CQF Exercises 3.1 Solutions

In problem sheet 3.1 question 2 reference is made to working in another exam - which is no longer used. To assist you here is the full solution of the same problem, but in the absence of dividends.

2. The Black–Scholes Equation (BSE) in the absence of dividends is

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$

Find all separable solutions of the form $V(S,t) = \Phi(S)\,\Psi(t)$.

Seek separable solution $V(S,t) = \Psi(t)\,\Phi(S)$
So

$$\frac{\partial V}{\partial t} = \Psi'\Phi, \quad \frac{\partial V}{\partial S} = \Phi'\Psi \Rightarrow \frac{\partial^2 V}{\partial S^2} = \Phi''\Psi$$

and we substitute in BSE and re-arrange so that all functions of S are on one side and all functions of t on the other, giving:

$$\frac{\Psi'}{\Psi} = \frac{-\frac{1}{2}\sigma^2 S^2 \Phi'' - rS\Phi' + r\Phi}{\Phi} = \text{constant} \ \ (\text{e.g. } c)\,.$$

(when using Fourier modes we usually take the constant to be $-\lambda^2$ which gives an eigenvalue problem.)

This now gives us two ODE's:

Firstly a 1st order ODE : $\Psi' = c\Psi \rightarrow \Psi = k\exp(ct)$
Secondly a 2nd order Cauchy-Euler equation:

$$\frac{1}{2}\sigma^2 S^2 \Phi'' + rS\Phi' + (c-r)\,\Phi = 0$$

So we look for the existence of solution of the form:

$$\Phi(S) = S^d$$

which upon substituting in the above gives a quadratic in d

$$d^{\,2} + \left(\frac{2r}{\sigma^2} - 1\right)d - \frac{2}{\sigma^2}(r-c) = 0$$

$$d_{\pm} = \frac{1}{2}\left(1 - \frac{2r}{\sigma^2}\right) \pm \frac{1}{2}\sqrt{\left(\frac{2r}{\sigma^2} - 1\right)^2 + 4 \times \frac{2}{\sigma^2}(r-c)}$$

Let us concentrate on the terms inside the square root sign, i.e.

$$\begin{aligned}\left(\frac{2r}{\sigma^2}-1\right)^2+4\times\frac{2}{\sigma^2}(r-c) &= \frac{4r^2}{\sigma^4}+1-\frac{4r}{\sigma^2}+\frac{8r}{\sigma^2}-\frac{8c}{\sigma^2} \\ &= 4\left(\frac{r^2}{\sigma^4}+\frac{1}{4}-\frac{r}{\sigma^2}+\frac{2r}{\sigma^2}-\frac{2c}{\sigma^2}\right) \\ &= 4\left(\frac{r^2}{\sigma^4}+\frac{1}{4}+\frac{r}{\sigma^2}-\frac{2c}{\sigma^2}\right) \\ &= 4\left(\left(\frac{r}{\sigma^2}+\frac{1}{2}\right)^2-\frac{2c}{\sigma^2}\right)\end{aligned}$$

now put back in the square root

$$d_\pm=\frac{1}{2}\left(1-\frac{2r}{\sigma^2}\right)\pm\frac{1}{2}\sqrt{4\left(\left(\frac{r}{\sigma^2}+\frac{1}{2}\right)^2-\frac{2c}{\sigma^2}\right)}$$

hence pulling the 4 out of the square root becomes a 2 and cancels with the half to give

$$d_\pm=\frac{1}{2}\left(1-\frac{2r}{\sigma^2}\right)\pm\sqrt{\left(\frac{r}{\sigma^2}+\frac{1}{2}\right)^2-\frac{2c}{\sigma^2}}$$

<u>3 cases to consider</u>:

(1) Solution for distinct roots - $\Phi(S)=aS^{d_+}+bS^{d_-}$

$$V(S,t)=\exp(ct)\,S^{\frac{1}{2}-\frac{r}{\sigma^2}}\left[AS^{d_+}+BS^{d_-}\right]\qquad A,\ B\text{ - constants}$$

where

$$d_+=\sqrt{\left(\frac{r}{\sigma^2}+\frac{1}{2}\right)^2-\frac{2c}{\sigma^2}}\ ;\qquad d_-=-\sqrt{\left(\frac{r}{\sigma^2}+\frac{1}{2}\right)^2-\frac{2c}{\sigma^2}}$$

(2) Repeated Root - $\Phi(S)=S^{\frac{1}{2}-\frac{r}{\sigma^2}}[a+b\log S]$
Now $\left(\frac{r}{\sigma^2}+\frac{1}{2}\right)^2=\frac{2c}{\sigma^2}\to c=\frac{\sigma^2}{2}\left(\frac{r}{\sigma^2}+\frac{1}{2}\right)^2$ therefore

$$V(S,t)=\exp\left(\frac{\sigma^2}{2}\left(\frac{r}{\sigma^2}+\frac{1}{2}\right)^2t\right)S^{\frac{1}{2}-\frac{r}{\sigma^2}}[\varepsilon+\zeta\log S]\qquad \varepsilon,\ \zeta\text{ - constants}$$

(3) Complex Roots i.e. $\frac{2c}{\sigma^2}>\left(\frac{r}{\sigma^2}+\frac{1}{2}\right)^2$ - $d_+=\alpha+i\beta$; $d_-=\alpha-i\beta$

$$\Phi(S)=S^\alpha\left[A\cos(\beta\ln S)+B\sin(\beta\ln S)\right]$$

where

$$\alpha = \left(\frac{1}{2} - \frac{r}{\sigma^2}\right); \quad \beta = \sqrt{\left|\left(\frac{r}{\sigma^2} + \frac{1}{2}\right)^2 - \frac{2c}{\sigma^2}\right|}$$

$$V\left(S,t\right) = \exp\left(ct\right) S^{\frac{1}{2}-\frac{r}{\sigma^2}} \left[A\cos\left(\beta \ln S\right) + B\sin\left(\beta \ln S\right)\right]$$