CQF 1.3 Probability & Introduction to Stochastic Calculus

Throughout this problem sheet, you may assume that X is a Brownian Motion (Weiner Process) and dX is its increment.

1. Let ϕ be a random variable which follows a standardised normal distribution, i.e. $\phi \sim N(0,1)$. If $\mathbb{E}[X]$ and $\mathbb{V}[X]$ are used to denote the Expectation and Variance of x in turn, calculate

(a) $\mathbb{E}\left[\phi^2\right]$

(b) $\mathbb{E}[\psi]$

(c) $\mathbb{V}[\psi]$

where $\psi = \sqrt{dt}\phi$. dt is a small time-step.

2. Consider the probability density function $p(x)$

$$p(x) = kx^2 \exp\left(-\lambda x^2\right), \qquad -\infty < x < \infty,$$

where $\lambda\,(>0)$ and k are both constants. Show that

$$k = \frac{2\lambda^{3/2}}{\sqrt{\pi}}.$$

Deduce that the odd moments of $p(x)$ are all zero, i.e.,

$$E\left[x^{2n+1}\right] = 0, \qquad n = 0,\ 1,\ 2, \ldots.$$

3. Using the formula below for stochastic integrals, for a function $F(X(\tau),\tau)$,

$$\int_0^t \frac{\partial F}{\partial X} dX(\tau) = F(X(t),t) - F(X(0),0) - \int_0^t \left(\frac{\partial F}{\partial \tau} + \frac{1}{2}\frac{\partial^2 F}{\partial X^2}\right) d\tau$$

show that we can write

a. $\displaystyle\int_0^t X(\tau)\,dX(\tau) = \tfrac{1}{2}X^2(t) - \tfrac{1}{2}t$

b. $\displaystyle\int_0^t \tau\,dX(\tau) = tX(t) - \int_0^t X(\tau)\,d\tau$

c. $\int_0^t X^2(\tau)\; dX(\tau) = \frac{1}{3}X^3(t) - \int_0^t X(\tau)\, d\tau$

4. Use Itô's lemma to obtain a SDE for each of the following functions:

(a) $f\ (X) = X^n$

(b) $y(X) = \exp(X)$

(c) $g(X) = \ln X$

(d) $h(X) = \sin X + \cos X$