Intra day high frequency trading : from empirical evidences to quantitative optimization

Charles-Albert Lehalle, PhD
clehalle@cheuvreux.com

Head of Crédit Agricole Cheuvreux Quantitative Research

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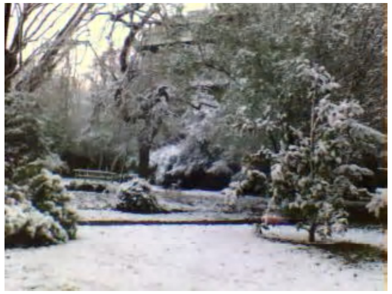
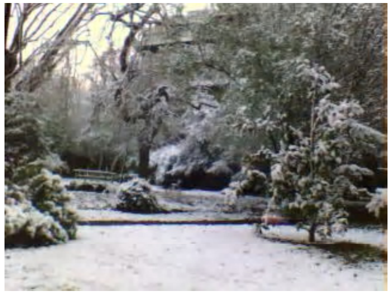
Pre high-frequency world

All seems to be smooth and regular, like a landscape covered by snow.



Post high-frequency world

Now we discover some singularities: the world is not as linear as we thought.



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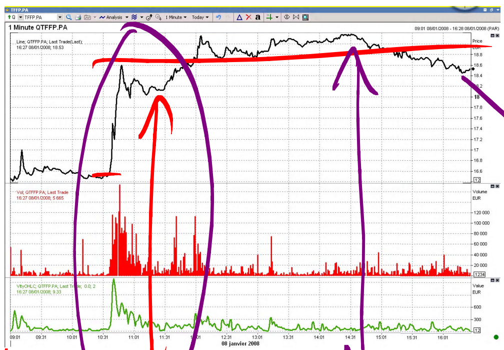
Here we will not focus on the fair value estimation process, but on how to deal with the price formation process.

That is real



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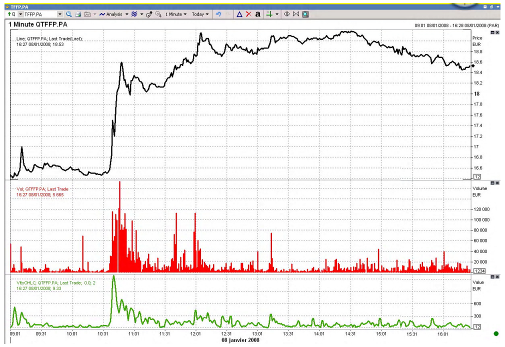

This is the effect of the announcement by president Sarkozy that public TV channels will no more be allowed to sell advertising...

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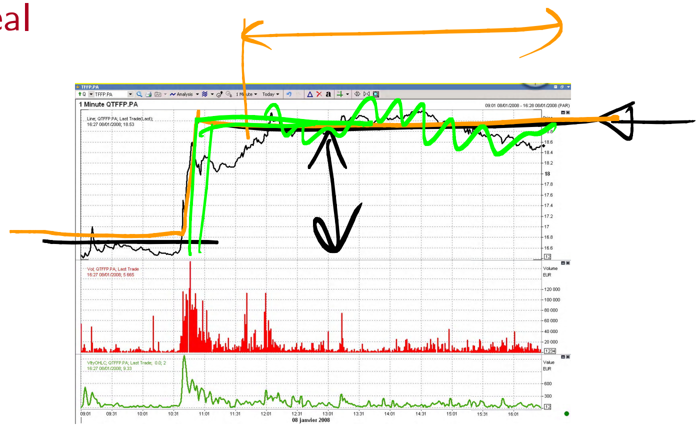

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The value of the main french private Channels jumped immediately.

Effets of such a change

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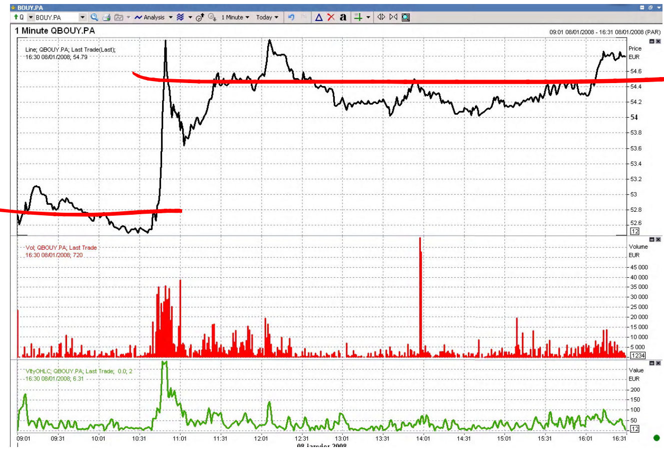
4. Closed loops within the worldwide markets (and other classes of assets)

This vision is the Capital Asset Pricing Model (CAPM) one: a worldwide market driven by few systematic factors. In this framework some temporary explorations are conducted by isolated stocks or sectors (specifics explorations).
Extra day arbitrageurs focus on those aspects. At intra day scale, we try to understand and detect when and how auction mechanisms of market places are able to absorb such trajectories.

Propagation - correlations

5 min



CHEUVREUX

Propagation - zooming out: 30 minutes

Propagation - zooming out: 1 hour

Propagation - zooming out: daily

Propagation - sector: daily media

Contenu

evidences

Work

The principles of an auction market

Some agents wants to buy, others want to sell. Three main roles:

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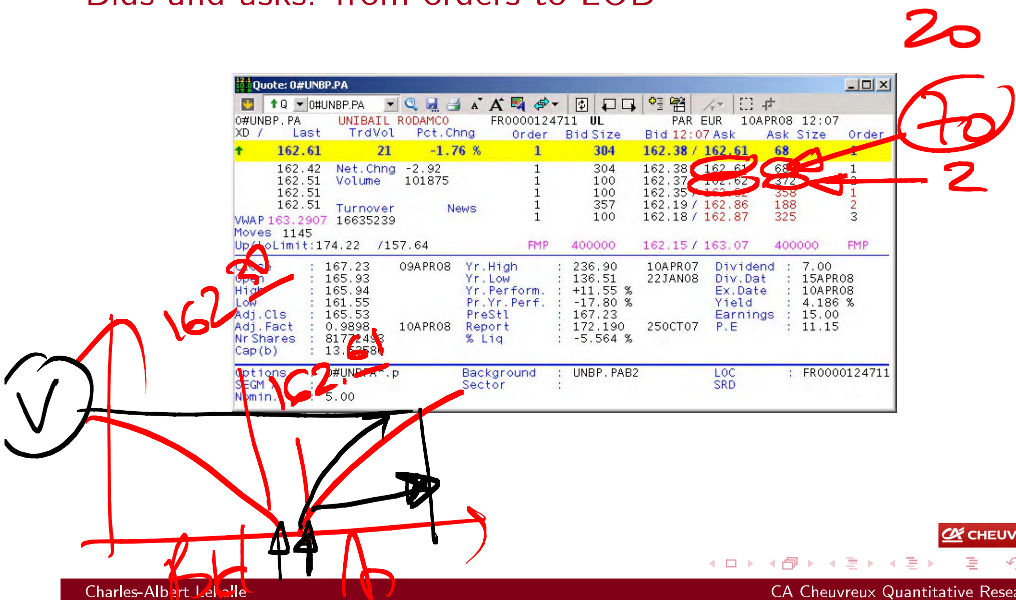
Each agent place orders at given prices for given quantities, when they match: a transaction occurs

Bids and asks: from orders to LOB



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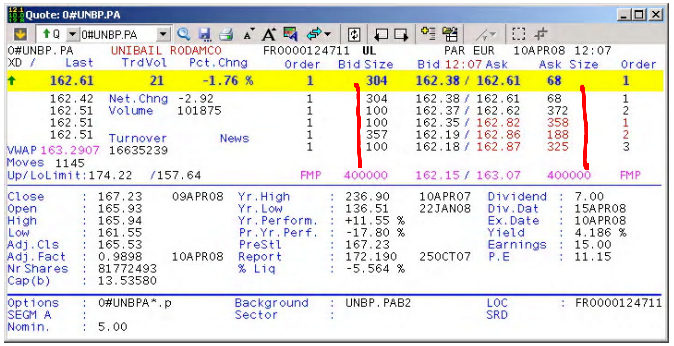
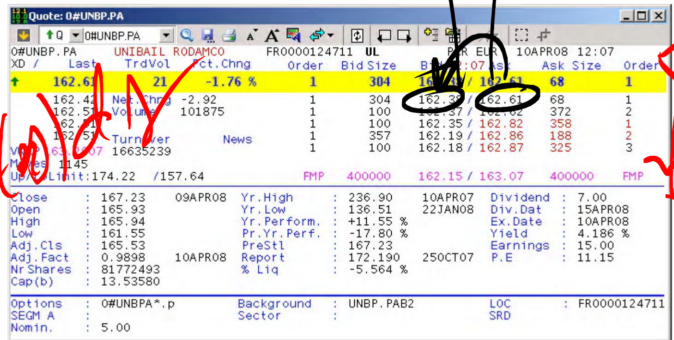


Price is a function of quantity!

Bids and asks: from orders to LOB

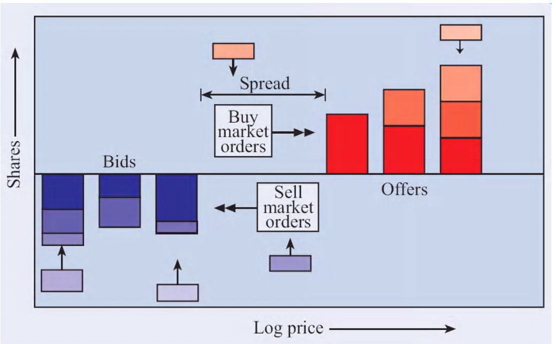
162.61 − 162.35



F^-_{LOB}

Price is a function of quantity!
Notations: $f^{\pm}_{\rm LOB}$ for densities, $F^{\pm}_{\rm LOB}$ for cumulated quantities

From orders to LOB

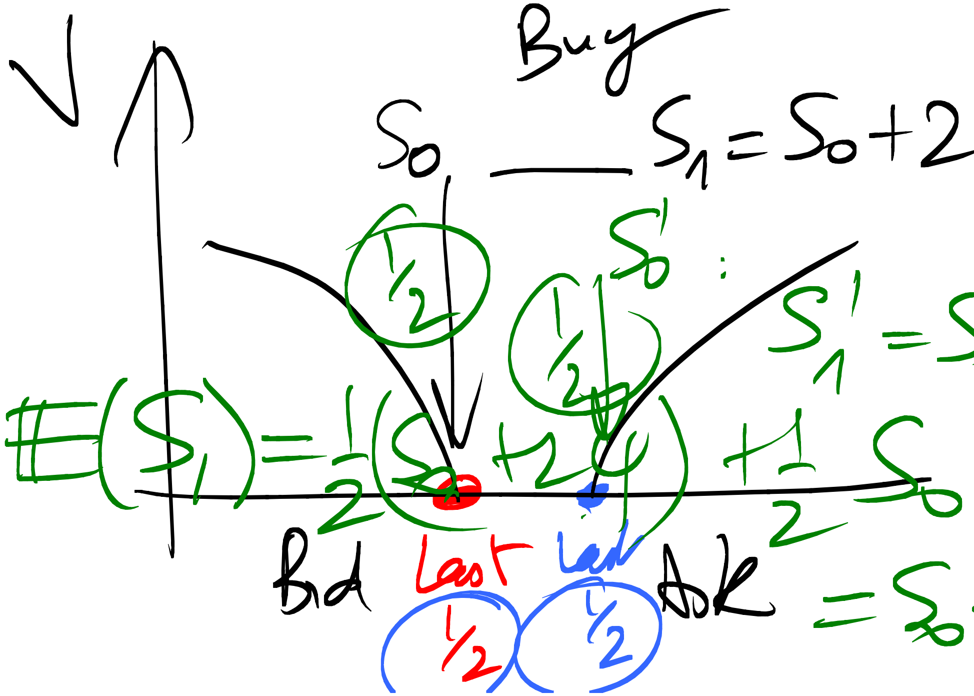


Statistical theory of the continuous double auction - Eric Smith , J Doyne Farmer, Laszlo Gillemot and Supriya Krishnamurthy - Santa Fe Institute - September 2003

Buy

S_0 —— $S_1 = S_0 + 2\psi$

S_0' : $S_1' = S_0'$



$$\mathbb{E}(S_1) = \frac{1}{2}(S_0 + 2\psi) + \frac{1}{2} S_0 = S_0 + \psi$$

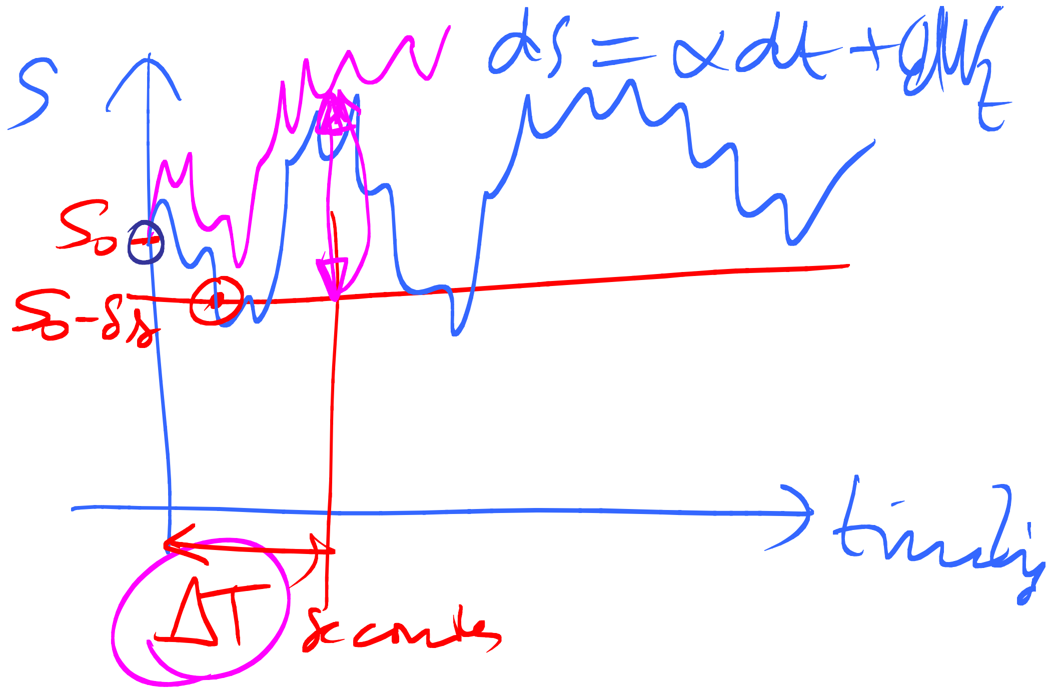
Continuous auctions Actors

Liquidity provider

- Wait to be executed (not executed for sure)
- Execution price is better than the last quoted one
- (sometimes) pays less fees

Liquidity consumer

- Executed for sure
- Pays around half a bid-ask spread more than the last quoted price
- Pays *Price Impact* for large quantities



S
$dS = \alpha dt + \sigma dW_t$
S_0
$S_0 - \delta_S$
timing
ΔT
seconds

The free option myth

If I decide to place a *Buy* order (one share) at a price of $S_0 - \delta S$ as a liquidity provider during ΔT seconds, and then to buy at $S_{\Delta T}$ as a consumer (if I have not been executed): it's *as simple* as a barrier option.

A very special one: k.o. option with barrier level $B =$ the strike K.

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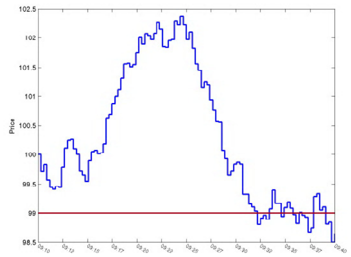

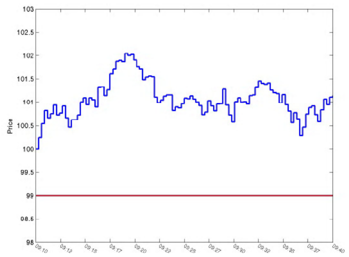

Should I stay or should I go?

Let q_T be the probability to cross the barrier between 0 and T (τ_B is the first time of crossing), bayes decomposition gives us:

$$\mathbf{E}(S) = \mathbf{E}(S|\tau_B < T)\,\underbrace{\mathbf{P}(\tau_B < T)}_{q_T} + \mathbf{E}(S|\tau_B \geq T)\mathbf{P}(\tau_B \geq T)$$

$$E(S) = E(S \mid \tau_B < \Delta T) \cdot q_T + E(S \mid \tau_B \geq T) \cdot (1 - q_T)$$

crossing time

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Volume

$$\Rightarrow \boldsymbol{E}(S|\tau_B \geq T) = \frac{S_0 - B\,q_T}{1 - q_T}$$

microst.

Finally, the payoff of such a strategy is (at this time scale, risk free interest rate can be neglected):

$$\boldsymbol{E}(S_T \text{ if } \tau_B < T,\ B \text{ otherwise}) = S_0$$

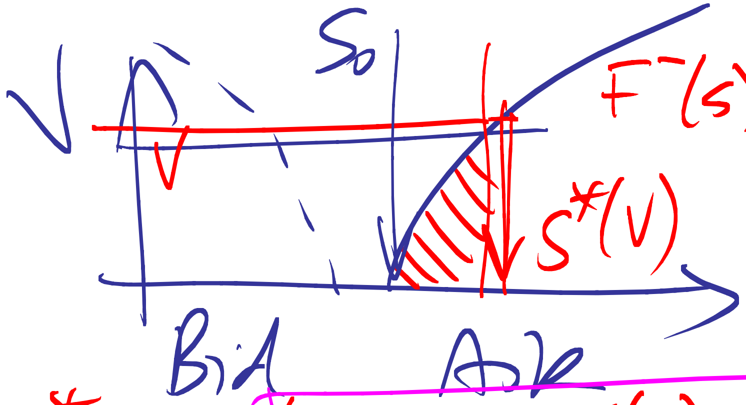
$\Rightarrow$ All the complexity of intra day trading comes from market microstructure and price / volume dependencies.

$$V = \int_{S_0}^{S_0+\delta S(V)} f^{-}(s)\, ds$$

$$\partial_V \Rightarrow 1 = \partial_V f^{-}(S) \cdot f(S_0 + \delta S V)$$

answer

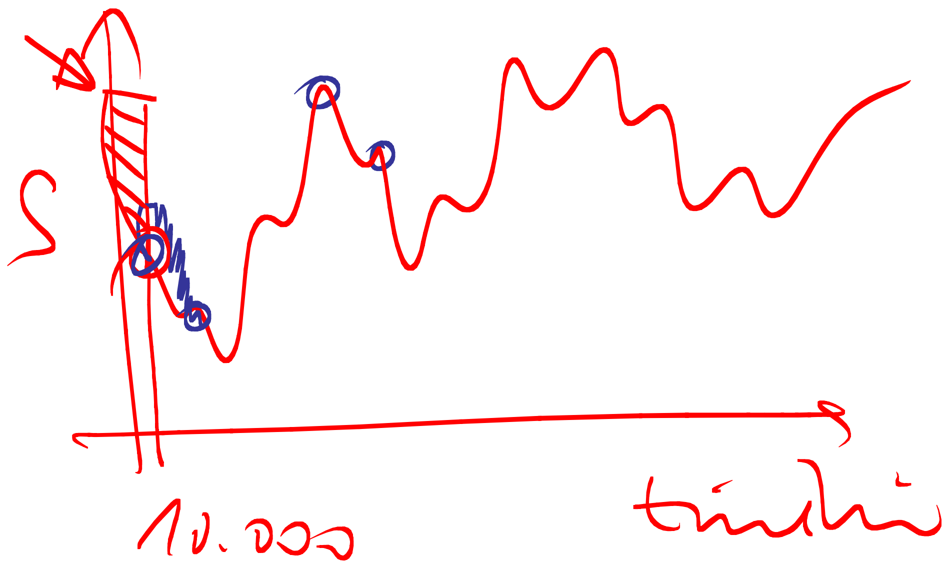
$$S(V) = \text{Market Impact of tiny trade}$$

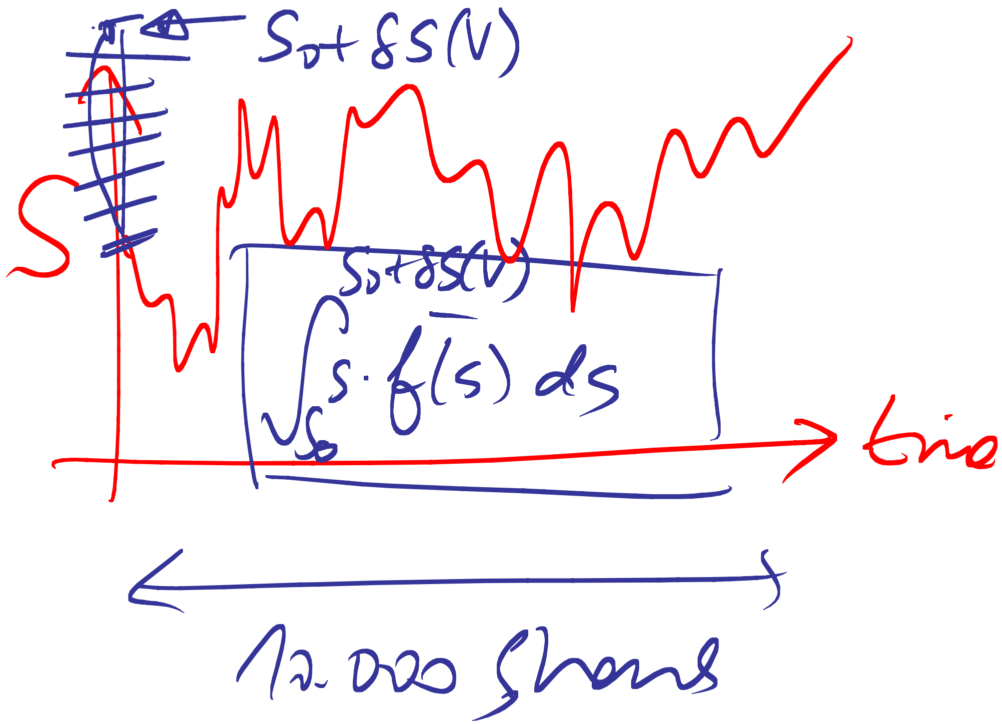


$$F^-(S) = \int_{S_0}^{S} f^-(s)\,ds$$

$S^*(V)$ /

$$\boxed{V = \int_{S_0}^{S^*(V)} f^-(s)\,ds}$$


S
10.000



$S_0 + \delta S(V)$
S
$S_0 + \delta S(V)$
$\int_{S_0}^{S} s \cdot f(s)\, ds$
time
12.000 Shares

My first continuous auctions formula

Let $f^{-}_{\mathrm{LOB}}(s)$ be the volume available in the LOB by sellers (ask prices) at price s (resp. $f^{+}_{\mathrm{LOB}}(s)$ for bids quantities). When I want to buy a quantity V of shares of this instrument, I will generate a Market Impact of S_{MI} implicitly defined by the equation (S_0 is the last quoted price):

$$V = \int_{s=S_0}^{S_{MI}} f^{-}_{\mathrm{LOB}}(s)\, ds \tag{1}$$

and my Volume Weightened Average Price (VWAP) will be:

$$\boxed{\mathrm{VWAP} = \frac{1}{V} \int_{s=S_0}^{S_{MI}} s \cdot f^{-}_{\mathrm{LOB}}(s)\, ds} \tag{2}$$

One fondamental order book equation

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$$f^{-}_{\mathrm{LOB}}(S_{PMI}(v)) = \frac{1}{\partial_v S_{MI}(v)} \qquad (3)$$

$$S_{MI}(V) = S_0 + \eta \cdot V$$

$$\partial S_{MI}(V) = \eta \cdot \Rightarrow f^{-}(S) = \frac{1}{\eta}$$

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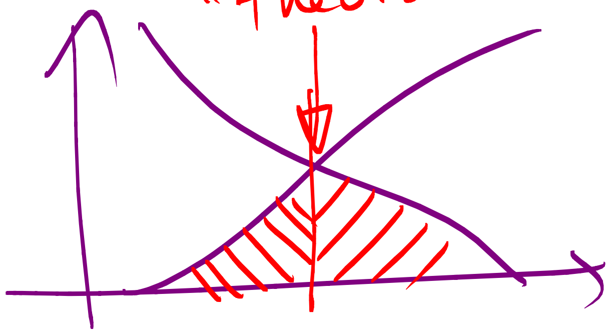
"*The big secret...Quantitative finance is one of the easiest branches of mathematics*"

Paul Wilmott on his blog, April 2008...

Fixing auctions

The rules are different for a fixing auction.

"theoretical" price



0 → 5min

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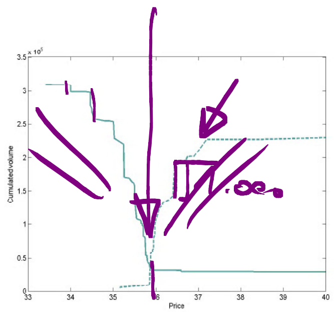


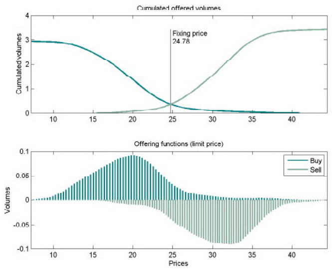
- during the *pre fixing* phase, orders are not crossed to generate transactions
- a "*theoretical price*" is published
- at the end (5 minutes later): the fixed price is the natural Pareto equilibrium

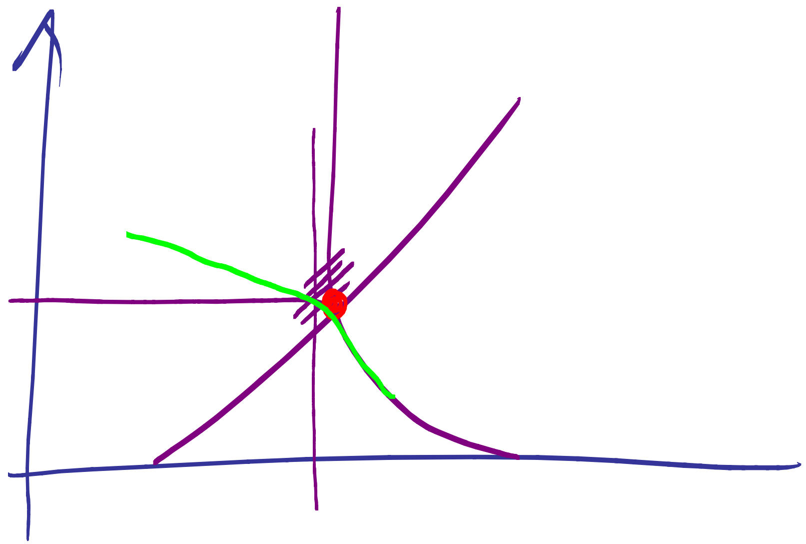
Fixing price

The fixing price S^* is such that

$$= \int_{-\infty}^{S^*} f^-(s)\,ds$$

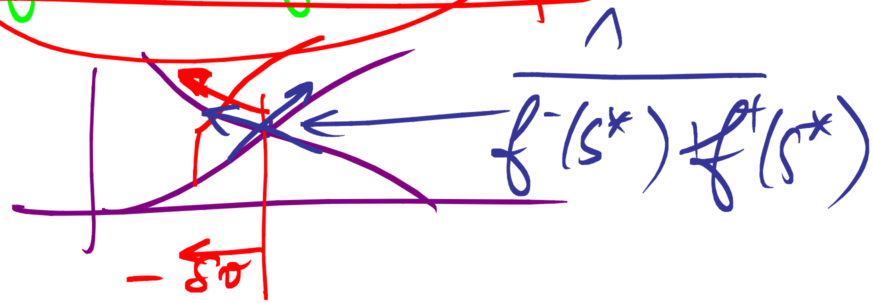
$$\int_{+\infty}^{S^*} f^-(s)\,ds \quad = \qquad F^+_{\mathrm{LOB}}(S^*) = F^-_{\mathrm{LOB}}(S^*) \tag{4}$$





$$\delta s \cdot \left[f^-(s^*) + f^+(s^*) \right] = -\delta v + o(\delta s)$$

$$\boxed{\delta s = \frac{-\delta v}{f^-(s^*) + f^+(s^*)}} + o(\delta s)$$



$$F^-(\tilde{s}^*) + \boxed{\underbrace{\delta v}_{\text{small}} \cdot 1_{\tilde{s}^* > s^*}} + \qquad = 1$$

$$\parallel$$

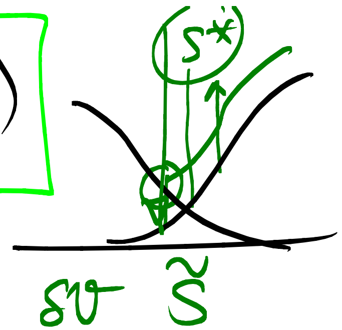
$$F^+(\tilde{s}^*) = \cancel{F^+(s^*)} - f(s^*)\,\delta s + o(\delta s)$$

$$\cancel{F^-(s^*)} + f(s^*)\cdot\delta s + o(\delta s) +$$

$$\delta s \cdot \left[f^-(s^*) + f^+(s^*)\right] = -\delta v + o(\delta s)$$

$$\boxed{F^+(s^*) = F^-(s^*)}$$

Former eq. price



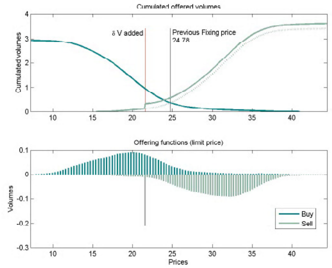
$$\tilde F^-(\tilde s^*) = F(\tilde s^*) + \delta\vartheta + 1_{\tilde s^* > \tilde s}$$

$$\|$$

$$\tilde F^+(\tilde s^*) = F^+(\tilde s^*)$$

$\tilde s^*$ new eq.

If I put a quantity δV into the fixing orderbook at a price $\tilde{S}$ into F^{-}_{LOB}.



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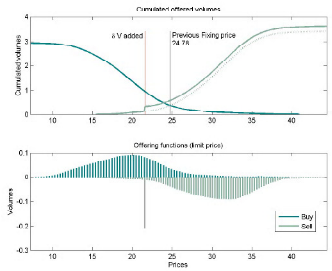

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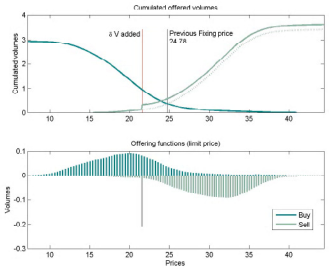

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$$F^{+}_{\text{LOB}}(\bar{S}) = F^{-}_{\text{LOB}}(\bar{S}) + \delta V \cdot \mathbf{1}_{\tilde{S} < S^{*}} \tag{5}$$

One fundamental Fixing equation

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$$F^{+}_{\text{LOB}}(S^{*}) - f^{+}_{\text{LOB}}(S^{*})\cdot\delta s = F^{-}_{\text{LOB}}(S^{*}) + f^{-}_{\text{LOB}}(S^{*})\cdot\delta s + \delta V + o(\delta s)$$

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and finally:

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It is a matter of slopes. The effect of adding volume far from the previous equilibrium price S^* is mainly related to local properties of the orderbook around S^*.

From price impact to market impact

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- (your trading tactic will be optimised on an order-by-order basis)
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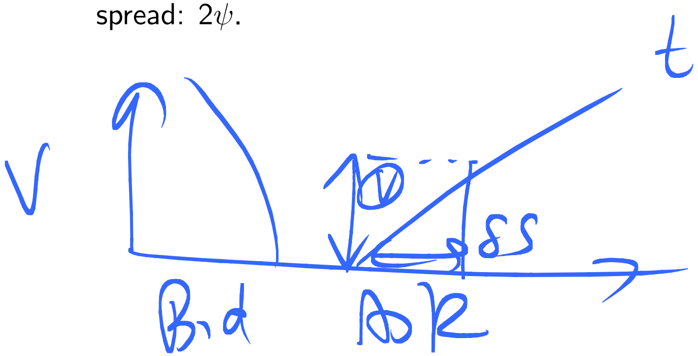
Common sense for Market Impact model

- Crossing the Bid-Ask spread. If the last trade was on the opposite side to my order (once over two trades...), I pay the spread: 2ψ.



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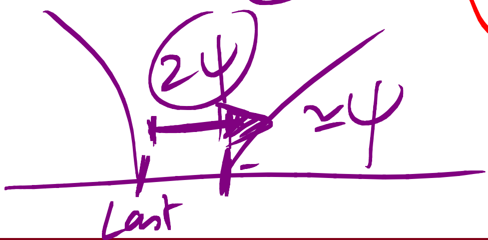

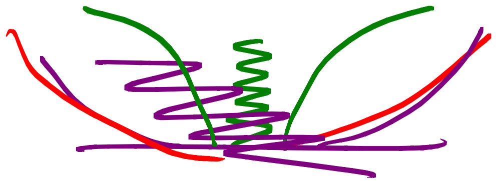

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- Dynamical market depth. The more the volatility of the price diffusion is high, the more the LOB is structurally empty: M.I. increases with σ.
- Usual quantities: the more my volume v is large compared to the "*usual*" traded volume $V_{\Delta T}$ (during this time interval ΔT), the more my M.I. increases.

Qualitative Market Impact model

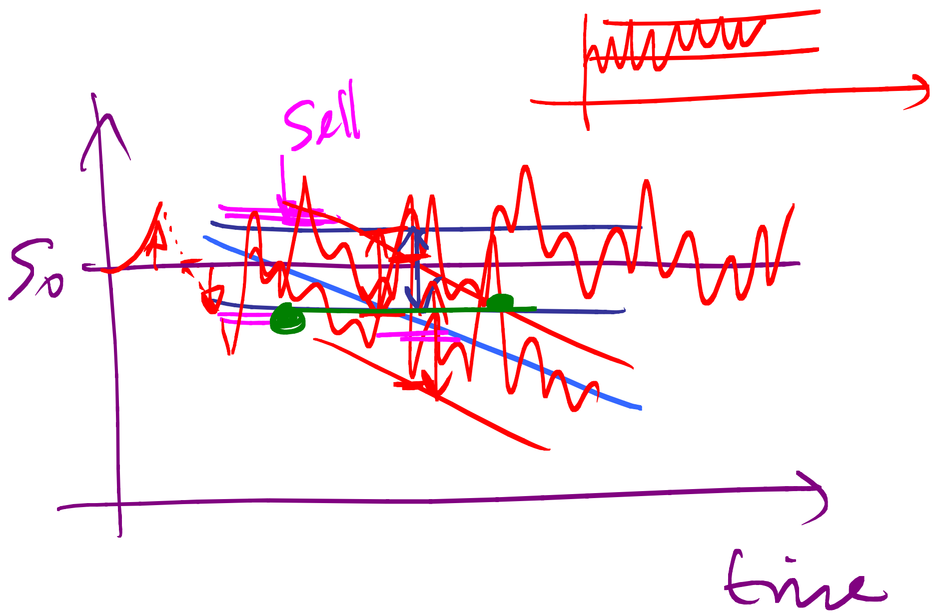
Basic Market Impact model

$$\mathrm{MI}(v) = \psi + F\left(\sigma, \frac{v}{\sqrt{\Delta T}}\right) \tag{7}$$

v

$\dfrac{v}{\sqrt{\Delta T}}$

where $F(\cdot,\cdot)$ increases with respect to its two inputs.



Sell
S_0
time

$$MI(\nu) = \boxed{\psi + F\left(\sigma, \frac{\nu}{V_{\Delta T}}\right)}$$

Qualitative Market Impact model

Lasry & Lions

Basic Market Impact model

$$\mathrm{MI}(v) = \psi + F\left(\sigma, \frac{v}{V_{\Delta T}}\right) \tag{7}$$

where $F(\cdot,\cdot)$ increases with respect to its two inputs.

Usually σ is on the risk side of the equation, because market impact is subtracted to your return, we already see that in a quantitative trading optimisation framework σ is (at least) on the return side...

Capture statistical invariants needs datasets

- If you have a proprietary datasets with your own orders (with enough trades into it), use them.

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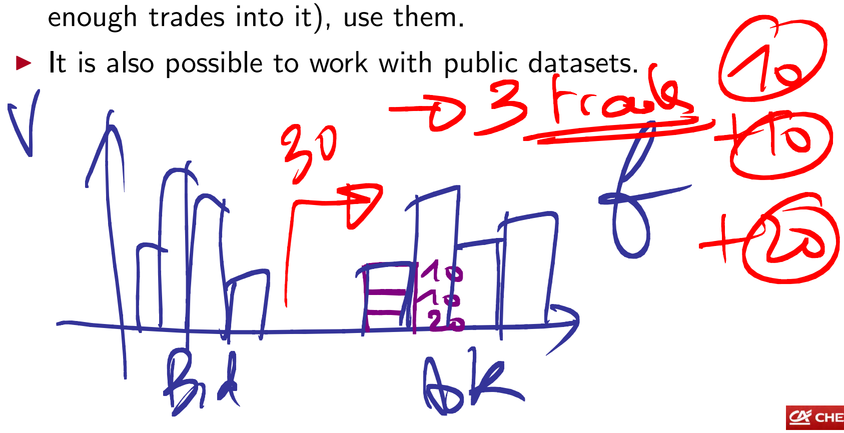

Capture statistical invariants needs datasets

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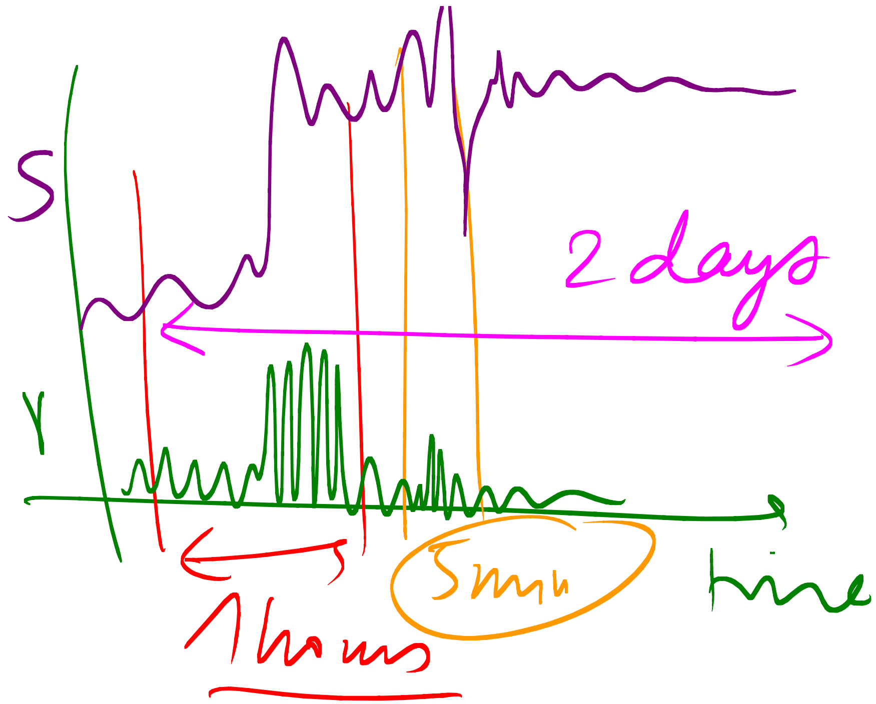

S
2 days
r
1 hours
5 min
time

Buy $S = \tilde{S} + MI(v)$

Sell $S = \tilde{S} - MI(v)$

$MI(v) = \Upsilon_n + K \cdot \sigma_n \cdot \sqrt{\frac{v}{V_n}}$

20 min

estimate

- order-by-order Market microstructure

Market Impact "trade-by-trade scale"

LOB

$$MI(v) = \psi + f\left(\sigma, \frac{v}{V}\right)$$

- Market Models

5min 15 - 1 hours time Scale

Permanent Model

non-stationary

Dynamics

Capture statistical invariants needs datasets

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What do we need?

As seen is equation (7), we need to capture σ, usual volumes, and half bid-ask spread.

$$MI(v) = \psi + F(\sigma, \frac{v}{V})$$

What means "*capture statistical invariants*"

- No risk-neutral measure available,

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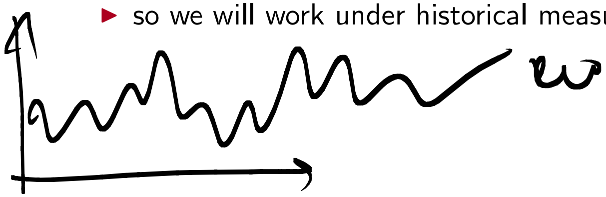

What means "*capture statistical invariants*"

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What means "*capture statistical invariants*"

EON.DE

1 day

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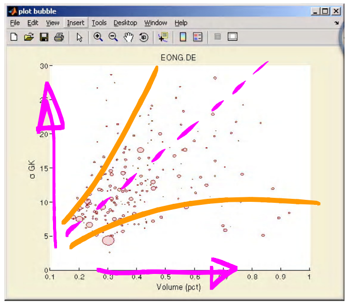
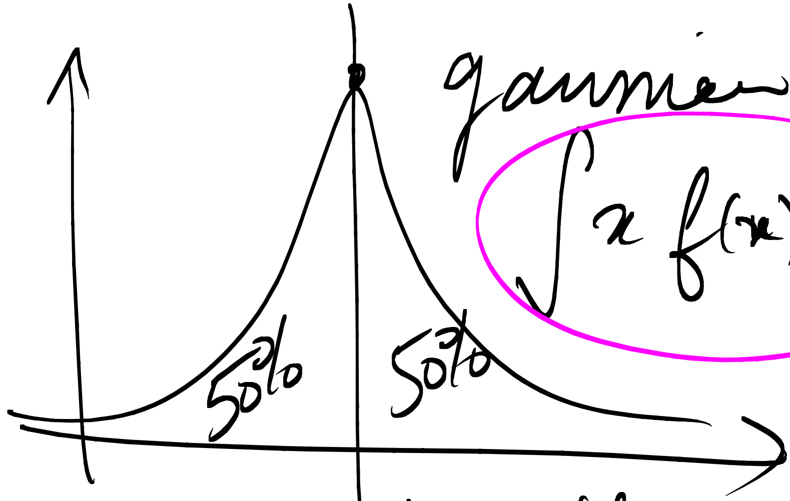


CHEUVREUX



gaussien

$$\int x\, f(x)\, dx$$

median Volume /20 min

= mean/expér.

$$\exp\left[-\frac{1}{2\sigma^2}\left[y^2-2\mu y+\mu^2-2\sigma^2 y\right]\right)$$

$$\left[y-(\sigma^2+\mu)\right]^2+\mu^2-(\sigma^2+\mu)^2$$

$$\int_{-\infty}^{\infty}\frac{e^{\frac{1}{2}\sigma^2+\mu}}{\sqrt{2\pi}\,\sigma}\,e^{-\frac{\left[y-(\sigma^2+\mu)\right]^2}{2\sigma^2}}\,dy = 1$$

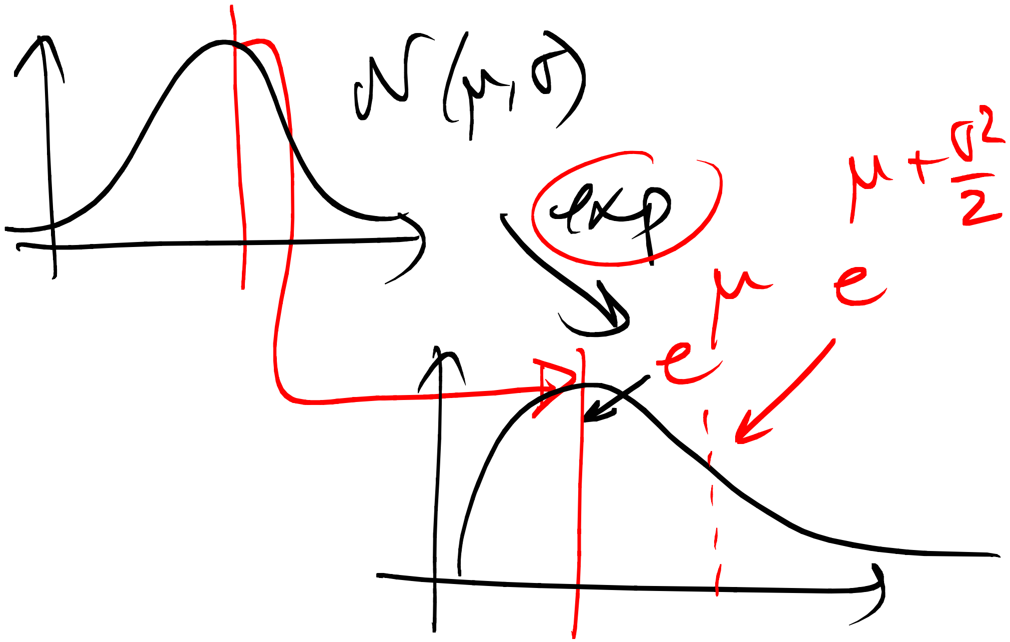
$\sigma^2+\mu$

$$\int_0^{\infty} \frac{x}{\sqrt{2\pi}\cdot\sigma\cdot x} e^{-\frac{(\ln(x)-\mu)^2}{2\sigma^2}} dx$$

$$y = \ln x \Rightarrow dy = \frac{dx}{x}$$

$$\int_{-\infty}^{+\infty} \frac{e^{y} \cdot e^{-\frac{(y-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\,\sigma} dy$$

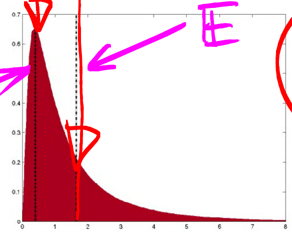
$$\exp\left[-\frac{1}{2\sigma^2}\left[y^2 - 2\mu y + \mu^2 - 2\sigma^2 y\right]\right]$$



$\mathcal{N}(\mu, \sigma)$
exp
e^{μ}
$e^{\mu + \frac{\sigma^2}{2}}$

Volumes: do not use "average" on log normal distributions

mode
median



E

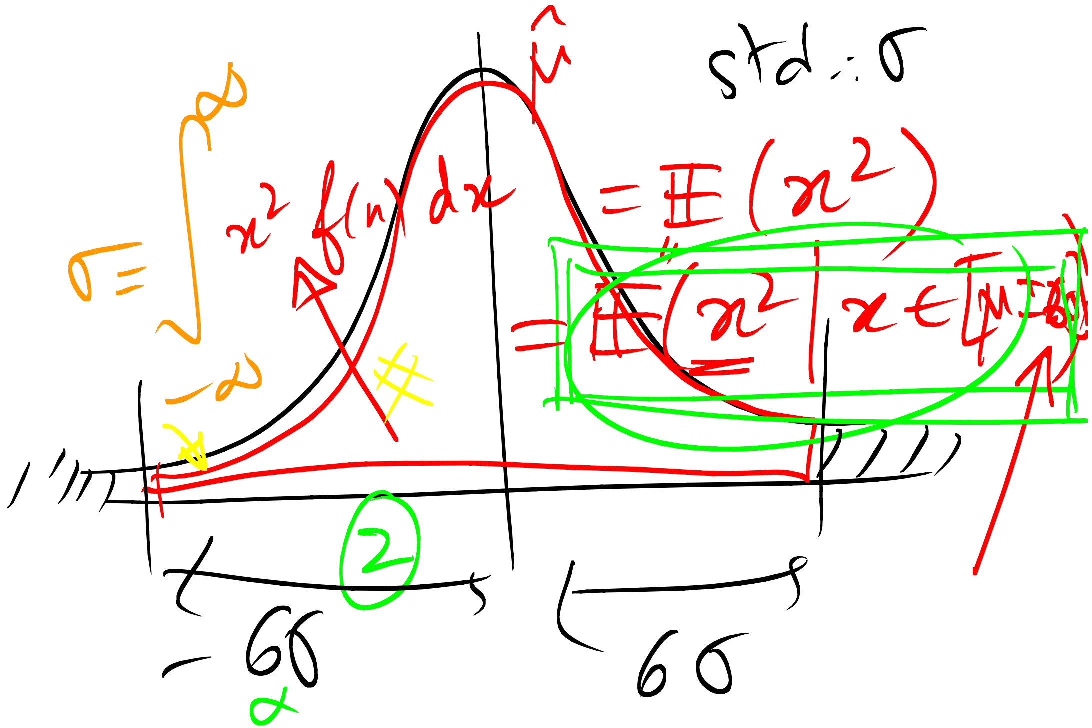
VaR

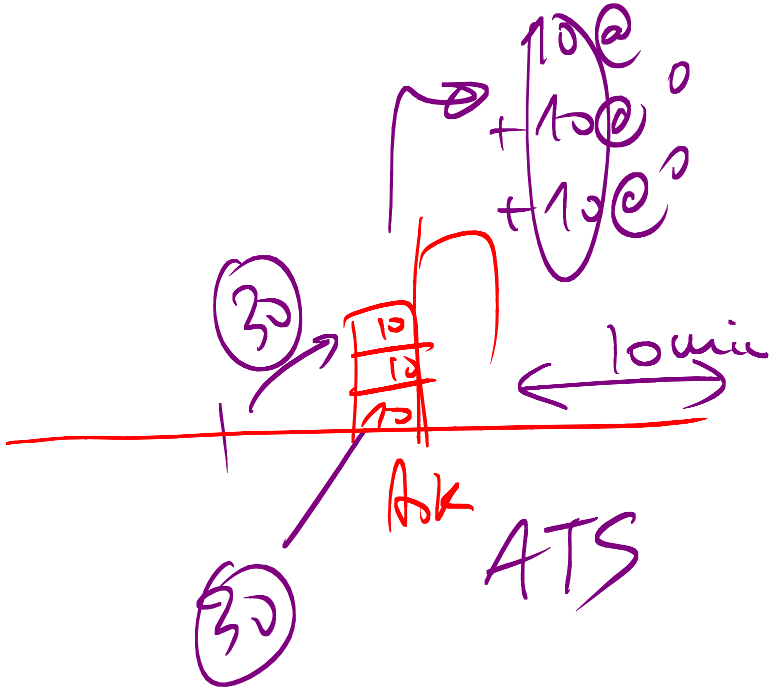
Density of a log normal distribution:

$$f(x) = \frac{1}{x\sigma\sqrt{2\pi}} \exp - \frac{(\ln(x) - \mu)^2}{2\sigma^2} \quad (8)$$

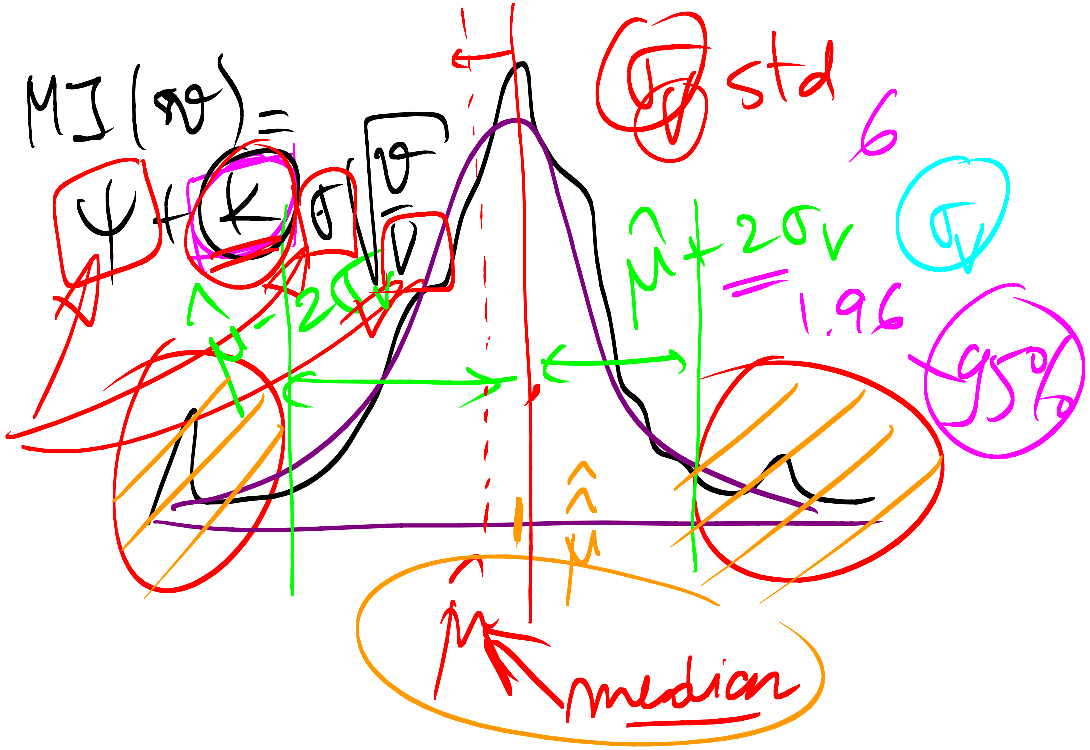
So (the median is not equal to the expectation):

$$\boxed{M = \boldsymbol{E}(X) = \exp \mu + \frac{\sigma^2}{2}}$$


std : σ
$\hat{\mu}$
$\sigma = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \mathbb{E}(x^2)$
$= \mathbb{E}(x^2 \mid x \in [\mu \pm 6\sigma])$
-6σ
6σ
2
α

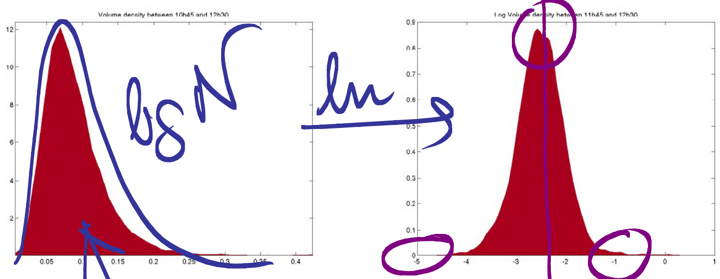

30
30
10
10
10
Ask
ATS
10min

$$MI(\vartheta) = \psi + k\,\sigma\sqrt{\frac{\vartheta}{\nu}}$$



Five months of FTE.PA, between 11h45 and 12h30

45min



symm.
median.

$$MI(\nu) = \psi + F\left(\sigma, \frac{\nu}{\nu}\right)$$

Log normality: what do we really need

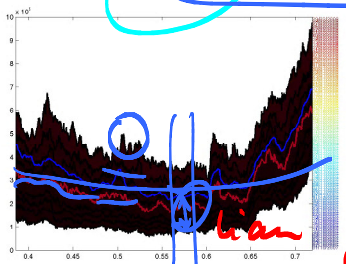
V_m

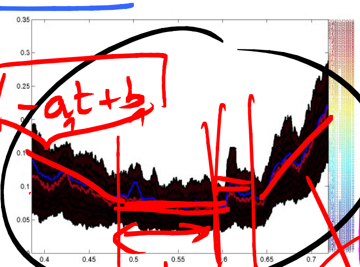
It is easy to see that $\boldsymbol{P}(X > M) \simeq 30\%$. Do we want the mean value, or the median?

$\mathbb{E}(\ \cdot\ )$

Five months of FTE.PA (05/07 to 03/08) capital vs proportions

volume during 15min





9h time 17h30

b_2 b_3

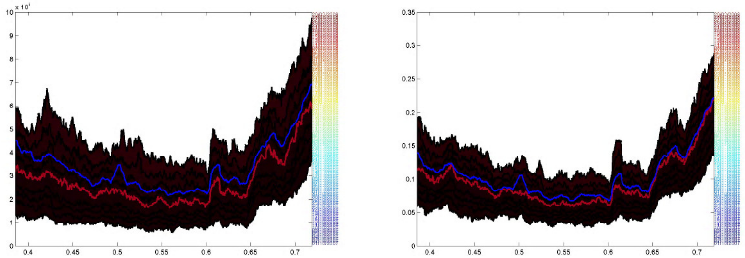
median

$a_4 t + b_4$

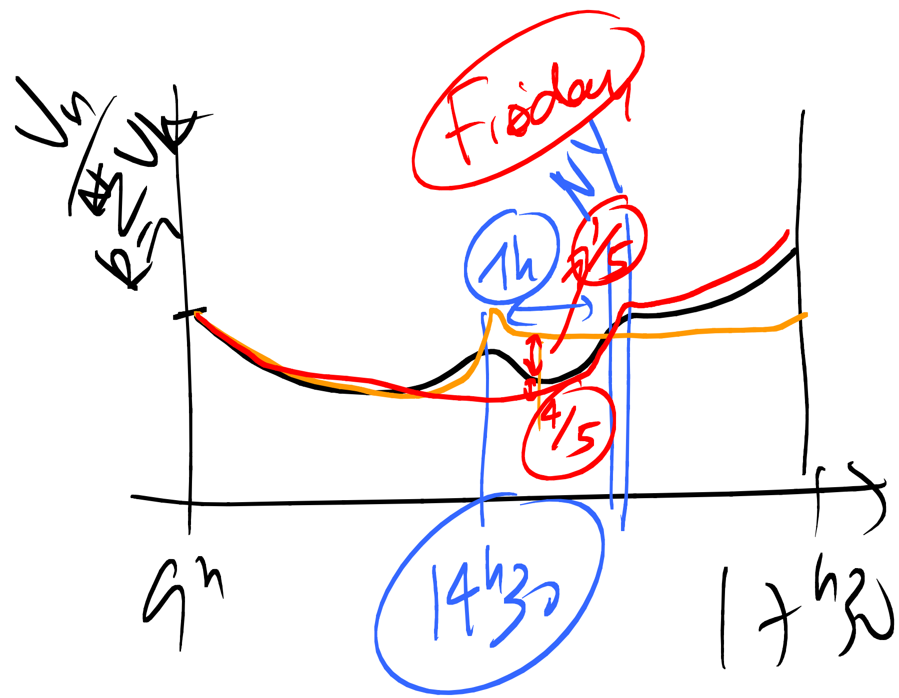
Log normality: what do we really need

It is easy to see that $\boldsymbol{P}(X > M) \simeq 30\%$. Do we want the mean value, or the median?

Five months of FTE.PA (05/07 to 03/08) capital vs proportions

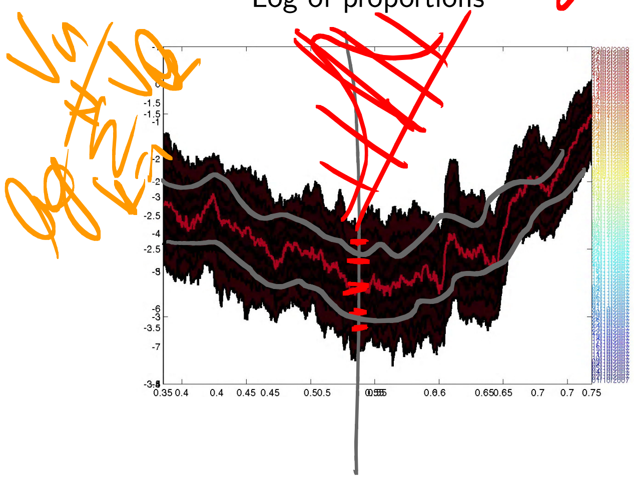


If you really want the mean, use robust mean (expectation using median and truncated variance) instead of average!

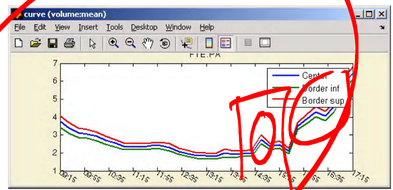

Friday
1h
4/5
14h30
9h
17h30
t

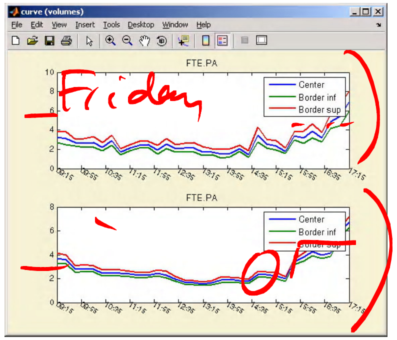
log $\sim N(\mu, \sigma)$

Log of proportions



Do not mix heterogeneous effects





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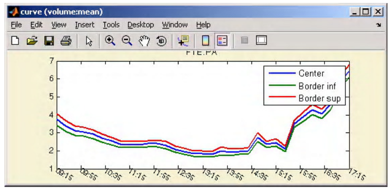
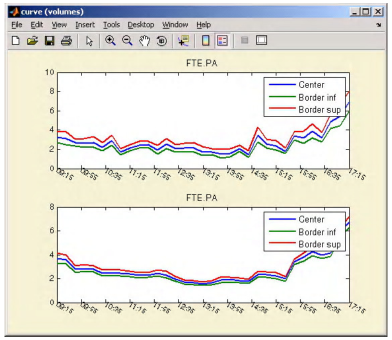
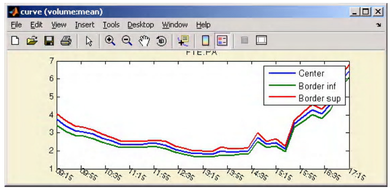
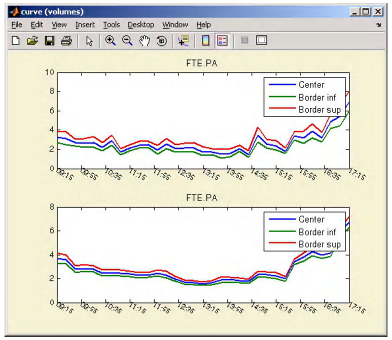




A large part of the variance comes from mixing Fridays with other days.

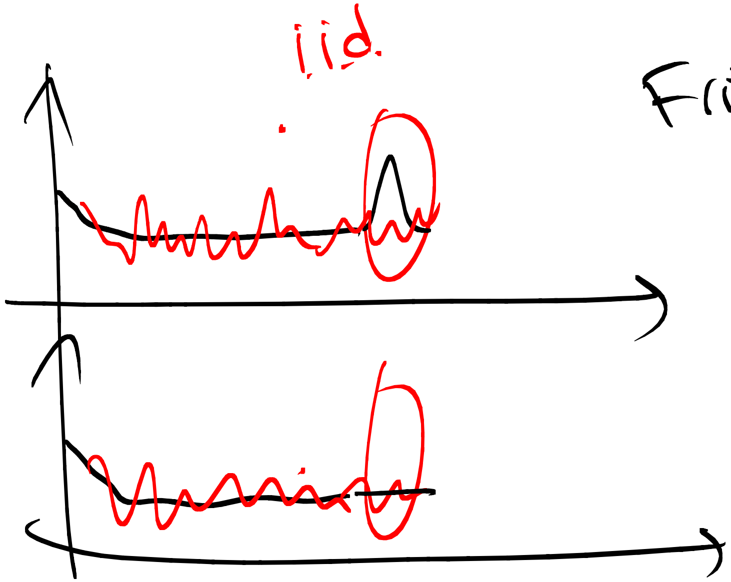
Do not mix heterogeneous effects



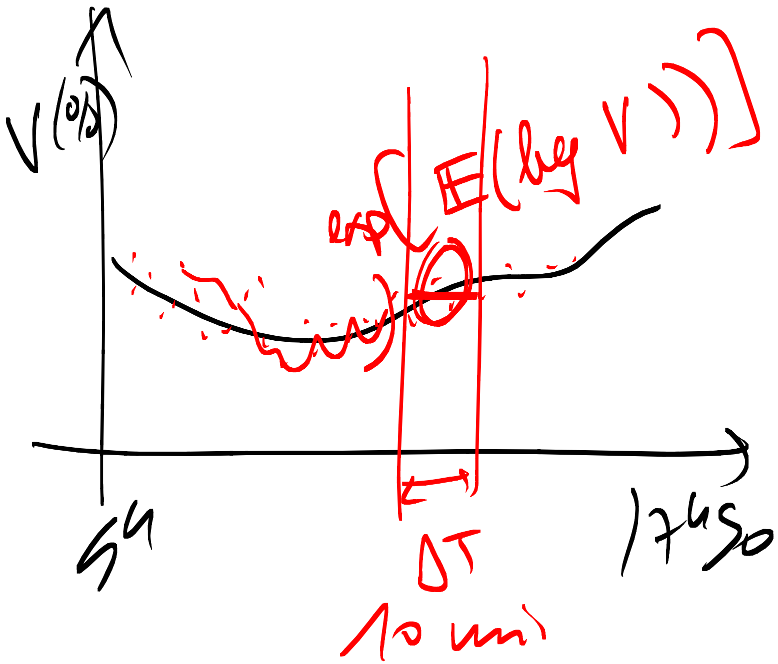


A large part of the variance comes from mixing Fridays with other days.

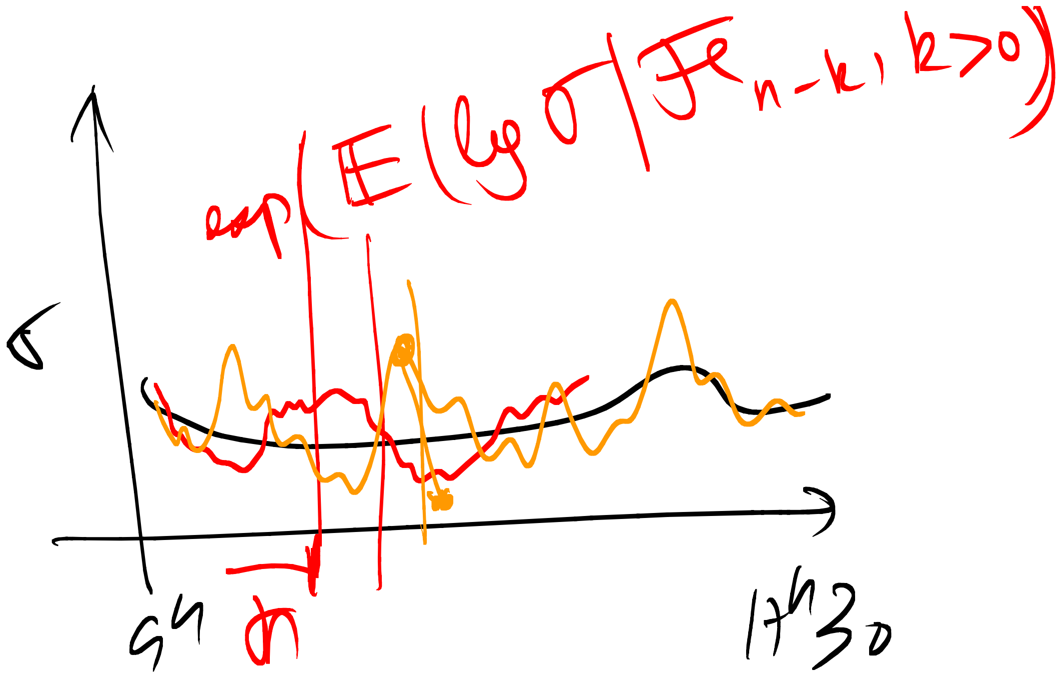
You can use auto correlations to obtain more robust estimators.



Friday?



V (%)
exp[E(log V)]
ΔT
10 min



exp(E(log σ | F_{n-k}, k>0))
σ
9h
m
17h30

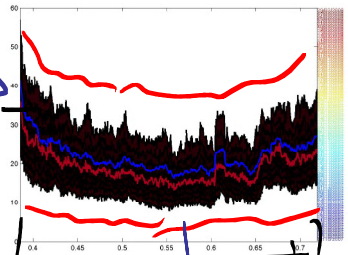
Volatility : do only add squared volatilities !

We all know that $\mathcal{N}(0,\sigma_1)+\mathcal{N}(0,\sigma_2)\sim\mathcal{N}(0,\sqrt{\sigma_1^2+\sigma_2^2})$.

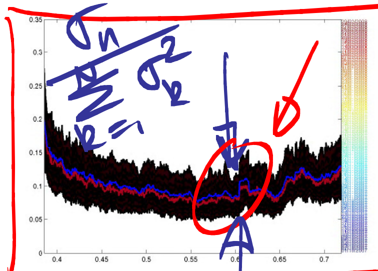
Five months of FTE.PA (05/07 to 03/08) σ_{GK} vs proportions

$\sigma(\boxed{\%})$

S



9h time 17h30



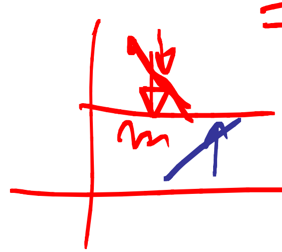
Volatility are not "*as lognormal*" as volumes are. Mainly, they are "*far less i.i.d.*"...

$$d(\sigma^2) = \gamma(m - \sigma^2)\,dt + k\sigma\, dW_t$$

σ_t

$\sigma^2 > m$ $\quad$ $\sigma^2 < m$

$$= -\gamma c\, dt + k\sigma\, dW_t$$



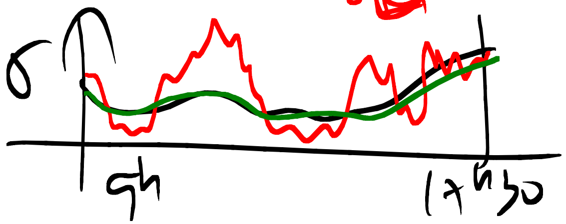
$$+\gamma c\, dt + k\sigma\, dW_t$$

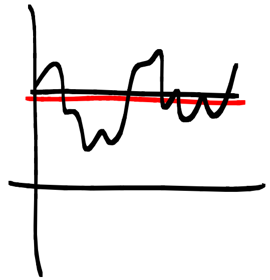
Intra day volatility modelling

$\sigma > 0$

Because of the Markovian aspect of σ intra day, we would like to model it throught diffusion process.
A natural candidate is the C.I.R. model:

$$d(\sigma^2) = \gamma \cdot (m - \sigma_t^2)\, dt + \xi \sigma_t\, dWt \tag{9}$$





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$$d(\sigma^2) = \gamma \cdot (m - \sigma_t^2)\, dt + \kappa \sigma_t \, dWt \tag{9}$$

The point is that m and κ should be time dependent here...

Intra day volatility estimation

- Proportion of volatility seems to be more stable by volatility itself

Intra day volatility estimation

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- How could we use proportion of volatility?

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- But measuring volatility itself is difficult

Intra day volatility estimation

Jean Jacod
& Delattre

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- How could we use ~~proportion of volatility?~~
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Our prices are discretized (rounded?) on a price grid

δS

1 cent €

Intra day volatility estimation

- Proportion of volatility seems to be more stable by volatility itself
- How could we use proportion of volatility?
- But measuring volatility itself is difficult

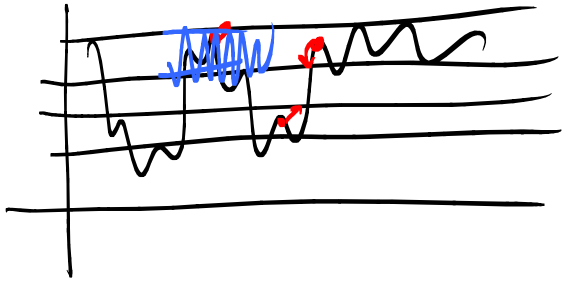
Our prices are discretized (rounded?) on a price grid

A lot of interesting papers have been written on this subject: Jacod, Delattre, At-Sahalia, Zhang, Mykland, Shepard, Rosenbaum

A "*simple*" model:

$$X_{n+1} = X_n + \sigma\sqrt{\delta t}\,\xi_n\,,\; S_n = X_n + \varepsilon \qquad (10)$$

Noise

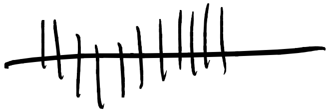


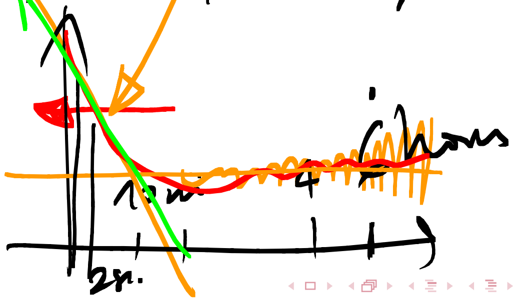
A "*simple*" model:

$$X_{n+1} = X_n + \sigma\sqrt{\delta t}\,\xi_n\,,\quad S_n = X_n + \varepsilon \tag{10}$$

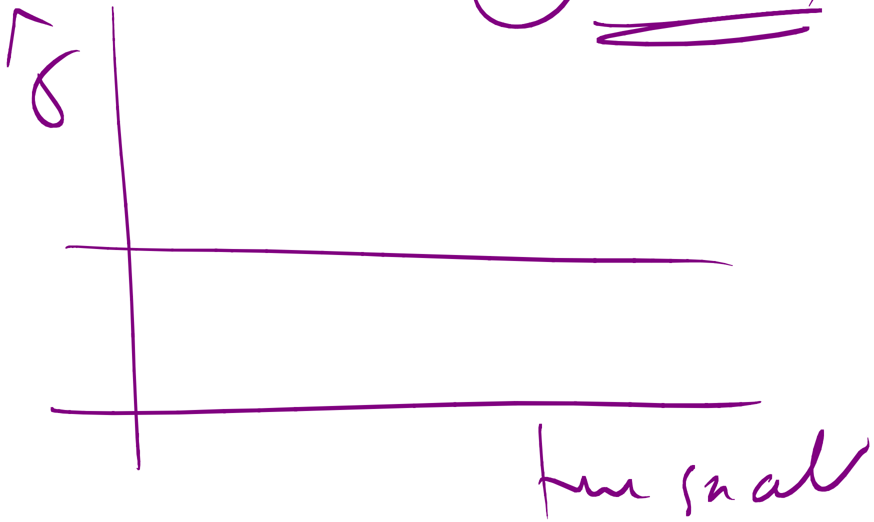
$$\sum_n (S_{n+1} - S_n)^2 = 2nE(\varepsilon^2) + O(\sqrt{n})$$

tick-by-tick /





$n \cdot \mathbb{E}(\varepsilon) + \sigma$



A "*simple*" model:

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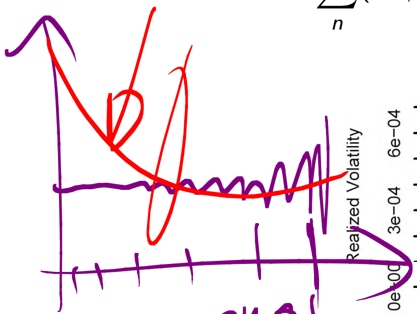
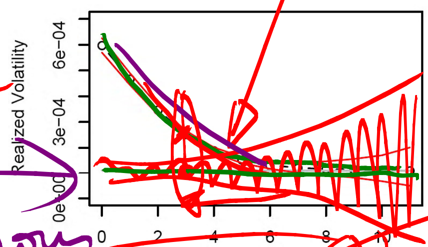
discretiz.



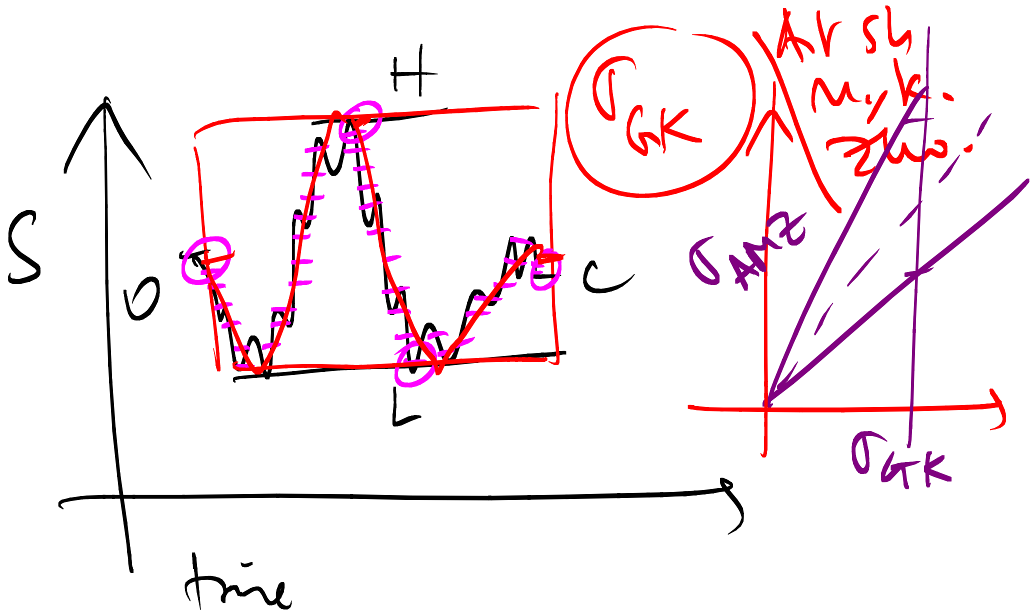


30 mn? 2 hours

thanks to Mathieu Rosenbaum for this plot

The simplest intra day volatility estimator

- *On the Estimation of Security Price Volatility from Historical Data2*, Mark B. Garman and Michael J. Klass, 1980

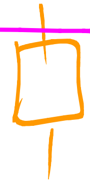

S
O
H
L
C
time
σ GK
Arr sh
σ AMZ
σ GK

The simplest intra day volatility estimator

- *On the Estimation of Security Price Volatility from Historical Data2*, Mark B. Garman and Michael J. Klass, 1980
- Uses four points (a full candle) rather than only one

$$\hat{\sigma}^2_{GK} = \frac{(S_h - S_l)^2}{2} - (2\ln(2) - 1)(S_o - S_c)^2 \tag{11}$$

4 point OHLC



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$$\hat{\sigma}^2_{GK} = \frac{(S_h - S_l)^2}{2} - (2\ln(2) - 1)(S_o - S_c)^2 \qquad (11)$$

- More robust to micro structure effects

Bid-ask spread : when microstructure limits the use of diffusion models

If only one microstructure effect should be kept, it would be the Bid-Ask spread:

- Sell price $\neq$ Buy price

Bid-ask spread : when microstructure limits the use of diffusion models

If only one microstructure effect should be kept, it would be the Bid-Ask spread:

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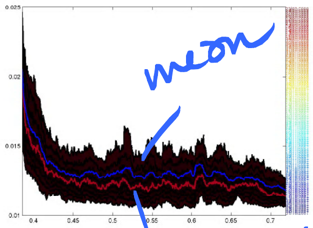
- Volume has an influence on the price

Bid-ask spread : when microstructure limits the use of diffusion models

If only one microstructure effect should be kept, it would be the Bid-Ask spread:

- Sell price $\neq$ Buy price
- Volume has an influence on the price
- Volatility estimations are not so simple

ε



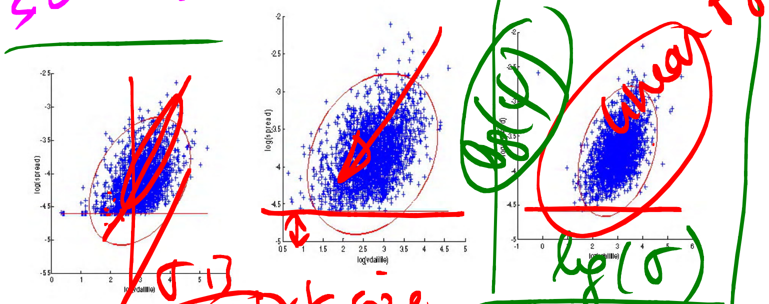
$+ n\,\mathbb{E}(\varepsilon^2)$

What about real statistics?

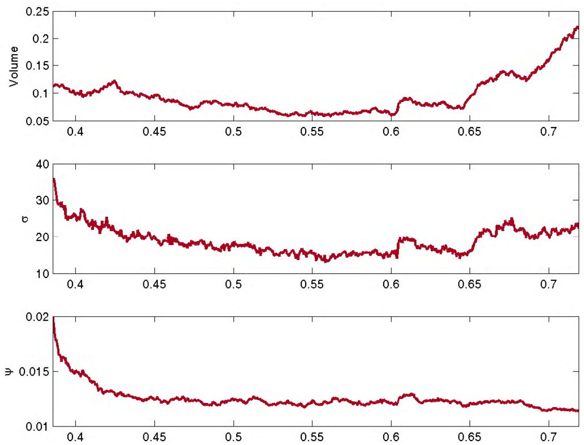
liquidity

not only by viscosity

"*Spread effect*" on three stocks

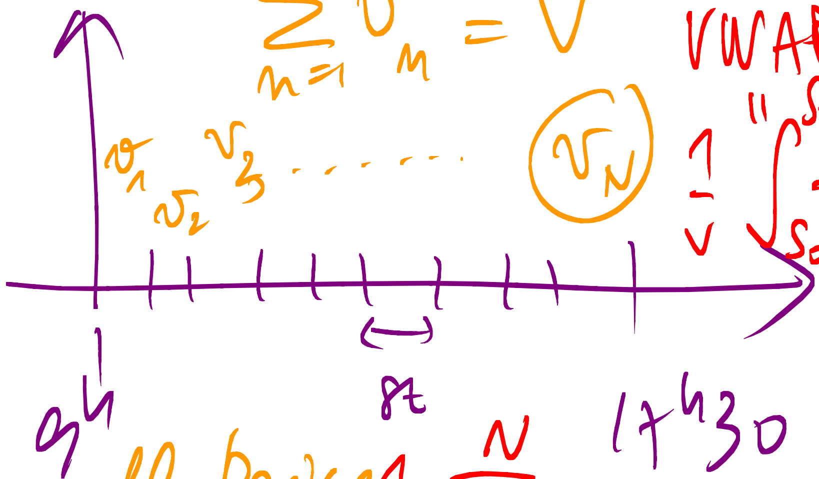


"high" tick size


0.25
0.2
0.15
0.1
0.05
Volume
0.4
0.45
0.5
0.55
0.6
0.65
0.7
40
30
20
10
σ
0.02
0.015
0.01
ψ

Trade a given volume V (to buy)

$$\sum_{n=1}^{N} v_n = V$$



VWAP

$$= \frac{1}{V}\int_{S_0}^{S_0+\delta S(V)} x\, f^+(x)\,dx$$

Overall price $\frac{1}{V}\sum_{n=1}^{N} S_n V_n$

$$MI(n) = \underbrace{\psi_n + k \cdot \sigma_n}_{\text{to be estimated}} \sqrt{\frac{v}{V_n}}$$

$n\delta t$ $\quad$ $(n+1)\delta t$

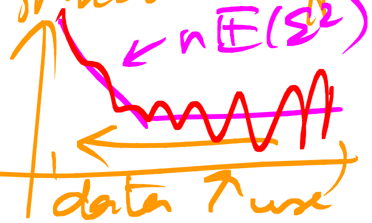
$\approx$ 20 min

ψ

σ $\quad$ V $\quad$ $\sim \log N$

microstructure effect

$n\,E(\varepsilon^2)$

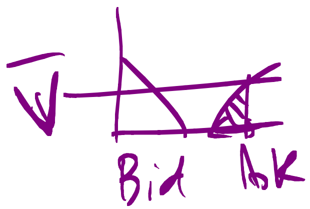


$\psi = G(\sigma, \ldots, V)$

"min tick size fact"

— Microstructure effects .

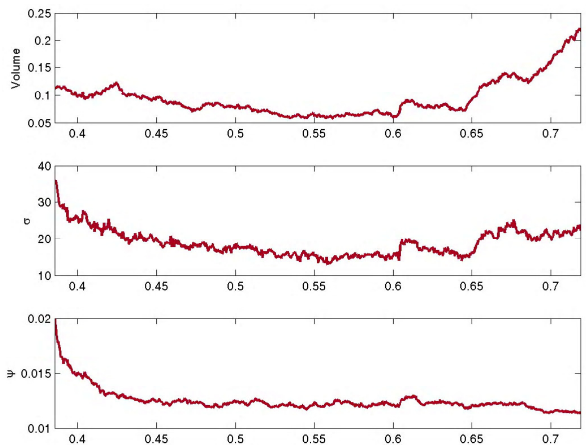
LOB, Consumers ≠ providers



MI $\frac{1}{2}$ Bid-Ask spread

— $MI(v) = \psi + F\left(\sigma, \frac{v}{V}\right)$

usual



Give me two of the three curves, I will give you the missing one...

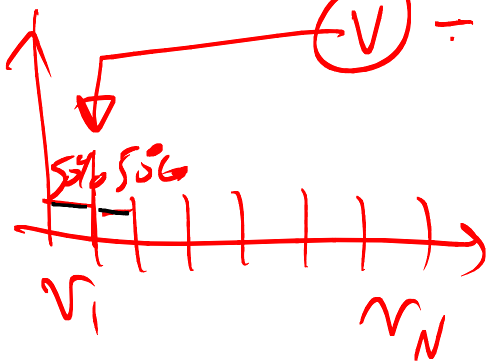
$H(S, V)$ = new price after Buying V

$$\boxed{\underline{H}(\underline{H}(S, V_1), V_2) \leq \underline{H}(S, V_1 + V_2)}$$

Market impact (volume driven) demands to trade slowly

As suggested by equation (7), a good way to reduce your Market Impact is to trade on a larger time interval ΔT.

$V = v_1, \ldots, v_N = (100\%, 0, 0, \ldots, 0)$



$MI(v) = \psi + F\left(\sigma, \frac{V}{V}\right)$

10 min
20 min
$\sim 2x$.

Market impact (volume driven) demands to trade slowly

As suggested by equation (7), a good way to reduce your Market Impact is to trade on a larger time interval ΔT.

$$\left.\begin{array}{l} v \equiv \mathrm{cst} \\ V_{\Delta T} \nearrow \text{ with } \Delta T \nearrow \end{array}\right\} \Rightarrow \psi + F\left(\sigma, \frac{v}{V_{\Delta T}}\right) \searrow \text{ with } \Delta T \nearrow$$

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$\Rightarrow$ trade as slowly as possible!

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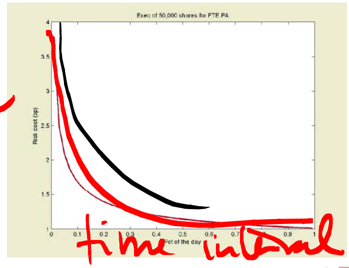
$$k \cdot \sigma \cdot \sqrt{\frac{v}{V}}$$

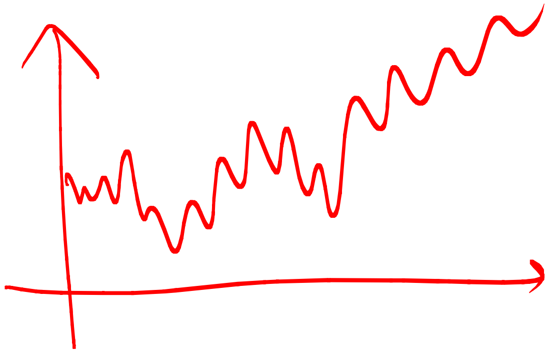
Price



time interval

Market risk (volatility driven) demands to trade fast

The more you wait and the more you tale market risk.



Market risk (volatility driven) demands to trade fast

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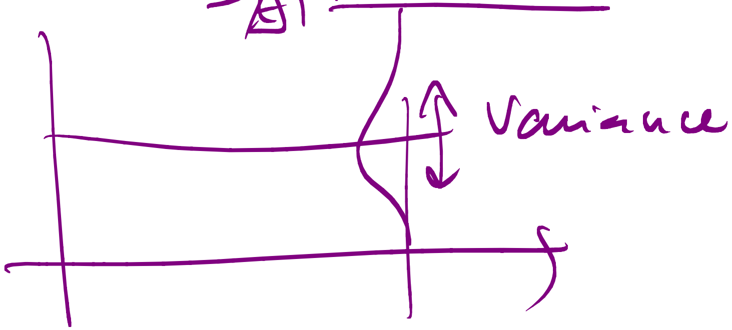
For an arithmetic Brownian diffusion, the amplitude of the risk is:

$$S_{\Delta T} = \boxed{\alpha \cdot \sigma\sqrt{\Delta T}}$$



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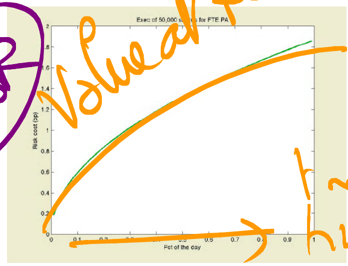


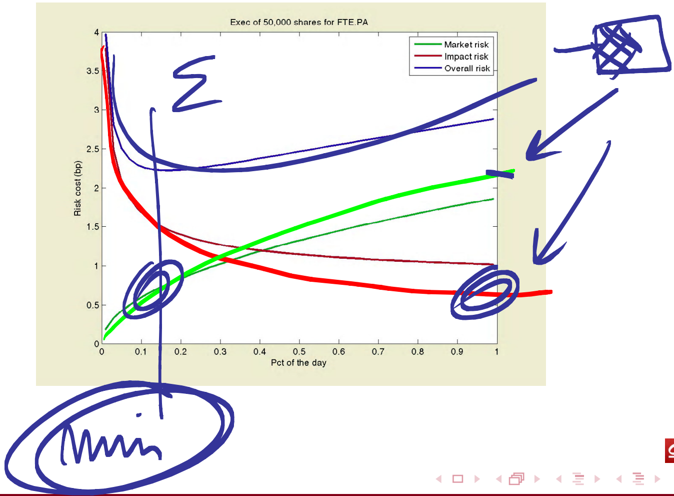
Risk

Volume at Risk

trading time

Those two effects are mixed

$v_1 + \dots + v_N = V$



Continuous auctions mechanisms

Two main behaviours:

- Liquidity provider: provide orders to the orderbook (a quantity at a given price for a defined side), my orders are consolidated inside the "*limit order book*".
- Liquidity consumer: give orders (*at best*, *price limited*, or more complex) which consume one or more orders of the LOB.

Each market place (*Dark Pools*, *Multilateral Trading Facilities*, etc) has this set of specific orders, like *iceberg*, *pegged* or *totally hidden*).

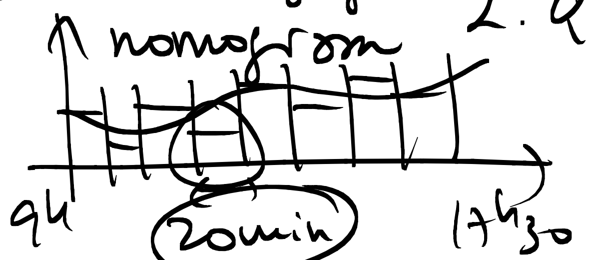
Trading framework

Layer 1: Strategy

1. timescale
2. quanta

TZ & NR.

nomogram



Layer 2 tactical one

v_m

20min

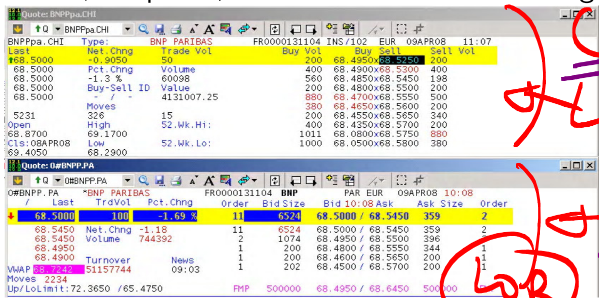
liquidity provider

Multilateral trading facilities (MTF)

1st oct. 2007

MiFID

Chi-X, turquoise, etc... are here or are coming



Chi-X
LOB
EuroNext
LOB
Turquoise

More order types: totally hidden, pegged, etc...

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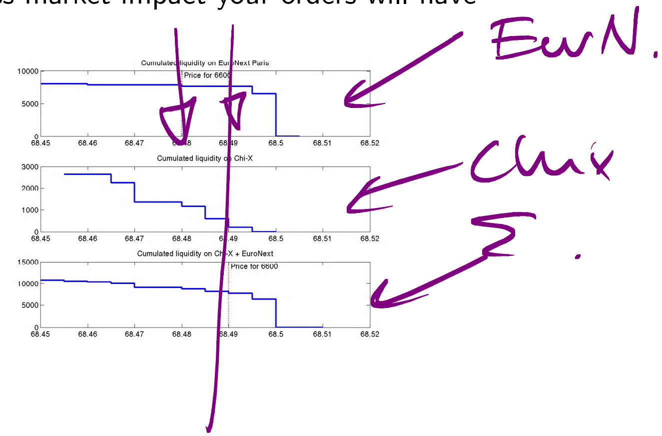
The more liquidity pools you can address,
the less market impact your orders will have



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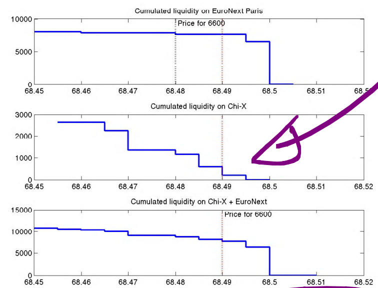


Chi-X

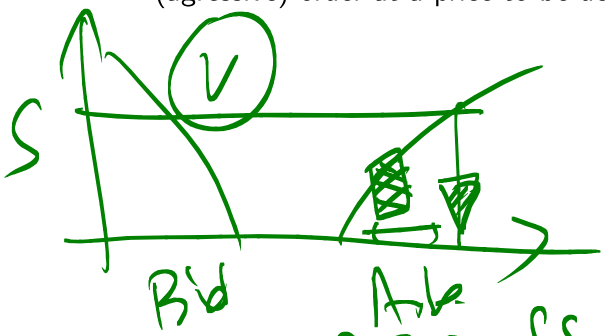
EuroMillion

Beware of the latency issues!

Dark pools

Euro Millenium

- Light pools garantee to executed the full quatity of your (agressive) order at a price to be defined,



fs incertain

CHEUVREUX

Dark pools

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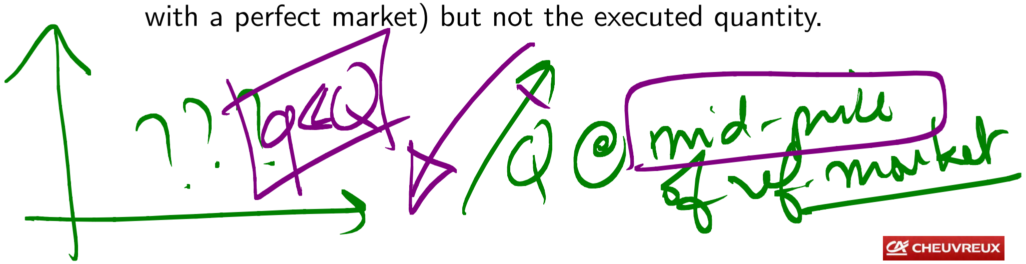
- Dark pools garantee your execution price (often in relationship with a perfect market) but not the executed quantity.



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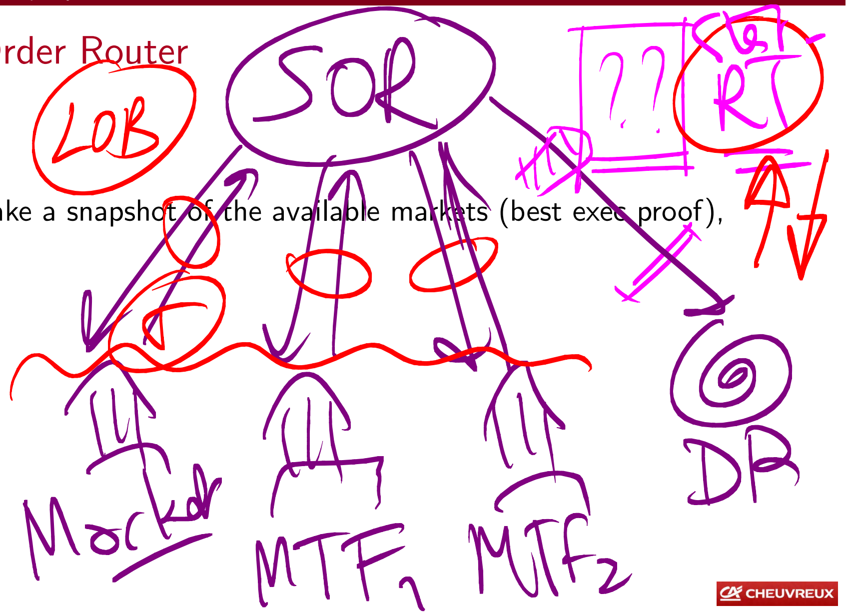
Illuminating the new dark influence on trading and U.S. market structure, Carl Carrie, **JOT** Winter 2008

Smart Order Router

- take a snapshot of the available markets (best exec proof),



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Smart Order Router

10% → DP
94%
6% exec

- take a snapshot of the available markets (best exec proof),
- take your decision (split)
- what about Dark pools (no market data)?
- need of statistical tables

What does mean VWAP of an order in such a context?

Equations

I have a volume V^* to buy from 0 to T ; I will assume a regular time grid of width δt (n from 1 to $N = [T/\delta t]$).

δt

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$$S_{n+1} = S_n + \alpha\,\delta t + \sigma_{n+1}\sqrt{\delta t}\;\xi_{n+1} \tag{12}$$

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My market impact is additive qnd is given by a $\eta_n(v_n)$ function.

$$W = \sum_{n=1}^{N} v_n \left(S_n + \eta_n(v_n) \right)$$

$$= v_1 S_1 + v_2 S_2 + \cdots \quad + \overbrace{\eta \cdot}^{M.I}$$

$$S_0 + \sigma_1 \xi_1 \cdot \sqrt{\delta t}$$

$$S_0 + \sigma_1 \xi_1 \cancel{\sqrt{\delta t}} + \sigma_2 \xi_2 \sqrt{\delta t} + M.I.$$

$$x_n = \sum_{k=n}^{N} v_k$$

$$V^* \cdot S_0 + \sum \sigma_1 \xi_1 x_n + \widetilde{\;}$$

$$V^* \cdot S_0 + \sigma_1 \xi_1 (v_1 + v_2 + \cdots) + \sigma_2 \xi_2 (v_2 + \cdots v_N)$$

Equations

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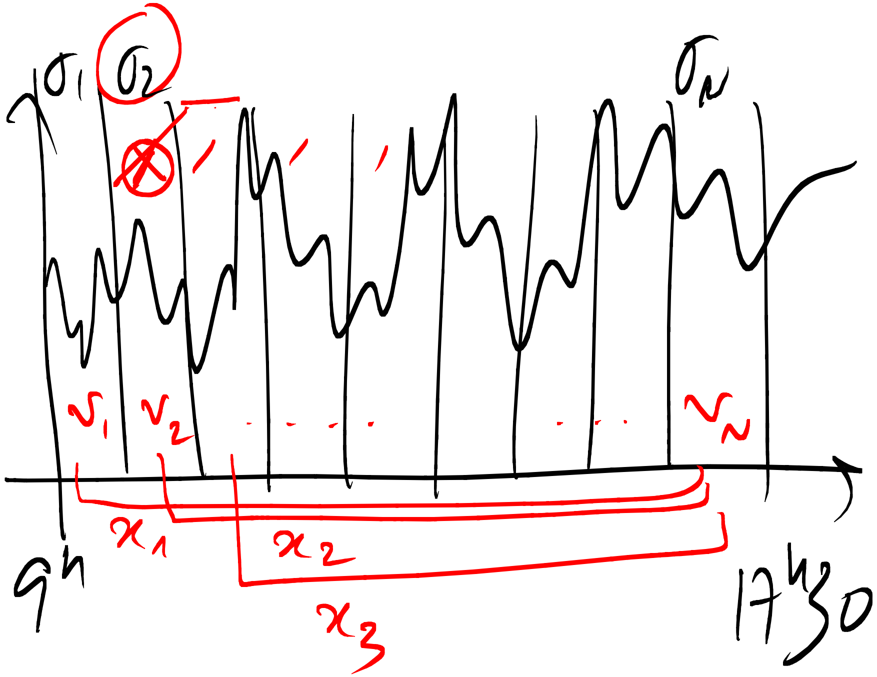
So my total price is:

$$\boxed{W = \sum_{n=1}^{N} v_n \cdot \underbrace{(S_n + \eta_n(v_n))}_{\tilde{S}_n(v_n)}} \tag{13}$$

CA CHEUVREUX

The classical rewriting

$$x_n = \sum_n^N v_n,\ x_1 = V,\ x_{N+1} = 0$$



σ1 σ2 σN
v1 v2 vN
x1 x2 x3
9h
17h30

$$W = \sum \left[x_n \boxed{\sigma_n \cdot \xi_n} + v_n \cdot \underbrace{\eta_n(v_n)}_{\psi + k \cdot \sigma_n \left(\frac{v_n}{V_n}\right)^{\gamma}} \right]$$

$$\underline{\mathbb{E}(W)}$$

$$\mathbb{V}(W)$$

$$\xi_n \cdot k \, \sigma_n^2 \left(\frac{v_n}{V_n}\right)^{\gamma}$$

The classical rewriting

x_n = the remaining V - the execut

$$x_n = \sum_n^N v_n, \; x_1 = V, \; x_{N+1} = 0$$

Market

$$W = V \cdot S_0 + \underbrace{\sum_{n=1}^{N} x_n \sigma_n \sqrt{\delta t}\, \xi_n}_{\text{Market risk}} + \overbrace{\sum_{n=1}^{N} v_n \cdot \eta_n(v_n)}^{\text{Market}} + \underbrace{V \cdot \alpha T}_{\text{trend.}} \tag{14}$$

S_0

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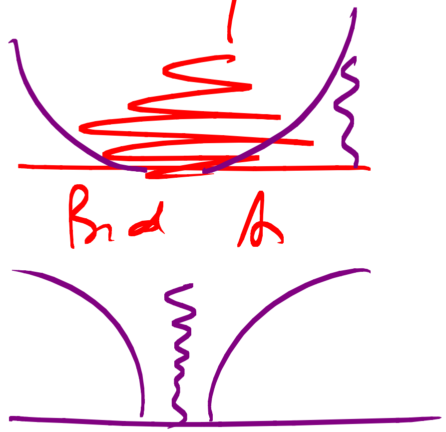
$$v_n = \frac{\frac{v_n}{\rho_n}}{\sum_{k=1}^{N} \frac{v_k}{\rho_k}} \cdot V^{*}$$

$$\psi + F\left(\rho, \frac{v}{V}\right)$$



$$\boxed{\eta_n(v_n) + v_n \cdot \eta'_n(v_n) = \lambda}$$

$$\eta_n(v_n) = \eta \cdot \sigma_n \cdot \frac{v_n}{\bar{V}_n} \quad \Big| \quad v_n = \frac{\lambda \cdot \bar{V}_n}{2\eta \cdot \sigma_n}$$

$$2\eta\sigma_n \frac{v_n}{\bar{V}_n} = \lambda$$

$$\lambda = \frac{V^*}{\sum \frac{\bar{V}_n}{\sigma_n}} \cdot 2\eta$$

$$\sum v_n = \boxed{\frac{\lambda}{2\eta} \sum \frac{\bar{V}_n}{\sigma_n} = V^*}$$

Cost minimization

$$W = \sum x_n \sigma_n \xi_n + \sum v_n \eta_n(v_n)$$

$$\mathbb{E}(W \mid \sigma, V, \psi) = \sum v_n \cdot \eta_n(v_n)$$

$$\sum_{n=1}^{N} v_n = V^*$$

Lagrangian ML.

$$\boxed{\eta_n(v_n) + v_n \cdot \eta_n'(v_n) = \lambda}$$

Cost minimization

$$\boldsymbol{E}(W|\sigma, V, \alpha) = V \cdot S_0 + \sum_{n=1}^{N} v_n \cdot \eta_n(v_n) + V \cdot \alpha\, T$$

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Linear market impact: $\eta(v_n) = \eta \cdot v_n / V_n$

Cost minimization

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Trade regularly, follow the market...

$$\mathbb{E}\left[\sum_{n=1}^{N} x_n \sigma_n \xi_n\right]^2 \quad \mathbb{E}(\xi^2) = 1$$

$$\xi \sim N(0,1)$$

$$\sum_{k \neq k} x_n \cdot x_k \, \sigma_n \cdot \sigma_k \, \xi_n \perp \xi_k$$

Brownian Motion

When $n \neq k$ $\mathbb{E}(\xi_n \cdot \xi_k) = 0$

$$= x_n^2 \sigma_n^2 \quad ②$$

Risk minimization

$$W = \sum \left(x_n \cdot \sigma_n \xi_n + v_n \cdot \eta_n(v_n) \right)$$

$$\mathbb{E}(W) = \cdot$$

$$\mathbb{V}\left(W \middle| \sigma_n, \bar{V}_n, \ldots\right) = \sum_{n=1}^{N} x_n^2 \sigma_n^2$$

$$\mathbb{E}\left[\mathbb{E}(W) - \sum x_n \cdot \sigma_n \xi_n + v_n \cdot \eta_n(v_n) \right]^2$$

$$\text{Min} : x_1 = V, x_2 = 0, \ldots 0$$

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Risk minimization

$$\mathbf{V}(W|\sigma, V, \alpha) = \sum_{n=1}^{N} x_n^2\, \sigma_n^2\, \delta t$$

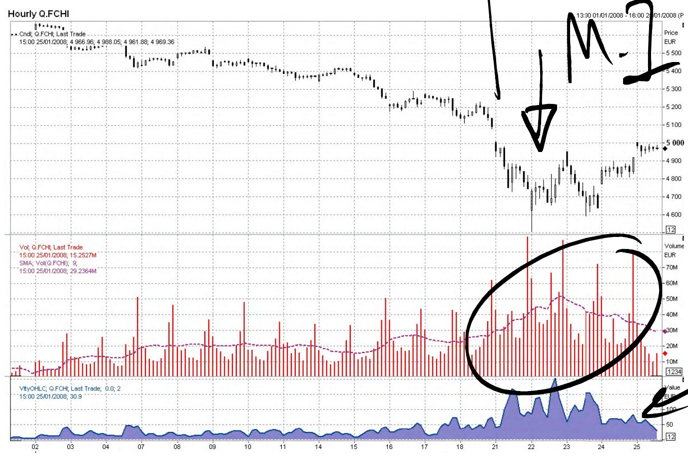
Risk minimization

$$\boxed{\mathbf{V}(W|\sigma, V, \alpha) = \sum_{n=1}^{N} x_n^2 \, \sigma_n^2 \, \delta t}$$

Trade **all** your quantity during the first time interval!

A market impact effect?

M.I



σ

$$\boxed{f^{-}\left(\frac{v}{h}\right) = \frac{1}{\partial_r f_{MI}(v)}} \Rightarrow \psi + k\,\sigma \cdot \left(\frac{v}{V}\right)^{\gamma}$$

$$h''$$

$$\gamma \cdot k \cdot \sigma \cdot \frac{v^{\gamma-1}}{V}$$

$$v = \left(\frac{h-\psi}{k\sigma}\right)^{1/\gamma} \cdot V$$

$$\gamma \cdot k \cdot \sigma \left(\frac{h-\psi}{k\sigma}\right)^{\frac{\gamma-1}{\gamma}}$$

Market Impact main model

deriv → $f^{-}(v) = \frac{1}{\partial_v f_m(v)}$

WB

With ψ the half spread, σ the volatility (in currency), and V a constant homogeneous with a quantity of shares.
The most common model is:

$$\boxed{f_m(v) = \psi + \kappa\sigma\cdot\left(\frac{v}{V}\right)^{\gamma}} \quad (15)$$

$$\alpha\cdot\psi + k\cdot\boxed{\sigma^{m}}\left(\frac{v}{V}\right)^{\beta}$$

This M.I. model put inside equation (3) gives:

$$f_{\text{LOB}}^{-}\left(S_0+\psi+\kappa\,\sigma\left(\frac{v}{V}\right)^{\gamma}\right)\cdot\kappa\sigma\gamma\frac{v^{\gamma-1}}{V^{\gamma}}=1$$

This M.I. model put inside equation (3) gives:

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using $\boxed{h = \cancel{\psi +}\, \kappa\sigma\,(v/V)^{\gamma}}$; i.e. $v = V\left((h - \phi/2)/\kappa\sigma\right)^{1/\gamma}$

$MI(v) = \psi + k \cdot \sigma \cdot \left(\frac{v}{V}\right)^{\gamma}$

This M.I. model put inside equation (3) gives:

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using $h = \psi + \kappa\sigma\,(v/V)^{\gamma}$; i.e. $\boxed{v = V\,((h-\phi/2)/\kappa\sigma)^{1/\gamma}}$
we obtain:

$$(16) \qquad \boxed{f_c(S_0 + h) = \frac{V}{\kappa\sigma\gamma\left(\frac{h-\psi}{\kappa\sigma}\right)^{\frac{\gamma-1}{\gamma}}}}$$

LOB

$+ \, 0$

Equation (16) implies that $\gamma \leq 1$, besides:

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A square root market impact

A square root market impact implies a linear LOB.

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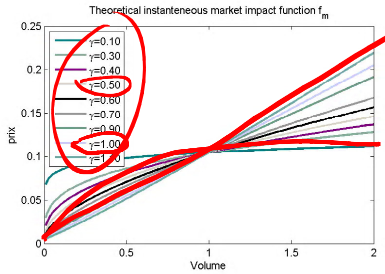
A square root market impact

A square root market impact implies a linear LOB.

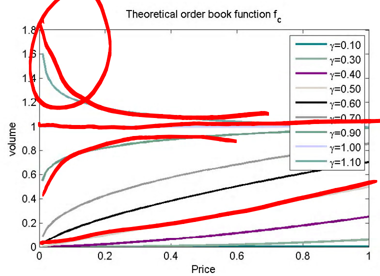
With $\gamma = 1/2$, we obtain

$$f_c(S_0 + h) = 2(h - \psi)V$$



$MI = f(\gamma)$

LOB density

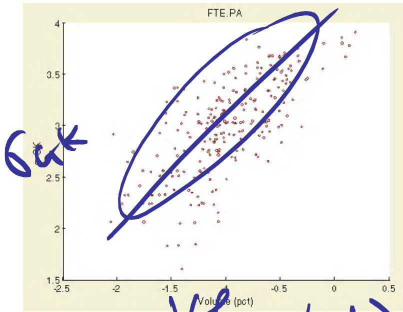


$\gamma = \frac{1}{2}$

Some evidences: FTE.PA 20/11/2007

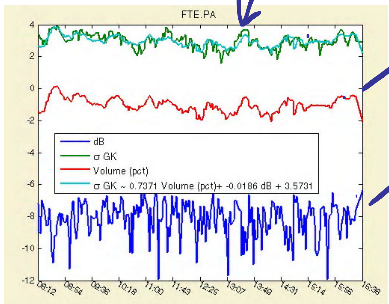
one day

σ_{GK}



σ_{GK}

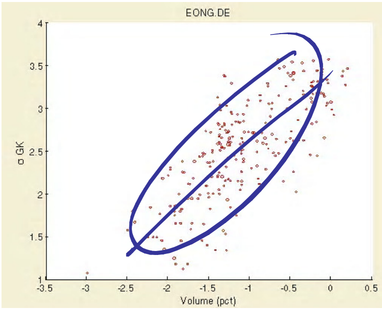
Volume(pct)

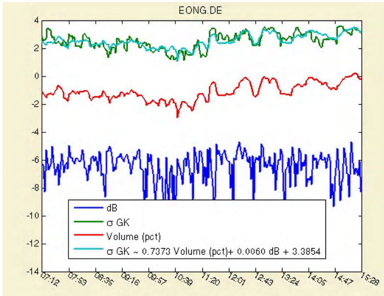


%V.

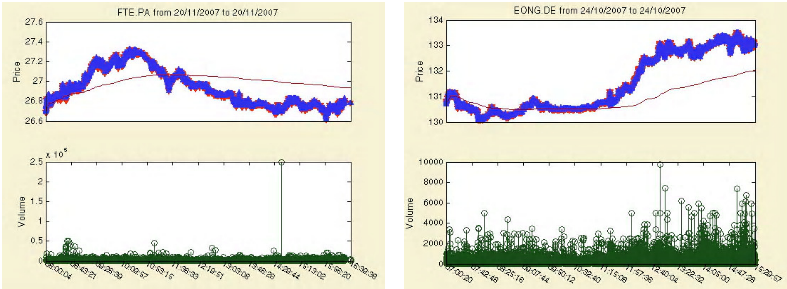
ε

Some evidences: EONG.DE 24/10/2007





FTE and EOND trading data



$$f_{LOB}^{-1} = \frac{1}{\partial_v \, f_{n}(v)} = \psi + \beta \,.$$

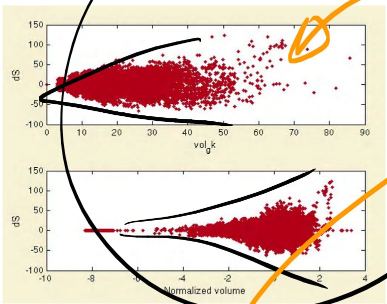
6 months 30 sec by 30 sec data for one stock

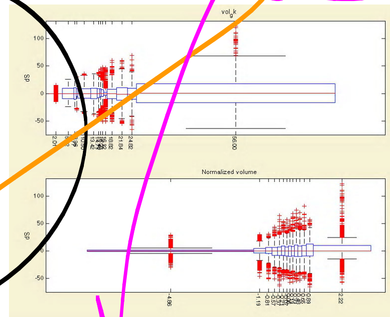
$$M.I(v) = k.\sigma^{\alpha}\left(\frac{v}{V}\right)^{\gamma}$$

ρ_v

σ

MI





$$\left(\frac{v}{V}\right)$$

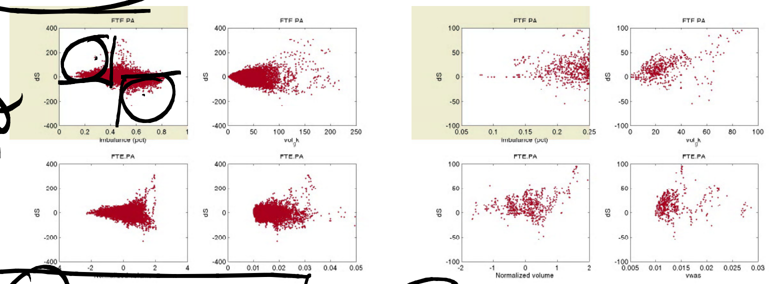
$$\log MI(v) = \log k + \alpha \log \sigma + \gamma . \log \rho_v$$

CA CHEUVREUX

Standard pitfalls

20 min

- Do not use naively absolute value on δS (have a look at imbalance)



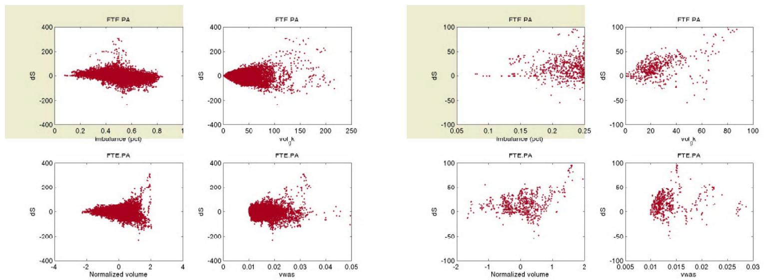
20% more than usual

$\simeq 1,2$

20%

Standard pitfalls

- Do not use naively absolute value on δS (have a look at imbalance)



- Really think about choosing the right ΔT

Simplest case: constant volatility and volumes

Simplest case: constant volatility and volumes

$$W = \sum x_n \cdot \sigma_n \cdot \xi_n + \sum v_n \cdot \eta_n(v_n)$$

$\mathbb{E}(\,|\sigma_n\,\cdot\,)$

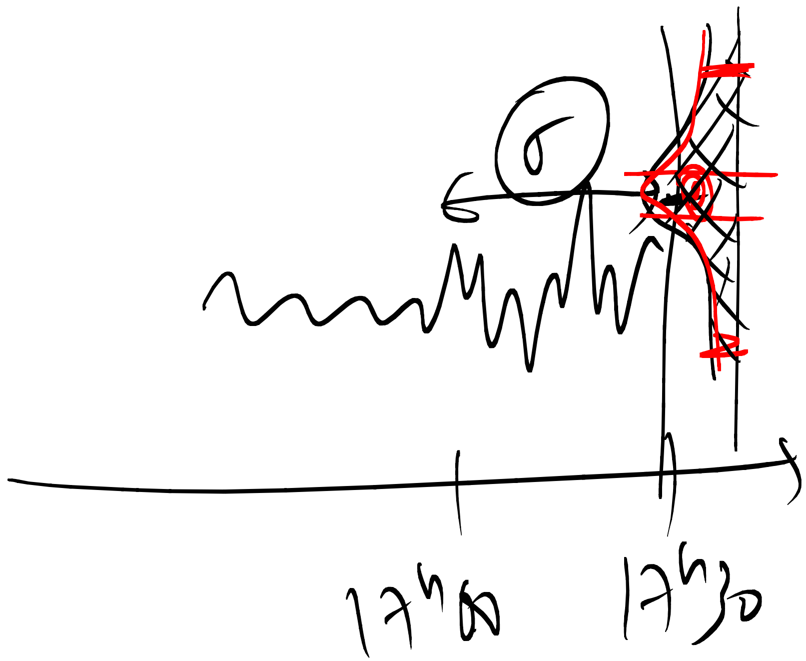
$$\boldsymbol{E}(W|\sigma, V, \alpha) = V \cdot S_0 + \sum_{n=1}^{N} v_n \cdot \eta_n(v_n) + V \cdot \alpha\, T$$

$\mathbb{V}(\,|\,\cdot)$

$$\mathbf{V}(W|\sigma, V, \alpha) = \sum_{n=1}^{N} x_n^2\, \sigma_n^2\, \delta t$$

$$\min \mathbb{E}(\cdot) + \lambda\, \mathbb{V}(\cdot)$$

risk aversion



17h00
17h30

Simplest case: constant volatility and volumes

$$\boldsymbol{E}(W|\sigma, V, \alpha) = V \cdot S_0 + \sum_{n=1}^{N} v_n \cdot \eta_n(v_n) + V \cdot \alpha\, T$$

$$\mathbf{V}(W|\sigma, V, \alpha) = \sum_{n=1}^{N} x_n^2\, \sigma_n^2\, \delta t$$

We will minimise (read Almgren, Chriss)

$$\mathcal{C}_1(v) = \boldsymbol{E}(W|\sigma, V, \alpha) + \boldsymbol{\lambda} \cdot \mathbf{V}(W|\sigma, V, \alpha) \tag{17}$$

for a *Buy* order.

$$2\eta(x_n - x_{n+1})\frac{\sigma_n}{V_n} - 2\eta(x_{n-1} - x_n)\frac{\sigma_{n-1}}{V_{n-1}} + 2\lambda x_n \sigma_n^2 = 0$$

linear

$$x_{n+1}\left(\cancel{2}\eta\frac{\sigma_n}{V_n}\right) = \left(\cancel{2}\eta\left[\frac{\sigma_n}{V_n} + \frac{\sigma_{n-1}}{V_{n-1}}\right] + 2\lambda\sigma_n^2\right) x_n - \cancel{2}\eta\frac{\sigma_{n-1}}{V_{n-1}} x_{n-1}$$

$$\frac{V_t}{\sigma}$$

$$x_{n+1} = \left(\frac{\lambda}{\eta}\sigma_n^2\frac{V_n}{\sigma_n} + 1 + \frac{\sigma_{n-1}}{\sigma_n}\cdot\frac{V_n}{V_{n-1}}\right)x_n - \frac{\lambda}{\eta}\frac{\sigma_{n-1}}{\sigma_n}\cdot\frac{V_n}{V_{n-1}}\cdot x_{n-1}$$

$$\mathbb{E}(\cdot) = \sum \eta_n(v_n) \cdot v_n \quad , \quad \eta_n(v_n) = \sigma_n \frac{v_n}{V_n} \cdot \eta$$

$$\mathbb{V}(\cdot) = \sum x_n^2 \sigma_n^2 \quad \Big| \quad v_n = x_n - x_{n+1}$$

$$\sum \eta (x_n - x_{n+1})^2 \cdot \frac{\sigma_n}{V_n} + \lambda \sum x_n^2 \sigma_n^2 ,$$

$$\uparrow \text{"}n+1\text{"} = n$$

$$\partial_n(\,) = 2\eta\,(x_n - x_{n+1}) \frac{\sigma_n}{V_n} - 2\eta (x_{n-1} - x_n) \frac{\sigma_{n-1}}{V_{n-1}}$$

$$+ 2\lambda x_n \sigma_n^2 = 0$$

Simplest case: constant volatility and volumes

$\mathbb{E}(W|\sigma, V, \alpha)$

σ

$$\boldsymbol{E}(W|\sigma, V, \alpha) = V \cdot S_0 + \sum_{n=1}^{N} v_n \cdot \eta_n(v_n) + V \cdot \alpha\, T$$

$$\mathbf{V}(W|\sigma, V, \alpha) = \sum_{n=1}^{N} x_n^2\, \sigma_n^2\, \delta t$$

We will minimise (read Almgren, Chriss)

$$\boxed{\mathcal{C}_1(v) = \boldsymbol{E}(W|\sigma, V, \alpha) + \lambda \cdot \mathbf{V}(W|\sigma, V, \alpha)} \tag{17}$$

for a *Buy* order.
λ is a risk aversion parameter.

Solution for constant volatility and volumes

No need for Lagrangian multipliers (thanks to the $v \longrightarrow x$ change of variable)

(18)

$$x_{n+1} = \left(1 + \frac{\sigma_{n-1}}{\sigma_n} \cdot \frac{V_n}{V_{n-1}} + \lambda \sigma_n V_n \, \delta t\right) x_n - \frac{\sigma_{n-1}}{\sigma_n} \cdot \frac{V_n}{V_{n-1}} \cdot x_{n-1}$$

$x_{n+1} = a x_n + b x_{n-1}$

$X^2 - aX + b = 0$, $X_{+/-}$

$$\boxed{x_n = \underline{A} X_+^n + \underline{B} X_-^n}$$

$\boxed{2 \text{ point}}$

$x_0 = V^*$

$x_N = 0$

Solution for constant volatility and volumes

No need for Lagrangian multipliers (thanks to the $v \longrightarrow x$ change of variable)

(18)

$$x_{n+1} = \left(1 + \frac{\sigma_{n-1}}{\sigma_n} \cdot \frac{V_n}{V_{n-1}} + \underset{\eta}{\lambda} \sigma_n V_n \, \delta t\right) x_n - \frac{\sigma_{n-1}}{\sigma_n} \cdot \frac{V_n}{V_{n-1}} \cdot x_{n-1}$$

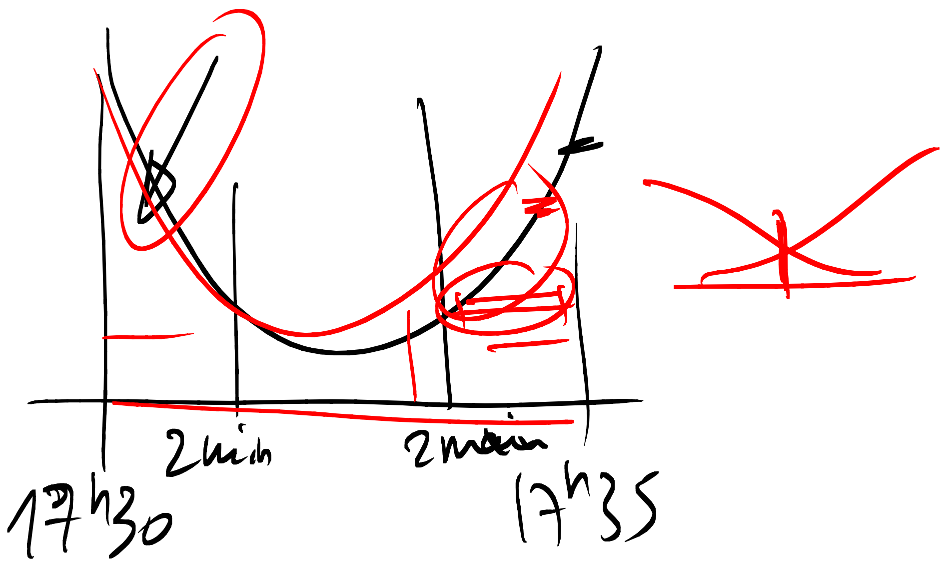
And when $\sigma_n \equiv \sigma$, $V_n \equiv \bar{V}$, we obtain this linear equation:

(19)

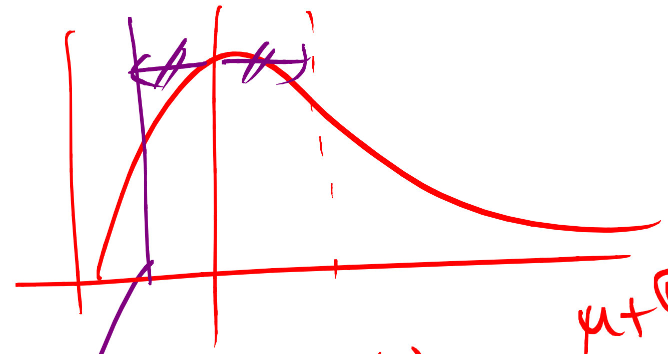
$$\boxed{x_{n+1} = (2 + \underset{\eta}{\lambda} \sigma \bar{V} \, \delta t) \cdot x_n - x_{n-1}}$$

its "*initial conditions*" are $x_1 = V$, $x_{N+1} = 0$

CHEUVREUX


2min
2min
17h30
17h35

$$MI(\psi) = \psi + K \sigma \left(\frac{M}{V}\right)^{\gamma}$$



e^{μ}

$$\mathbb{E}(V) = e^{\mu+\frac{\sigma^2}{2}}$$

$$\mathbb{E}\left(\frac{1}{V}\right) = e^{\mu-\frac{\sigma^2}{2}} \neq e^{-\mu}$$

$$\neq e^{-\mu+\frac{\sigma^2}{2}}$$

$$E\left(\frac{1}{Y}\right) = \int \frac{1}{\sqrt{2\pi}\,\sigma} e^{\underbrace{-y - \frac{(y-\mu)^2}{2\sigma^2}}} dy = e^{\mu - \frac{\sigma^2}{2}}$$

$$-\frac{1}{2\sigma^2}\left[\underbrace{2\sigma^2 y + y^2 - 2\mu y + \mu^2}\right]$$

$$\left[\underbrace{y - (\mu - \sigma^2)}\right]^2 + \underbrace{\mu^2 - (\mu - \sigma^2)^2}$$

$$-\sigma^4 + 2\mu\sigma^2$$

$$E\left(\frac{1}{Y}\right) = \exp\left(-\frac{\sigma^4 - 2\mu\sigma^2}{2\sigma^2}\right) = -\left(\frac{\sigma^2}{2} - \mu\right)$$

$$V_n \sim \log N(\mu, \sigma)$$

$$\mathbb{E}\left(\frac{1}{V_n}\right) = \int_0^{\infty} \frac{1}{x^2\sqrt{2\pi}\,\sigma} \cdot e^{-\frac{(\ln x - \mu)^2}{2\sigma^2}} dx$$

$$y = \ln x \Rightarrow dy = \frac{dx}{x}$$

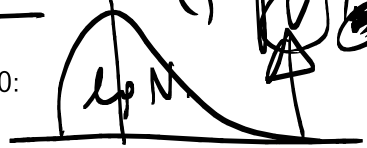
$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-y} e^{-\frac{(y-\mu)^2}{2\sigma^2}} dy$$

Simple case: random volumes

$$\mathbb{E}\left(\eta\sigma\sqrt{\frac{v}{V}}\,\Big|\,\sigma\right) = \eta\sigma\sqrt{\frac{v}{V}}$$

median

$$W' = W - V \cdot S_0 \text{ and } \alpha \equiv 0:$$



$$\mathbb{E}(\cdot) = \mathbb{E}\Big(\sum x_n \sigma_n \xi_n + \sigma_n \cdot \underbrace{\eta_n(v_n)}_{\eta\sigma\cdot\frac{v_n}{V_n}}\Big)$$

$$= \sum v_n^2\, \eta \cdot \sigma_n\, \mathbb{E}\left(\frac{1}{V_n}\right) = \frac{1}{V_n}$$

$$\mathbb{E}\left(\sum_{n=1}^{N} \eta\, \sigma_n v_n^2 \left(\frac{1}{V_n} - \frac{1}{\bar{V}_n}\right)\right)^2$$

$$\sum_{\substack{n=1\\k}}^{N} \eta\, \sigma_n \sigma_k v_n^2 v_k^2 \left(\frac{1}{V_n} - \frac{1}{\bar{V}_n}\right)\left(\frac{1}{V_k} - \frac{1}{\bar{V}_k}\right)$$

$$\mathbb{E}\left(f(V_k) \cdot f(V_n)\right) \quad k \neq n$$

$$W = \sum x_n \sigma_n \xi_n + \eta \cdot \sigma_n \frac{\sigma_n^2}{V_n}$$

$$\mathbb{E}(W) = \eta \sum \sigma_n v_n^2 \frac{1}{\overline{V_n}} \qquad \frac{1}{V_n} \prod_{k \neq n} \frac{1}{V_k} \qquad 2$$

$$\mathbb{V}(W) = \mathbb{E}\left(\sum x_n \sigma_n \xi_n + \eta \sigma_n v_n^2 \left(\frac{1}{V_n} - \frac{1}{\overline{V}} \right) \right)^2$$

$$= \mathbb{E}\left(\quad \right)^2 + \quad$$

$$= \sum x_n^2 \sigma_n^2 + \sum \eta^2 \sigma_n^2 v_n^4 \mathbb{V}\left(\frac{1}{V_n} \right) + \cancel{0}$$

Simple case: random volumes

$\bar{V}_n := \mathbb{E}\left(\frac{1}{V_n}\right)^{-1}$

$W' = W - V \cdot S_0$ and $\alpha \equiv 0$:

$$\mathbf{E}(W|\sigma, \alpha = 0) = \eta \sum_{n=1}^{N} (x_n - x_{n+1})^2 \sigma_n \mathbf{E}\left(\frac{1}{V_n}\right) \tag{20}$$

What about $\mathbf{E}(1/V_n)$?

$\mathbb{V}(W|\sigma) =$

Simple case: random volumes

No close Formula! V_n Random

$W' = W - V \cdot S_0$ and $\alpha \equiv 0$:

$$\mathbf{E}(W|\sigma,\alpha=0) = \eta \sum_{n=1}^{N} (x_n - x_{n+1})^2 \sigma_n \mathbf{E}\left(\frac{1}{V_n}\right) \tag{20}$$

What about $\mathbf{E}(1/V_n)$? If $V_n \perp V_k$ when $n \neq k$

$$\mathbf{V}(W|\sigma,\alpha=0) = \sum_{n=1}^{N} x_n^2 \sigma_n^2 \delta t + \eta^2 \sum_{n=1}^{N} (x_n - x_{n+1})^4 \sigma_n^2 \cdot \mathbf{V}\left(\frac{1}{V_n}\right)$$

else...

$C_2(V)$ "$\sum \eta (x_n - x_{n+1})^2 \sigma_n \frac{1}{V_n} + \lambda \sum x_n^2 \sigma_n^2 + \lambda \eta^2 \sum (x_t - x_{t+1})^4 \sigma_n^2 \frac{1}{V_n}$"

$$\frac{\partial(x_{..})}{\partial x_n} = 2\eta a\left[x_n - x_{n+1} - (x_{n-1} - x_n)\right] + \lambda 2 x_n \sigma_n^2 + 4\eta^2 \nabla(a)\left[(x_n - x_{n+1})^3 - (x_{n-1} - x_n)^3\right] = 0$$

median σ_n^2 $\nabla(a)$

x_{n-1}

x_n

$\downarrow$

X

x_{n+1}

$x_0 = V$

$x_1 = 0$

$\vdots$

$x_N = 0$

$$\eta \cdot a\left[2x_n - x_{n-1} - X\right] + \lambda x_n \sigma_n^2 + 4\eta^2 \nabla(a)\left[(x_n - X)^3 + (x_{n-1} - X)^3\right] = 0$$

$$C_3(v_{..}) = \sum \eta\, a\, v_n^2 + \lambda \cdot \left[\sum x_n^2 \sigma_n^2 + v_n^4\, \eta^2\, \nabla(a) \right]$$

$x_n - x_{n+1}$

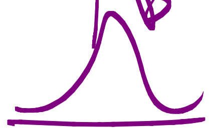
$$C_3(x_{..}) = \sum \eta \cdot a\, (x_n - x_{n+1})^2 + \lambda \left[\sum x_n^2 \sigma_n^2 + \eta^2 (x_n - x_{n+1})^4\, \nabla(a) \right]$$

$${}_n\partial(x_{..}) = 2\eta\, a \left[x_n - x_{n+1} - (x_{n-1} - x_n) \right] + \lambda\, 2\, x_n \sigma_n^2 + 4\eta^2\, \nabla(a) \left[(x_n - x_{n+1})^3 - (x_{n-1} - x_n)^3 \right]$$

$$\boxed{W = \sum x_n \sigma_n \xi_n + \eta \cdot v_n^2 \cdot a}$$

$$\mathbb{E}(W) = a\eta \cdot v_n^2$$

$$\mathbb{V}(W|a) = \sum x_n^2 \sigma_n^2$$



Bootstrap

$$\mathbb{V}(W) = \sum x_n^2 \sigma_n^2 + \sum \eta^2 v_n^4 \underbrace{\mathbb{V}(a)}_{n}$$

$$W = \sum x_n \sigma_n \xi_n + v_n \cdot \underbrace{\eta_n(v_n)}_{k \cdot \sigma_n \frac{v_n}{V_n} \cdot v_n}$$

$$\boxed{V_n = a \cdot \sigma_n}$$

No more usual Volume !!

$$\Rightarrow W = \sum x_n \sigma_n \xi_n + k \cdot a \cdot v_n^2$$

Useful case: stochastic volatility

Useful case: stochastic volatility

$$\frac{\sigma_n}{V_n} \rightsquigarrow \frac{\sigma_n}{\overline{V_n}} \rightsquigarrow \mathbb{E}\left(\frac{\sigma_n}{V_n}\right)$$

$$\boldsymbol{E}(W|\cancel{\sigma}, \alpha = 0) = \eta \sum_{n=1}^{N} (x_n - x_{n+1})^2 \boldsymbol{E}\left(\frac{\sigma_n}{V_n}\right) \tag{21}$$

Independence between σ_n and V_n can simplify things...

$$\frac{1}{\overline{V_n}} = \mathbb{E}\left(\frac{1}{V_n}\right)$$

Useful case: stochastic volatility

$$\boldsymbol{E}(W|\sigma,\alpha=0)=\eta\sum_{n=1}^{N}(x_n-x_{n+1})^2\boldsymbol{E}\left(\frac{\sigma_n}{V_n}\right) \tag{21}$$

Independence between σ_n and V_n can simplify things...

$$\mathbf{V}(W|\alpha=0)=\sum_{n=1}^{N}x_n^2\underbrace{\boldsymbol{E}(\sigma_n^2)}_{\int_{n\,\delta t}^{(n+1)\,\delta t}\sigma_t^2\,dt}\delta t+\eta^2\sum_{n=1}^{N}(x_n-x_{n+1})^4\cdot\mathbf{V}\left(\frac{\sigma_n^2}{V_n}\right)$$

Towards stochastic control

Lions & 70's Bensoussan

$u_t = \min \mathbb{E}(y(t \to T))$

Let use a continuous framework (read Almgren and Lorenz); v rate of buying: $dx = -v . dt$

$$
(22) \qquad \begin{cases} dx = -v\,dt & \text{remaining shares} \\ dy = (s + \eta v)\,v\,dt & \text{spent money} \\ ds = \alpha\,dt + \sigma\,dW & \text{stock price} \end{cases}
$$

$dy = (s + \eta \cdot v) \cdot v \cdot dt$ $\qquad$ $ds = \alpha\,dt + \sigma dW$

We want to minimize $\boxed{\mathbf{E}(y(T))}$ so that $x(T) = 0$, let u be the *value function of the solution* (min from t). $t \to T$

$$
d\begin{pmatrix} x \\ y \\ s \end{pmatrix} = \boxed{\begin{pmatrix} -v \\ (s+\eta \cdot v)v \\ \alpha \end{pmatrix}} dt + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma \end{bmatrix} dW
$$

$$\boxed{\min_{v}\left(-v\cdot\partial_x u + v(s+\eta\cdot v)\partial_y u\right)}$$

$$-\partial_x u + s\,\partial_y u + 2v^{*}\cdot\eta\cdot\partial_y u = 0$$

$$\boxed{v^{*} = \frac{\partial_x u - s\cdot\partial_y u}{2\eta\,\partial_y u}}$$

u^{*}

$=$

$\partial_t u = \frac{\partial u}{\partial t}$

Hamilton-Jacobi-Bellman:

∂

$\frac{\partial^2 u}{\partial s \partial s}$

$$\partial_t u + \min_v \{-v\partial_x u + v(s + \eta v)\partial_y u + \boxed{\alpha \partial_s u + \frac{1}{2}\sigma^2 \partial_s^2 u}\} = 0 \quad (23)$$

$$\partial_t u + \min_{v} \left(\left\langle \begin{pmatrix} -v \\ v(s+\eta \cdot v) \\ \alpha \end{pmatrix}, \begin{pmatrix} \partial_x u \\ \partial_y u \\ \partial_s u \end{pmatrix} \right\rangle + \frac{1}{2}\sigma^2 \partial_s^2 u \right) = 0$$

$$\partial_t u + \alpha \partial_s u + \frac{1}{2}\sigma^2 \partial_s^2 u + \boxed{\min_{v} \left(-v \cdot \partial_x u + v(s+\eta \cdot v)\partial_y u \right)} = 0$$

$$\partial_t u + \alpha \cdot \partial_s u + \frac{1}{2}\sigma^2 \cdot \partial_s^2 u - v^* \cdot \partial_x u + v^*(s + \eta \cdot v^*) \cdot \partial_y u = 0$$

u^*

Hamilton-Jacobi-Bellman:

$$\partial_t u + \min_v \{-v\partial_x u + v(s+\eta v)\partial_y u + \alpha \partial_s u + \frac{1}{2}\sigma^2 \partial_s^2 u\} = 0 \tag{23}$$

Solve

$$\min_u \{\eta v^2 \partial_y u + v(s\partial_y u - \partial_x u)\}$$

Hamilton-Jacobi-Bellman:

$$\partial_t u + \min_v \{-v\partial_x u + v(s+\eta v)\partial_y u + \alpha\partial_s u + \frac{1}{2}\sigma^2\partial_s^2 u\} = 0 \tag{23}$$

Solve

$$\min_u \{\eta v^2 \partial_y u + v(s\partial_y u - \partial_x u)\}$$

$$v^* = \frac{s\partial_y u - \partial_x u}{2\eta\partial_y u} \tag{24}$$

replace v^* into equation (23), and solve in u. Then replace in (24)... done.

Hamilton-Jacobi-Bellman: $u = \min \mathbb{E}\, Y(\tau)$ $\boxed{t \to T}$

$$(23) \quad \partial_t u + \underbrace{\min_v \{-v\partial_x u + v(s + \eta v)\partial_y u + \alpha \partial_s u + \frac{1}{2}\sigma^2 \partial_s^2 u\}} = 0$$

Solve

$$\min_u \{\eta v^2 \partial_y u + v(s\partial_y u - \partial_x u)\}$$

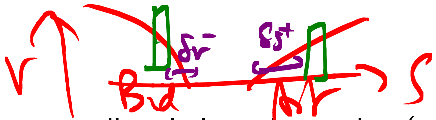
$$(24) \quad v^* = \frac{s\partial_y u - \partial_x u}{2\eta \partial_y u}$$

replace v^* into equation (23), and solve in u. Then replace in (24)... done.

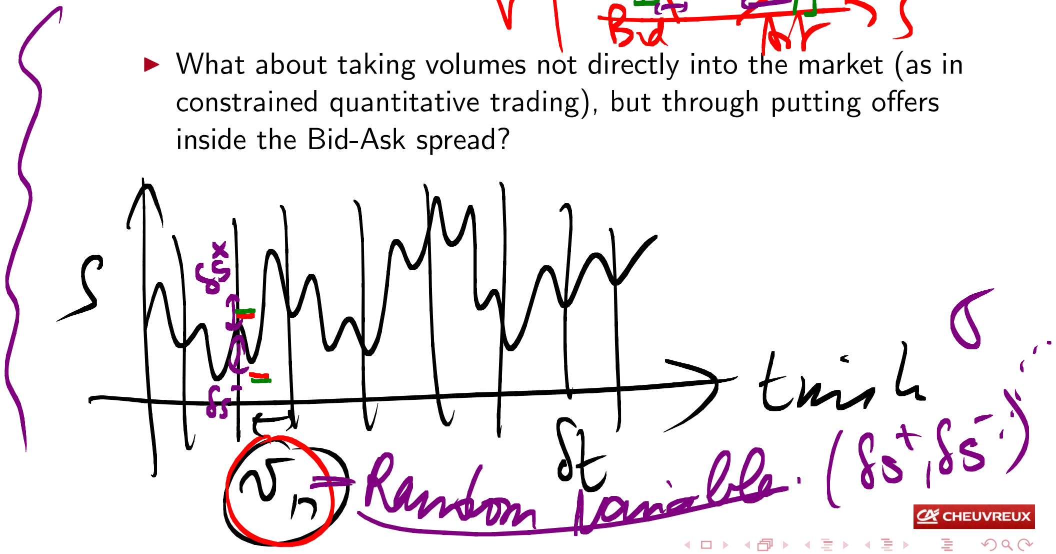
"The big secret...Quantitative finance is one of the easiest branches of mathematics"

Paul Wilmott on his blog, the 1st of April 2008...

Market making



- What about taking volumes not directly into the market (as in constrained quantitative trading), but through putting offers inside the Bid-Ask spread?



Market making

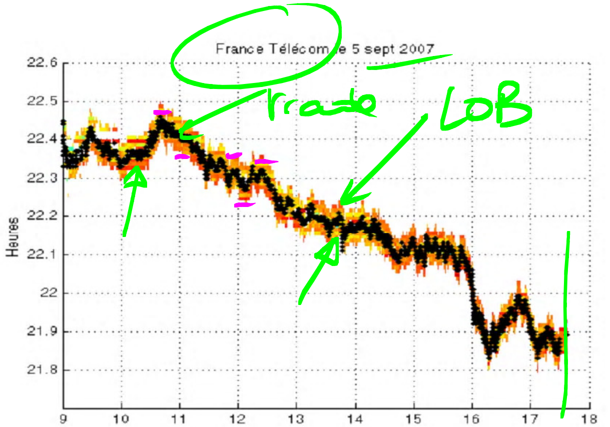
- What about taking volumes not directly into the market (as in constrained quantitative trading), but through putting offers inside the Bid-Ask spread?
- At time $n\,\delta t$ the half spread is ψ, the mid price is $\tilde{S}$, the LOB is known, you decide to offer quantities at prices $\tilde{S} - \delta s^{-}$ and $\tilde{S} - \delta s^{+}$

Market making

- What about taking volumes not directly into the market (as in constrained quantitative trading), but through putting offers inside the Bid-Ask spread?
- At time $n\,\delta t$ the half spread is ψ, the mid price is $\tilde{S}$, the LOB is known, you decide to offer quantities at prices $\tilde{S} - \delta s^-$ and $\tilde{S} - \delta s^+$
- You "*control*" your trading not by quantities but by limit prices $\delta s^\pm$
- The usual v_n are now Random variables conditioned by the distance between your offer and the market LOB
- More probability theory... less closed form formula...

LOB data

S



9h

17h30

Portfolio trading

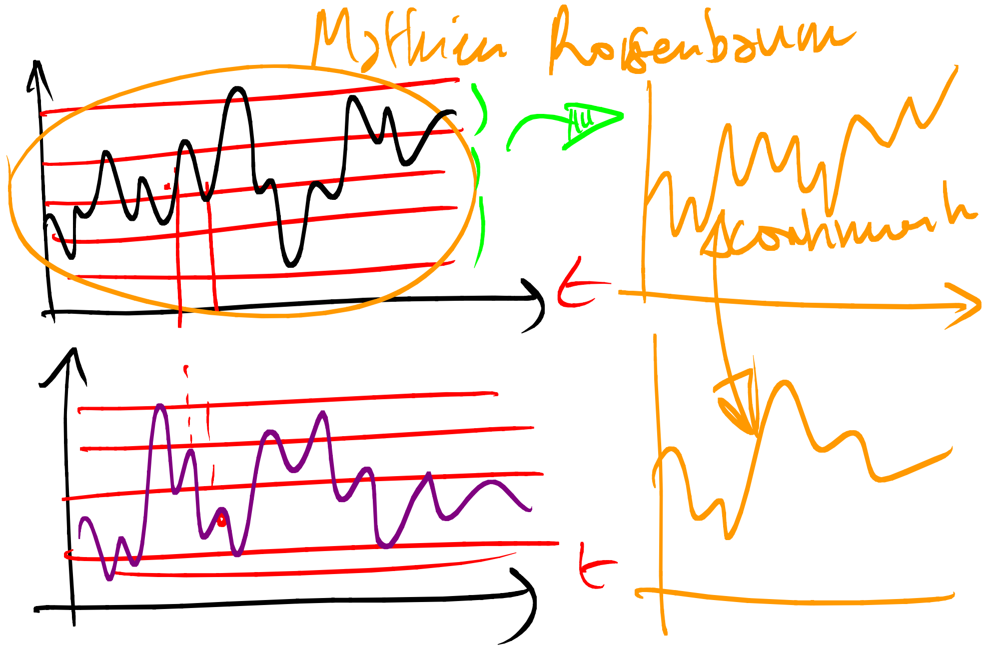
Equation (12) is now

(25) $$S_{n+1}^{(k)} = S_n^{(k)} + \alpha^{(k)}\,\delta t + \sigma_{n+1}^{(k)}\sqrt{\delta t}\,\xi_{n+1}^{(k)}$$

$V_n^{(k)}$

$\sigma_{n+1}^{(k')}$

$V_n^{(k')}$


Matthias Rosenbaum
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Portfolio trading

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- Innovations $\xi^{(k)}$ are correlated, volume, trends and volatility are also correlated...
- Seems more difficult (correlations are difficult to estimate on financial data)

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- Seems more difficult (correlations are difficult to estimate on financial data)
- Similarly to volatility, how can we estimate tick-by-tick correlations? What about synchronisation of high freq datasets ? (see Zhang, Hayashi, Yoshida)
- But Central Limit Theorem should help...

Portfolio trading

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$$S_{n+1}^{(k)} = S_n^{(k)} + \alpha^{(k)}\,\delta t + \sigma_{n+1}^{(k)}\sqrt{\delta t}\,\xi_{n+1}^{(k)} \tag{25}$$

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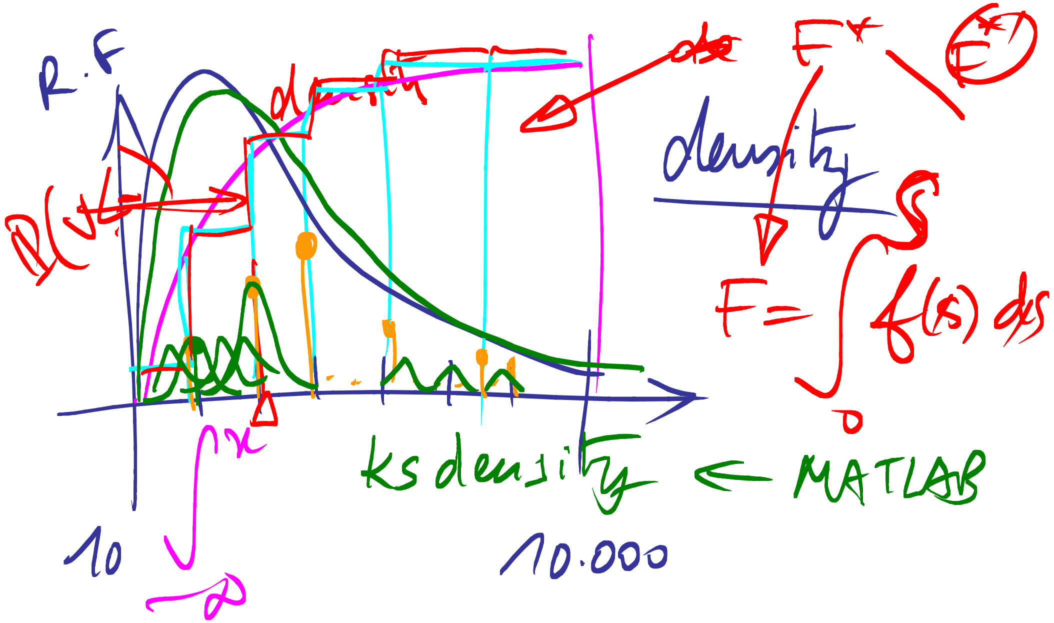
- But Central Limit Theorem should help...

Anyway multi stocks view is very useful: Market Impact can be seen as a relative move between two usually highly correlated stocks.


R.F
density
F = ∫ f(s) ds
ksdensity ← MATLAB
10
10.000

Implicit market impact models

2 dim
3 ...

- Based on non parametric statistics
- Kernel / local methods (read Devroye, Lugosi)
  Let $K(x)$ a kernel function (symmetric positive measure with its max at zero), we know that:

$$f^*(x) = \frac{1}{N \cdot h(N)} \sum_{n=1}^{N} K\left(\frac{x - X_n}{h(N)}\right) \tag{26}$$

  is a good (non parametric) estimate of the density of the law of $(X_n)_{n \in \{1,\dots,N\}}$; in local regressions, such kernels are used to weight observations

Implicit market impact models

$\psi + k \cdot \sigma \left(\frac{v}{V}\right)^{\gamma}$

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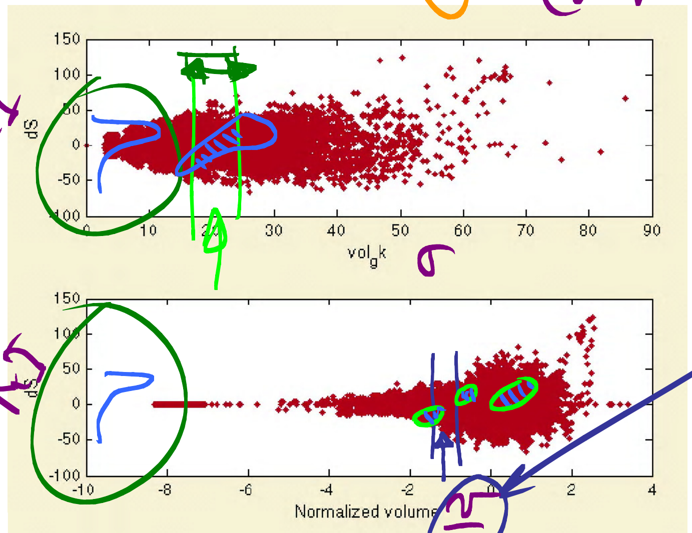
  is a good (non parametric) estimate of the density of the law of $(X_n)_{n\in\{1,\ldots,N\}}$; in local regressions, such kernels are used to weight observations
- It is like "*smoothly*" look around similar market conditions in the past, going back to an estimation framework:

$$\boxed{\mathrm{MI}(v) = \boldsymbol{E}\,(\delta S \,|\, v, \text{ market conditions})}$$

Direct use of such models

$MI = K \cdot \sigma \left(\frac{v}{V}\right)^{\gamma}$ parameters



Interpretation of risk aversion

- Equation (17) involves a risk aversion parameter λ.
- What about the risk aversion parameter of asset allocation theory?

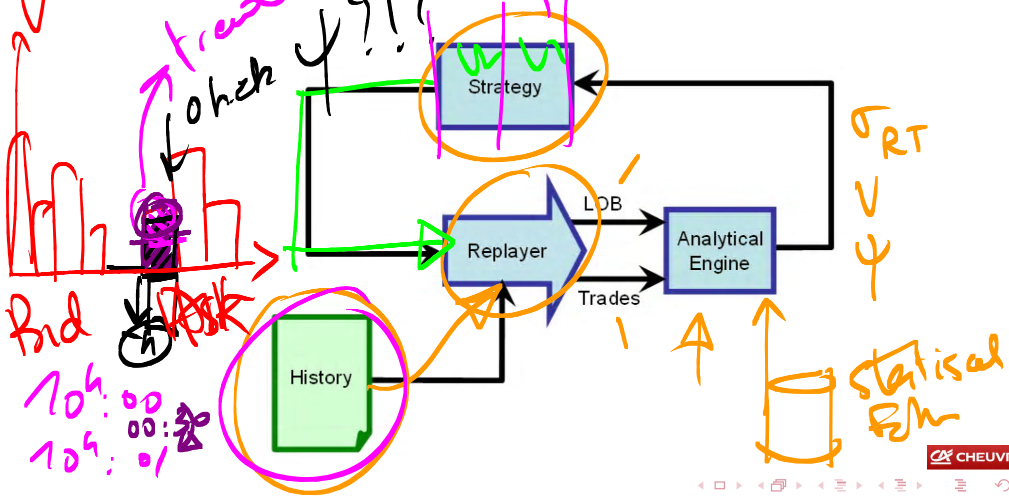
Interpretation of risk aversion

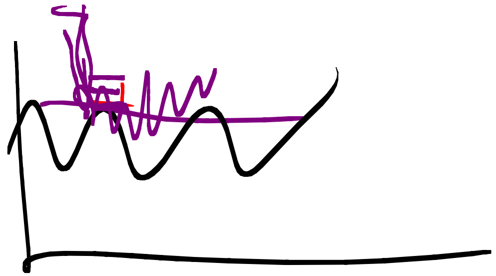
- Equation (17) involves a risk aversion parameter λ.
- What about the risk aversion parameter of asset allocation theory?
- For portfolio trading, this is a real question
- Will you use the same risk aversion parameter in you Markowitz / Black-Litterman allocation and your trading strategy?

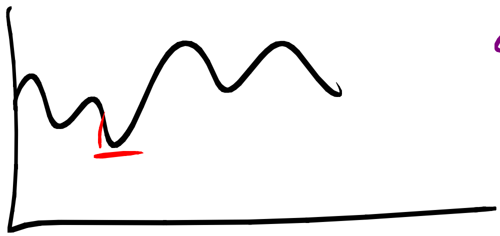
(read Engle, Ferstenberg)

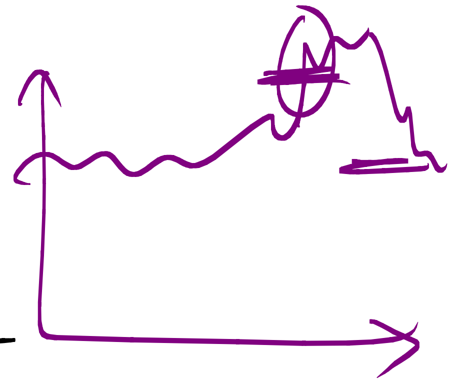
Focus on back tests

"Naive" backtest framework









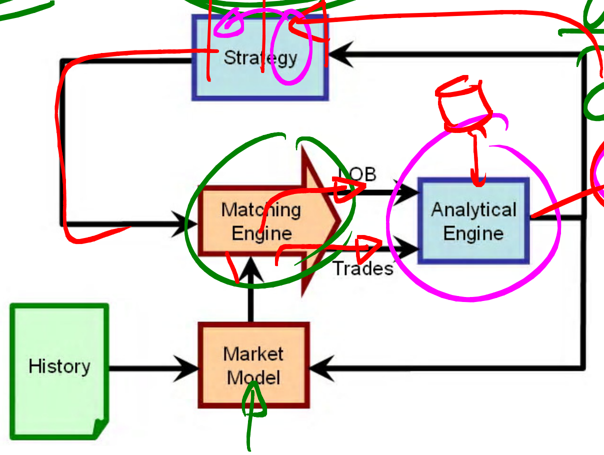
Focus on back tests

$\frac{\partial \text{Payoff}}{\partial \psi}$

"*Really*" backtest framework

$\frac{\partial \text{Payoff}}{\partial \psi} \to \nu$

$\frac{\partial \Pi}{\partial \psi}$



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Conclusion (1/2)

- Intra day trading is affected by
  - Intra day historical invariance (volumes, volatility, spread nomograms)
  - Dependencies / correlations between those variables
  - Price discretisation (bid-ask bounce can be greater that volatility)

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  - Intra day historical invariance (volumes, volatility, spread nomograms)
  - Dependencies / correlations between those variables
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- We have seen how such effects can be captured using (not straight forward) statistics
  - Take into account the tick size (volatility and spread)
  - Identify trading contexts to build dedicated intra day nomograms (Friday effect)
  - Do not forget to look at auto correlations

Conclusion (1/2)

- There is a quantitative framework to deal with all those effects
  - Optimise to find the balance between two main effects: Market Impact demands to trade slowly and Market Risk demands to trade fast
  - Be sure to have accurate estimates for the parameters of your equations

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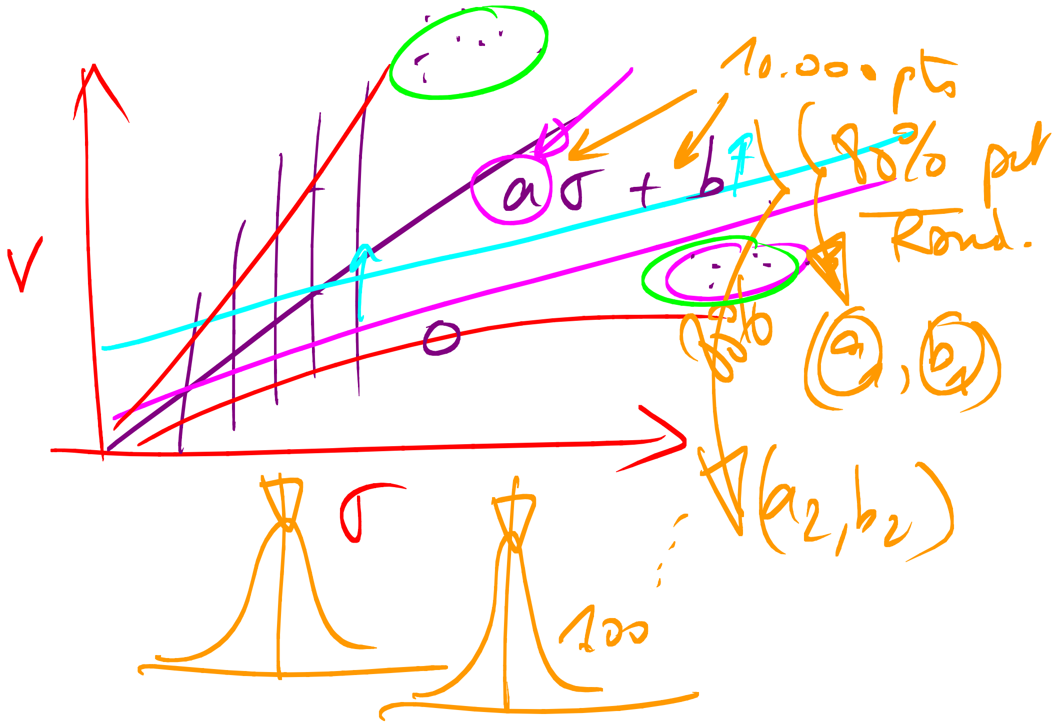

V
σ
aσ + b
10.000 pts
(80% pt Rand.
(a1, b1)
(a2, b2)
100

Conclusion

- Intra day trading is affected by micro structure effects
- We have seen how such effects can be captured using (not straight forward) statistics
- There is a quantitative framework to deal with all those effects