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Libor Market Model

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Model of the Yield Curve

- developed by Brace, Gatarek and Musiela - BGM - (1997), Jamshidian (1997) and Miltersen, Sandmann and Sondermann (1997)
- multi-factor model. i.e. more than one source of uncertainty
- we are able to choose the volatility structure
- requires a good degree of computational effort (Monte-Carlo Simulation)

Model of the Yield Curve (continued)

- similar model to the Heath, Jarrow and Morton (HJM) model
- HJM is expressed in terms of instantaneous rates and these are not directly observable in the market
- because of this, its difficult to calibrate HJM to prices of actively traded instruments
- LIBOR Market Model is expressed in terms of observable and discrete forward rates

The Yield Curves

The primary, i.e. important, curves:

- Spot or Zero (sometimes called Bullet, Discount) curve.

1. Observable rates are seen as Interbank cash Deposit Rates (Depos).
2. Maximum term for these observable rates is about a year.
3. The (implied) curve, however, can be constructed much further out.
4. LIBOR rates are set on these types of deposits.

The Yield Curves (continued)

- Futures / Forward Curve.
  1. Most liquid short term interest rate futures markets are on 3-month LIBOR based deposits.
  2. Futures contracts are standardised.
  3. Non-standardised equivalent of these futures are Forward Rate Agreements (FRAs).
  4. Observable rates are seen in liquid markets out to about 3 years.
  5. Again, the implied curve can be constructed to much further out.

The Yield Curves (continued)

- Swap curve.
    1. Observable out to 30 years (and beyond).
    2. Usually built from LIBOR based spot and futures curves but not always.

  There are many other curves. Government and corporate bond (credit), non-LIBOR based, inflation-based, etc.

  However, given that they are based on the same reference rate, if you know one you can calculate the other two.

  Assuming no arbitrage!

Yield Curve versus Traded Instruments

Yield Curves can be constructed but they cannot be traded! Traded instruments are functions of yields calculated from the curves.

Discrete Forward Yields

The LIBOR Market Model describes the dynamics of discrete forward rates as follows:

Let $F(t; T_i, T_{i+1})$ be the forward rate seen at time t, for the period between T_i and T_{i+1} where $i \geq 0, i \in I$

The compounding period is τ_i where $\tau_i = T_{i+1} - T_i$ and is expressed as fractions of a (365 or 360 day) year.

This means that if you contract to lend money at today's quoted forward rate, F_i, you must deposit the agreed notional at time T_i and will receive, at time T_{i+1}, the notional times $(1 + \tau_i F_i(t))$.

We will abbreviate $F(t; T_i, T_{i+1})$ to $F_i(t)$ or just F_i

The discount factor over the forward period τ_i is just $\dfrac{1}{(1+\tau_i F_i(t))}$

This is how the markets trade forwards as well!

Discrete Yield Curves

The Market Model describes the dynamics of observable market rates. The LIBOR Market Model is based on discrete forward rates.

Relationship between Spot Rates and Forward Rates

Assuming discrete compounding and by using arbitrage arguments we can see that:

$$(1 + T_i^r r_i)(1 + \tau_i F_i(t)) = (1 + T_{i+1}^r r_{i+1}) \qquad 1$$

where r_i is the spot, or zero-coupon, rate

$T_i^r = T_i - t$ and is expressed as fractions of a (365 or 360 day) year.

Also, $t = 0$ so $T_i^r = T_i$

We have defined the price, at time t, of a zero-coupon bond which matures at time T_i (in units of years) as:

$$Z(t; T_i) = \frac{1}{(1 + T_i r_i)}$$

Therefore, from (1), we have:

$$1 + \tau_i F_i(t) = \frac{Z(t; T_i)}{Z(t; T_{i+1})} \qquad 2$$

The First Forward Rate

Note that when $i = 0$,

$$1 + \tau_0 F_0(t) = 1 + T_1 r_1$$

Implies,

$$F_0(t) = r_1$$

This just says that the first forward rate we need to build our forward curve starts at time zero. But this is the definition of the first zero-coupon or spot rate. So, therefore, they are equivalent.

Numeraires

You might read, in some textbooks and articles, about the concept of numeraires.

A numeraire defines the units in which traded security prices are measured.

Example: If we choose the numeraire as the price of Citigroup stock, all securities will be measured relative to the Citigroup price. If Citi trades at $46 and, say, Microsoft at $29, Microsoft's traded price is 0.63 units of Citigroup.

Up to now, in the continuous models that you have been studying, you have assumed, implicitly or explicitly, that the numeraire is a continuously compounded money market account – i.e. a risk free investment.

It is a security that is worth Π at time zero and earns the risk-free rate r. It follows the process:

$$d\Pi = r\Pi dt$$

Note the drift of Π depends on r (which itself can be stochastic) and its volatility is zero.

It has no dX term.

The fact that the numeraire has a drift dependent on r and no source of uncertainty allows us to price instruments as if we are in the risk-neutral world.

Change of Numeraire

We cannot use this instrument in the discretely hedged world or we will bring in uncertainty in the form of the unknown path of continuous rates between time T_{i+1} and T_i.

It is a subtle adjustment, since the numeraire will still be a risk-free money market investment. However, rather than continuously compounded and reinvested, the account – sometimes known as a rolling CD – will pay and compound interest over the period T_i to T_{i+1}. At time T_{i+1}we will reinvest the proceeds for a further period, $T_{i+1}$to T_{i+2}. And so on.

Change of Numeraire (continued)

The rate we will use is the zero rate observed at T_i, corresponding to the maturity T_{i+1} Since that rate holds for the entire period we don't have to worry about what rates do over the period. We are firmly back in the risk-neutral world with risk-free rates –for traded assets – and zero volatility – i.e. no dX term – for the numeraire.

Some textbooks call this the rolling-forward risk neutral world.

Its pretty clear that the usual continuous risk-neutral world is just the discrete world case in the limit as $T_i \to T_{i+1}$.

Forward Rate Dynamics

Let us assume that each of the n discretely compounded forward rates, $F_i(t)$, evolves according to the lognormal stochastic differential equation:

$$dF_i = \mu_i(F,t)F_i dt + \sigma_i(F,t)F_i dX_i \qquad 3$$

We can see that the traded assets, the $Z_i s$, are functions of the $F_i s$. From (2) we have

$$Z(t;T_{i+1}) = \frac{Z(t;T_i)}{1+\tau_i F_i(t)} \qquad \text{4a}$$

This is equivalent to

$$Z(t;T_i) = Z_i = \frac{Z(t;T_j)}{1+\tau_j F_j(t)} \text{ where } j = i-1 \qquad \text{4b}$$

This is to say that the zero rate, Z_i, is a function of forward rates F_j where $j < i$. It is not a function of forward rates, F_i.

Lets just look at the derivation of the $Z_i(t;T_i)s$, noting that the Z_is depend on variables $F_j(0 \leq j \leq i-1)$.

Using a Taylor Series Expansion we have,

$$dZ_i = \frac{\partial Z_i}{\partial t}dt + \sum_{j=0}^{i-1} \frac{\partial Z_i}{\partial F_j}dF_j + \frac{1}{2}\sum_{j=0}^{i-1}\sum_{k=0}^{i-1} \frac{\partial^2 Z_i}{\partial F_j \partial F_k}dF_j dF_k + \frac{1}{2}\sum_{j=0}^{i-1} \frac{\partial^2 Z_i}{\partial F_j \partial t}dF_j dt + \ldots \qquad 5$$

We will ignore terms in dt of order greater than one.

From Ito's Lemma we know that:

dX_j is of order $dt^{1/2}$. Therefore the 4th term drops out.

and

$$dF_j dF_k = \sigma_j \sigma_k F_j F_k \rho_{jk} dt \qquad 6$$

$$\text{where } \rho_{jk} dt = dX_j dX_k$$

Inserting (3) and (6) into (5) we have

$$dZ_i = \left(\frac{\partial Z_i}{\partial t} + \sum_{j=0}^{i-1} \frac{\partial Z_i}{\partial F_j}\mu_j F_j + \frac{1}{2}\sum_{j=0}^{i-1}\sum_{k=0}^{i-1}\frac{\partial^2 Z_i}{\partial F_j \partial F_k}\sigma_j \sigma_k F_j F_k \rho_{jk}\right)dt + \sum_{j=0}^{i-1}\frac{\partial Z_i}{\partial F_j}\sigma_j F_j dX_j \qquad 7$$

We know the zero-coupon bonds are traded and in our risk-neutral world evolve as,

$$dZ_i = rZ_i dt + Z_i \sum_{j=0}^{i-1} a_{ij} dX_j \qquad 8$$

where $Z_i = Z(t; Z_i)$

Note that since the $Z_i s$ are traded assets and their drift rates are all r – the risk-free rate, we are clearly in a risk-neutral world.

Equating the coefficients of dt and dX_j in equations (6) and (7) we have

$$rZ_i = \frac{\partial Z_i}{\partial t} + \sum_{j=0}^{i-1} \frac{\partial Z_i}{\partial F_j}\mu_j F_j + \frac{1}{2}\sum_{j=0}^{i-1}\sum_{k=0}^{i-1} \frac{\partial^2 Z_i}{\partial F_j \partial F_k}\sigma_j \sigma_k F_j F_k \rho_{jk}$$

and

$$a_{ij} = \frac{\partial Z_i}{\partial F_j}\frac{\sigma_j F_j}{Z_i}$$

From (4a) we have

$$Z_i = \frac{Z(t;T_{i-1})}{1+\tau_{i-1}F_{i-1}(t)}$$

Therefore,

$$\frac{\partial Z_i}{\partial F_j} = \frac{-\tau_j Z(t;T_i)}{1+\tau_j F_j(t)} \text{ for all } j < i \qquad 9$$

and

$$a_{ij} = \frac{-\tau_j Z(t;T_i)}{1+\tau_j F_j(t)} \frac{\sigma_j F_j}{Z(t;T_i)} = -\frac{\sigma_j F_j \tau_j}{1+\tau_j F_j(t)} \text{ for all } j < i$$

From (7) we have

$$rZ_i = \frac{\partial Z_i}{\partial t} + \sum_{j=0}^{i-1} \frac{\partial Z_i}{\partial F_j} \mu_j F_j + \frac{1}{2} \sum_{j=0}^{i-1} \sum_{k=0}^{i-1} \frac{\partial^2 Z_i}{\partial F_j \partial F_k} \sigma_j \sigma_k F_j F_k \rho_{jk}$$

From (4a) we have,

$$Z_i = (1 + \tau_i F_i) Z_{i+1}$$

Therefore, from Ito's lemma we can write

$$dZ_i = (1 + \tau_i F_i) dZ_{i+1} + \tau_i Z_{i+1} dF_i + \tau_i \sigma_i F_i Z_{i+1} \sum_{j=1}^{i} a_{i+1,j} \rho_{ij} dt \text{ where } \rho_{jk} dt = dX_j dX_k$$

Equating the *dt* terms we have,

$$rZ_i = (1 + \tau_i F_i) rZ_{i+1} + \tau_i Z_{i+1} \mu_i F_i + \tau_i \sigma_i F_i Z_{i+1} \sum_{j=1}^{i} a_{i+1,j} \rho_{ij} dt$$

Therefore,

$$\mu_i = \sigma_i \sum_{j=1}^{i} \frac{\sigma_j F_j \tau_j}{1 + \tau_j F_j} \rho_{ij} dt$$

Thus substituting back into (3), we have,

$$dF_i = F_i \sum_{j=1}^{i} \frac{\tau_j F_j \sigma_j \sigma_i \rho_{ij}}{1 + \tau_j F_j} dt + \sigma_i F_i dX_i$$

From Ito's Lemma we can state,

$$d\ln F_i(t) = (\sigma_i(t)\sum_{j=1}^{i}\frac{\tau_j F_j(t)\sigma_j(t)\rho_{ij}}{1+\tau_j F_j(t)} - \frac{\sigma_i(t)^2}{2})dt + \sigma_i(t)dX_i$$

If we assume that $F_j(t) = F_j(t_k)$ and $\sigma_j(t) = \sigma_j(t_k)$ for $t_k < t < t_{k+1}$, as an approximation, we can write

$$F_i(t_{k+1}) = F_i(t_k)\exp[(\sigma_i(t_{i-k-1})\sum_{j=k+1}^{i}\frac{\tau_j F_j(t_k)\sigma_j(t_{j-k-1})\rho_{ij}}{1+\tau_j F_j(t_k)} - \frac{\sigma_i(t_{i-k-1})^2}{2})\tau_k + \sigma_i(t_{i-k-1})\epsilon_i\sqrt{\tau_k}]$$

where $\epsilon_i \sim N(0,1)$

Now all we need to do is measure the forward rates (F_i), their volatilities (σ_i), and their correlations (ρ_{ij}). Then we can do Monte-Carlo simulation.

Forward Rate Volatilities

These can be estimated from the volatilities of caps – or more specifically caplets.

If we assume that σ_i is just a function of of the number of whole accrual periods between the next reset date and time T_i then we can see that σ_i will just be piecewise constant, i.e. a step function, over the accrual intervals.

σ_i can be estimated from the volatilties used to value caplets. If we assume that ζ_i is the volatility for the caplet that corresponds to the period between time T_i and T_{i+1}, then we have

$$\zeta_i^2 T_i = \sum_{j=1}^{i} \sigma_j^2(t_{j-1})\tau_{j-1}$$

Extended to several factors

If the source of uncertainity in the F_is comes from more than one factor and there are p independent factors, we can write,

$$dF_i = F_i \sum_{k=1}^{i} \frac{\tau_k F_i \sum_{q=1}^{p} \sigma_{k,q} \sigma_{i,q} \rho_{jk,q}}{1 + \tau_k F_k(t)} dt + F_i \sum_{q=1}^{p} \sigma_{i,q} dX_{i,q}$$