CQF Module 1.4 Solved Problems

Stochastic Differential Equations and Itô's Lemma

1. Use Itô's lemma to show that

$$d\cos(X(t)) = \alpha\cos(X(t))\,dt + \beta\sin(X(t))\,dX$$

&

$$d\sin(X(t)) = \alpha\sin(X(t))\,dt - \beta\cos(X(t))\,dX$$

and determine the constants α & β.

Put

$$\left.\begin{array}{l} F = \cos(X(t)) \\ G = \sin(X(t)) \end{array}\right\} \Rightarrow \text{ Itô gives}$$

$$\left.\begin{array}{rl} dF &= \dfrac{\partial F}{\partial X}dX + \frac{1}{2}\dfrac{\partial^2 F}{\partial X^2}dt = -\sin(X)dX - \frac{1}{2}\cos(X)\,dt \\ dG &= \dfrac{\partial G}{\partial X}dX + \frac{1}{2}\dfrac{\partial^2 G}{\partial X^2}dt = \cos(X)dX - \frac{1}{2}\sin(X)\,dt \end{array}\right\}$$

comparing with earlier expressions gives

$$\alpha = -\tfrac{1}{2}; \quad \beta = -1$$

2. Consider the stochastic differential equation

$$dG(t) = a(G,t)\,dt + b(G,t)\,dX.$$

Find $a(G,t)$ and $b(G,t)$ where

(a) $G(t) = X^2(t)$

(b) $G(t) = 1 + t + \exp(X(t))$

(c) $G(t) = f(t)\,X(t)$, where f is a bounded and continuous function.

We use Itô's lemma on a function $G(X(t),t)$:

$$dG = \frac{\partial G}{\partial X}dX + \left(\frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial X^2}\right)dt.$$

a.

$$dG = 2X\,dX + dt = 2\sqrt{G}dX + dt.$$

Therefore

$$a(G,t) = 1 \text{ and } b(G,t) = 2\sqrt{G}$$

b.

$$dG = \exp\left(X\left(t\right)\right) dX + \left(1 + \tfrac{1}{2}\exp\left(X\left(t\right)\right)\right) dt.$$

Rearranging the formula for $G(t)$ we have $\exp(X(t)) = G(t) - 1 - t$, and so

$$dG = \underbrace{\left(G\left(t\right) - 1 - t\right)}_{b(G,t)} dX + \underbrace{\tfrac{1}{2}\left(1 + G\left(t\right) - t\right)}_{a(G,t)} dt.$$

c.

$$dG = f\left(t\right) dX + X\left(t\right)\frac{df}{dt} dt = f\left(t\right) dX + \frac{G\left(t\right)}{f\left(t\right)}\frac{df}{dt} dt$$

therefore

$$a\left(G,t\right) = \frac{G\left(t\right)}{f\left(t\right)}\frac{df}{dt} \text{ and } b\left(G,t\right) = f\left(t\right)$$

which gives us additional information that $f(t)$ should be non-zero. Obviously must differentiable or no solution exists.

3. The change in a share price $S(t)$ satisfies

$$dS = A\left(S,t\right) dX + B\left(S,t\right) dt,$$

for some functions A and B. If $f = f(S,t)$, then Itô's lemma gives the following stochastic differential equation

$$df = \left(\frac{\partial f}{\partial t} + B\frac{\partial f}{\partial S} + \frac{1}{2}A^2\frac{\partial^2 f}{\partial S^2}\right) dt + A\frac{\partial f}{\partial S} dX.$$

Can A and B be chosen so that a function $g = g(S)$ has a change which has zero drift, but non-zero diffusion? State any appropriate conditions.

A function $g(S)$ will satisfy the shorter SDE

$$dg = \left(B\frac{dg}{dS} + \frac{1}{2}A^2\frac{d^2 g}{dS^2}\right) dt + A\frac{dg}{dS} dX.$$

For $g(S)$ to have a zero drift but non-zero diffusion, we require the condition

$$B\frac{dg}{dS} + \frac{1}{2}A^2\frac{d^2 g}{dS^2} = 0.$$

i.e.

$$\frac{dg}{dS} + \frac{1}{2}\frac{A^2}{B}\frac{d^2 g}{dS^2} = 0$$

We can find a solution to this problem if $\frac{A^2}{B}$ is independent of time.

4. Show that $F = \arcsin\left(2aX\left(t\right) + \sin F_0\right)$ is a solution of the stochastic differential equation

$$dF = 2a^2\left(\tan F\right)\left(\sec^2 F\right)dt + 2a\left(\sec F\right)dX,$$

where $F_0 = F\left(0\right),\ X\left(0\right) = 0$ and a is a constant. **Hint: you may find the following useful**

$$\frac{d}{dx}\arcsin ux = \frac{u}{\sqrt{1-u^2x^2}}$$

$F = \arcsin\left(2aX\left(t\right) + \sin F_0\right)$ implies $\sin F = 2aX\left(t\right) + \sin F_0$ hence

$$\frac{dF}{dX} = \frac{2a}{\sqrt{1-\left(2aX+\sin F_0\right)^2}} = 2a\left\{1-\left(2aX+\sin F_0\right)^2\right\}^{-1/2}$$

$$\frac{d^2F}{dX^2} = \frac{\left(2a\right)^2\left(2aX\left(t\right)+\sin F_0\right)}{\left\{1-\left(2aX+\sin F_0\right)^2\right\}^{3/2}}$$

So Itô gives

$$dF = \frac{2a}{\sqrt{1-\left(2aX+\sin F_0\right)^2}}dX + \frac{1}{2}\frac{\left(2a\right)^2\left(2aX\left(t\right)+\sin F_0\right)}{\left\{1-\left(2aX+\sin F_0\right)^2\right\}^{3/2}}dt$$

We know $\cos^2 F + \sin^2 F = 1 \Longrightarrow \cos F = \sqrt{1-\sin^2 F} = \sqrt{1-\left(2aX+\sin F_0\right)^2}$. Some more trig.

$$\sec F = \frac{1}{\cos F} = \frac{1}{\sqrt{1-\left(2aX+\sin F_0\right)^2}}$$

and

$$\left(\tan F\right)\left(\sec^2 F\right) = \frac{\sin F}{\cos F}\frac{1}{\cos^2 F} = \frac{\sin F}{\cos^3 F} = \frac{2aX+\sin F_0}{\left\{1-\left(2aX+\sin F_0\right)^2\right\}^{3/2}}$$

which gives

$$dF = 2a^2\left(\tan F\right)\left(\sec^2 F\right)dt + 2a\left(\sec F\right)dX.$$

5. Show that

$$\int_0^t X\left(\tau\right)\left(1-e^{-X^2\left(\tau\right)}\right)dX\left(\tau\right) = F\left(X\left(t\right)\right) + \int_0^t G\left(X\left(t\right)\right)d\tau$$

where the functions F and G should be determined.
We can do this two ways. Perform Itô's lemma and then integrate or use the stochastic integral version as given in the first question. Comparing

$$\int_0^t X(\tau)\left(1-e^{-X^2(\tau)}\right)dX(\tau)=\overline{F}(X(t))+\int_0^t G(X(t))\,d\tau$$

with

$$\int_0^t \frac{\partial F}{\partial X}dX(\tau)=F(X(t),t)-F(X(0),0)+\int_0^t-\left(\frac{\partial F}{\partial \tau}+\tfrac{1}{2}\frac{\partial^2 F}{\partial X^2}\right)d\tau$$

suggests that

$$\frac{\partial F}{\partial X}=X(\tau)\left(1-e^{-X^2(\tau)}\right)$$

so integrating over $[0,t]$ gives $\overline{F}(X(t),t)$, which we will do by substitution, i.e. put $u=X^2$ which gives

$$F(X(t),t)-F(X(0),0)=\tfrac{1}{2}X^2(t)+\tfrac{1}{2}e^{-X^2(t)}-\tfrac{1}{2}.$$

Also knowing $\frac{\partial F}{\partial X}$ allows us to easily obtain $\frac{\partial^2 F}{\partial X^2}=2X^2(t)\,e^{-X^2(t)}-e^{-X^2(t)}+1$. Hence

$$G(X(t))=-\tfrac{1}{2}\frac{\partial^2 F}{\partial X^2}=-\tfrac{1}{2}\left(1-e^{-X^2(t)}\right)-X^2(t)\,e^{-X^2(t)}$$

and we have shown

$$\int_0^t X(\tau)\left(1-e^{-X^2(\tau)}\right)dX(\tau)=\overline{F}(X(t))+\int_0^t G(X(t))\,d\tau$$

where

$$\begin{aligned}\overline{F}(X(t),t)&=\tfrac{1}{2}X^2(t)+\tfrac{1}{2}e^{-X^2(t)}-\tfrac{1}{2}\\ G(X(t))&=-\tfrac{1}{2}\left(1-e^{-X^2(t)}\right)-X^2(t)\,e^{-X^2(t)}.\end{aligned}$$