 4y - 2}{3x - 4y - 3}$

(iv) $2y' + y = (x - 1)\,y^3$

(v) $(x + 3y - 1)\,dx + (3x - 2y + 4)\,dy = 0$

3. (i) Calculate $\left(\sqrt{3} + i\right)^{25}$

(ii) Find $\sin 5\theta$ and $\cos 5\theta$ in terms of powers of $\sin\theta$ and $\cos\theta$

(iii) Find $\sin^5\theta$ and $\cos^5\theta$ in terms of $\sin n\theta$ and $\cos n\theta,\ n \in \mathbb{N}$

(iv) Find <u>all</u> the roots of $x^6 - 1$.

4. If $z = x + iy$, solve the following equation

$$\cos z = 4$$