

Exercise for Session 4.5

CQF

Exercise 1:

The objective of the exercise is to check that the following fact is true:

Fact 1. *If a process $Y(t)$ is a martingale under \mathbb{Q} and $\eta_t = \frac{d\mathbb{Q}}{d\mathbb{P}}$, then the process $M(t) = Y(t)\eta_t$ is a martingale under \mathbb{P} .*

We will focus on the case where both $Y(t)$ and $\eta(t)$ are modelled as diffusions processes with respective dynamics

$$dY(t) = f(t, Y(t))dt + g(t, Y(t))dX(t)$$

and

$$\frac{d\eta(t)}{\eta(t)} = -\theta(t)dX(t)$$

where $X(t)$ is a standard Brownian motion under the \mathbb{P} measure.

Questions -

- (i). Knowing that $Y(t)$ is a martingale under \mathbb{Q}^θ , express the drift function $f(\cdot)$ in terms of the diffusion function $g(\cdot)$ and of the process $\theta(t)$.
- (ii). Apply the Itô product rule to show that $M(t) = Y(t)\eta_t$ is a martingale under \mathbb{P} .

Exercise 2: (Optional)

Derive formula (25) on slide 80

$$C(t) = B(t, U)N[d_1(B(t, U), t, T)] - KB(t, T)N[d_2(B(t, U), t, T)] \quad (1)$$

where

$$\begin{aligned} d_1(b, t, T) &= \frac{\ln\left(\frac{b}{K}\right) - \ln B(t, T) + \frac{1}{2}v_U(t, T)}{v_U(t, T)} \\ d_2(b, t, T) &= d_1 - v_U(t, T) \\ v_U^2(t, T) &= \int_t^T (b(s, U) - b(s, T))^2 ds \end{aligned}$$

Start from the forward asset pricing formula given in equation (24), on slide 79,

$$C(t) = B(t, T) \mathbf{E}^{\mathbb{P}^T} [(F_B(T, T, U) - K)^+ | \mathcal{F}_t] \quad (2)$$

where the dynamics of the forward price $F_B(t, T, U)$ is given in equations (22) and (23) on slide 78.

Hints:

1. you could use an approach similar to the derivation of the Black-Scholes formula presented in Section 3.3 of Lecture 3.3 (slides 63-75);
2. Note that the random variable $Y(T) = \int_t^T (b(s, U) - b(s, T)) dX^T(s)$ is Normally distributed with mean 0 and variance $v_U^2(t, T)$.