

Know Your Weapon

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Market Formula (Bachelier-Thorp)

$$c = Se^{(b-r)T} N(d_1) - Xe^{-rT} N(d_2)$$

$$p = Xe^{-rT} N(-d_2) - Se^{(b-r)T} N(-d_1)$$

Where:

$$d_1 = \frac{\ln(S/X) + (b + \sigma_{X,T}^2/2)T}{\sigma_{X,T} \sqrt{T}}$$
$$d_2 = d_1 - \sigma_{X,T} \sqrt{T}$$

S = Asset price

X = Strike

T = Years to maturity

r = risk - free - rate

b = cost - of - carry

$\sigma_{X,T}$ = volatility that can be different for each strike and maturity

See Haug 2007 "Derivatives Models on Models" chapter 2

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Black-Scholes-Merton

$$c = Se^{(b-r)T} N(d_1) - Xe^{-rT} N(d_2)$$

$$p = Xe^{-rT} N(-d_2) - Se^{(b-r)T} N(-d_1)$$

Where:

$$d_1 = \frac{\ln(S/X) + (b + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

S = Asset price

X = Strike

T = Years to maturity

r = risk - free - rate

b = cost - of - carry

σ = volatility

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Delta Greeks

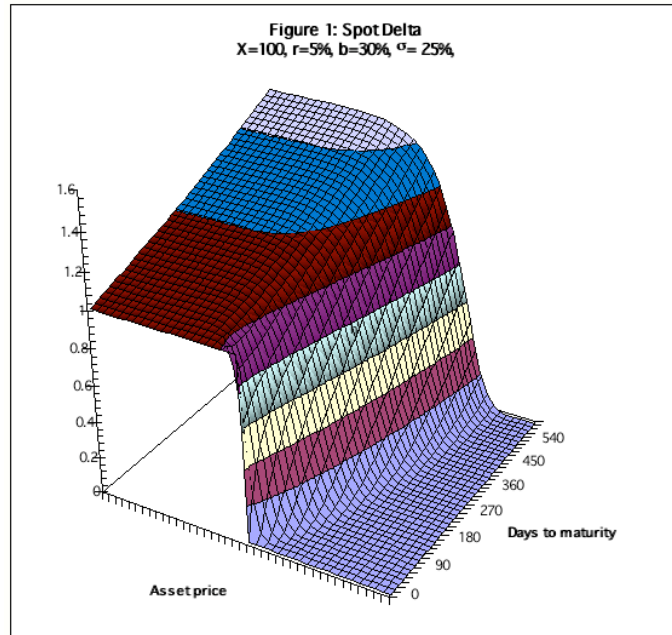
- Delta
- Delta mirror strikes
- Strike from delta
- Elasticity

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Delta higher than one

$$\Delta_{call} = \frac{\partial c}{\partial S} = e^{(b-r)T} N(d_1)$$

$$\Delta_{put} = \frac{\partial p}{\partial S} = -e^{(b-r)T} N(-d_1)$$



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Delta Mirror Strikes

$$X_P = \frac{S^2}{X_C} e^{(2b+\sigma^2)T}, \quad X_C = \frac{S^2}{X_P} e^{(2b+\sigma^2)T}$$

Special case delta symmetric straddle (Wystrup(1999)):

$$X_C = X_P = S e^{(b+\sigma^2/2)T}$$

Delta symmetric asset: $S = X e^{(-b-\sigma^2/2)T}$

At this strike the delta is $\Delta_C = \frac{e^{(b-r)T}}{2}, \quad \Delta_P = -\frac{e^{(b-r)T}}{2}$

$$c = \frac{S e^{(b-r)T}}{2} - X^{-rT} N(-\sigma\sqrt{T}), \quad p = X^{-rT} N(\sigma\sqrt{T}) - \frac{S e^{(b-r)T}}{2}$$

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Strikes from delta

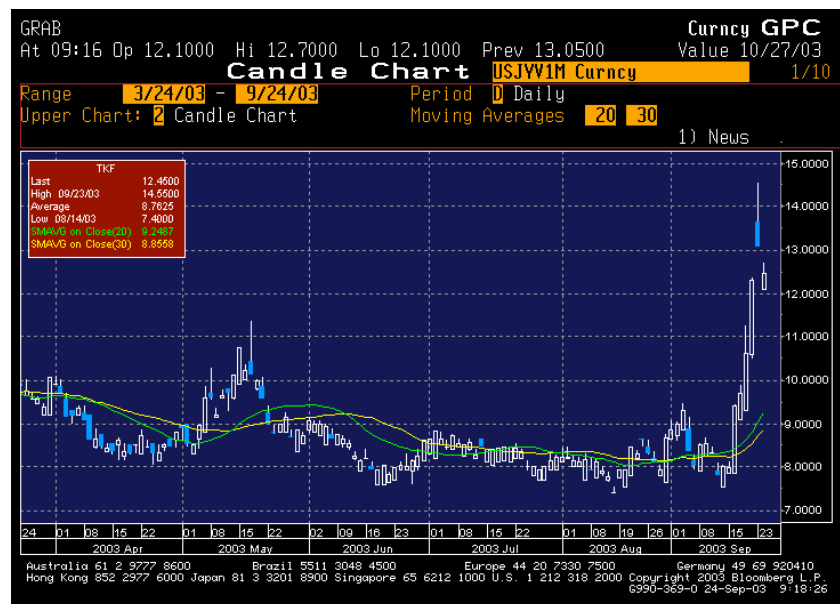
Wystrup(1999):

$$X_C = S \exp[N^{-1}(\Delta_C e^{(r-b)T})\sigma\sqrt{T} + (b + \sigma^2/2)T]$$

$$X_P = S \exp[N^{-1}(-\Delta_P e^{(r-b)T})\sigma\sqrt{T} + (b + \sigma^2/2)T]$$

Robust and accurate approximation of inverse cumulative normal distribution needed, Moro(1995).

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$$\text{DdeltaDvol} \quad \frac{\partial c}{\partial S \partial \sigma} = \frac{\partial p}{\partial S \partial \sigma} = \frac{-e^{(b-r)T} d_2}{\sigma} n(d_1)$$

Maximal value at

$$S_L = Xe^{-bT - \sigma \sqrt{T} \sqrt{4 + T\sigma^2} / 2}$$

Minimal value at

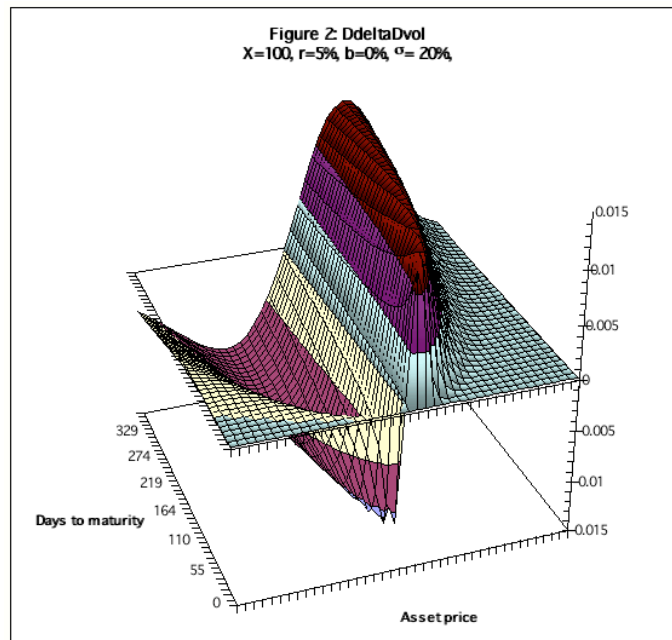
$$S_U = Xe^{-bT + \sigma \sqrt{T} \sqrt{4 + T\sigma^2} / 2}$$

Minimal value at

$$X_L = Se^{bT - \sigma \sqrt{T} \sqrt{4 + T\sigma^2} / 2}$$

Maximal value at

$$X_U = Se^{bT + \sigma \sqrt{T} \sqrt{4 + T\sigma^2} / 2}$$



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Useful Tools

- A library
- Paper and pencil
- Mathematica
- Maple
- Matlab (?)
- Others ?

Implementation:

VBA, VB, C/C++, Java....

you name it

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Elasticity

$$\Lambda_{call} = \Delta_{call} \frac{S}{call}, \quad \Lambda_{put} = \Delta_{put} \frac{S}{put}$$

Option volatility: $\sigma_O \approx \sigma |\Lambda|$ Compound options

Option Beta, expected return satisfy the CAPM equation (Merton-71):

$$E[return] = r + E[r_m - r] \beta_i$$

$$\beta_C = \frac{S}{call} \Delta_C \beta_S = \Lambda_C \beta_S, \quad \beta_P = \frac{S}{put} \Delta_P \beta_S = \Lambda_P \beta_S$$

Option Sharp ratios

$$\frac{\mu_O - r}{\sigma_O} = \frac{\mu_S - r}{\sigma}$$

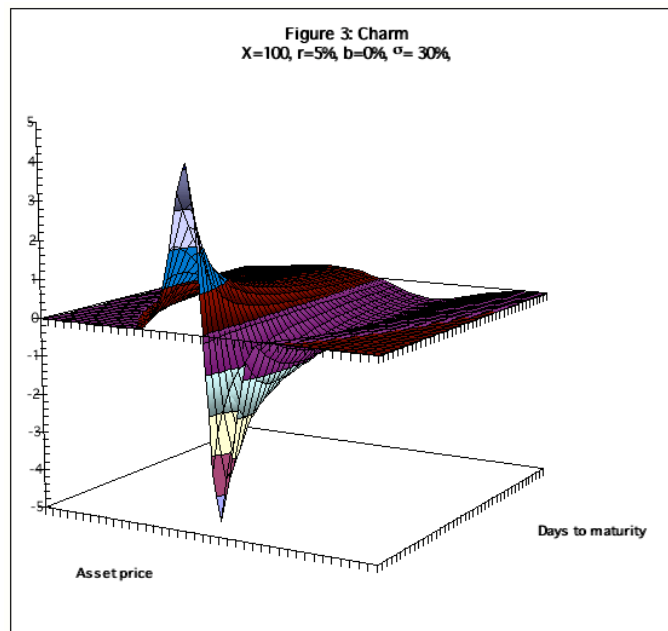
Smile?

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Charm

$$\frac{\partial \Delta_C}{\partial T} = -e^{(b-r)T} \left[n(d_1) \left(\frac{b}{\sigma \sqrt{T}} - \frac{d_2}{2T} \right) + (b-r) N(d_1) \right]$$

$$\frac{\partial \Delta_P}{\partial T} = -e^{(b-r)T} \left[n(d_1) \left(\frac{b}{\sigma \sqrt{T}} - \frac{d_2}{2T} \right) - (b-r) N(-d_1) \right]$$



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Gamma Greeks

- Gamma
- Saddle gamma
- GammaP
- Gamma symmetry
- DGammaDVol
- DGammaDspot
- DGammaDTime

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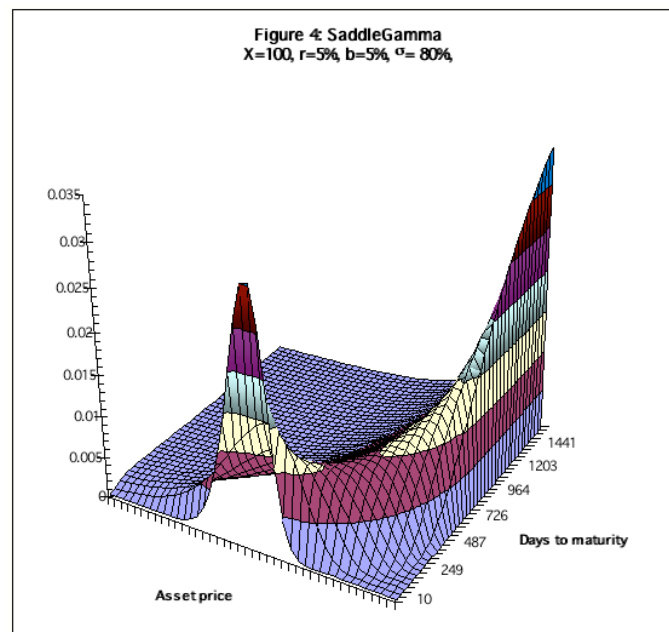
Saddle Gamma

Alexander Adamchuk www.wilmott.com

$$T_{\Gamma} = \frac{1}{2(\sigma^2 + b)}$$

$$S_{\bar{\Gamma}} = Xe^{(-b-3\sigma^2/2)T_S}$$

$$\Gamma_S = \frac{e^{(b-r)T} \sqrt{\frac{e}{\pi}} \sqrt{\frac{b}{\sigma^2} + 1}}{X}$$



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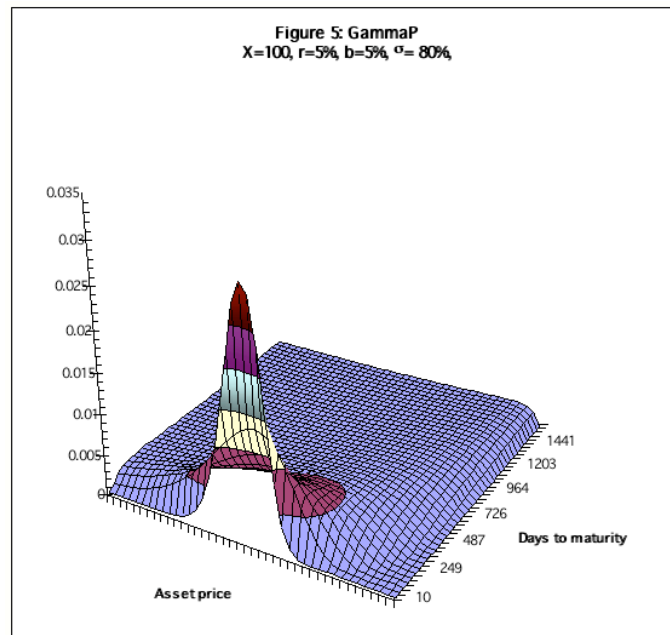
GammaP

$$\Gamma_P = \Gamma \frac{S}{100}$$

Max GammaP at

$$S = Xe^{(-b-\sigma^2/2)T}$$

$$X = Se^{(b+\sigma^2/2)T}$$

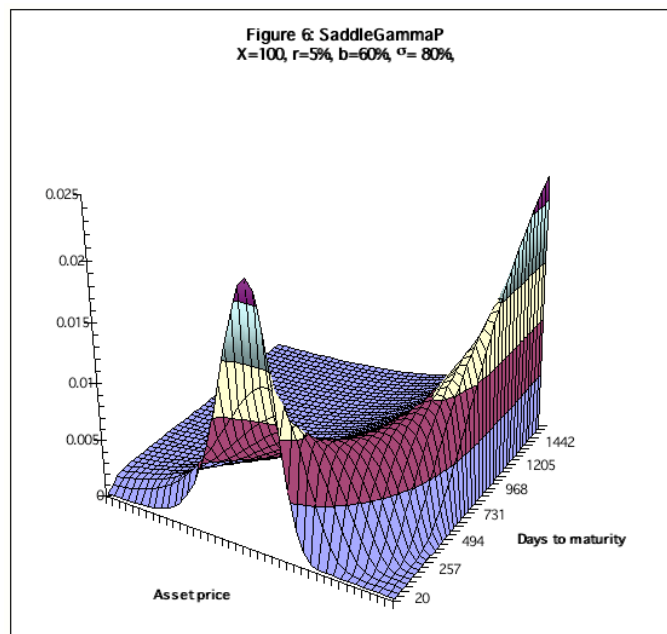


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Saddle GammaP

•Spot gamma

•Forward gamma



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Gamma-symmetry

Put-call symmetry Bates(1991) and Carr and Bowie (1994):

$$c(S, X, T, r, b, \sigma) = \frac{X}{Se^{bT}} p\left(S, \frac{(Se^{bT})^2}{X}, T, r, b, \sigma\right)$$

Gamma-symmetry

$$\Gamma(S, X, T, r, b, \sigma) = \frac{X}{Se^{bT}} \Gamma\left(S, \frac{(Se^{bT})^2}{X}, T, r, b, \sigma\right)$$

Also gives vega and cost-of-carry symmetry

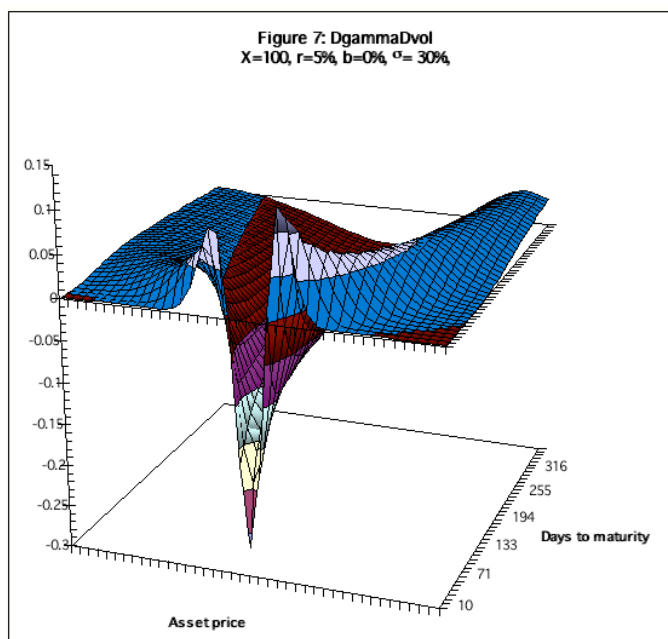
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$$\text{DgammaDvol} \quad \frac{\partial \Gamma}{\partial \sigma} = \Gamma \left(\frac{d_1 d_2 - 1}{\sigma} \right) \quad \frac{\partial \Gamma_P}{\partial \sigma} = \Gamma_P \left(\frac{d_1 d_2 - 1}{\sigma} \right)$$

Positive outside interval

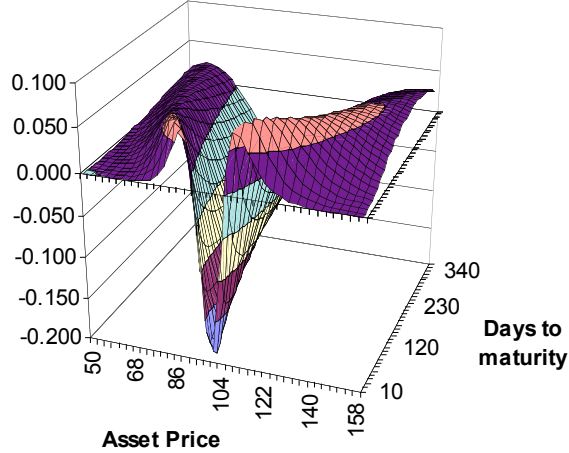
$$S_L = Xe^{-bT - \sigma \sqrt{T} \sqrt{4 + T\sigma^2}} / 2$$

$$S_U = Xe^{-bT + \sigma \sqrt{T} \sqrt{4 + T\sigma^2}} / 2$$



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Merton Jump-Diffusion
Vol 30%, Jumps 3, Vol form Jumps 40%



	Call
Asset price (S)	80.00
Strike price (X)	100.00
Time to maturity (T)	0.25
Risk-free rate (r)	5.00%
Volatility (σ)	30.00%
Jumps per year (λ)	3.00
Percent of total volatility (γ)	40.00%
Value	0.5255

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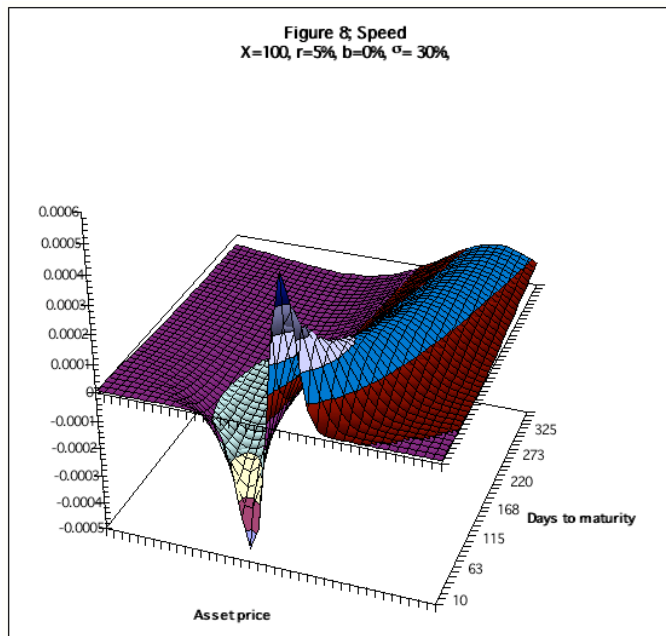
Speed (DgammaDspot)

$$\frac{\partial^3 c}{\partial S^3} = -\frac{\Gamma \left(1 + \frac{d_1}{\sigma \sqrt{T}} \right)}{S}$$

$$SpeedP = -\Gamma \frac{d_1}{S}$$

Speed is used by
Fouque, Papanicolaou,
and Sircar (2000) as part
of stochastic vol model

Figure 8: Speed
X=100, r=5%, b=0%, σ= 30%

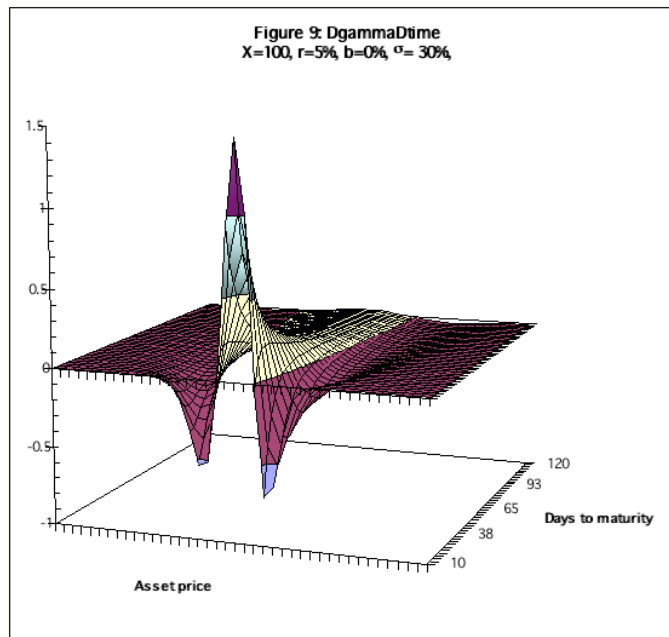


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DgammaDtime

$$\frac{\partial \Gamma}{\partial T} = \Gamma \left(r - b + \frac{bd_1}{\sigma\sqrt{T}} + \frac{1 - d_1d_2}{2T} \right)$$

$$\frac{\partial \Gamma_P}{\partial T} = \Gamma_P \left(r - b + \frac{bd_1}{\sigma\sqrt{T}} + \frac{1 - d_1d_2}{2T} \right)$$



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Numerical Greeks

- More robust (?)
- Model independent
- Faster to implement (?)

Two-sided finite difference

$$\Delta_C \approx \frac{c(S + \Delta S, X, T, r, b, \sigma) - c(S - \Delta S, X, T, r, b, \sigma)}{2\Delta S}$$

Backward derivative

$$\Theta \approx \frac{c(S, X, T, r, b, \sigma) - c(S, X, T - \Delta T, r, b, \sigma)}{\Delta T}$$

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Numerical Greeks

$$\Delta_C \approx \frac{c(S + \Delta S, X, T, r, b, \sigma_1) - c(S - \Delta S, X, T, r, b, \sigma_2)}{2\Delta S}$$

$$\Theta \approx \frac{c(S, X, T, r, b, \sigma_1) - c(S, X, T - \Delta T, r, b, \sigma_2)}{\Delta T}$$

Gamma and other second derivatives, central finite difference

$$\Gamma \approx \frac{c(S + \Delta S, \dots) - 2c(S, \dots) + c(S - \Delta S, \dots)}{\Delta S^2}$$

Speed and other third order derivatives, central finite difference

$$Speed \approx \frac{1}{\Delta S^3} [c(S + 2\Delta S, \dots) - 3c(S + \Delta S, \dots) + 3c(S, \dots) - c(S - \Delta S, \dots)]$$

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Know Your Weapon Part 2

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Numerical Greeks

What about mixed derivatives? For example DdeltaDvol and Charm

$$DdeltaDvol \approx \frac{1}{4\Delta S\Delta\sigma} [c(S + \Delta S, \dots, \sigma + \Delta\sigma) - c(S + \Delta S, \dots, \sigma - \Delta\sigma) - c(S - \Delta S, \dots, \sigma + \Delta\sigma) + c(S - \Delta S, \dots, \sigma - \Delta\sigma)]$$

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Vega “Greeks”

- Vega
- Vega maximum
- VegaP
- Vega symmetry
- Vega Leverage
- DVegaDvol
- DVegaDtime

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$$\text{Vega} \quad \frac{\partial c}{\partial \sigma} = Se^{(b-r)T} n(d_1) \sqrt{T}$$

Vega local max

$$S = Xe^{(-b+\sigma^2/2)T}$$

$$X = Se^{(b+\sigma^2/2)T}$$

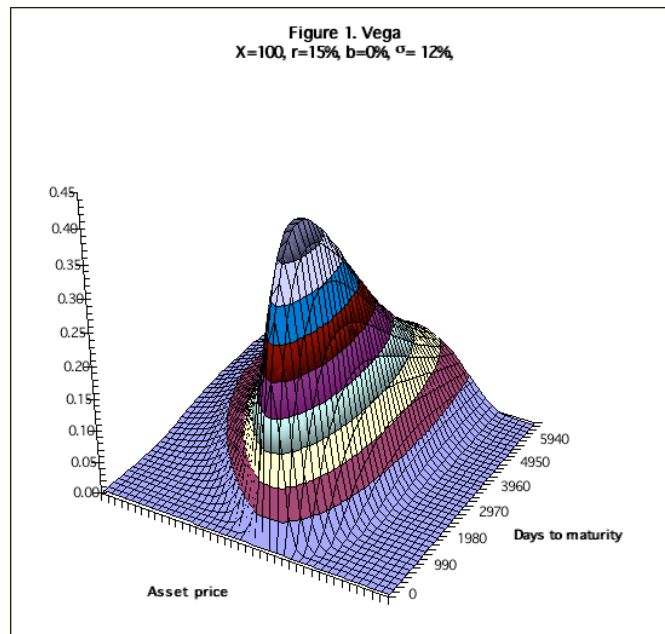
Global maximum

$$T_V = \frac{1}{2r}$$

$$S_V = Xe^{(-b+\sigma^2/2)T_V}$$

$$= Xe^{\frac{-b+\sigma^2/2}{2r}}$$

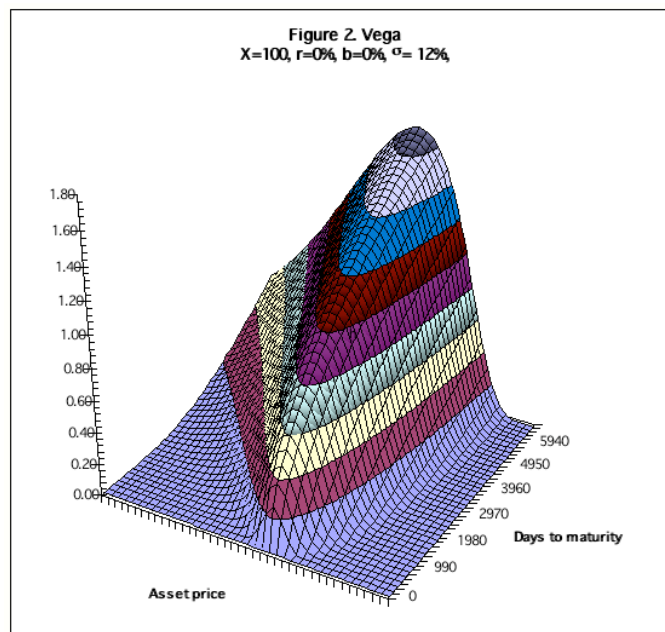
$$\text{Vega}(S_V, T_V) = \frac{X}{2\sqrt{re\pi}}$$



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Why the Vega top?

Discounting at some point will dominate over volatility effect (Vega).



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Vega-symmetry

Put-call symmetry Bates(1991) and Carr and Bowie (1994):

$$c(S, X, T, r, b, \sigma) = \frac{X}{Se^{bT}} p(S, \frac{(Se^{bT})^2}{X}, T, r, b, \sigma)$$

Vega-symmetry

$$Vega(S, X, T, r, b, \sigma) = \frac{X}{Se^{bT}} Vega(S, \frac{(Se^{bT})^2}{X}, T, r, b, \sigma)$$

Also gives gamma and cost-of-carry symmetry

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Vega-gamma relationship

Taleb(1997):

$$Vega = \Gamma \sigma S^2 T$$

Vega from delta

$$Vega = Se^{(b-r)T} \sqrt{T} n[N^{-1}(e^{(r-b)T} | \Delta |)]$$

Gamma from delta

$$\Gamma = \frac{e^{(b-r)T} n[N^{-1}(e^{(r-b)T} | \Delta |)]}{S \sigma \sqrt{T}}$$

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VegaP

Vega gives dollar change in option value for one percent point change in implied volatility. VegaP gives dollar change in option value for percentage move in volatility.

$$VegaP = \frac{\sigma}{10} S e^{(b-r)T} n(d_1) \sqrt{T}$$

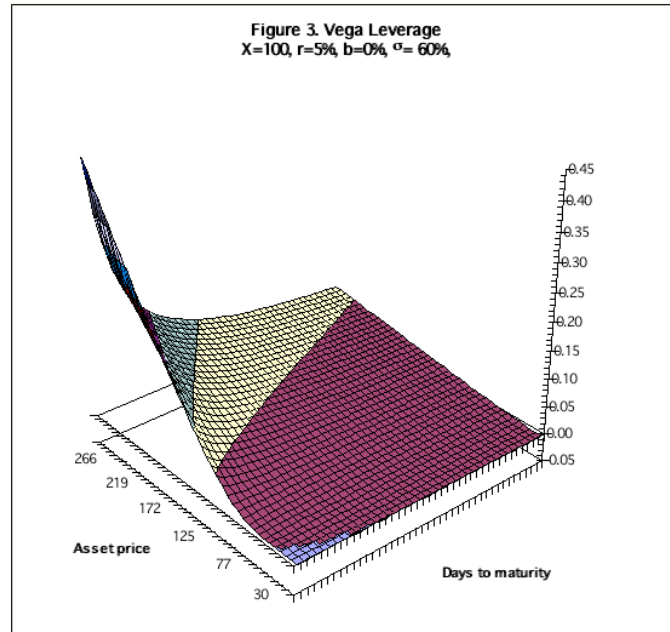
VegaP makes much more sense when comparing sensitivity to changes in Implied volatility.

If you want to speculate on an increase in implied volatility what type of options offers the most bang for the bucks?

Vega leverage

Percent change in option value for percent point change in implied volatility.

$$Vega \frac{\sigma}{call}, \quad Vega \frac{\sigma}{put}$$



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DvegaDvol Vomma/Volga $\frac{\partial^2 c}{\partial \sigma^2} = Vega \left(\frac{d_1 d_2}{\sigma} \right)$

Positive outside

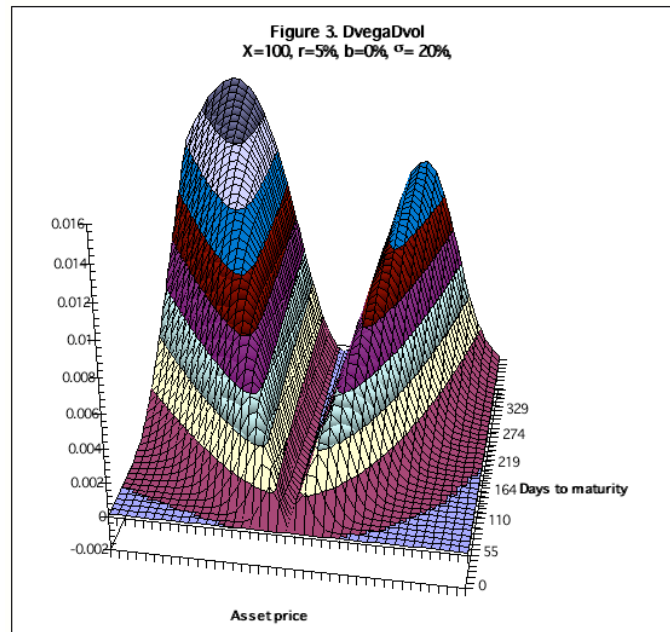
$$S_L = Xe^{(-b-\sigma^2/2)T}$$

$$S_U = Xe^{(-b+\sigma^2/2)T}$$

Positive outside

$$X_L = Se^{(b-\sigma^2/2)T}$$

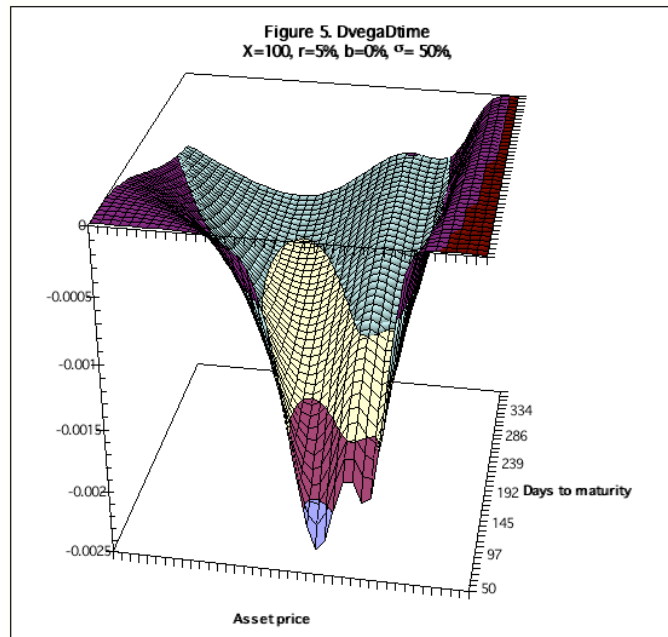
$$S_U = Se^{(b+\sigma^2/2)T}$$



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DvegaDtime

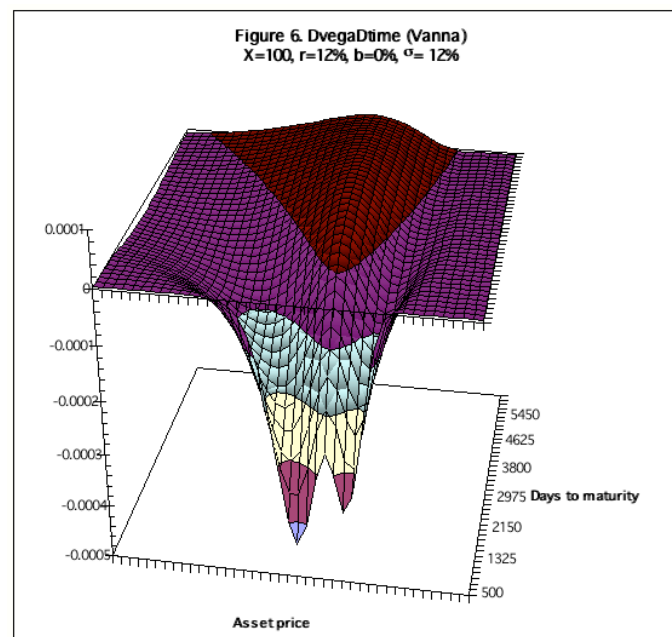
$$\frac{\partial^2 c}{\partial \sigma \partial T} = Vega \left(r - b + \frac{bd_1}{\sigma \sqrt{T}} - \frac{1 + d_1 d_2}{2T} \right)$$



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DvegaDtime

$$\frac{\partial^2 c}{\partial \sigma \partial T} = Vega \left(r - b + \frac{bd_1}{\sigma \sqrt{T}} - \frac{1 + d_1 d_2}{2T} \right)$$



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Theta

$$\Theta_C = -\frac{\partial c}{\partial T} = -\frac{Se^{(b-r)T}n(d_1)}{2\sqrt{T}} - (b-r)Se^{(b-r)T}N(d_1) - rXe^{-rT}N(d_2)$$

$$\Theta_C = -\frac{\partial c}{\partial T} = -\frac{Se^{(b-r)T}n(d_1)\sigma}{2\sqrt{T}} + (b-r)Se^{(b-r)T}N(-d_1) + rXe^{-rT}N(-d_2)$$

Drift-less theta

$$\theta_C = \theta_P = -\frac{Sn(d_1)}{2\sqrt{T}}$$

Theta symmetry

$$\theta(S, X, T, 0, 0, \sigma) = \frac{X}{S} \theta\left(S, \frac{S^2}{X}, T, 0, 0, \sigma\right)$$

Bleed-offset volatility

$$\frac{\Theta}{Vega}$$

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Rho

$$\rho_C = \frac{\partial c}{\partial r} = TXe^{-rT}N(d_2), \quad \rho_P = \frac{\partial p}{\partial r} = -TXe^{-rT}N(-d_2)$$

In case of options on futures (b=0)

$$\rho_C = \frac{\partial c}{\partial r} = -Tc, \quad \rho_P = \frac{\partial p}{\partial r} = -Tp$$

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Probability “Greeks”

Risk neutral probability of ending up in-the-money

$$\xi_C = N(d_2) > 0, \quad \xi_P = N(-d_2) > 0$$

Strike-delta

$$\frac{\partial c}{\partial X} = -e^{-rT} N(d_2), \quad \frac{\partial p}{\partial X} = e^{-rT} N(-d_2)$$

Probability mirror strikes

$$X_P = \frac{S^2}{X_C} e^{(2b-\sigma^2)T}, \quad X_C = \frac{S^2}{X_P} e^{(2b-\sigma^2)T}$$

Probability neutral straddle

$$X_C = X_P = S e^{(b-\sigma^2/2)T}$$

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Probability “Greeks”

Strikes from probability

$$X_C = S \exp[-N^{-1}(p_i)\sigma\sqrt{T} + (b - \sigma^2/2)T]$$

$$X_P = S \exp[N^{-1}(-p_i)\sigma\sqrt{T} + (b - \sigma^2/2)T]$$

Risk neutral probability density

$$RND = \frac{\partial^2 c}{\partial X^2} = \frac{\partial^2 p}{\partial X^2} = \frac{n(d_2)e^{-rT}}{X\sigma\sqrt{T}}$$

Probability neutral straddle

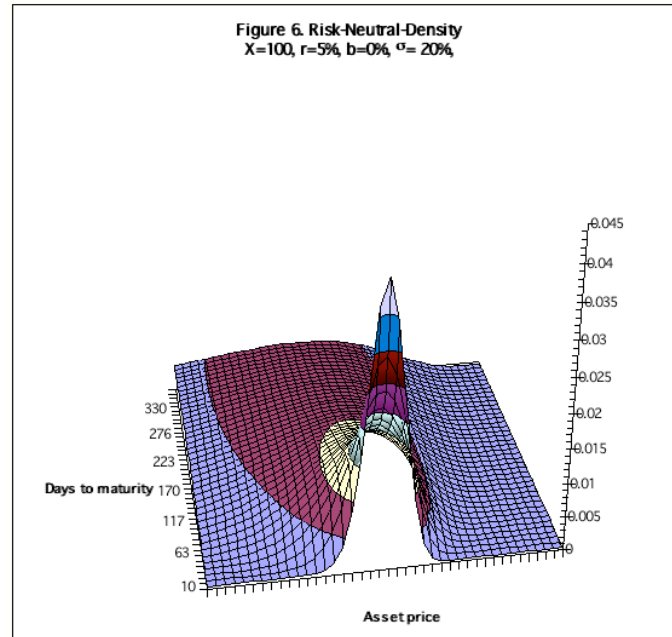
$$X_C = X_P = S e^{(b-\sigma^2/2)T}$$

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Risk neutral probability density

$$RND = \frac{\partial^2 c}{\partial X^2} = \frac{\partial^2 p}{\partial X^2} = \frac{n(d_2)e^{-rT}}{X\sigma\sqrt{T}}$$

Breeden and
Litzenberger (1978)



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Probability “Greeks”

Risk neutral probability of ever being in-the-money

$$p_C = (X/S)^{\mu+\lambda} N(-z) + (X/S)^{\mu-\lambda} N(-z + 2\lambda\sigma\sqrt{T})$$

$$p_P = (X/S)^{\mu+\lambda} N(z) + (X/S)^{\mu-\lambda} N(z - 2\lambda\sigma\sqrt{T})$$

where

$$z = \frac{\ln(X/S)}{\sigma\sqrt{T}} + \lambda\sigma\sqrt{T}, \quad \mu = \frac{b - \sigma^2/2}{\sigma^2}, \quad \lambda = \sqrt{\mu^2 + \frac{2r}{\sigma^2}}$$

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