Part A

1. Which of the following are possible reason(s) to run a short or long position in a credit default swap

(D) all of the above

2. By which set of the following terms is a credit default swap classified?

(B) reference entity, settlement mechanism, term, premium and credit event definition

3. The spread to Libor paid or received in a total return swap is a function of which of the following?

(D) insufficient information to answer (any required profit margin the amount and the value of the reference asset)

4. What is "jump-to-default risk"?

(B) sudden default of the reference name in the market in the very near future, as opposed to a gradual credit deterioration

Part B

1. Find hazard rate(intensity) of the following function.

(a)

$$h(t) = \frac{f(t)}{R(t)} = \frac{\lambda e^{-\lambda t}}{e^{-\lambda t}} = \lambda$$

(b)

$h(x|\alpha,\beta) = \alpha\beta(\alpha x)^{\beta-1}$, where scale parameter $\alpha > 0$ and shape parameter $\beta > 0$

2. The Probability Generating Function of a discrete random variable X is defined to be the generating function $G(s) = E(s^X)$ of its probability mass function.

(a)

If X has p.g.f. G(s), then

$$\mathbf{E}(X) = G'(1);$$

more generally, the kth factorial moment is

$$\mu^{(k)} = E(X(X-1)\ldots(X-k+1)) = G^{(k)}(1);$$

and, in particular,

$$\text{var}(X) = G''(1) + G'(1) - (G'(1))^2$$

as s = 1

$$G'(s) = \sum_{k=1}^{\infty} P(X=k)s^k, \text{ so that } G'(s) = \sum_{k=1}^{\infty} P(X=k)\, k\, s^{k-1}$$

Hence $\quad G'(1) = \sum_{k=1}^{\infty} P(X=k)k = EX$

$$G''(s) = \sum_{k=1}^{\infty} P(X=k)\, k\,(k-1)\, s^{k-2}$$

Hence $\quad G''(1) = \sum_{k=1}^{\infty} P(X=k)\, k\,(k-1) = E(X(X-1) = EX^2 - EX$

and $\quad \text{var}(X) = EX^2 - (EX)^2 = G''(1)) + EX - (EX)^2$

$$= G''(1) + G'(1) - (G'(1))^2$$

(b)

$X \sim \text{Poisson}(\lambda)$

$$G_X(s) = \sum_{k=0}^{\infty} e^{-\lambda}\frac{\lambda^k}{k!} s^k = e^{-\lambda}\sum_{k=0}^{\infty}\frac{(\lambda s)^k}{k!} = e^{-\lambda} e^{\lambda s} = e^{\lambda(s-1)}$$

(c)

$$G'(s) = \lambda e^{\lambda(s - 1)} \quad \text{if } s = 1$$

then

$$G'(1) = \lambda \quad \text{if } \lambda = 1$$

Then

$$G'(1) = 1$$

So

Expected number of default is one

(d)

Let $N(t)$ be Poisson process with intensity $\lambda > 0$: Then $M(t) := N(t) - \lambda t$ is a martingale (compensated Poisson process) (we note that $EN(t)= \lambda t$):

$$\begin{aligned} E[N(t) - \lambda t \mid F(s)] &= E[N(t) - N(s) + N(s) - \lambda t + \lambda s - \lambda s \mid F(s)] \\ &= N(s) - \lambda s + E(N(t) - N(s) + N(s) - \lambda t + \lambda s) \\ &= N(s) - \lambda s \end{aligned}$$

where we've used independency increments of $N(t)$ and that $EN(t) = \lambda t$:

Let $F(s) \in F(t)$ if $0 \leq s \leq t$ be given. Because $N(t) - N(s)$ is independent of $F(s)$ and has expected value $\lambda(t - s)$, we have

$$\begin{aligned} E[M(t) \mid F(s)] &= E[M(t) - M(s) \mid F(s) + E\{M(s) \mid F(s)] \\ &= E[N(t) - N(s) - \lambda(t - s) \mid F(s)] + M(s) \\ &= E[N(t) - N(s)] - \lambda(t - s) + M(s) \\ &= M(s) \end{aligned}$$

3) In the context of the Merton (1974) model, at any time t the firm assets Vt are assumed to be sum of its debt Dt and its equity Et,

$$Vt = Et + Dt:$$

a)
The fact that the firm can only default at time T. This assumption is important to be able to treat the firm's equity as a vanilla European call option, and therefore apply the Black-Scholes pricing formula.

b)

Asset Value (million)	£100
Principal Value (million)	£60
Risk -free rate	5%
Volatility	30%
Time to maturity	1
d1	2.019
d2	1.719
N(-d1)	0.022
N(-d2)	0.043
Debt (Do) (million)	£56.80
YTM	5.47%
Credit Spread	0.47%
N(d1)	0.978
N(d2)	0.957
Equity (Eo) (million)	£43.20

4. Suppose that the probability of company A defaulting in one year is 10% and the probability of company B defaulting in one year is 15%. Assuming default correlation is 30%, calculate the probability that both company default in one year by using bivariate Gaussian copula.

$Q_A = 0.1$, $Q_B = 0.15$, $\rho_{AB} = 0.3$

$$u_A = N^{-1}(0.1) = -1.28155$$

$$u_B = N^{-1}(0.15) = -1.03643$$

$$M(-1.28155, -1.03643, 0.3) = 0.029781$$

$$\beta_{AB}(1) = \frac{0.029781 - 0.2 \text{ x } 0.15}{\sqrt{(0.2-0.2^2)}\sqrt{(0.15-0.15^2)}} = 0.137979$$

A	0.1
B	0.15
ρ	0.3
Ua	-1.28155
Ub	-1.03643
Variable 1	-1.28155
Variable 2	-1.03643
Correlation	0.3
M(a,b,ρ)	0.029781
$\beta_{AB}(1)$	0.137979

5) Suppose there is a risky bond V (r, t; p), where interest rate is stochastic with SDE

$$dr = u(r, t)dt + w(r, t)dx,$$

and the risk of default is governed by Poisson Process with intensity p. Now consider the risky bond is hedged by a risk-free bond Z(r, t), in particular the hedging portfolio can be written as

$$\Pi = V(r, p, t) - \Delta Z(r, t)$$

$$d\Pi = dV - \Delta Z =$$

$$= \left(\frac{\partial V}{\partial t} + \frac{1}{2} w^2 \frac{\partial^2 V}{\partial r^2}\right) dt + \frac{\partial V}{\partial r} dr - \Delta\left(\left(\frac{\partial Z}{\partial t} + \frac{1}{2} w^2 \frac{\partial^2 Z}{\partial r^2}\right) dt + \frac{\partial Z}{\partial r}\right)$$

$$= \left(\frac{\partial V}{\partial t} + \frac{1}{2} w^2 \frac{\partial^2 V}{\partial r^2} - \Delta\left(\frac{\partial Z}{\partial t} + \frac{1}{2} w^2 \frac{\partial^2 Z}{\partial r^2}\right)\right) dt + \left(\frac{\partial V}{\partial r} - \Delta\frac{\partial Z}{\partial r}\right) dr$$

Chose $\Delta = \frac{\partial V}{\partial r} / \frac{\partial Z}{\partial r}$ to eliminate risk. The value of the heging portfolio will suddenly jump -V if default occurs with probability μdt, to sum up

$$d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2} w^2 \frac{\partial^2 V}{\partial r^2} - \Delta\left(\frac{\partial Z}{\partial t} + \frac{1}{2} w^2 \frac{\partial^2 Z}{\partial r^2}\right)\right)dt - \mu V dt$$

Set $d\Pi = r\Pi dt$, leads to

$$\frac{\partial V}{\partial t} + \frac{1}{2} w^2 \frac{\partial^2 V}{\partial r^2} - (r + p)V = \frac{\partial V}{\partial r} / \frac{\partial Z}{\partial r} \left(\frac{\partial Z}{\partial t} + \frac{1}{2} w^2 \frac{\partial^2 Z}{\partial r^2} - rZ\right)$$

Which is

$$\frac{\frac{\partial V}{\partial t} + \frac{1}{2} w^2 \frac{\partial^2 V}{\partial r^2} - (r+p)V}{\frac{\partial V}{\partial r}} = \frac{\frac{\partial Z}{\partial t} + \frac{1}{2} w^2 \frac{\partial^2 Z}{\partial r^2} - rZ}{\frac{\partial V}{\partial r}}$$

The only way this equation holds is that both sides are independent of V and Z(similar argument in interest rate model), and suppose both sides equal to a function a(r; t) = w(r, t)λ(r, t) - u(r; t).

Thus on default we have

$$d\Pi = -\theta V(r, p, t) + O(dt^{\frac{1}{2}})$$

we lose the bond but recover 1 – θ.

The pricing equation will be

$$\frac{\partial V}{\partial t} + \frac{1}{2} w^2 \frac{\partial^2 V}{\partial r^2} + (u + \lambda w)\frac{\partial V}{\partial r} - (r + (1 - \theta)p)V = 0$$

The fundamental risky pricing formula is

$$V(t, T) = E[e^{-\int_t^T r_{s\,ds}}\ \mathbf{1}_{\{\tau>T\}} \mid F_t] = E[E(e^{-\int_t^T r_{s\,ds}}\ \mathbf{1}_{\{\tau>T\}} \mid F_T) \mid F_t]$$

$$= E[e^{-\int_t^T r_{s\,ds}}\ E(\mathbf{1}_{\{\tau>T\}} \mid F_T) \mid F_t]= E[e^{-\int_t^T r_{s\,ds}}\ e^{-\int_t^T p_{s\,ds}} \mid F_T]$$

$$= E[e^{-\int_t^T (r_s + p_s)ds} \mid F_t]$$

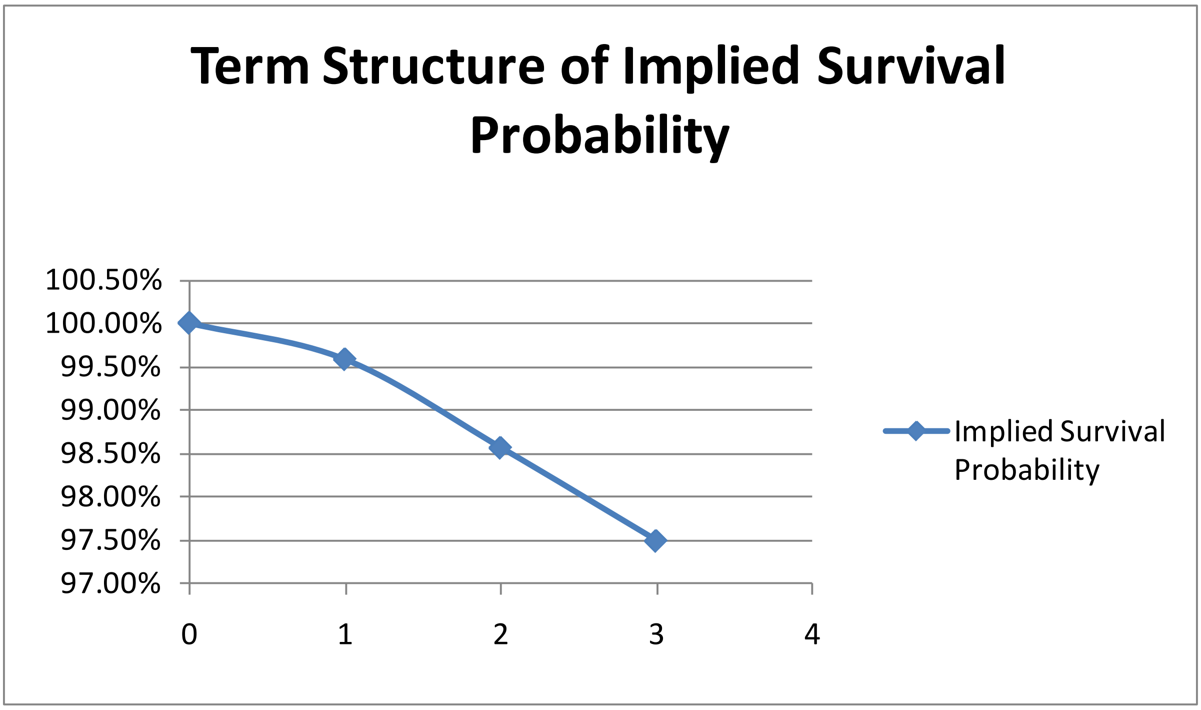
6)

The following spreadsheet shows the term structure of CDS spreads and the current discount factors $Z(t; T)$ for a company. Assume that the CDS premium is paid annually (once a year) and a recovery rate of 50%.

PROBABILITY OF DEFAULT

Recovery Rate	50%

TIME (Years)	dt	MARKET SPREAD	DF	IMPLIED SURVIVAL PROB	first term	second term	sum	quotient	first term	last term
0				100.00%						
1	1	21.00	0.9801	99.58%						
2	1	36.00	0.9513	98.56%	- 0.0015		- 0.0015	0.4791	- 0.0031	0.9887
3	1	42.00	0.9151	97.49%	- 0.0020	0.0009	- 0.0011	0.4614	- 0.0025	0.9774



$P(T_1) = 99.58\%$
$P(T_2) = 98.56\%$
$P(T_3) = 97.49\%$

7)

a)
The Mezzanine note has 3% subordination, therefore it is impacted once the portfolio suffers 0.03 * 125 = 3.75 or 4 defaults. If we assume 40% recovery the loss per default is 0.6 * 0.48 million. The subordination is 3% of the portfolio or 3,000,000, which is eaten after 3,000,000/480,000 = 6.25 or 6 defaults. Therefore assuming a recovery rate, this affords 2 more defaults as additional protection to the Mezzanine note.

b)
Assume that the portfolio of reference assets consists of m financial instruments and the asset return until time t (further we omit t for simplicity of the notations) of the i-th issuer in the portfolio, Ai, is assumed to be of the form:

$$A_i = wZ + \sqrt{1 - w^2}\varepsilon_i \qquad (1)$$

where Z, ε_i, i = 1, ..., m, are independent standard normally distributed random variables. Then, conditionally on the common market factor Z, the asset returns of the different issuers are independent. Note, that due to the stability of normal distributions under convolution the asset return A_i follows a standard normal distribution as well.

According to Merton's approach we assume that default occurs when the asset return of obligor i crosses the threshold C_i, which is implied by the obligor's default probability q_i = 3%:

$$q_i = P\,[A_i \leq C_i] = \Phi(C_i)\ ,$$

where Φ is the standard normal distribution function.

The model is calibrated to observable market prices of credit default swaps, i.e. the default thresholds are chosen so that they produce risk neutral default probabilities implied by quoted credit default swap spreads:

$$C_i = \Phi^{-1}(qi) = \text{NORMSINV}(3\%) = -1.88$$

According to equation (1), the i-th issuer defaults if

$$\varepsilon_i \leq \frac{C_i - w_i Z}{\sqrt{1 - w_i^2}}$$

Then the probability that the i-th issuer defaults conditional on the factor Z is

$$p_i(Z) = \Phi\left(\frac{C_i - w_i Z}{\sqrt{1 - w_i^2}}\right)$$

If we assume that the portfolio is homogeneous, i.e. w_i = w and C_i = -1.88 for all i and the notional amounts and recovery R are the same for all issuers, then the default probability of all issuers in the portfolio conditional on Z is given by

$$p(Z) = \Phi\left(\frac{-1.88 - wZ}{\sqrt{1 - w^2}}\right)$$

c)

The binomial distribution converges towards the Poisson distribution as the number of trials goes to infinity while the product np remains fixed. Therefore the Poisson distribution with parameter $\lambda = np$ can be used as an approximation to $\beta(n, p)$ of the binomial distribution if n is sufficiently large and p is sufficiently small. According to two rules of thumb, this approximation is good if $n \geq 20$ and $p \leq 0.05$, or if $n \geq 100$ and $np \leq 10$.

$$\beta\,(F(1|Z), 125)$$

As $F(1|Z) < p$ and $125 > n$, K which has Poisson distribution follows binomial distribution.

d)

Z=NORMSDIST(0.1)	0.540		U	3%
ω	0.3		D	7%
ε	0.33			
$A_i = wZ + \sqrt{(1 - w^2)}\varepsilon_i$	0.48	the asset return until time t		
F(t=1,Z=0.54)	0.0162	the probability that the i-th issuer defaults conditional on the factor Z		
$q_i = P[A_i \leq d_i] = \Phi(d_i)$	0.0301	the obligor's default probability qi		
$d_i = \Phi^{-1}(q_i)$	-1.88	default treshhold		
$\varepsilon \leq F(t=1,Z)$	1	the i-th issuer defaults if		
L_i = (1 - 40%)*800000	480000	Loss Given Default for security i		
$P(\tau \leq t) = \Phi(A_i)$	0.6838			
$\tau_i = F^{-1}(\Phi(A_i))$	0.48	default time of issuer i		