CQF Exercises 4.2

dX is the usual increment of Brownian motion

1. The bond pricing equation, derived is given by

$$\frac{\partial V}{\partial t} + \frac{1}{2} w^2 \frac{\partial^2 V}{\partial r^2} + (u - \lambda w) \frac{\partial V}{\partial r} - rV = 0.$$

   A bond has payoff at maturity $t = T$ of one unit, i.e.

$$V(r, T) = 1$$

   Solve the above equation for $V(r, T)$ given that w is constant and

$$(u - \lambda w) = 1.$$

   Hint: we know the solution has the form $V(r, t) = \exp(A(t) - rB(t))$.

2. The interest rate r is assumed to be satisfied by a SDE $dr = dX$. By hedging with a bond of different maturity derive the bond pricing equation

$$\frac{\partial V}{\partial t} + \frac{1}{2} \frac{\partial^2 V}{\partial r^2} - a(r, t) \frac{\partial V}{\partial r} - rV = 0,$$

where $a(r, t)$ is an arbitrary function. Assuming that a is a function of t only and a bond has payoff at maturity $t = T$ of one unit, i.e.

$$V(r, t; T) = 1$$

find a solution of the form

$$V(r, t) = \exp(A(t) + rB(t))$$

where $A(t)$ can be written as

$$A(t) = -\int_t^T \left[a(s)(s - T) + \beta (s - T)^2 \right] ds$$

and determine the constant β.

3. What final condition (payoff) should be applied to the bond pricing equation for a swap, cap, floor, zero-coupon bond and a bond option?

4. Consider the bond pricing equation

$$\frac{\partial B}{\partial t}+\frac{1}{2}w^2\frac{\partial^2 B}{\partial r^2}+(u-\lambda w)\frac{\partial B}{\partial r}-rB=0,$$

where $dr=(u-\lambda w)\,dt+wdX$ is the risk-neutral spot rate. Suppose this risk-neutral model is defined by

$$dr=ar^2dt+br^{3/2}dX,$$

where a and b are constants. We wish to use this to price a new type if interest rate derivative called a "perpetual bond" whose value is

$$\max(r-E,0)$$

and which can be exercised at any time , where $E>0$ is the exercise price. Show that this price is given by

$$B=\frac{E}{\alpha_1-1}$$

where

$$\alpha_1=\frac{-\left(a-b^2/2\right)+\sqrt{\left(a-b^2/2\right)^2+2b^2}}{b^2}.$$

5. Consider the Vasicek model for the spot rate r with mean rate $\overline{r}$ and reversion rate γ Suppose $\gamma=0.1$, $\overline{r}=0.1$,and standard deviation $\sigma=20\%$. Price a Zero Coupon Bond that matures in year 10, if the spot rate is 10%. (Very much a spreadsheet based problem). **Hint: You can use the definitions of** $A(t)$ and $B(t)$ given in the Wilmott book.

6. In class we derived a two factor interest rate model with the BPE given by

$$\frac{\partial V}{\partial t}+\frac{1}{2}w^2\frac{\partial^2 V}{\partial r^2}+\rho wq\frac{\partial^2 V}{\partial r\partial l}+\frac{1}{2}q^2\frac{\partial^2 V}{\partial l^2}+(u-\lambda_r w)\frac{\partial V}{\partial r}+(p-\lambda_l q)\frac{\partial V}{\partial l}-rV=0.$$

where the two state variables evolve according to

$$\begin{aligned} dr &= udt+wdX_1 \\ dl &= pdt+qdX_2. \end{aligned}$$

Given that $u-\lambda_r w=0=p-\lambda_l q$ and $w=q=\sqrt{a+br+cl}$, where $a,\ b$ and c are constants, derive a set of equations and boundary conditions for $A,\ B$ and C such that a bond V is of the form

$$V=\exp\left(A(t)+rB(t)+lC(t)\right)$$

is a solution of the BPE with redemption value

$$V(r, l, t; T) = 1.$$

You are not required to solve these equations.