CQF Final Examination For The Award Of Distinction

December 2008

There are three sections and nine questions in this examination.

Full marks can be obtained from complete answers to four questions.

You are required to do at least one question from each section. The fourth can be chosen from any section.

If you answer more than four, your best four will be taken and graded.

The use of calculators is permitted in this examination.

You may assume throughout this examination that dX is an increment in a standard Brownian motion $X(t)$:

$$\begin{aligned} \mathbb{E}[dX] &= 0 \\ \mathbb{E}\left[dX^2\right] &= dt. \end{aligned}$$

Section A

1. a) Consider a random variable Y, that is normally distributed such that $Y \sim N(\mu, \sigma^2)$. If the Moment Generating Function (MGF) is denoted by $M_Y(\theta)$, show that for this distribution

$$M_Y(\theta) = e^{\theta\mu + \frac{1}{2}\theta^2\sigma^2}.$$

Now define the k^{th} moment m_k of the random variable Y by

$$m_k = \left.\frac{d^k}{d\theta^k} M_Y(\theta)\right|_{\theta=0} \quad ; \quad k = 0, 1, 2, \ldots$$

Use this to obtain the first four moments of a standard normal random variable $\phi \sim N(0,1)$ and hence calculate the **skew** and **kurtosis** for ϕ.

b) The random number generator in Excel, RAND(), produces uniformly distributed random variables over $[0,1]$, written Unif$[0,1]$.

Show that if we generate a number N of this random variable then the algorithm

$$\sqrt{\frac{12}{N}}\left(\sum_{1}^{N} RAND() - \frac{N}{2}\right) \tag{1}$$

produces a single standardized Normal $\phi \sim N(0,1)$. Expression (1) should be derived. Further, show that (1) is consistent with the Central Limit Theorem.

2. In the following, $X(t)$ is a standard Brownian motion.
Part A Which of the following processes are Martingales?

a) $Y(t) = X(t) + 4t$

b) $Y(t) = X^2(t)$

c) $Y(t) = t^2 X(t) - 2\int_0^t sX(s)ds$

d) $Y(t) = X_1(t)X_2(t)$ where $X_1(t)$ and $X_2(t)$ are two standard Brownian motions with correlation ρ so that $dX_1(t)dX_2(t) \to \rho dt$. Does the answer depend on the value of ρ?

Part B
Define the process $Y(t)$ by the stochastic differential equation

$$dY(t) = f(t, Y(t))dt + dX(t)$$

with $Y(0) = y_0$, and $f(t, Y(t))$ is some given function satisfying the Novikov condition. Consider in addition the exponential martingale $M(t)$ given by

$$M(t) = \exp\left\{-\int_0^t f(s, Y(s))ds - \frac{1}{2}\int_0^t f^2(s, Y(s))dX(s)\right\}$$

Show that the process $Z(t) = M(t)Y(t)$ is a martingale.

3. Suppose that the process S evolves according to Geometric Brownian motion

$$dS = \mu S dt + \sigma S dX.$$

Show that

$$d(\log S) = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dX.$$

Now deduce that the expected value of $S(t)$ at time $t > 0$, given S_0 at time zero, is

$$\mathbb{E}\left[S(t)|\, S_0\right] = S_0 e^{\mu t}.$$

Show that if $V = S^n$, for constant n, then V follows

$$\frac{dV}{V} = n\left(\mu + \tfrac{1}{2}(n-1)\sigma^2\right) dt + n\sigma dX.$$

Hence or otherwise find a general expression for

$$\mathbb{E}\left[S^n(t)|\, S_0\right].$$

(b) We wish to compute $m(t) = \mathbb{E}\left[e^{aX(t)}\right]$, where a is a given constant. First, use Itô to show that

$$m(t) = 1 + \frac{a^2}{2}\int_0^t m(s)ds$$

Next take the derivative with respect to t to deduce an ODE for $m(t)$. Finally, solve the ODE to evaluate the expectation $m(t)$.

Section B

4. **(a)**

   Briefly explain the following:

   i. The difference between hedging, speculation and arbitrage.

   ii. The difference between writing a call option and buying a put option.

   iii. What happens when an investor shorts a share (also discuss dividends).

   iv. What is the difference between a straddle and strangle?

(b)

A stock is currently worth £40. It is also known that at the end of a six month term this asset price will have risen to £45 or fallen to £38. The risk-free interest rate is 5% per annum.

Using the Binomial Method, what is the value of a six month European call option with a strike price of £40?

(c)

A binary Call option has payoff $H(S-E)$ at expiry $t=T$ and a binary Put has payoff $H(E-S)$ where H is the Heaviside step function defined by

$$\begin{cases} H(Y)=1 & Y>0 \\ H(Y)=\frac{1}{2} & Y=0 \\ H(Y)=0 & Y<0 \end{cases}$$

A customer wants to enter into a position which consists of a Binary Call plus a Binary Put with the same exercise price E and expiry time T. If the interest rate is r what is the fair price of this position?

5. Consider a Markowitz world with a three asset risky economy where the covariance matrix of expected returns is given by

$$\begin{pmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_2^2 & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_3^2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}.$$

Firstly show that the covariance matrix is strictly positive definite in the sense that

$$\begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_2^2 & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_3^2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} > 0$$

if $\begin{pmatrix} x & y & z \end{pmatrix} \neq \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}$. **(It is sufficient to show that the resulting quadratic form is a sum of perfect squares.)**

Deduce that the inverse of the covariance matrix is

$$\begin{pmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_2^2 & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_3^2 \end{pmatrix}^{-1} = \frac{1}{2} \begin{pmatrix} 5 & -3 & 1 \\ -3 & 3 & -1 \\ 1 & -1 & 1 \end{pmatrix}$$

Assume that the expected returns on the risky assets are respectively,

$$\begin{pmatrix} R_1 \\ R_2 \\ R_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

Using the method of Lagrange optimisation, deduce that the boundary of the opportunity set is given by

$$\sigma_{\sqcap}^2(\rho) = \frac{1}{4}\left(3\rho^2 - 8\rho + 8\right),$$

where ρ is the prescribed level of expected return and $\sigma(\rho)$ is the minimal level of risk corresponding to the level of expected return.

By differentiating $\sigma_{\sqcap}^2$, show that the efficient frontier is given by

$$\sigma_{\sqcap}^2 = \frac{1}{4}\left(3\rho^2 - 8\rho + 8\right) \text{ for } \rho \geq \frac{4}{3},$$

and identify this on a diagram.

6. **(a)** Consider the Itô integral of the form

$$\int_0^T f(t, X(t))\, dX(t) = \lim_{N \to \infty} \sum_{i=0}^{N-1} f(t_i, X_i)(X_{i+1} - X_i).$$

The interval $[0, T]$ is divided into N partitions with end points

$$t_0 = 0 < t_1 < t_2 < \ \ldots\ldots \ < t_{N-1} < t_N = T,$$

where the length of an interval $t_i - t_{i+1}$ tends to zero as $N \to \infty$.

Using Itô's lemma show that

$$3\int_0^T X(t)^2\, dX(t) = X(T)^3 - X(0)^3 - 3\int_0^T X(t)\, dt.$$

Show from the definition of the Itô integral that the result can also be found by initially writing the integral

$$3\int_0^T X^2 dX = \lim_{N \to \infty} 3\sum_{i=0}^{N-1} X_i{}^2 (X_{i+1} - X_i)$$

Hint: You may use $3b^2(a-b) = a^3 - b^3 - 3b(a-b)^2 - (a-b)^3$**. Full details of all working should be given.**

(b) Consider the diffusion process for the spot rate r which evolves according to the stochastic differential equation

$$dr = -ar dt + b dX.$$

Both a and b are constants. Write down (**not derive**) the forward Fokker-Planck equation for the transition probability density function $p(r', t')$ for this process, where a primed variable refers to a future state/time.

By solving the Fokker-Planck equation which you have obtained, obtain the **steady state** probability distribution $p_\infty(r')$, which is given by

$$p_\infty = \sqrt{\frac{a}{b^2 \pi}} \exp\left(-\tfrac{a}{b^2} r'^2\right).$$

Section C

7.

(a) An asset S follows the lognormal random walk

$$dS = \mu S dt + \sigma S dX$$

and we wish to value a derivative that pays off at expiry T an amount which is a function of the path taken by the asset between time zero and expiry.

Assuming that an option value V thus depends on S, t and a quantity

$$I = \int_0^t f(S, \tau)\, d\tau,$$

where f is a specified function and r the risk free interest rate, derive the pricing equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + f(S, t)\frac{\partial V}{\partial I} + rS\frac{\partial V}{\partial S} - rV = 0 \qquad (7)$$

for the function $V(S, I, t)$.

(b) For an arithmetic strike Asian call option the payoff at time T is

$$\max\left(S - \tfrac{1}{T}\int_0^T S(\tau)\, d\tau, 0\right)$$

and for a put option the payoff is

$$\max\left(\tfrac{1}{T}\int_0^T S(\tau)\, d\tau - S, 0\right).$$

Write down the corresponding partial differential equation for this call option $V_C(S, I, t)$ and put option $V_P(S, I, t)$, and hence verify that

$$V_C(S, I, t) - V_P(S, I, t) = S\left(1 - \tfrac{1}{rT}\left(1 - e^{-r(T-t)}\right)\right) - \tfrac{1}{T}e^{-r(T-t)}\int_0^t S(\tau)\, d\tau$$

Briefly outline the method of *upwind differencing* when applied to solving equations of the form (7), when using finite difference methods. Why is such a technique necessary when solving (7) numerically?

8. (a) Briefly explain how the Explicit Finite Difference Method can be used to price American call options.

(b) Consider a perpetual American **put** option $V(S)$, which satisfies the Euler problem

$$\frac{1}{2}\sigma^2 S^2 \frac{d^2 V}{dS^2} + (r - D) S \frac{dV}{dS} - rV = 0, \quad S^* < S < \infty,$$

$$V(S) = E - S, \quad 0 \le S \le S^*,$$

$$V(S^*) = E - S^*, \quad \frac{dV}{dS}(S^*) = -1, \quad \lim_{S \to \infty} V(S) \to 0,$$

where $S \ge 0$ is the spot price, $E > 0$ is the strike, $S^* > 0$ is the optimal exercise boundary, $\sigma > 0$ is the constant volatility, $r > 0$ is the constant interest rate and D is the dividend yield.
Show that $V(S) = S^\alpha$ is a solution of the differential equation above provided that

$$\frac{1}{2}\sigma^2\alpha^2 + \left(r - D - \frac{1}{2}\sigma^2\right)\alpha - r = 0$$

Show that one of the roots of this equation, α^-, is always negative. Hence deduce that for $S > S^*$,

$$V(S) = -\frac{S^*}{\alpha^-}\left(\frac{S}{S^*}\right)^{\alpha^-}, \quad S^* = \frac{\alpha^-}{\alpha^- - 1}E.$$

Show that

$$\sigma^2\alpha^- + \left(r - D - \frac{1}{2}\sigma^2\right) = -\sqrt{\left(r - D - \frac{1}{2}\sigma^2\right) + 2r\sigma^2} < 0.$$

9. **Part A**

Consider the following mean-reverting Ornstein-Uhlenbeck process U_t, which satisfies the stochastic differential equation

$$dU_t = -\theta U_t dt + \sigma dX.$$

The drift θ and diffusion σ are constant. Show that by using an integrating factor method,

$$U_t = \alpha \exp\left(-\theta t\right) + \sigma \left(X_t - \theta \int_0^t \exp\left(\theta\left(s-t\right)\right) X_s ds \right)$$

where $U\left(0\right) = \alpha$.

Part B

The two factor interest rate model with the Bond Pricing Equation (BPE) is given by

$$\frac{\partial V}{\partial t} + \frac{1}{2} w^2 \frac{\partial^2 V}{\partial r^2} + \rho w q \frac{\partial^2 V}{\partial r \partial l} + \frac{1}{2} q^2 \frac{\partial^2 V}{\partial l^2} + \left(u - \lambda_r w\right) \frac{\partial V}{\partial r} + \left(p - \lambda_l q\right) \frac{\partial V}{\partial l} - rV = 0,$$

where the two state variables evolve according to

$$\begin{aligned} dr &= udt + wdX_1 \\ dl &= pdt + qdX_2. \end{aligned}$$

The Brownian motions are correlated with $\mathbb{E}\left[dX_1 dX_2\right] = \rho dt$.

Given that $u - \lambda_r w = 0 = p - \lambda_l q$ and $w = q = \sqrt{a + br + cl}$, where $a,\ b$ and c are constants, derive a set of first order equations and boundary conditions for $A,\ B$ and C such that a bond V is of the form

$$V = \exp\left(A\left(t;T\right) - rB\left(t;T\right) - lC\left(t;T\right)\right),$$

is a solution of the BPE with redemption value

$$V\left(r, l, T; T\right) = 1.$$

You are not required to solve these equations.