1. Definitions

Definition 1 The **simple forward rate** from S to T contracted at time t, also called the **LIBOR forward rate**, is defined as

$$L(t,S,T) = -\frac{B(t,T) - B(t,S)}{(T-S)B(t,T)}.$$

The **simple spot rate** from t to T, also called the **LIBOR spot rate**, is defined as

$$L(t,T) = L(t;t,T) = -\frac{B(t,T) - 1}{(T-t)B(t,T)}.$$

The **continuously compounded forward rate** from S to T contracted at time t is defined as

$$R(t,S,T) = -\frac{\ln B(t,T) - \ln B(t,S)}{T-S}.$$

The **continuously compounded spot rate** from t to T is defined as

$$R(t,T) = R(t,t,T) = -\frac{\ln B(t,T)}{T-t}.$$

Definition 2 In a continuous time model the **instantaneous forward rate** with maturity T contracted at time t is defined by

$$f(t,T) = \lim_{dt \to 0} R(t,T,T+dt) = -\frac{\partial \ln B(t,T)}{\partial T},$$

and the **short rate** at time t is defined by

$$r(t) = f(t,t).$$

The rate $f(t,T)$ can be interpreted as the riskless interest rate contracted at t over the infinitesimal time interval from T to $T+dT$. Similarly, $r(t)$ can be interpreted as the riskless interest rate contracted at t over the infinitesimal time interval from t to $t+dt$.

Clearly, for $t \leq S \leq T$ we have

$$B(t,T) = B(t,S)e^{-\int_S^T f(t,u)du}.$$

Definition 3 The **money market account** process is defined by

$$A_t = e^{\int_0^t r(u)du},$$

that is,

$$\begin{aligned} dA(t) &= r(t)A(t)dt, \\ A(0) &= 1. \end{aligned}$$

2. Discrete Heath-Jarrow-Morton Model

We shall consider the discrete **one-factor** version of the HJM model, which is based on a (not necessarily recombinant) binomial tree. Multi-factor models are also possible, which are built using trees with more than two branches at each node. In the one-factor case considered here the tree of bond prices looks like this (presented here with time horizon $T = 3$):

$$
\begin{array}{ccccccc}
 & & & & \begin{array}{l} B(2,2,uu)=1 \\ B(2,3,uu) \end{array} & \rightarrow & B(3,3,uu)=1 \\
 & & \begin{array}{l} B(1,1,u)=1 \\ B(1,2,u) \\ B(1,3,u) \end{array} & \nearrow & & & \\
 & \nearrow & & \searrow & \begin{array}{l} B(2,2,ud)=1 \\ B(2,3,ud) \end{array} & \rightarrow & B(3,3,ud)=1 \\
\begin{array}{l} B(0,0)=1 \\ B(0,1) \\ B(0,2) \\ B(0,3) \end{array} & & & & & & \\
 & & & & \begin{array}{l} B(2,2,du)=1 \\ B(2,3,du) \end{array} & \rightarrow & B(3,3,du)=1 \\
 & \searrow & \begin{array}{l} B(1,1,d)=1 \\ B(1,2,d) \\ B(1,3,d) \end{array} & \nearrow & & & \\
 & & & \searrow & \begin{array}{l} B(2,2,dd)=1 \\ B(2,3,dd) \end{array} & \rightarrow & B(3,3,dd)=1
\end{array}
$$

The fact that a non-recombinant tree is used means that the bond prices may be path dependent.

2.1 Risk Neutral Probabilities

The fundamental theorem of asset pricing provides a necessary and sufficient condition for the lack of arbitrage, namely the existence of a non-degenerate measure that turns all discounted asset prices (here all discounted bond prices) into martingales. Discounting is by means by the short rate $r(t)$ determined by the bond prices with one time step to maturity,

$$r(t) = -\frac{\ln B(t, t+1)}{\tau}.$$

At each time t and each node ω_t of the tree we therefore have the risk neutral probability $p(t, \omega_t) \in (0, 1)$ attached to the 'up' branch, and $1 - p(t, \omega_t) \in (0, 1)$ attached to the 'down' branch such that for each $S > t + 1$

$$
\begin{aligned}
p(t, \omega_t) &= \frac{B(t, S, \omega_t) - B(t+1, S, \omega_t d)e^{-\tau r(t,\omega_t)}}{B(t+1, S, \omega_t u)e^{-\tau r(t,\omega_t)} - B(t+1, S, \omega_t d)e^{-\tau r(t,\omega_t)}} \\
&= \frac{B(t, S, \omega_t) - B(t+1, S, \omega_t d)B(t, t+1, \omega_t)}{B(t+1, S, \omega_t u)B(t, t+1, \omega_t) - B(t+1, S, \omega_t d)B(t, t+1, \omega_t)}.
\end{aligned} \tag{1}
$$

The tree with the risk neutral probabilities attached to their corresponding branches and short rates specifies at each node looks like this:

$$
\begin{array}{ccccccc}
 & & & \begin{array}{l} p(1,u) \\ \nearrow \end{array} & \begin{array}{l} B(2,2,uu)=1 \\ B(2,3uu) \end{array} & \rightarrow & B(3,3,uu)=1 \\
 & \begin{array}{l} p(0) \\ \nearrow \end{array} & \begin{array}{l} B(1,1,u)=1 \\ B(1,2,u) \\ B(1,3,u) \end{array} & r(1,u) & & & \\
 & & & \begin{array}{l} \searrow \\ 1-p(1,u) \end{array} & \begin{array}{l} B(2,2,ud)=1 \\ B(2,3,ud) \end{array} & \rightarrow & B(3,3,ud)=1 \\
\begin{array}{l} B(0,0)=1 \\ B(0,1) \\ B(0,2) \\ B(0,3) \end{array} & r(0) & & & & & \\
 & & & \begin{array}{l} p(1,d) \\ \nearrow \end{array} & \begin{array}{l} B(2,2,du)=1 \\ B(2,3,du) \end{array} & \rightarrow & B(3,3,du)=1 \\
 & \begin{array}{l} \searrow \\ 1-p(0) \end{array} & \begin{array}{l} B(1,1,d)=1 \\ B(1,2,d) \\ B(1,3,d) \end{array} & r(1,d) & & & \\
 & & & \begin{array}{l} \searrow \\ 1-p(1,d) \end{array} & \begin{array}{l} B(2,2,dd)=1 \\ B(2,3,dd) \end{array} & \rightarrow & B(3,3,dd)=1
\end{array}
$$

Observe that the prices for the bond with the longest maturity (maturity T) together with the short rates (or, equivalently, together with the prices of the bond with time 1 to maturity, that is, the shortest possible time to maturity) as shown in the tree

$$\begin{array}{cccccc}
 & & & & B(2,3,uu) & \rightarrow \quad B(3,3,uu)=1 \\
 & & & \nearrow & & \\
 & & B(1,3,u) & r(1,u) & & \\
 & \nearrow & & \searrow & & \\
 & & & & B(2,3,ud) & \rightarrow \quad B(3,3,ud)=1 \\
B(0,3) & r(0) & & & & \\
 & & & & B(2,3,du) & \rightarrow \quad B(3,3,du)=1 \\
 & \searrow & & \nearrow & & \\
 & & B(1,3,d) & r(1,d) & & \\
 & & & \searrow & & \\
 & & & & B(2,3,dd) & \rightarrow \quad B(3,3,dd)=1
\end{array}$$

determine all the remaining bond prices for all other maturities S at each node of the tree. Indeed, the bond prices with maturity T together with the short rates determine the risk neutral probabilities by (1). Since the bond prices at maturity S are known to be equal to 1, the risk neutral probabilities can then be used to compute the bond prices prior to maturity S using the martingale property

$$B(t,S,\omega_t) = [p(t,\omega_t)B(t+1,S,\omega_t u) + (1-p(t,\omega_t))B(t+1,S,\omega_t u)]\, e^{-\tau r(t,\omega_t)}. \tag{2}$$

The following example illustrates this procedure.

Example 4 Let $\tau = \frac{1}{12}$, let $T = 3$ and let the short rates and the prices for the bond with maturity 3 be as in the following tree:

$$\begin{array}{ccccc}
 & & & & B(2,3,uu)=0.9991 \\
 & & & \nearrow & \\
 & & B(1,3,u)=0.9962 & r(1,u)=2.40\% & \\
 & \nearrow & & \searrow & \\
 & & & & B(2,3,ud)=0.9960 \\
B(0,3)=0.9927 & r(0)=4.11\% & & & \\
 & & & & B(2,3,du)=0.9986 \\
 & \searrow & & \nearrow & \\
 & & B(1,3,d)=0.9959 & r(1,d)=3.00\% & \\
 & & & \searrow & \\
 & & & & B(2,3,dd)=0.9969
\end{array}$$

First, we compute the risk neutral probabilities $p(t)$ and $q(t) = 1 - p(t)$ using (1):

$$\begin{array}{ccccc}
 & & & p(1,u)=0.70787 & B(2,3,uu)=0.9991 \\
 & & & \nearrow & \\
 & p(0)=0.68609 & B(1,3,u)=0.9962 & r(1,u)=2.40\% & \\
 & \nearrow & & \searrow & \\
 & & & q(1,u)=0.29213 & B(2,3,ud)=0.9960 \\
B(0,3)=0.9927 & r(0)=4.11\% & & & \\
 & & & p(1,d)=0.87816 & B(2,3,du)=0.9986 \\
 & \searrow & & \nearrow & \\
 & q(0)=0.31391 & B(1,3,d)=0.9959 & r(1,d)=3.00\% & \\
 & & & \searrow & \\
 & & & q(1,d)=0.12184 & B(2,3,dd)=0.9969
\end{array}$$

Then we compute the prices of the bond maturing at time step 2 using (2), starting with the price $B(2,2) = 1$ at maturity and moving backwards through the tree:

$$\begin{array}{ccccc}
 & & & p(1,u)=0.70787 & B(2,2,uu)=1 \\
 & & & \nearrow & B(2,3,uu)=0.9991 \\
 & p(0)=0.68609 & B(1,2,u)=0.9980 & r(1,u)=2.40\% & \\
 & & B(1,3,u)=0.9962 & & \\
 & \nearrow & & \searrow & B(2,2,ud)=1 \\
 & & & q(1,u)=0.29213 & B(2,3,ud)=0.9960 \\
B(0,2)=0.9944 & r(0)=4.11\% & & & \\
B(0,3)=0.9927 & & & & \\
 & & & p(1,d)=0.87816 & B(2,2,du)=1 \\
 & \searrow & B(1,2,d)=0.9975 & \nearrow & B(2,3,du)=0.9986 \\
 & q(0)=0.31391 & B(1,3,d)=0.9959 & r(1,d)=3.00\% & \\
 & & & \searrow & B(2,2,dd)=1 \\
 & & & q(1,d)=0.12184 & B(2,3,dd)=0.9969
\end{array}$$

Finally we compute the prices of the bond maturing at time step 1 using (2) once again, starting with $B(1,1)=1$ and moving backwards through the tree:

$$
\begin{array}{lllll}
 & & & p(1,u)=0.70787 & B(2,2,uu)=1 \\
 & & B(1,1,u)=1 & \nearrow & B(2,3,uu)=0.9991 \\
 & p(0)=0.68609 & B(1,2,u)=0.9980 & r(1,u)=2.40\% & \\
 & \nearrow & B(1,3,u)=0.9962 & \searrow & B(2,2,ud)=1 \\
B(0,1)=0.9966 & & & q(1,u)=0.29213 & B(2,3,ud)=0.9960 \\
B(0,2)=0.9944 & r(0)=4.11\% & & & \\
B(0,3)=0.9927 & & & p(1,d)=0.87816 & B(2,2,du)=1 \\
 & \searrow & B(1,1,d)=1 & \nearrow & B(2,3,du)=0.9986 \\
 & q(0)=0.31391 & B(1,2,d)=0.9975 & r(1,d)=3.00\% & \\
 & & B(1,3,d)=0.9959 & \searrow & B(2,2,dd)=1 \\
 & & & q(1,d)=0.12184 & B(2,3,dd)=0.9969
\end{array}
$$

2.2 Arbitrage

As we have seen, the prices of bonds with different maturities are closely interrelated. If the relationship is broken, it means there is an arbitrage opportunity.

To spot arbitrage we can compute the risk neutral probabilities

$$
\begin{aligned}
p(t,S,\omega_t) &= \frac{B(t,S,\omega_t)e^{\tau r(t,\omega_t)} - B(t+1,S,\omega_t d)}{B(t+1,S,\omega_t u) - B(t+1,S,\omega_t d)} \\
&= \frac{B(t,S,\omega_t)/B(t,t+1,\omega_t) - B(t+1,S,\omega_t d)}{B(t+1,S,\omega_t u) - B(t+1,S,\omega_t d)}
\end{aligned} \tag{3}
$$

at each node using bonds of different maturities $S = t+2,\ldots,T$. Here we have introduced the argument S in $p(t,S,\omega_t)$ to indicate that this has been computed using bonds with maturity S. We know that lack of arbitrage means that the risk neutral probabilities should in fact be independent of S,

$$p(t,\omega_t) = p(t,S,\omega_t) \quad \text{for each } S > t+1, \tag{4}$$

and non-degenerate,

$$0 < p(t,S,\omega_t) < 1 \quad \text{for each } S > t+1. \tag{5}$$

If (3) produces values $p(t,S,\omega_t)$ that violate (4) or (5), it means that there is arbitrage in the model.

Example 5 Is there arbitrage in the following tree of bond prices?

$$
\begin{array}{lllll}
 & & & & B(2,3,uu)=0.9952 \\
 & & B(1,2,u)=0.9914 & \nearrow & \\
 & & B(1,3,u)=0.9838 & \searrow & \\
B(0,1)=0.9954 & \nearrow & & & B(2,3,ud)=0.9915 \\
B(0,2)=0.9860 & & & & \\
B(0,3)=0.9788 & & & & B(2,3,du)=0.9921 \\
 & & B(1,2,d)=0.9902 & \nearrow & \\
 & \searrow & B(1,3,d)=0.9818 & \searrow & \\
 & & & & B(2,3,dd)=0.9901
\end{array}
$$

To answer this question we compute the risk neutral probabilities at each node. At the root node at time 0 there are two bonds with maturity greater than 1, and we can compute two values $p(0,2)$ and $p(0,3)$ from (3). At time 1 there is only one bond with maturity greater than 2, leading to a single value $p(1,3,u)$ at node u and a single value $p(1,3,d)$ at node d, as in the following tree, where $q(t, S, \omega_t) = 1 - p(t, S, \omega_t)$:

$$
\begin{array}{cccccc}
 & \begin{array}{l} p(0,2) = 0.2971 \\ p(0,3) = 0.7616 \\ \nearrow \end{array} & \begin{array}{l} B(1,2,u) = 0.9914 \\ B(1,3,u) = 0.9838 \end{array} & \begin{array}{l} p(1,3,u) = 0.2243 \\ \nearrow \\ \searrow \\ q(1,3,u) = 0.7757 \end{array} & \begin{array}{l} B(2,3,uu) = 0.9952 \\ \\ B(2,3,ud) = 0.9915 \end{array} \\
\begin{array}{l} B(0,1) = 0.9954 \\ B(0,2) = 0.9860 \\ B(0,3) = 0.9788 \end{array} & & & & \\
 & \begin{array}{l} \searrow \\ q(0,2) = 0.7029 \\ q(0,3) = 0.2384 \end{array} & \begin{array}{l} B(1,2,d) = 0.9902 \\ B(1,3,d) = 0.9818 \end{array} & \begin{array}{l} p(1,3,d) = 0.7100 \\ \nearrow \\ \searrow \\ q(1,3,d) = 0.2900 \end{array} & \begin{array}{l} B(2,3,du) = 0.9921 \\ \\ B(2,3,dd) = 0.9901 \end{array}
\end{array}
$$

Because $p(0,2) \neq p(0,3)$ we know that there is arbitrage in this tree of bond prices.

To utilise the arbitrage opportunity detected in Example 5 we can replicate the time 1 prices of the bond maturing at time 2 by a portfolio consisting of x bonds maturing at time 3 and y bonds with 1 time step to maturity. The construction is similar to that of a portfolio replicating a derivative security in the CRR model, and is based on solving the following system of equations for x and y:

$$
\begin{aligned}
xB(1,3;u) + yB(1,1) &= B(1,2,u), \\
xB(1,3;d) + yB(1,1) &= B(1,2,d).
\end{aligned}
$$

The value of this portfolio at time 0 is

$$xB(0,3) + yB(0,1),$$

which should be compared to the bond price $B(0,2)$. If

$$xB(0,3) + yB(0,1) < B(0,2),$$

then arbitrage can be achieved by shorting the bond maturing at time 2 and taking a long position in the portfolio consisting of x bonds maturing at time 3 and y bonds maturing at time 1. If

$$xB(0,3) + yB(0,1) > B(0,2),$$

then arbitrage can be achieved by the opposite strategy, shorting the portfolio and taking a long position in the bond maturing at 2.

Example 6 For the data in Example 5 we write down the system of equations for the portfolio replicating the time 1 prices of the bond maturing at time 2:

$$
\begin{aligned}
0.9838x + y &= 0.9914, \\
0.9818x + y &= 0.9902.
\end{aligned}
$$

The solution is

$$x = 0.6, \quad y = 0.40112.$$

The time 0 value of the portfolio consisting of $x = 0.6$ bonds maturing at time 3 and $y = 0.40112$ bonds maturing at time 1 is

$$
\begin{aligned}
0.9788x + 0.9954y &= 0.9788 \times 0.6 + 0.9954 \times 0.40112 \\
&= 0.98655.
\end{aligned}
$$

This is greater than $B(0,2) = 0.9860$, leading to the following arbitrage strategy:

- At time 0:
  - sell short 0.6 bonds maturing at time 3;
  - sell short 0.40112 bonds maturing at time 1;
  - buy 1 bond maturing at time 2.

  The cash balance of these transactions will be

$$0.6 \times 0.9788 + 0.40112 \times 0.9954 - 1 \times 0.9860 \cong 0.00055.$$

- At time 1 close all bond positions, that is:
  - buy 0.6 bonds maturing at time 3;
  - pay the face value of 1 for each of the 0.40112 bonds maturing at time 1;
  - sell 1 bond maturing at time 2.

  The cash balance of these transactions will be, at node u

$$-0.6 \times 0.9838 - 0.40112 \times 1 + 1 \times 0.9914 = 0,$$

  and at node d

$$-0.6 \times 0.9818 - 0.40112 \times 1 + 1 \times 0.9902 = 0.$$

The positive balance of 0.00055 raised at time 0 is the arbitrage profit.

2.3 From Volatilities Through Forward Rates to Bond Prices

We know that bond prices $B(t, S)$ and the forward rates $f(t, S)$ are related to one another according to the formulae

$$f(t, S) = -\frac{\ln B(t, S+1) - \ln B(t, S)}{\tau}$$

for any t and S such that $0 \leq t \leq S < T$, and

$$B(t, S) = e^{-\tau \sum_{i=t}^{S-1} f(t,i)} \tag{6}$$

for any t and S such that $0 \leq t \leq S \leq T$. Because of this, as an alternative to specifying the tree of bond prices

$$\begin{array}{ccccccc}
 & & & & \begin{array}{l} B(2,2,uu)=1 \\ B(2,3,uu) \end{array} & \to & B(3,3,uu)=1 \\
 & & \begin{array}{l} B(1,1,u)=1 \\ B(1,2,u) \\ B(1,3,u) \end{array} & \begin{array}{c} \nearrow \\ \searrow \end{array} & & & \\
 & \nearrow & & & \begin{array}{l} B(2,2,ud)=1 \\ B(2,3,ud) \end{array} & \to & B(3,3,ud)=1 \\
\begin{array}{l} B(0,0)=1 \\ B(0,1) \\ B(0,2) \\ B(0,3) \end{array} & & & & & & \\
 & \searrow & & & \begin{array}{l} B(2,2,du)=1 \\ B(2,3,du) \end{array} & \to & B(3,3,du)=1 \\
 & & \begin{array}{l} B(1,1,d)=1 \\ B(1,2,d) \\ B(1,3,d) \end{array} & \begin{array}{c} \nearrow \\ \searrow \end{array} & & & \\
 & & & & \begin{array}{l} B(2,2,dd)=1 \\ B(2,3,dd) \end{array} & \to & B(3,3,dd)=1
\end{array}$$

we can, equivalently, specify the forward rates tree

$$
\begin{array}{ccccc}
 & & f(1,1,u) & \nearrow & f(2,2,uu) \\
 & \nearrow & f(1,2,u) & \searrow & f(2,2,ud) \\
f(0,0) & & & & \\
f(0,1) & & & & \\
f(0,2) & & & & \\
 & \searrow & f(1,1,d) & \nearrow & f(2,2,du) \\
 & & f(1,2,d) & \searrow & f(2,2,dd)
\end{array}
$$

(Here we use $T=3$ for simplicity.)

In the discrete HJM model the forward rates are assumed to satisfy the recursive relationships

$$
\begin{aligned}
f(t+1,S,\omega_t u) &= f(t,S,\omega_t)+\mu(t,S,\omega_t)\tau+\sigma(t,S,\omega_t)\sqrt{\tau}, \\
f(t+1,S,\omega_t d) &= f(t,S,\omega_t)+\mu(t,S,\omega_t)\tau-\sigma(t,S,\omega_t)\sqrt{\tau},
\end{aligned} \tag{7}
$$

at each node ω_t for $t<T-2$, given certain functions $\mu(t,S,\omega_t)$ and $\sigma(t,S,\omega_t)$. In addition, it is assumed that the risk neutral probability at each node ω_t is

$$p(t,\omega_t)=\frac{1}{2}.$$

The effect of these assumptions is that the conditional expectation and variance of $f(t+1,S)$ at time t are

$$
\begin{aligned}
E\left(f(t+1,S)\right|\mathcal{F}_t) &= f(t,S)+\mu(t,S)\tau, \\
\operatorname{Var}\left(f(t+1,S)\right|\mathcal{F}_t) &= \sigma(t,S)^2\tau.
\end{aligned}
$$

Let us express the risk neutral probabilities in terms of the μ's and σ's:

$$
\begin{aligned}
p(t,S) &= \frac{B(t,S)-B(t+1,S,d)B(t,t+1)}{B(t+1,S,u)B(t,t+1)-B(t+1,S,d)B(t,t+1)} \\
&= \frac{e^{-\tau\sum_{i=t}^{S-1}f(t,i)}-e^{-\tau\sum_{i=t+1}^{S-1}f(t+1,i,d)}e^{-\tau f(t,t)}}{e^{-\tau\sum_{i=t+1}^{S-1}f(t+1,i,u)}e^{-\tau f(t,t)}-e^{-\tau\sum_{i=t+1}^{S-1}f(t+1,i,d)}e^{-\tau f(t,t)}} \\
&= \frac{e^{-\tau\sum_{i=t+1}^{S-1}f(t,i)}-e^{-\tau\sum_{i=t+1}^{S-1}f(t+1,i,d)}}{e^{-\tau\sum_{i=t+1}^{S-1}f(t+1,i,u)}-e^{-\tau\sum_{i=t+1}^{S-1}f(t+1,i,d)}} \\
&= \frac{e^{\tau^2\sum_{i=t+1}^{S-1}\mu(t,i)}-e^{\tau^{3/2}\sum_{i=t+1}^{S-1}\sigma(t,i)}}{e^{-\tau^{3/2}\sum_{i=t+1}^{S-1}\sigma(t,i)}-e^{\tau^{3/2}\sum_{i=t+1}^{S-1}\sigma(t,i)}}.
\end{aligned}
$$

Since $p(t,S)=\frac{1}{2}$, we find that

$$e^{\tau^2\sum_{i=t+1}^{S-1}\mu(t,i)}=\cosh\tau^{3/2}\sum_{i=t+1}^{S-1}\sigma(t,i).$$

This equality holds for all t and S such that $1\le t+1<S\le T$. This means that all μ's are determined by the σ's. Indeed, taking $S=t+1,t+2,\ldots,T$, we obtain the sequence of equalities

$$
\begin{aligned}
e^{\tau^2\mu(t,t+1)} &= \cosh\tau^{3/2}\sigma(t,t+1), \\
e^{\tau^2[\mu(t,t+1)+\mu(t,t+2)]} &= \cosh\tau^{3/2}\left[\sigma(t,t+1)+\sigma(t,t+2)\right], \\
&\vdots \\
e^{\tau^2[\mu(t,t+1)+\cdots+\mu(t,T-1)]} &= \cosh\tau^{3/2}\left[\sigma(t,t+1)+\cdots+\sigma(t,T-1)\right],
\end{aligned} \tag{8}
$$

from which we can compute $\mu(t,t+1), \mu(t,t+2), \dots, \mu(t,T-1)$ one-by-one in terms of $\sigma(t,t+1), \sigma(t,t+2), \dots, \sigma(t,T-1)$.

It follows from the above that the entire tree of forward rates $f(t,S)$ (or, equivalently, of bond prices $B(t,S)$) is determined by the initial term structure of forward rates

$$f(0,0), f(0,1), \dots, f(0,T-1)$$

and the volatilities

$$\sigma(t,t+1), \sigma(t,t+2), \dots, \sigma(t,T-1)$$

for each $t = 0, 1, \dots, T-2$.

The procedure of constructing the discrete HJM model is this:

- Take the initial term structure of forward rates $f(0,S)$ for $S = 0, \dots, T-1$ (or the bond prices $B(0,S)$ for $S = 1, \dots, T$). These are today's forward rates/bond prices quoted by the exchanges.

- Estimate the volatilities $\sigma(t,S)$, for example, from historical data or as implied volatilities from selected interest rate derivatives.

- Compute the μ's using (8) from the σ's.

- Compute the tree of forward rates $f(t,S)$ from the initial rates $f(0,S)$ and the μ's and σ's using (7).

- Compute the tree of bond prices $B(t,S)$ from the tree of forward rates $f(t,S)$ using (6).

Having completed this construction, you will be ready for pricing any interest rate derivatives based on this model.

Example 7 We shall construct all the forward rates $f(t,S)$ and bond prices $B(t,S)$ at all nodes of a tree with $T = 3$ steps given that $\tau = 1$ and

$$f(0,0) = 2.9635\%, \quad f(0,1) = 2.9478\%, \quad f(0,2) = 2.9609\%,$$
$$\sigma(0,1) = \sigma(0,2) = \sigma(1,2,u) = \sigma(1,2,d) = 0.01.$$

First of all, we compute the μ's using (8):

$$\begin{array}{lcl}
 & & \mu(1,2,u) = 4.9999 \times 10^{-5} \\
\begin{array}{l}\mu(0,1) = 4.9999 \times 10^{-5} \\ \mu(0,2) = 1.4999 \times 10^{-4}\end{array} & \begin{array}{c}\nearrow \\ \searrow\end{array} & \\
 & & \mu(1,2,d) = 4.9999 \times 10^{-5}
\end{array}$$

Next, we compute the forward rates $f(t,S)$ using (7):

$$\begin{array}{lclcl}
 & & & & f(2,2,uu) = 0.049809 \\
 & & \begin{array}{l}f(1,1,u) = 0.039528 \\ f(1,2,u) = 0.039759\end{array} & \begin{array}{c}\nearrow \\ \searrow\end{array} & \\
 & \nearrow & & & f(2,2,ud) = 0.029809 \\
\begin{array}{l}f(0,0) = 0.029635 \\ f(0,1) = 0.029478 \\ f(0,2) = 0.029609\end{array} & & & & \\
 & \searrow & & & f(2,2,du) = 0.029809 \\
 & & \begin{array}{l}f(1,1,d) = 0.019528 \\ f(1,2,d) = 0.019759\end{array} & \begin{array}{c}\nearrow \\ \searrow\end{array} & \\
 & & & & f(2,2,dd) = 0.009809
\end{array}$$

Finally, we compute the bond prices $B(t,S)$ using (6):

$$\begin{array}{lclcl}
 & & & & B(2,3,uu) = 0.95141 \\
 & & \begin{array}{l}B(1,2,u) = 0.96124 \\ B(1,3,u) = 0.92377\end{array} & \begin{array}{c}\nearrow \\ \searrow\end{array} & \\
 & \nearrow & & & B(2,3,ud) = 0.97063 \\
\begin{array}{l}B(0,1) = 0.97080 \\ B(0,2) = 0.94260 \\ B(0,3) = 0.91510\end{array} & & & & \\
 & \searrow & & & B(2,3,du) = 0.97063 \\
 & & \begin{array}{l}B(1,2,d) = 0.98066 \\ B(1,3,d) = 0.96147\end{array} & \begin{array}{c}\nearrow \\ \searrow\end{array} & \\
 & & & & B(2,3,dd) = 0.99024
\end{array}$$

2.4 Interest Rate Derivatives in Discrete HJM Model

The pricing and hedging of interest rate derivatives in the discrete HJM model is very similar to hedging and pricing options in the CRR model, except that bonds are used in place of stock as the underlying securities, and discounting is done by means of the short rate. However, the types of derivatives can be quite different: apart from the usual calls and puts, there are derivatives such as callable bonds, swaps, caps, caplets, floors, or floorlets.

We begin with a simple example of a European call written on a zero-coupon bond. Such an option only makes sense if the exercise time of the option is before the maturity of the bond. This is because the bond value is deterministic at maturity, and following this the bond will no longer exist.

Example 8 In a model with three time steps, $T = 3$, one month each, $\tau = \frac{1}{12}$, and with bond prices as in the following tree, we consider a call on the bond maturing at 3 with strike price 0.9965 and expiry time 2:

$$\begin{array}{ccccc} & & & \nearrow & B(2,3,uu)=0.9991 \\ & & B(1,3,u)=0.9962 & r(1,u)=2.40\% & \\ & \nearrow & & \searrow & B(2,3,ud)=0.9960 \\ B(0,3)=0.9927 & r(0)=4.11\% & & & \\ & \searrow & & \nearrow & B(2,3,du)=0.9986 \\ & & B(1,3,d)=0.9959 & r(1,d)=3.00\% & \\ & & & \searrow & B(2,3,dd)=0.9969 \end{array}$$

The prices $C(t)$ of the option are computed exactly as in the CRR model, starting with the payoffs at time 2 and sweeping backwards through the tree:

$$\begin{array}{ccccc} & & & \nearrow & \begin{array}{l} C(2,uu)=0.0026 \\ B(2,3,uu)=0.9991 \end{array} \\ & & \begin{array}{l} C(1,u)=0.001837 \\ B(1,3,u)=0.9962 \end{array} & r(1,u)=2.40\% & \\ & \nearrow & & \searrow & \begin{array}{l} C(2,ud)=0 \\ B(2,3,ud)=0.9960 \end{array} \\ \begin{array}{l} C(0)=0.001849 \\ B(0,3)=0.9927 \end{array} & r(0)=4.11\% & & & \\ & \searrow & & \nearrow & \begin{array}{l} C(2,du)=0.0021 \\ B(2,3,du)=0.9986 \end{array} \\ & & \begin{array}{l} C(1,d)=0.001888 \\ B(1,3,d)=0.9959 \end{array} & r(1,d)=3.00\% & \\ & & & \searrow & \begin{array}{l} C(2,dd)=0.0004 \\ B(2,3,dd)=0.9969 \end{array} \end{array}$$

Call options can be used by bond writers to allow them to buy the bond back at a prescribed price prior to maturity. A bond that carries this provision is called a **callable bond**. In effect, we have the schematic formula

$$\text{callable bond} = \text{bond} - \text{call option}.$$

In particular, the price of a bond should be reduced by that of the associated call option to obtain the price of the callable bond.

A bond may be callable at a single fixed date only, in which case the associated call option will be of European type, or it may be possible to call the bond at any time up to maturity, which will correspond to an American call option.

Example 9 A callable bond that can be bought back at time 2 for 0.9965 can be constructed using the call option described in Example 8. If the bond and option prices are $B(0, 3) = 0.9927$ and $C(0) = 0.001849$ as in Example 8, then the price of the callable bond will be

$$B(0, 3) - C(0) = 0.9927 - 0.001849 = 0.99085.$$

Other examples of interest derivatives include caps and caplets, and floors and floorlets.

A **cap** is a provision attached to a floating rate bond that specifies the maximum coupon rate paid, called the **strike rate**, in each period over the lifetime of the loan. The strike rate is specified as a simple rate and is to be compared with the LIBOR rate. If the LIBOR rate exceeds the strike rate in any period, a coupon at the strike rate is paid for that period. Otherwise, a coupon at the LIBOR rate is paid.

A **caplet** is a similar provision applying to a single specified time period, that is, a single coupon payment. A caplet can be regarded as a European option written on the floating rate. A cap can be thought of and priced as a series of caplets.

A **floor** is similar to a cap and a **floorlet** is similar to a caplet, except that the strike rate specifies the minimum rather than the maximum coupon rate paid.

Example 10 For the tree of unit bond prices

$$
\begin{array}{lcl}
 & & B(2,3,uu) = 0.9905 \\
 & \begin{array}{l} B(1,2,u) = 0.9948 \\ B(1,3,u) = 0.9848 \end{array} & \\
 & & B(2,3,ud) = 0.9875 \\
\begin{array}{l} B(0,1) = 0.9901 \\ B(0,2) = 0.9828 \\ B(0,3) = 0.9726 \end{array} & & \\
 & & B(2,3,du) = 0.9908 \\
 & \begin{array}{l} B(1,2,d) = 0.9913 \\ B(1,3,d) = 0.9808 \end{array} & \\
 & & B(2,3,dd) = 0.9891
\end{array}
$$

with a one-month time step, $\tau = \frac{1}{12}$, consider a floating coupon bond with face value $F = 100$ maturing at time step 3. The bond has the par value 100 at time 0. The coupons are

$$C_t = \frac{1 - B(t-1,t)}{B(t-1,t)} F.$$

The cash flow for the writer of the bond is this:

$$
\begin{array}{ccc}
 & & -100.52272 \\
 & -0.99990 & \\
 & & -100.52272 \\
100 & & \\
 & & -100.87764 \\
 & -0.99990 & \\
 & & -100.87764
\end{array}
$$

Consider a caplet with strike rate $\lambda = 8.0267\%$ that applies at time 1. (This simple rate corresponds to a continuously compounded short rate of 8%.) The coupon payment is therefore capped at $\lambda \tau F = 0.66889$ at time 1. The caplet is worth $0.99990 - 0.66889 = 0.33101$ at both nodes at time 1. This can be regarded as a payoff of a European option with expiry time 1, the value of which at time 0 can be found in the usual manner by discounting and computing the risk neutral expectation. The result is shown in the tree

$$
\begin{array}{cc}
 & 0.33101 \\
0.32773 & \\
 & 0.33101
\end{array}
$$

The cash flow for the writer of the bond with the caplet attached is

$$
\begin{array}{ccc}
 & & -100.52272 \\
 & -0.66889 & \\
 & & -100.52272 \\
99.67227 & & \\
 & & -100.87764 \\
 & -0.66889 & \\
 & & -100.87764
\end{array}
$$

Now consider a caplet with the same strike rate $\lambda = 8.0267\%$ that applies at time 2 rather than 1. It caps the coupon payment at $\lambda \tau F = 0.66889$ at time 2. This will affect the coupons at nodes du and dd, but not at uu or ud. The caplet's value is $0.87764 - 0.66889 = 0.20875$ at nodes du and dd, and it is 0 at uu and ud. This can be regarded as a payoff of a European option with expiry at time 2, which can then be priced in the usual manner as shown here

$$
\begin{array}{ccccc}
 & & & \nearrow & 0 \\
 & \nearrow & 0 & & \\
 & & & \searrow & 0 \\
0.12677 & & & & \\
 & & & \nearrow & 0.20875 \\
 & \searrow & 0.20693 & & \\
 & & & \searrow & 0.20875
\end{array}
$$

The cash flow for the writer of the bond with this caplet attached is

$$
\begin{array}{ccccc}
 & & & \nearrow & -100.52272 \\
 & \nearrow & -0.99990 & & \\
 & & & \searrow & -100.52272 \\
99.87323 & & & & \\
 & & & \nearrow & -100.66889 \\
 & \searrow & -0.99990 & & \\
 & & & \searrow & -100.66889
\end{array}
$$

Finally, consider a cap with strike rate $\lambda = 8.0267\%$. It caps all coupon payments at $\lambda \tau F = 0.66889$. The cap can be considered as the superposition of the two caplets considered above. The time 0 value of the cap is therefore the sum of the two caplet values $0.32773 + 0.12677 = 0.45450$. The cash flow for the writer of the bond with the cap attached is

$$
\begin{array}{ccccc}
 & & & \nearrow & -100.52272 \\
 & \nearrow & -0.66889 & & \\
 & & & \searrow & -100.52272 \\
99.54550 & & & & \\
 & & & \nearrow & -100.66889 \\
 & \searrow & -0.66889 & & \\
 & & & \searrow & -100.66889
\end{array}
$$

3. Continuous Time Short Rate Models

We assume that the dynamics of the short rate is given by

$$dr(t) = \mu(t)dt + \sigma(t)dW(t), \tag{9}$$

where $W(t)$ is Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$, where P is the market probability (the probability that applies in reality). We also assume that the time t unit bond prices for any maturity $T > 0$ are determined by the short rate $r(t)$

$$B(t,T) = F^T(t, r(t)),$$

where $F^T(t,r)$ is a deterministic function of $(t,r) \in \mathbb{R}^2$, sufficiently regular to allow all transformations that follow below. We must have

$$F^T(T,r) = 1$$

for any T, r in order that $B(T,T) = F^T(T, r(T)) = 1$. We also assume that the model is free of arbitrage.

What are the consequences of these assumptions? What restrictions do they impose on bond prices? Are bond prices uniquely determined by them?

3.1 Risk Premium

To answer these questions we shall first investigate the relationship between bonds with different maturities S and T. To this end we construct a portfolio consisting of $x^T(t)$ bonds with maturity T and $x^S(t)$ bonds with maturity S at time t. The value of such a portfolio is

$$V(t) = x^T(t)F^T(t, r(t)) + x^S(t)F^S(t, r(t)).$$

Portfolios of this kind will form a self-financing strategy if

$$dV(t) = x^T(t)dF^T(t, r(t)) + x^S(t)dF^S(t, r(t)).$$

Using the Itô formula, we can write

$$\begin{aligned} dF^T &= \left(F^T_t + \mu F^T_r + \frac{1}{2}\sigma^2 F^T_{rr} \right) dt + \sigma F^T_r dW \\ &= \alpha^T F^T dt + \sigma^T F^T dW, \end{aligned} \tag{10}$$

where the subscripts $_t$ and $_r$ denote partial derivatives with respect to t and r, and where

$$\alpha^T = \frac{F^T_t + \mu F^T_r + \frac{1}{2}\sigma^2 F^T_{rr}}{F^T}, \tag{11}$$

$$\sigma^T = \frac{\sigma F^T_r}{F^T}. \tag{12}$$

There is also a similar formula for dF^S. We can, therefore, write

$$\begin{aligned} dV &= x^T dF^T + x^S dF^S \\ &= \left(x^T F^T \alpha^T + x^S F^S \alpha^S \right) dt + \left(x^T F^T \sigma^T + x^S F^S \sigma^S \right) dW \\ &= \left(w^T \alpha^T + w^S \alpha^S \right) V dt + \left(w^T \sigma^T + w^S \sigma^S \right) V dW, \end{aligned}$$

where

$$w^T = \frac{x^T F^T}{V}, \quad w^S = \frac{x^S F^S}{V}$$

are the weights in the portfolio. Suppose that we want to construct a strategy with risk free increments dV. This can be achieved if

$$w^T \sigma^T + w^S \sigma^S = 0.$$

Since $w^T + w^S = 1$,we can solve the last two equations for w^T and w^S to get

$$w^T = -\frac{\sigma^S}{\sigma^T - \sigma^S}, \quad w^S = \frac{\sigma^T}{\sigma^T - \sigma^S}.$$

If the weights are given by these formulae, then

$$\begin{aligned} dV &= \left(w^T \alpha^T + w^S \alpha^S\right) V dt \\ &= \frac{\sigma^T \alpha^S - \sigma^S \alpha^T}{\sigma^T - \sigma^S} V dt. \end{aligned}$$

To avoid arbitrage, the growth rate $\frac{\sigma^T \alpha^S - \sigma^S \alpha^T}{\sigma^T - \sigma^S}$ of V must be that of the risk free asset, the money account $B(t) = e^{\int_0^t r(u)du}$,for which

$$dB = rBdt. \tag{13}$$

It follows that

$$\frac{\sigma^T \alpha^S - \sigma^S \alpha^T}{\sigma^T - \sigma^S} = r.$$

Transforming this equality, we obtain

$$\frac{\alpha^S - r}{\sigma^S} = \frac{\alpha^T - r}{\sigma^T}.$$

Observe that the right-hand side does not depend on S, while the left-hand side does not depend on T. As a result, neither side depends on T or S. We come to the important conclusion that the quantity

$$\lambda = \frac{\alpha^T - r}{\sigma^T}$$

does not depend on T, that is, it is the same for all bonds, irrespective of their maturity T. It is called the **market price of risk**. These conclusions can be summarised in the following proposition.

Proposition 11 *In a bond market such that*

$$B(t, T) = F^T(t, r(t))$$

for each $T > 0$, *where the short rate* $r(t)$ *satisfies* (9), *there exists an adapted process* $\lambda(t)$ *such that*

$$\lambda = \frac{\alpha^T - r}{\sigma^T} \tag{14}$$

for each $T > 0$.

3.2 Term Structure Equation

Substituting the expressions (11) and (12) for α^T and σ^T into

$$\lambda = \frac{\alpha^T - r}{\sigma^T},$$

we find after a few transformations that

$$F_t^T + (\mu - \sigma\lambda) F_r^T + \frac{1}{2}\sigma^2 F_{rr}^T - rF^T = 0 \tag{15}$$

This partial differential equation, which should be considered together with he final condition

$$F^T(T, r) = 1, \tag{16}$$

is called the **term structure equation**.

Proposition 12 *In a bond market such that*

$$B(t, T) = F^T(t, r(t))$$

for each $T > 0$*, where the short rate* $r(t)$ *satisfies* (9)*,* F^T *will satisfy the term structure equation* (15) *with final condition* (16)*.*

An important conclusion of what we have done so far is that the dynamics (9) of the short rate alone (that is, the knowledge of $\mu(t)$ and $\sigma(t)$) does not determine the bond prices uniquely. However, as soon as we also know $\lambda(t)$, the bond prices are uniquely given as the solution of the term structure equation (15) with the final condition (16). Indeed, we do not need to know all three processes $\mu(t), \sigma(t), \lambda(t)$ but only the two processes $\sigma(t)$ and $\mu(t) - \sigma(t)\lambda(t)$ to have all the bond prices determined uniquely.

It also follows that, given the dynamics of the short rate

$$dr(t) = \mu(t)dt + \sigma(t)dW(t)$$

and the dynamics of the prices of a single bond (say the bond with the longest maturity T)

$$dB(t, T) = \alpha^T(t)B(t, T)dt + \sigma^T(t)B(t, T)dW(t),$$

which follows from (10), the bond prices $B(t, S)$ for all maturities $S < T$ are determined uniquely. This is so because the dynamics of $B(t, T)$ together with that of $r(t)$ determine the risk premium $\lambda(t)$ via (14), and this, in turn, fixes the prices of bonds for all maturities $S < T$.

3.3 Risk Neutral Probability

So far we have used the market probability P. We shall now study the risk neutral probability P^* that turns all discounted bond prices $B(t)^{-1}B(t, T)$ into martingales for each $T > 0$.

First, we consider the discounted bond prices $B(t)^{-1}B(t, T)$ under probability P. Using (10) for the stochastic differential $dF^T(t, r(t))$ and the fact that (13) implies

$$dB(t)^{-1} = -B(t)^{-2}dB(t) = -B(t)^{-2}r(t)B(t)dt = -r(t)B(t)^{-1}dt,$$

we compute

$$\begin{aligned}
d\left[B(t)^{-1}B(t,T)\right] &= d\left[B(t)^{-1}F^T(t,r(t))\right] \\
&= B^{-1}dF^T + F^T dB^{-1} \\
&= (F_t^T + \mu F_r^T + \frac{1}{2}\sigma^2 F_{rr}^T)B^{-1}dt + \sigma F_r^T B^{-1}dW - F^T r B^{-1}dt \\
&= (F_t^T + (\mu - \sigma\lambda)\, F_r^T + \frac{1}{2}\sigma^2 F_{rr}^T - rF^T)B^{-1}dt + \sigma F_r^T B^{-1}\,(\lambda dt + dW) \\
&= \sigma F_r^T B^{-1}\,(\lambda dt + dW)\,.
\end{aligned}$$

The last equality follows by the term structure equation (15).

For $B(t)^{-1}B(t,T)$ to be a martingale, the drift term λdt would need to disappear. This can be achieved by applying the Girsanov theorem to change P to a probability measure P^* under which the process

$$W^*(t) = \int_0^t \lambda(u)du + W(t)$$

becomes a Brownian motion. We note that P^* does not depend on T since λ does not depend on T either. We then have

$$d\left[B(t)^{-1}B(t,T)\right] = \sigma F_r^T B^{-1}dW^*, \tag{17}$$

which means that $B(t)^{-1}B(t,T)$ is a martingale under the new probability measure P^*, and this is so for any maturity $T > 0$. We have, therefore, constructed a risk neutral probability P^*. It is sometimes called the **spot martingale measure**.

3.4 Feynman-Kac Formula

Because $B(t)^{-1}B(t,T)$ is a martingale under P^*, we have

$$\mathbb{E}^{P^*}\left(B(T)^{-1}B(T,T)|\mathcal{F}_t\right) = B(t)^{-1}B(t,T).$$

Multiplying both sides by $B(t)$ and using the fact that $B(T,T) = 1$, we obtain the following representation for bond prices:

$$\begin{aligned}
B(t,T) &= \mathbb{E}^{P^*}\left(B(T)^{-1}B(t)|\mathcal{F}_t\right) \\
&= \mathbb{E}^{P^*}\left(e^{-\int_t^T r(u)du}\middle|\,\mathcal{F}_t\right), \qquad (18)
\end{aligned}$$

where $\mathcal{F}_t$ is the filtration generated by W^*. In particular,

$$B(0,T) = \mathbb{E}^{P^*}\left(e^{-\int_0^T r(u)du}\right).$$

This is often referred to as the **Feynman-Kac formula**, and has many important uses, in particular, in Monte Carlo numerical methods.

3.5 Short Rate Dynamics under Risk Neutral Probability

We can write (9) as

$$\begin{aligned}
dr(t) &= \mu(t)dt + \sigma(t)dW(t) \\
&= \left[\mu(t) - \sigma(t)\lambda(t)\right]dt + \sigma(t)\left[\lambda(t)dt + dW(t)\right] \\
&= \left[\mu(t) - \sigma(t)\lambda(t)\right]dt + \sigma(t)dW^*(t).
\end{aligned}$$

We have seen in Section 3.2 that all bond prices are uniquely determined by the two processes $\mu(t) - \sigma(t)\lambda(t)$ and $\sigma(t)$. This means that all bond prices are in fact uniquely determined by the dynamics of the short rate under the risk neutral probability P^* (but not necessarily by the dynamics of the short rate under P).

Based on this observation, a number of models have been proposed, which **specify the dynamics of the short rate under the risk neutral probability** P^*. In other words, these models specify a particular form of the processes $\mu(t) - \sigma(t)\lambda(t)$ and $\sigma(t)$. They come under the general name of **short rate models**. Below we shall study some specific examples of short rate models.

3.5.1 Merton Model

In 1973 Merton proposed to model the short rate as

$$dr(t) = adt + \sigma dW^*(t). \tag{19}$$

The model has two parameters $a, \sigma > 0$.

In this case the bond prices $B(t,T)$ can be computed explicitly with the help of the Feynman-Kac formula. For simplicity, we shall do this for $t = 0$, but the argument can easily be extended to any t.

To this end we shall first compute the distribution of the random variable $\int_0^T r(t)dt$ under P^*. Form (19) we have

$$r(t) = r(0) + at + \sigma W^*(t).$$

Since

$$d\left[(T-t)\,W^*(t)\right] = -W^*(t)dt + (T-t)\,dW^*(t),$$

it follows that

$$\int_0^T r(t)dt = Tr(0) + \frac{a}{2}T^2 + \sigma\int_0^T W^*(t)dt = Tr(0) + \frac{a}{2}T^2 + \sigma\int_0^T (T-t)\,dW^*(t).$$

The stochastic integral $\int_0^T (T-t)dW^*(t)$ has the normal distribution under P^* with mean 0 and variance $\int_0^T (T-t)^2 dt = \frac{1}{3}T^3$.As a result, $\int_0^T r(u)du$ is normally distributed under P^* with mean $Tr(0) + \frac{1}{2}aT^2$ and variance $\frac{1}{3}\sigma^2 T^3$.

We know that if a random variable X has normal distribution with mean m and variance s^2, then e^{-X} has expectation $e^{-m+s^2/2}$. It follows by the Feynman-Kac formula that

$$B(0,T) = \mathbb{E}^{P^*}\left(e^{-\int_0^T r(u)du}\right) = e^{-Tr(0)-\frac{1}{2}aT^2+\frac{1}{6}\sigma^2T^3}.$$

By a relatively simple extension of this argument we can also obtain

$$B(t,T) = \mathbb{E}^{P^*}\left(e^{-\int_t^T r(u)du}\middle|\,\mathcal{F}_t\right) = e^{-(T-t)r(t)-\frac{1}{2}a(T-t)^2+\frac{1}{6}\sigma^2(T-t)^3}.$$

Calibration To use a model of this kind in practice (in particular, to price derivative securities) we need to know the parameter values a and σ. It is important to understand that it does not make any sense at all to use the dynamics of the short rate

$$dr(t) = adt + \sigma dW^*(t)$$

under the risk neutral probability P^* because this is usually not the real market probability. Under the market probability P we have

$$dr(t) = (a + \sigma\lambda(t))\, dt + \sigma dW(t).$$

This could in fact be used to estimate σ, which is one of the parameters we need. We could also estimate $a + \sigma\lambda(t)$, but this is of little use in finding the value of a if we don't know $\lambda(t)$.

On the other hand, we do know the initial term structure, that is, the market prices of bonds $B^{\text{market}}(0, T)$ for any maturity T that is traded. We can therefore calibrate the model by choosing the parameters a, σ so that the theoretical values $B(0, T) = e^{-Tr(0)-\frac{1}{2}aT^2+\frac{1}{6}\sigma^2T^3}$ match the market values $B^{\text{market}}(0, T)$ as closely as possible. This is usually done using the least squares method.

3.5.2 Vasiček Model

The Merton Model is simple enough but unrealistic in that it implies unlimited growth of the risk free rate $r(t)$. The short rate model

$$dr(t) = (a - br(t))\, dt + \sigma dW^*(t),$$

put forward by Vasiček in 1977 deals with the problem by introducing a feature called **mean reversion** of the short rate. The model has three parameters $a, b, \sigma > 0$. The first term next to dt causes the short rate to be puled back towards the value $\frac{a}{b}$, the stronger the farther it is from this value.

In this case the equation for $r(t)$ is a stochastic differential equation. First of all, we need to try to solve it. Observe that

$$\begin{aligned} d\left[e^{bt}r(t)\right] &= be^{bt}r(t)dt + e^{bt}dr(t) \\ &= be^{bt}r(t)dt + e^{bt}(a - br(t))\, dt + \sigma e^{bt}dW^*(t) \\ &= ae^{bt}dt + \sigma e^{bt}dW^*(t). \end{aligned}$$

It follows that

$$\begin{aligned} e^{bt}r(t) &= r(0) + a\int_0^t e^{bu}du + \sigma\int_0^t e^{bu}dW^*(u) \\ &= r(0) + \frac{a}{b}\left(e^{bt} - 1\right) + \sigma\int_0^t e^{bu}dW^*(u), \end{aligned}$$

so that

$$r(t) = e^{-bt}r(0) + \frac{a}{b}\left(1 - e^{-bt}\right) + \sigma e^{-bt}\int_0^t e^{bu}dW^*(u).$$

It follows that

$$\begin{aligned} \int_0^T r(t)dt &= r(0)\int_0^T e^{-bt}dt + \frac{a}{b}\int_0^T\left(1 - e^{-bt}\right)dt + \sigma\int_0^T e^{-bt}\left[\int_0^t e^{bu}dW^*(u)\right]dt \\ &= r(0)\frac{1 - e^{-Tb}}{b} + a\int_0^T \frac{1 - e^{-b(T-t)}}{b}dt + \sigma\int_0^T e^{-bt}\left[\int_0^t e^{bu}dW^*(u)\right]dt. \end{aligned}$$

We need to study the last term, containing an iterated integral. There is a version of the Fubini theorem that allows swapping the order of the stochastic integral with respect to $dW^*(u)$ and the ordinary integral with respect to dt, but we haven't covered this in our course and therefore need to work a little bit harder around the problem. By the Itô formula

$$d\left[e^{bu}W^*(u)\right] = be^{bu}W^*(u)du + e^{bu}dW^*(u),$$

so that

$$\int_0^t e^{bu}dW^*(u) = e^{bt}W^*(t) - b\int_0^t e^{bu}W^*(u)du.$$

Using the Fubini theorem to swap the order of the integrals with respect to du and dt below, we obtain

$$\begin{aligned}
\int_0^T e^{-bt}\left[\int_0^t e^{bu}dW^*(u)\right]dt &= \int_0^T W^*(t)dt - b\int_0^T e^{-bt}\left[\int_0^t e^{bu}W^*(u)du\right]dt \\
&= \int_0^T W^*(t)dt - b\int_0^T e^{bu}W^*(u)\left[\int_u^T e^{-bt}dt\right]du \\
&= \int_0^T W^*(t)dt - b\int_0^T e^{bu}W^*(u)\left(-\frac{e^{-Tb} - e^{-bu}}{b}\right)du \\
&= \int_0^T W^*(t)dt - \int_0^T W^*(u)\left(-e^{-b(T-u)} + 1\right)du \\
&= \int_0^T W^*(u)e^{-b(T-u)}du.
\end{aligned}$$

Since

$$d\left[W^*(u)e^{-b(T-u)}\right] = e^{-b(T-u)}dW^*(u) + be^{-b(T-u)}W^*(u)du,$$

we therefore have

$$\begin{aligned}
\int_0^T e^{-bt}\left[\int_0^t e^{bu}dW^*(u)\right]dt &= \int_0^T W^*(u)e^{-b(T-u)}du \\
&= \frac{1}{b}W^*(T) - \frac{1}{b}\int_0^T e^{-b(T-u)}dW^*(u) \\
&= \int_0^T \frac{1 - e^{-b(T-u)}}{b}dW^*(u),
\end{aligned}$$

and finally

$$\begin{aligned}
\int_0^T r(t)dt &= r(0)\frac{1 - e^{-Tb}}{b} + a\int_0^T \frac{1 - e^{-b(T-t)}}{b}dt + \sigma\int_0^T e^{-bt}\left[\int_0^t e^{bu}dW^*(u)\right]dt \\
&= r(0)\frac{1 - e^{-Tb}}{b} + a\int_0^T \frac{1 - e^{-b(T-t)}}{b}dt + \sigma\int_0^T \frac{1 - e^{-b(T-u)}}{b}dW^*(u).
\end{aligned}$$

As a result, $-\int_0^T r(t)dt$ has the normal distribution with respect to P^* with mean

$$m = -r(0)\frac{1 - e^{-Tb}}{b} - a\int_0^T \frac{1 - e^{-b(T-t)}}{b}dt$$

and variance

$$s^2 = \sigma^2\int_0^T \left(\frac{1 - e^{-b(T-u)}}{b^2}\right)^2 du.$$

By the Feynman-Kac formula

$$B(0,T) = \mathbb{E}^{P^*}\left(e^{-\int_0^T r(u)du}\right) = e^{m+s^2/2} = e^{-r(0)\frac{1-e^{-Tb}}{b} - a\int_0^T \frac{1-e^{-b(T-t)}}{b}dt + \frac{\sigma^2}{2}\int_0^T \left(\frac{1-e^{-b(T-t)}}{b}\right)^2 dt}.$$

It is not very difficult to extend the above argument to obtain

$$B(t,T) = e^{-r(t)\frac{1-e^{-b(T-t)}}{b} - a\int_t^T \frac{1-e^{-b(T-u)}}{b}du + \frac{\sigma^2}{2}\int_t^T \left(\frac{1-e^{-b(T-u)}}{b}\right)^2 du}$$

for any $0 \leq t \leq T$.

3.5.3 Affine Term Structure Models

Observe that in both the Merton model and the Vasiček model the bond prices can be written as $B(t,T) = e^{m(t,T)-n(t,T)r(t)}$ with $m(t,T)$ and $n(t,T)$ being deterministic functions. Namely,

$$\begin{aligned} n(t,T) &= T - t \\ m(t,T) &= -a\int_t^T n(u,T)du + \frac{\sigma^2}{2}\int_t^T n(u,T)^2 du \\ &= -\frac{1}{2}a(T-t)^2 + \frac{1}{6}\sigma^2 (T-t)^3 \end{aligned}$$

in the Merton model, and

$$\begin{aligned} n(t,T) &= \frac{1-e^{-b(T-t)}}{b} \\ m(t,T) &= -a\int_t^T n(u,T)du + \frac{\sigma^2}{2}\int_t^T n(u,T)^2 du \\ &= -a\int_t^T \frac{1-e^{-b(T-u)}}{b}du + \frac{\sigma^2}{2}\int_t^T \left(\frac{1-e^{-b(T-u)}}{b}\right)^2 du \end{aligned}$$

in the Vasiček model.

In general, models of this kind, where

$$B(t,T) = e^{m(t,T)-n(t,T)r(t)}$$

for some deterministic functions $m(t,T)$ and $n(t,T)$, are called **affine term structure models**. The value of such models lies in their relative simplicity. In general, finding the deterministic functions $m(t,T)$ and $n(t,T)$ can be reduced to solving ordinary differential equations, resulting in computational efficiency of such models.

4. Continuous Time Heath-Jarrow-Morton Model

The principal difference between the short rate models studied above and the Heath-Jarrow-Morton (HJM) method is that the latter begins by specifying the dynamics for the entire term structure of forward rates $f(t,T)$ for all maturity dates T, rather than just the dynamics of the short rate $r(t)$.

This approach offers much more flexibility than short rate models and a perfect match of the model to the initial term structure.

The dynamics of forward rates under the market probability P is specified as

$$df(t,T) = \mu(t,T)dt + \sigma(t,T)dW(t),$$

where $W(t)$ is a Brownian motion under the market probability P. Note that the stochastic differential $df(t,T)$ applies to the t variable, whereas T is treated as a parameter.

The first question we need to address is that of lack of arbitrage in a model of this kind. To this end we shall compute $dB(t,T)$. If we can represent it in the form

$$dB(t,T) = B(t,T)\alpha^T(t)dt + B(t,T)\sigma^T(t)dW(t)$$

for some processes $\alpha^T(t)$ and $\sigma^T(t)$, then we can use the same argument as in Section 3.1 to conclude that the expression

$$\frac{\alpha^T(t) - r(t)}{\sigma^T(t)} = \lambda(t) \tag{20}$$

defining the risk premium $\lambda(t)$ should be independent of T.

Let us recall the relationship between the forward rates $f(t,T)$ and unit bond prices $B(t,T)$: $B(t,T) = e^{-\int_t^T f(t,u)du}$.Using the Itô formula, we can express the stochastic differential $dB(t,T)$ (with respect to the t variable, with T fixed) as

$$dB(t,T) = -B(t,T)d\int_t^T f(t,u)du + \frac{1}{2}B(t,T)\left(d\int_t^T f(t,u)du\right)^2.$$

Since

$$\begin{aligned}
d\int_t^T f(t,u)du &= -f(t,t)dt + \int_t^T (df(t,u))\,du \\
&= -r(t)dt + \int_t^T (\mu(t,u)dt + \sigma(t,u)dW(t))\,du \\
&= -r(t)dt + \left(\int_t^T \mu(t,u)du\right)dt + \left(\int_t^T \sigma(t,u)du\right)dW(t) \\
&= -r(t)dt + \tilde{\mu}(t,T)dt + \tilde{\sigma}(t,T)dW(t),
\end{aligned}$$

where

$$\begin{aligned}
\tilde{\mu}(t,T) &= \int_t^T \mu(t,u)du, \\
\tilde{\sigma}(t,T) &= \int_t^T \sigma(t,u)du,
\end{aligned}$$

it follows that

$$\begin{aligned}
dB(t,T) &= B(t,T)\left(r(t) - \tilde{\mu}(t,T) + \frac{1}{2}\tilde{\sigma}(t,T)^2\right)dt - B(t,T)\tilde{\sigma}(t,T)dW(t) \\
&= B(t,T)a^T(t)dt + B(t,T)\sigma^T(t)dW(t),
\end{aligned}$$

where

$$\alpha^T(t) = r(t) - \tilde{\mu}(t,T) + \frac{1}{2}\tilde{\sigma}(t,T)^2,$$
$$\sigma^T(t) = -\tilde{\sigma}(t,T).$$

Substituting this into (20), we obtain the condition

$$\tilde{\mu}(t,T) = \frac{1}{2}\tilde{\sigma}(t,T)^2 + \lambda(t)\tilde{\sigma}(t,T),$$

or, on differentiating with respect to T,

$$\mu(t,T) = \sigma(t,T)\tilde{\sigma}(t,T) + \lambda(t)\sigma(t,T). \tag{21}$$

This condition must be satisfied to ensure lack of arbitrage.

The dynamics of the forward rate can also be expressed as

$$\begin{aligned} df(t,T) &= \mu(t,T)dt + \sigma(t,T)dW(t) \\ &= \mu(t,T)dt + \sigma(t,T)\left(-\lambda(t)dt + dW^*(t)\right) = \alpha(t,T)dt + \sigma(t,T)dW^*(t), \end{aligned}$$

where $\alpha(t,T) = \mu(t,T) - \lambda(t)\sigma(t,T)$ and

$$W^*(t) = \int_0^t \lambda(u)du + W(t)$$

is a Brownian motion with respect to the risk neutral measure P^*, see Section 3.3.

Condition (21) for the lack of arbitrage can now be written as

$$\alpha(t,T) = \sigma(t,T)\tilde{\sigma}(t,T).$$

The conclusion is that to specify the forward rate dynamics under the risk neutral probability, we need to prescribe the diffusion coefficients $\sigma(t,T)$ for the forward rates. The drifts $\alpha(t,T)$ under the risk neutral probability are then uniquely determined by the last equality. This is in close analogy with what we have seen in the discrete HJM model.

The use of the HJM model can be outlined as follows:

- Specify a model for the volatilities $\sigma(t,T)$ with some parameters $a_1, \ldots, a_n$.
- The drifts of the forward rates under the risk neutral probability are now given by

$$\alpha(t,T) = \sigma(t,T)\tilde{\sigma}(t,T).$$

- Observe today's term structure of market forward rates $f^{\mathrm{m}}(0,T)$.
- Compute the forward rates as

$$f(t,T) = f^{\mathrm{m}}(0,T) + \int_0^t \alpha(u,T)du + \int_0^t \sigma(u,T)dW^*(u).$$

- Compute the bond prices using

$$B(t,T) = e^{-\int_t^T f(t,u)du}.$$

- Calibrate the model of volatilities by finding the values of the parameters $a_1, \ldots, a_n$ from historical volatility estimates and/or from implied volatilities for selected derivative securities.
- Use the above results to compute the prices of any given derivative securities.

Observe that this procedure achieves a perfect fit to the initial term structure of forward rates.

Example 13 Let us consider an example to illustrate this procedure. We take $\sigma(t,T)$ to be constant, $\sigma(t,T) = \sigma$,where $\sigma > 0$ (obtaining a continuous time version of the Ho-Lee model). As a result,

$$\alpha(t,T) = \sigma \int_t^T \sigma du = \sigma^2 (T-t)$$

and

$$\begin{aligned} f(t,T) &= f^{\mathrm{m}}(0,T) + \int_0^t \alpha(u,T) du + \int_0^t \sigma(u,T) dW^*(u) \\ &= f^{\mathrm{m}}(0,T) + \sigma^2 \int_0^t (T-t)\, dt + \sigma W^*(t) = f^{\mathrm{m}}(0,T) + \sigma^2 t \left(T - \frac{t}{2}\right) + \sigma W^*(t). \end{aligned}$$

It follows that

$$\begin{aligned} B(t,T) &= e^{-\int_t^T f(t,u)du} = e^{-\int_t^T \left(f^{\mathrm{m}}(0,u) + \sigma^2 t\left(u - \frac{t}{2}\right) + \sigma W^*(t)\right)du} \\ &= e^{-\int_t^T f^{\mathrm{m}}(0,u)du} e^{-\frac{1}{2}\sigma^2 t T(T-t) - \sigma(T-t)W^*(t)}. \end{aligned}$$

5. Forward Measure and Change of Numeraire Technique

Our motivation in this section comes from the need to price options, particularly interest rate options.

Consider a European option with payoff X and expiry time T. The time 0 price of the option can be expressed as

$$C(0) = \mathbb{E}^{P^*}\!\left(\frac{X}{B(T)}\right),$$

where P^* is the risk neutral probability and $B(T) = e^{\int_0^T r(u)du}$is the money market account.

The difficulty in computing the expectation $\mathbb{E}^{P^*}\!\left(\frac{X}{B(T)}\right)$ lies in the fact that we would need to know and use the joint distribution of the random variables X and $B(T)$ under P^*. This can be achieved in many cases, but often involves rather complicated and tedious calculations.

The technique discussed here can simplify the task of computing $\mathbb{E}^{P^*}\!\left(\frac{X}{B(T)}\right)$ significantly. We define a new measure P^T with density

$$\frac{dP^T}{dP^*} = \frac{1}{B(T)B(0,T)}$$

with respect to P^*, and call it the T**-forward measure**. It is a probability measure because

$$P^T(\Omega) = \mathbb{E}^{P^*}\!\left(\frac{1}{B(T)B(0,T)}\right) = \frac{1}{B(0,T)}\mathbb{E}^{P^*}\!\left(e^{-\int_0^T r(u)du}\right) = \frac{B(0,T)}{B(0,T)} = 1.$$

The option price can now be expressed as

$$C(0) = \mathbb{E}^{P^*}\!\left(\frac{X}{B(T)}\right) = B(0,T)\mathbb{E}^{P^*}\!\left(\frac{X}{B(T)B(0,T)}\right) = B(0,T)\mathbb{E}^{P^T}\!(X)\,.$$

To use the last formula we do not need to compute $B(0,T)$, as this is the current price given by the market. We only need to compute $\mathbb{E}^{P^T}\!(X)$, for which we need to know the distribution of X under the forward measure P^T, rather than the joint distribution of X and $B(T)$ under P^*. In many cases computing $\mathbb{E}^{P^T}\!(X)$ turns out much easier than computing $\mathbb{E}^{P^*}\!\left(\frac{X}{B(T)}\right)$.

The formula for the option price $C(0)$ in terms of the expectation under P^T is a consequence of the following general so-called **change or numeraire** property.

Theorem 14 *$\frac{X(t)}{B(t)}$ is a martingale under P^* if and only if $\frac{X(t)}{B(t,T)}$ is a martingale under P^T.*

Remark 15 In this context $B(t)$ is called the **numeraire** under P^*, and $B(t,T)$ the **numeraire** under P^T.

Proof First we verify that for any random variable Y

$$\mathbb{E}^{P^T}\!(Y|\,\mathcal{F}_t) = \mathbb{E}^{P^*}\!\left(\frac{B(t)}{B(t,T)}\frac{Y}{B(T)}\,\middle|\,\mathcal{F}_t\right). \tag{22}$$

Indeed, for any $\mathcal{F}_t$-measurable random variable η

$$\begin{aligned}
\mathbb{E}^{P^T}\!(Y\eta) &= \frac{1}{B(0,T)}\mathbb{E}^{P^*}\!\left(\frac{Y\eta}{B(T)}\right) = \frac{1}{B(0,T)}\mathbb{E}^{P^*}\!\left(\mathbb{E}^{P^*}\!\left(\frac{Y\eta}{B(T)}\,\middle|\,\mathcal{F}_t\right)\right)\\
&= \frac{1}{B(0,T)}\mathbb{E}^{P^*}\!\left(\frac{B(T,T)}{B(T)}\mathbb{E}^{P^*}\!\left(\frac{B(t)}{B(t,T)}\frac{Y\eta}{B(T)}\,\middle|\,\mathcal{F}_t\right)\right) = \mathbb{E}^{P^T}\!\left(\mathbb{E}^{P^*}\!\left(\frac{B(t)}{B(t,T)}\frac{Y}{B(T)}\,\middle|\,\mathcal{F}_t\right)\eta\right),
\end{aligned}$$

which proves (22).

Now suppose that $\frac{X(t)}{B(t)}$ is a martingale under P^*. Then

$$\begin{aligned}
\mathbb{E}^{P^T}\!\left(\frac{X(t)}{B(t,T)}\,\middle|\,\mathcal{F}_s\right) &= \frac{B(s)}{B(s,T)}\mathbb{E}^{P^*}\!\left(\frac{1}{B(T)}\frac{X(t)}{B(t,T)}\,\middle|\,\mathcal{F}_s\right) = \frac{B(s)}{B(s,T)}\mathbb{E}^{P^*}\!\left(\mathbb{E}^{P^*}\!\left(\frac{1}{B(T)}\,\middle|\,\mathcal{F}_t\right)\frac{X(t)}{B(t,T)}\,\middle|\,\mathcal{F}_s\right)\\
&= \frac{B(s)}{B(s,T)}\mathbb{E}^{P^*}\!\left(\frac{B(t,T)}{B(t)}\frac{X(t)}{B(t,T)}\,\middle|\,\mathcal{F}_s\right)\\
&= \frac{B(s)}{B(s,T)}\mathbb{E}^{P^*}\!\left(\frac{X(t)}{B(t)}\,\middle|\,\mathcal{F}_s\right) = \frac{B(s)}{B(s,T)}\frac{X(s)}{B(s)} = \frac{X(s)}{B(s,T)},
\end{aligned}$$

which means that $\frac{X(t)}{B(t,T)}$ is a martingale under P^T. The proof of the implication in the opposite direction is similar. ■

Now observe that if $C(t)$ is the time t price of an option with payoff X and expiry time T, then $\frac{C(t)}{B(t)}$ is a martingale under P^*. It follows by the last theorem that $\frac{C(t)}{B(t,T)}$ is a martingale under P^T. As a consequence, the forward measure P^T can also be used to express the option price at any time t prior to expiry T, and not just at time 0:

$$C(t) = B(t,T)\frac{C(t)}{B(T,t)} = B(t,T)\mathbb{E}^{P^T}\!\left(\frac{C(T)}{B(T,T)}\,\middle|\,\mathcal{F}_t\right) = B(t,T)\mathbb{E}^{P^T}\!(X|\,\mathcal{F}_t)\,.$$