Interest Rate Theory

An Introduction

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- Recap on arbitrage theory.
- Martingale measures and the bond market.
- Short rate models
- Affine term structures
- Inverting the yield curve
- Forward rate models. Heath-Jarrow-Morton, Musiela.
- Change of numeraire
- Market Models

0

Arbitrage Theory

Pricing financial derivatives

Definition:
A **contingent claim** (derivative) with **delivery time** T, is a random variable

$$X \in \mathcal{F}_T.$$

"At $t = T$ the amount X is paid to the holder of the claim".

Example: (European Call Option)

$$X = \max\left[S_T - K, 0\right]$$

(S_T =stock price at time T)

Let X be a contingent T-claim.

Problem: What is an "reasonable" price process $\Pi\left[t; X\right]$ for X?

Philosophy

- The derivative is **defined in terms of** underlying.

- The derivative can be **priced in terms of** underlying price.

- **Consistent** pricing.

- **Relative** pricing.

- No **mispricing** between derivative and underlying.

- No **arbitrage possibilities**.

Financial Markets

Price Process:

$$S(t) = [S_0(t), ..., S_N(t)]$$

$S_i(t) =$ price of asset i at time t. ($S_0 > 0$)

Example: (Black-Scholes, $S_0 := B$, $S_1 := S$)

$$\begin{aligned} dS &= \alpha S dt + \sigma S dW, \\ dB &= rB dt. \end{aligned}$$

Portfolio:

$$h(t) = [h_0(t), ..., h_N(t)]$$

$h_i(t) =$ number of units of asset i at time t.

Value Process:

$$V_h(t) = \sum_{i=0}^{N} h_i(t) S_i(t) = h(t)S(t)$$

Self Financing Portfolios

Definition: (intuitive)
A portfolio is **self-financing** if there is no exogenous infusion or withdrawal of money. "The purchase of a new asset must be financed by the sale of an old one."

Definition: (mathematical)
A portfolio is **self-financing** if the value process satisfies

$$dV_h(t) = \sum_{i=0}^{N} h_i(t) dS_i(t)$$

Major insight:
If the price process S is a **martingale**, and if h is **self-financing**, then V_h is a **martingale**.

Arbitrage

A portfolio h is an **arbitrage strategy** if

- h is self financing
- $V_h(0) = 0$
- $P(V_h(T) > 0) = 1$

  or more precisely

$$
\begin{aligned}
P(V_h(T) \geq 0) = 1 \\
P(V_h(T) > 0) > 0
\end{aligned}
$$

Interpretation:
An arbitrage possibilty is a serious case of mis-pricing on the market.

Main Question: When is the market free of arbitrage?

Absence of Arbitrage

The market is arbitrage free

iff

There exists a probability measure $Q \sim P$ such that all normalized price processes are **Q-martingales**.

i.e.

$$Z(t) = \frac{S(t)}{S_0(t)} = [1, Z_1(t), ..., Z_N(t)]$$

is a Q martingale.

i.e.

$$E^Q [Z_i(s) | \mathcal{F}_t] = Z_i(t), \quad t \leq s$$

Choice of Numeraire

The **numeraire** price S_0 can be chosen arbitrarily. Typically we choose the **riskless asset**, i.e.

$$S_0(t) = B(t)$$

where

$$dB(t) = r(t)B(t)dt$$

$$B(t) = e^{\int_0^t r(s)ds}$$

B = The **money account** (a bank with short rate r).

In this case Q is called the "risk neutral" measure.

Pricing

Definition:
A **contingent claim** with **delivery time** T, is a random variable

$$X \in \mathcal{F}_T.$$

"At $t = T$ the amount X is paid to the holder of the claim".

Example: (European Call Option)

$$X = \max\left[S_T - K, 0\right]$$

Let X be a contingent T-claim.

Main Pricing Problem:
What is an arbitrage free price process $\Pi\left[t; X\right]$ for X?

Solution: The extended market

$$S_t, \quad \Pi\,[t; X]$$

must be free of arbitrage. In particular, the process $\frac{\Pi[t;X]}{B(t)}$ must be a martingale, under some martingale measure Q, i.e.

$$\frac{\Pi\,[t; X]}{B(t)} = E^Q \left[\left. \frac{\Pi\,[T; X]}{B(T)} \right| \mathcal{F}_t \right]$$

Pricing formula:

$$\Pi\,[t; X] = E^Q \left[\left. e^{-\int_t^T r(s)ds} \times X \right| \mathcal{F}_t \right]$$

Black-Scholes Model:

$$\Pi\left[t; X\right] = e^{-r(T-t)} E^Q \left[X \middle| \mathcal{F}_t \right]$$

Q-dynamics:

$$dS = rSdt + \sigma S d\tilde{W}.$$

Simple claims:

$$X = \Phi(S_T),$$

$$\Pi\left[t; X\right] = e^{-r(T-t)} E^Q \left[\Phi(S_T) \middle| \mathcal{F}_t \right]$$

Kolmogorov $\Rightarrow$

$$\Pi\left[t; X\right] = F(t, S_t).$$

$F(t,s)$ solves the Black-Scholes equation:

$$\left\{ \begin{array}{rcl} \frac{\partial F}{\partial t} + rs\frac{\partial F}{\partial s} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 F}{\partial s^2} - rF &=& 0, \\ F(T,s) &=& \Phi(s). \end{array} \right.$$

Risk neutral dynamics

- For every arbitrage free price process Π_t, the process

$$\frac{\Pi_t}{B_t}$$

is a Q-martingale.

- The Q-dynamics of Π_t are of the form:

$$d\Pi_t = r_t \Pi_t dt + dM_t$$

where M is a Q-martingale

Problem: What if there are several different martingale measures Q?

Hedging

Def: A portfolio is a **hedge** against X ("replicates X") if

- h is self financing
- $V_h(T) = X, \quad P - a.s.$

Def: The market is **complete** if every X can be hedged.

Pricing Formula:
If h replicates X, then a natural way of pricing X is

$$\Pi\left[t; X\right] = V_h(t)$$

When can we hedge?

Existence of hedge

$$\Updownarrow$$

Existence of stochastic integral representation

Theorem:

The market is complete

iff

the martingale measure Q is unique.

Main Results:

- The market is arbitrage free $\Leftrightarrow$ There exists a martingale measure Q

- The market is complete $\Leftrightarrow$ Q is unique.

- Every X must be priced by the formula
$$\Pi\left[t; X\right] = E^Q\left[e^{-\int_t^T r(s)ds} \times X \middle| \mathcal{F}_t \right]$$
for some choice of Q.

- In a non-complete market, different choices of Q will produce different prices for X.

- For a hedgeable claim X, all choices of Q will produce the same price for X:
$$\Pi\left[t; X\right] = V_h(t) = E^Q\left[e^{-\int_t^T r(s)ds} \times X \middle| \mathcal{F}_t \right]$$

Metatheorem:

Assume that
N = Number of risky assets.

R = Number of independent sources of randomness.

Then the following hold.

- The market is arbitrage free **iff** $R \geq N$.

- The market is complete **iff** $R \leq N$.

- The market is arbitrage free and complete **iff** $R = N$.

1

Bonds and Interest Rates

Basic definitions

Bonds: T-bond = Zero coupon bond, which pays 1 $ at time of maturity T.

$$
\begin{aligned}
p(t,T) &= \text{price, at time } t\text{, of a } T\text{-bond.} \\
p(T,T) &= 1.
\end{aligned}
$$

Main problem:
Determine the **term structure**, i.e. the structure of $\{p(t,T);\ 0 \le t \le T,\ T \ge 0\}$ on an arbitrage free bond market.
Determine arbitrage free prices of other interest rate derivatives (interest rate options, swap rates, caps, floors etc.)

Riskless Interest Rates

At time $\mathbf{t}$:

- Sell one S-bond.

- Buy exactly $p(t, S)/p(t, T)$ $T-$bonds.

- Zero net investment.

At time $\mathbf{S}$:

- Pay out 1$

At time $\mathbf{T}$:

- Receive $p(t, S)/p(t, T) \cdot 1$$.

Net effect

- Contract is made at at t.

- An investment of 1 at time S has yielded $p(t,S)/p(t,T)$ at time T.

- The equivalent constant **instantaneous rate**, R, is given by

$$e^{R\cdot(T-S)} \cdot 1 = \frac{p(t,S)}{p(t,T)}$$

  The equivalent constant **simple rate**, L, is given by

$$(1+L)(T-S) \cdot 1 = \frac{p(t,S)}{p(t,T)}$$

Instantaneous rates:

1. The **forward rate for** $[S, T]$ **contracted at** t is defined as
$$R(t; S, T) = -\frac{\log p(t, T) - \log p(t, S)}{T - S}.$$

2. The **spot rate**, $R(S, T)$, for the period $[S, T]$ is defined as
$$R(S, T) = R(S; S, T).$$

3. The **instantaneous forward rate with maturity** T**, contracted at** t is defined by
$$f(t, T) = -\frac{\partial \log p(t, T)}{\partial T} = \lim_{S \to T} R(t; S, T).$$

4. The **instantaneous short rate at time** t is defined by
$$r(t) = f(t, t).$$

Simple rates (LIBOR):

1. The simple (LIBOR) **forward rate for** $[S, T]$ **contracted at** t, is defined as
$$L(t; S, T) = -\frac{p(t, T) - p(t, S)}{(T - S)p(t, T)}.$$

2. The simple (LIBOR) **spot rate for** $[S, T]$ is defined as
$$L(S, T) = -\frac{p(S, T) - 1}{(T - S)p(S, T)}.$$

Bond Prices $\sim$ Forward Rates

$$p(t,T) = p(t,s) \cdot \exp\left\{-\int_s^T f(t,u)du\right\},$$

In particular

$$p(t,T) = \exp\left\{-\int_t^T f(t,s)ds\right\}.$$

Toolbox

Proposition:
If the forward rate dynamics under Q are given by

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW$$

Then the bond dynamics are given by

$$\begin{aligned} dp(t,T) &= p(t,T)\left\{r(t) + A(t,T) + \frac{1}{2}\|S(t,T)\|^2\right\}dt \\ &+ p(t,T)S(t,T)dW \end{aligned}$$

$$\begin{cases} A(t,T) &= -\int_t^T \alpha(t,s)ds, \\ S(t,T) &= -\int_t^T \sigma(t,s)ds \end{cases}$$

The Money Account

$$\begin{cases} dB(t) = r(t)B(t)dt, \\ B(0) = 1. \end{cases}$$

i.e.

$$B(t) = \exp\left\{\int_0^t r(s)ds\right\},$$

Model of a bank with stochastic short rate r.

2

Short Rate Models

Short Rate Models

Model: (Under the objective measure.)

P:

$$\begin{aligned} dr &= \mu(t,r)dt + \sigma(t,r)dW, \\ dB &= r(t)Bdt. \end{aligned}$$

Question: Are bond prices uniquely determined by the P-dynamics of r, and the requirement of an arbitrage free bond market?

NO!!

WHY?

1. **Meta Theorem:**

$$\begin{aligned} N &= 0, \text{ (No risky asset)} \\ R &= 1, \text{ (One source of randomness, } W) \end{aligned}$$

   Thus $M < R$. The market is incomplete.

2. **Martingale Measures:**
   If the money-account B is the only exogenously given asset, then **every** $Q \sim P$ is a martingale measure.
   The martingale measure is not unique, so the market is not complete.

3. **Hedging portfolios:**
   You are only allowed to invest your money in the bank, and then sit back and wait.

We have not enough underlying assets in order to price bonds.

- There is **not** a unique price for a **particular** $T-$bond.

- In order to avoid arbitrage, bonds of different maturities have to satisfy internal **consistency** relations.

- If we take **one** "benchmark" T_0-bond as given, then all other bonds can be priced **in terms of** the market price of the benchmark bond.

Martingale Modelling

- All prices are determined by the Q-dynamics of r.

- Model dr directly under Q!

Problem: Parameter estimation!

Martingale pricing

Q**-dynamics:**

$$dr = \mu(t,r)dt + \sigma(t,r)dW$$

$$\begin{aligned} \Pi[t;X] &= E^Q\left[e^{-\int_t^T r(s)ds} \times X \middle| \mathcal{F}_t\right] \\ p(t,T) &= E^Q\left[e^{-\int_t^T r(s)ds} \times 1 \middle| \mathcal{F}_t\right] \end{aligned}$$

The Case $\mathbf{X} = \Phi(r(T))$:

The price is given by

$$\Pi[t;X] = F(t, r(t))$$

$$\begin{cases} F_t + \mu F_r + \frac{1}{2}\sigma^2 F_{rr} - rF &= 0, \\ F(T,r) &= \Phi(r(T)). \end{cases}$$

(Term Structure Equation)

1. Vasiček

$$dr = (b - ar)\,dt + \sigma dV,$$

2. Cox-Ingersoll-Ross

$$dr = (b - ar)\,dt + \sigma\sqrt{r}dV,$$

3. Dothan

$$dr = ardt + \sigma rdV,$$

4. Black-Derman-Toy

$$dr = a(t)rdt + \sigma(t)rdV,$$

5. Ho-Lee

$$dr = a(t)dt + \sigma dV,$$

6. Hull-White (extended Vasiček)

$$dr = \{\Phi(t) - ar\}\,dt + \sigma dV,$$

Bond Options

European call on a T-bond with strike price K and delivery date S.

$$
\begin{aligned}
X &= \max\left[p(S,T) - K,\ 0\right] \\
X &= \max\left[F^T(S, r(S)) - K,\ 0\right]
\end{aligned}
$$

$$
\begin{aligned}
F^T_t + \mu F^T_r + \frac{1}{2}\sigma^2 F^T_{rr} - rF^T &= 0, \\
F^T(T, r) &= 1.
\end{aligned}
$$

$$
\Phi(r) = \max\left[F^T(S, r) - K,\ 0\right]
$$

$$
\begin{aligned}
F_t + \mu F_r + \frac{1}{2}\sigma^2 F_{rr} - rF &= 0, \\
F(S, r) &= \Phi(r(S)).
\end{aligned}
$$

$$
\Pi\left[t; X\right] = F(t, r(t))
$$

Affine Term Structures

Lots of equations!

Need analytic solutions.

Def:
We have an **Affine Term Structure** if

$$F(t, r; T) = e^{A(t,T) - B(t,T)r},$$

where A and B are deterministic functions.

Problem: How do we specify μ and σ in order to have an ATS?

Proposition: Assume that μ and σ are of the form

$$
\begin{aligned}
\mu(t,r) &= \alpha(t)r + \beta(t), \\
\sigma^2(t,r) &= \gamma(t)r + \delta(t).
\end{aligned}
$$

Then the model admits an affine term structure

$$F(t,r;T) = e^{A(t,T)-B(t,T)r},$$

where A and B satisfy the system

$$
\begin{cases}
B_t(t,T) = -\alpha(t)B(t,T) + \frac{1}{2}\gamma(t)B^2(t,T) - 1, \\
B(T;T) = 0.
\end{cases}
$$

$$
\begin{cases}
A_t(t,T) = \beta(t)B(t,T) - \frac{1}{2}\delta(t)B^2(t,T), \\
A(T;T) = 0.
\end{cases}
$$

Parameter Estimation

Suppose that we have chosen a specific model, e.g. H-W . How do we estimate the parameters $a,\ b,\ \sigma$?

Naive answer:
Use standard methods from statistical theory.

NONSENSE!!

- The parameters are Q-parameters.
- Our observations are **not** under Q, but under P.
- Standard statistical techniques can **not** be used.
- We need to know the martingale measure Q.
- Who determines Q?
- **The Market!**
- We must get **price information from the market** in order to estimate parameters.

Inverting the Yield Curve

Q-dynamics with parameter vector α:

$$dr = \mu(t, r; \alpha)dt + \sigma(t, r; \alpha)dV$$

$$\Downarrow$$

Theoretical term structure

$$\{p(0, T; \alpha);\ T \geq 0\}$$

Observed term structure:

$$\{p^{\star}(0, T);\ T \geq 0\}.$$

Want: A model such that **theoretical** prices fit the **observed** prices of today, i.e. choose parameter vector α such that

$$p(0, T; \alpha) \approx \{p^{\star}(0, T);\ \forall T \geq 0\}$$

Number of equations $= \infty$ (one for each T).
Number of unknowns $=$dim(α)

Need: Infinite dimensional parameter vector.

Hull-White

Q-dynamics:

$$dr = \{\Phi(t) - ar\}\, dt + \sigma dV(t),$$

$$p(t,T) = e^{A(t,T)-B(t,T)r(t)},$$

$$B(t,T) = \frac{1}{a}\left\{1 - e^{-a(T-t)}\right\}$$

The **instantaneous forward rate** at T, contracted at t is given by

$$f(t,T) = -\frac{\partial \log p(t,T)}{\partial T}.$$

Choose Φ to fit the observed forward rate curve!

Result: The Hull-White model can be fitted exactly to any observed initial term structure. The calibrated model takes the form

$$p(t,T) = \frac{p(0,T)}{p(0,t)} \times e^{C(t,r(t))}$$

where C is given by

$$B(t,T)f^{\star}(0,t) - \frac{\sigma^2}{2a^2}B^2(t,T)\left(1 - e^{-2aT}\right) - B(t,T)r(t)$$

Analytical formulas for bond-options.

Models Based on the Short Rate

Pro:

- Easy to model Markov structure for r.
- Analytical expressions for bond prices and derivatives.

Con:

- Inverting the yield curve can be hard.
- Hard to model a flexible volatility structure for forward rates.
- One factor models implies perfect correlation along the yield curve.

3

Forward Rate Models

Heath-Jarrow-Morton

Idea: Model the dynamics for the **entire yield curve**.

The yield curve itself (rather than the short rate r) is the explanatory variable.

Model forward rates. Use observed yield curve as boundary value.

Dynamics:

$$\begin{aligned} df(t,T) &= \alpha(t,T)dt + \sigma(t,T)dW(t), \\ f(0,T) &= f^{\star}(0,T). \end{aligned}$$

One SDE for every fixed maturity time T.

Existence of martingale measure

$$
\begin{aligned}
f(t,T) &= \frac{\partial \log p(t,T)}{\partial T} \\
p(t,T) &= \exp\left\{-\int_t^T f(t,s)ds\right\}
\end{aligned}
$$

Thus:

Specifying forward rates.

$$\Longleftrightarrow$$

Specifying bond prices.

Thus:

No arbitrage

$$\Downarrow$$

restrictions on α and σ.

$\mathbf{P}$-dynamics:

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)d\tilde{W}(t)$$

Look for Girsanov transformation $P \to Q$, s.t.

$\mathbf{Q}$-dynamics:

$$dp(t,T) = r(t)p(t,T)dt + p(t,T)v(t,T)d\tilde{W}(t)$$

Toolbox:

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)d\tilde{W}$$

$$\Downarrow$$

$$\begin{aligned} dp(t,T) &= p(t,T)\left\{r(t) + A(t,T) + \frac{1}{2}\|S(t,T)\|^2\right\}dt \\ &+ p(t,T)S(t,T)dW \end{aligned}$$

$$\begin{cases} A(t,T) &= -\int_t^T \alpha(t,s)ds, \\ \\ S(t,T) &= -\int_t^T \sigma(t,s)ds \end{cases}$$

Girsanov:

$$
\begin{aligned}
dL(t) &= L(t) + \varphi(t)d\tilde{W}(t), \\
L(0) &= 1.
\end{aligned}
$$

Q-dynamics:

$$
\begin{aligned}
dp(t,T) &= p(t,T)r(t)dt \\
&+ \left\{A(t,T) + \frac{1}{2}||S(t,T)||^2 + S(t,T)\varphi(t)\right\} dt \\
&+ p(t,T)S(t,T)dW(t),
\end{aligned}
$$

Proposition:
$\exists$ a martingale measure

$$\Updownarrow$$

$\exists$ process $g(t) = [\varphi_1(t), \cdots \varphi_d(t)]$ s.t.

$$A(t,T) + \frac{1}{2}||S(t,T)||^2 + S(t,T)\varphi(t) = 0, \quad \forall t, T$$

alternatively

$$\alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,s)ds - \sigma(t,T)t)\varphi_t, \quad \forall t, T$$

Martingale Modelling

Q-dynamics:

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW(t)$$

Specifying forward rates.

$$\Longleftrightarrow$$

Specifying bond prices.

Thus:

Specifying Q dynamics

$$\Downarrow$$

restrictions on α and σ.

Which?

$$\text{Martingale modelling}$$
$$\Updownarrow$$
$$P = Q$$
$$\Updownarrow$$
$$\varphi \equiv 0$$

$$\alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,s)ds - \sigma(t,T)\varphi(t), \quad \forall t, T$$

Theorem: (HJM drift Condition) Under the Q measure the following relation must hold

$$\alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,s)ds.$$

Moral: Volatility can be specified freely. The forward rate drift term is then uniquely determined.

Musiela parametrization

Parameterize forward rates by the time **to** maturity (x), rather than time **of** maturity (T).

Def:

$$r(t, x) = f(t, t + x).$$

Q-dynamics:

$$dr(t, x) = \mu(t, x)dt + \tau(t, x)dW.$$

What are the relations between μ and τ under Q?

Compare with HJM!

$$df(t, T) = \alpha(t, T)dt + \sigma(t, T)dW.$$

$$
\begin{aligned}
dr(t,x) &= d\left[f(t,t+x)\right] \\
&= df(t,t+x) + f_T(t,t+x)dt \\
&= \left\{\alpha(t,t+x) + r_x(t,x)\right\} dt + \sigma(t,t+x)dW
\end{aligned}
$$

$$
\begin{aligned}
\mu(t,x) &= \alpha(t,t+x) + r_x(t,x) \\
\tau(t,x) &= \sigma(t,t+x).
\end{aligned}
$$

HJM-condition:

$$
\alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,s)ds.
$$

Musiela equation:

$$
\begin{aligned}
dr(t,x) &= \left\{\frac{\partial}{\partial x}r(t,x) + \tau(t,x)\int_0^x \tau(t,y)dy\right\} dt \\
&+ \tau(t,x)dW
\end{aligned}
$$

When τ is **deterministic** this is a **linear** equation in infinite dimensional space. Connections to control theory.

Forward Rate Models

Pro:

- Easy to model flexible volatility structure for forward rates.

- Easy to include multiple factors.

Con:

- The short rate will typically not be a Markov process.

- Computational problems.

4

Change of numeraire

Change of Numeraire

(Geman, Jamshidian, El Karoui)

Valuation formula:

$$\Pi\left[t; X\right] = E^Q\left[e^{-\int_t^T r(s)ds} \times X \left|\mathcal{F}_t\right.\right]s$$

Hard to compute. Double integral.

Note: If X and r are **independent** then

$$\begin{aligned}\Pi\left[t; X\right] &= E^Q\left[e^{-\int_t^T r(s)ds} \left|\mathcal{F}_t\right.\right]s \cdot E^Q\left[X \left|\mathcal{F}_t\right.\right]s \\ &= p(t,T) \cdot E^Q\left[X\right| \mathcal{F}_t\left.\right].\end{aligned}$$

Nice! We do not have to compute $p(t,T)$. It can be observed directly on the market! Single integral!

Sad Fact: X and r are (almost) never independent!

Idea: Use T-bond (for a fixed T) as numeraire. Define the **T-forward measure** Q^T by the requirement that

$$\frac{\Pi(t)}{p(t,T)}$$

is a Q^T-martingale for every price process $\Pi(t)$.

Then

$$\frac{\Pi[t;X]}{p(t,T)} = E^T\left[\left.\frac{\Pi[T;X]}{p(T,T)}\right|\mathcal{F}_t\right]$$

$$\Pi[T;X] = X, \qquad p(T,T) = 1.$$

$$\Pi[t;X] = p(t,T)E^T[X|\mathcal{F}_t]\,s$$

Do such measures exist?.

"The forward measure takes care of the stochastics over the interval $[t, T]$."

Enormous computational advantages.

Useful for interest rate derivatives, currency derivatives and derivatives defined by several underlying assets.

General change of numeraire.

Idea: Use a fixed asset price process $S(t)$ as numeraire. Define the measure Q^S by the requirement that

$$\frac{\Pi(t)}{S(T)}$$

is a Q^S-martingale for every arbitrage free price process $\Pi(t)$.

Constructing $\mathbf{Q}^S$: Fix a T-claim X. From general theory:

$$\Pi[0; X] = E^Q\left[\frac{X}{B(T)}\right]$$

Assume that Q^S exists and denote

$$L(t) = \frac{dQ^S}{dQ}, \quad \text{on } \mathcal{F}_t$$

Then

$$\begin{aligned}\frac{\Pi[0; X]}{S(0)} &= E^S\left[\frac{\Pi[T; X]}{S(T)}\right] = E^S\left[\frac{X}{S(T)}\right] \\ &= E^Q\left[L(T)\frac{X}{S(T)}\right]\end{aligned}$$

Thus we have

$$\Pi[0; X] = E^Q\left[L(T)\frac{X \cdot S(0)}{S(T)}\right],$$

Natural candidate:

$$L(t) = \frac{dQ_t^S}{dQ_t} = \frac{S(t)}{S(0)B(t)}$$

Proposition:

$$\Pi(t)/B(t) \text{ is a } Q\text{-martingale.}$$
$$\Downarrow$$
$$\Pi(t)/S(t) \text{ is a } Q^\star\text{-martingale.}$$

Proof.

$$E^{\star}\left[\left.\frac{\Pi\,(t)}{S(t)}\right|\mathcal{F}_s\right]=\frac{E^{Q}\left[L(t)\frac{\Pi(t)}{S(t)}\Big|\,\mathcal{F}_s\right]}{L(s)}$$

$$=\quad\frac{E^{Q}\left[\frac{\Pi(t)}{B(t)S(0)}\Big|\,\mathcal{F}_s\right]}{L(s)}=\frac{\Pi\,(s)}{B(s)S(0)L(s)}$$

$$=\quad\frac{\Pi\,(s)}{S(s)}.\blacksquare$$

Result:

$$\Pi\left[t;X\right] = S(t)E^{S}\left[\left.\frac{X}{S(T)}\right|\mathcal{F}_t\right]$$

We can observe $S(t)$ directly on the market.

Example: $X = S(T) \cdot Y$

$$\Pi\left[t;X\right] = S(t)E^{S}\left[\left.Y\right|\mathcal{F}_t\right]$$

Several underlying:

$$X = \Phi\left[S_0(T), S_1(T)\right]$$

Assume Φ is linearly homogeous. Transform to Q^0.

$$\begin{aligned}\Pi\left[t; X\right] &= S_0(t)E^0\left[\left.\frac{\Phi\left[S_0(T), S_1(T)\right]}{S_0(T)}\right| \mathcal{F}_t\right] \\ &= S_0(t)E^0\left[\left.\varphi\left[Z(T)\right]\right| \mathcal{F}_t\right]\end{aligned}$$

$$\varphi\left[z\right] = \Phi\left[1, z\right], \qquad Z(t) = \frac{S_1(t)}{S_0(t)}$$

Exchange option:

$$X = \max\left[S_1(T) - S_0(T),\ 0\right]$$

$$\Pi\left[t; X\right] = S_0(t) E^0\left[\max\left[Z(T) - 1,\ 0\right] \middle| \mathcal{F}_t\right]$$

European Call on Z with strike price K. Zero interest rate.

Piece of cake!

Identifying the Girsanov Transformation

Assume Q-dynamics of S known as

$$dS(t) = r(t)S(t)dt + S(t)v(t)dW(t)$$

$$L(t) = \frac{S(t)}{S(0)B(t)}$$

Thus

$$dL(t) = L(t)v(t)dW(t).$$

The Girsanov kernel is given by the numeraire volatility $v(t)$.

Forward Measures

Use price of T-bond as numeraire.

$$L^T(t) = \frac{p(t,T)}{p(0,T)B(t)}$$

$$\begin{aligned} dp(t,T) &= r(t)p(t,T)dt + p(t,T)v(t,T)dW(t), \\ dL^T(t) &= L^T(t)v(t,T)dW(t) \end{aligned}$$

Result:

$$\Pi\left[t;X\right] = p(t,T)E^T\left[X\middle|\mathcal{F}_t\right]$$

Common Conjecture: "The forward rate is an unbiased estimator of the future spot rate:"

Lemma:

$$f(t,T) = E^T\left[r(T)\middle|\mathcal{F}_t\right]$$

A new look on option pricing

(Geman, El Karoui, Rochet)

European call on asset S with strike price K and maturity T.

$$X = \max\left[S(T) - K,\ 0\right]$$

$$\begin{aligned}\Pi\left[0; X\right] \;&=\; S(0) \cdot Q^S\left[S(T) \geq K\right] \\ &-\; K \cdot p(0,T) \cdot Q^T\left[S(T) \geq K\right]\end{aligned}$$

Market Models

Problems with infinitesimal rates:

- Infinitesimal rates can never be observed in real life.

- Calibration to cap- or swaption data is difficult.

Disturbing facts from real life:

- The market uses Black-76 to value caps and swaptions. For this you need to assume that

  - The short rate is constant.

  - The LIBOR rates are lognormally distributed.

- Logically inconsistent!

- Despite this, the market happily continues to use Black-76.

Project

- Construct a **logically consistent model** which (to some extent) justifies market practice.

- Construct an **arbitrage free** model with the property that caps, floors and/or swaptions are priced with a Black-76 type formula.

Main models

- **LIBOR market models** (Miltersen-Sandmann-Sondermann, Brace-Gatarek-Musiela)

- **Swap market models** (Jamshidian).

- Instead of modeling instantaneous rates, we model discrete **market rates**, such as
  - LIBOR rates (LIBOR market models)
  - Forward swap rates (swap market models).
- Under a suitable numeraire the market rates can be modeled lognormally.
- The market models with thus produce pricing formulas of the type Black-76.
- By construction the market models are very easy to calibrate to market data, i.e. to:
  - Caps and floors (LIBOR market model)
  - Swaptions (swap market model)
- Exotic derivatives has to be priced numerically.

Caps

Resettlement dates:

$$T_0 < T_1 < \ldots < T_n,$$

Tenor:

$$\alpha = T_{i+1} - T_i, \quad i = 0, \ldots, n-1.$$

Typically $\alpha = 1/4$, i.e. quarterly resettlement.

LIBOR forward rate for $[T_{i-1}, T_i]$:

$$L_i(t) = \frac{1}{\alpha} \cdot \frac{p_{i-1}(t) - p_i(t)}{p_i(t)}, \quad i = 1, \ldots, N.$$

where we use the notation

$$p_i(t) = p(t, T_i)$$

Definition:

A **cap** with **cap rate** R and **resettlement dates** $T_0, \ldots, T_n$ is a contract which at each T_i give the holder the amount

$$X_i = \alpha \cdot \max\left[L_i(T_{i-1}) - R, 0\right], \quad i = 1, \ldots, N$$

The cap is thus a portfolio of **caplets** $X_1, \ldots, X_n$.

Black-76:

The **Black-76** formula for the caplet

$$X_i = \alpha_i \cdot \max\left[L(T_{i-1}, T_i) - R, 0\right], \qquad (1)$$

is given by

$$\mathbf{Capl_i^B}(t) = \alpha \cdot p_i(t)\left\{L_i(t)N[d_1] - RN[d_2]\right\}$$

where

$$\begin{aligned} d_1 &= \frac{1}{\sigma_i\sqrt{T_i - t}}\left[\ln\left(\frac{L_i(t)}{R}\right) + \frac{1}{2}\sigma_i^2(T-t)\right], \\ d_2 &= d_1 - \sigma_i\sqrt{T_i - t}. \end{aligned}$$

- Black-76 presupposes that each LIBOR rate is lognormal.

- The constants $\sigma_1, \ldots, \sigma_N$ are known as the **Black volatilities**

Market price quotes

Market prices are quoted in terms of

- **Implied Black volatilities:**
  - **flat volatilities**
  - **spot volatilities** (also known as **forward volatilities**)

Market Price Data

For each $i = 1, \ldots, N$:

$$\mathbf{Cap_i^m(t)} \;=\; \text{market price of cap with resettlement dates } T_0, T_1, \ldots, T_i$$

Implied market prices of caplets:

$$\mathbf{Capl_i^m}(t) = \mathbf{Cap_i^m}(t) - \mathbf{Cap_{i-1}^m}(t),$$

with the convention $\mathbf{Cap_0^m}(t) = 0$

Defining Implied Black Volatility

Given market price data as above, the implied Black volatilities are defined as follows.

- The implied **flat volatilities** $\bar{\sigma}_1, \ldots, \bar{\sigma}_N$ are defined as the solutions of the equations

$$\mathbf{Cap_i^m}(t) = \sum_{k=1}^{i} \mathbf{Capl_k^B}(t; \bar{\sigma}_i), i = 1, \ldots, N. \tag{2}$$

- The implied **forward** or **spot** volatilities $\bar{\sigma}_1, \ldots, \bar{\sigma}_N$ are defined as solutions of the equations

$$\mathbf{Capl_i^m}(t) = \mathbf{Capl_i^B}(t; \bar{\sigma}_i), i = 1, \ldots, N. \tag{3}$$

The sequence $\bar{\sigma}_1, \ldots, \bar{\sigma}_N$ is called the volatility **term structure**.

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The sequence $\bar{\sigma}_1, \ldots, \bar{\sigma}_N$ is called the volatility **term structure**.

Theoretical Price of a Caplet

By risk neutral valuation:

$$\mathbf{Capl_i}(t) = \alpha E_t^Q \left[e^{-\int_0^{T_i} r(s)ds} \cdot \max\left[L_i(T_{i-1}) - R, 0\right] \right],$$

Better to use T_i forward measure

$$\mathbf{Capl_i}(t) = \alpha p_i(t) E^{T_i} \left[\max\left[L_i(T_{i-1}) - R, 0\right] \middle| \mathcal{F}_t \right],$$

The crucial point is the distribution of L_i under $Q^i = Q^{T_i}$

Important Fact: L_i is a **martingale** under Q^i

$$L_i(t) = \frac{1}{\alpha} \cdot \frac{p_{i-1}(t) - p_i(t)}{p_i(t)}$$

Idea: Model L_i as GBM under Q^i:

$$dL_i = \sigma_i L_i dW^i$$

LIBOR Market Model Definition

Define, for each i, the dynamics of L_i under Q^i as

$$dL_i(t) = L_i(t)\sigma_i(t)dW^i(t), \quad i = 1, \dots, N,$$

where $\sigma_1(t), \dots, \sigma_N(t)$ are **deterministic** and W^i is Q^i-Wiener.

The initial term structure $L_1(0), \dots, L_N(0)$ is observed on the market.

Pricing Caps in the LIBOR Model

$$L_i(T) = L_i(t) \cdot e^{\int_t^T \sigma_i(s) dW^i(s) - \frac{1}{2}\int_t^T \|\sigma_i(s)\|^2 ds}.$$

Lognormal!

Theorem: Caplet prices are given by

$$\mathbf{Capl_i}(t) = \alpha_i \cdot p_i(t) \left\{ L_i(t) N[d_1] - R N[d_2] \right\},$$

where

$$\begin{aligned} d_1 &= \frac{1}{\Sigma_i(t, T_{i-1})} \left[\ln\left(\frac{L_i(t)}{R}\right) + \frac{1}{2}\Sigma_i^2(t, T_{i-1}) \right], \\ d_2 &= d_1 - \Sigma_i(t, T_{i-1}), \end{aligned}$$

$$\Sigma_i^2(t, T) = \int_t^T \|\sigma_i(s)\|^2 ds.$$

Moral: Each caplet price is given by a Black-76 formula with Σ_i as the Black volatility.

Practical Handling of the LIBOR Model

We are standing at time $t = 0$.

- Collect implied caplet volatilities
$$\bar{\sigma}_1, \ldots, \bar{\sigma}_N$$
from the market.

- Choose model volatilities
$$\sigma_1(\cdot), \ldots, \sigma_N(\cdot)$$
such that
$$\bar{\sigma}_i = \frac{1}{T_i} \int_0^{T_{i-1}} \sigma_i^2(s) ds, \quad i = 1, \ldots, N.$$

- Now the model is calibrated.

- Use numerical methods to compute prices of exotics.

Terminal Measure dynamics

Define the Likelihood process η_i^j as

$$\eta_i^j(t) = \frac{dQ^j}{dQ^i}, \quad \text{on } \mathcal{F}_t$$

Can show that

$$d\eta_i^{i-1}(t) = \eta_i^{i-1}(t) \frac{\alpha_i L_i(t)}{1 + \alpha_i L_i(t)} \sigma_i(t) dW^i(t).$$

Girsanov gives us

$$dW^i(t) = \frac{\alpha_i L_i(t)}{1 + \alpha_i L_i(t)} \sigma_i^\star(t) dt + dW^{i-1}(t).$$

Proposition The Q^N dynamics of the LIBOR rates are

$$\begin{aligned} dL_i(t) &= -L_i(t) \left(\sum_{k=i+1}^{N} \frac{\alpha_k L_k(t)}{1 + \alpha_k L_k(t)} \sigma_k(t) \sigma_i^\star(t) \right) dt \\ &+ L_i(t) \sigma_i(t) dW^N(t), \end{aligned}$$

Messy!