1) $$B(S,t) = e^{-r(T-t)} N(d_2)$$

$$N'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{\phi^2}{2}} d\phi$$

a) $$\Delta = \frac{\partial B}{\partial S} = \frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}\, S} e^{-\frac{d_2^2}{2}}$$

Proof:

$$\Delta = \frac{\partial B}{\partial S} = e^{-r(T-t)} N(d_2) =$$

$$= e^{-r(T-t)} \frac{1}{\sqrt{2\pi}} e^{-\frac{d_2^2}{2}} \frac{\partial d_2}{\partial S} =$$

$$= \frac{e^{-r(T-t)}}{\sqrt{2\pi}} \cdot e^{-\frac{d_2^2}{2}} \cdot \frac{1}{\sigma\sqrt{T-t}\, S} = \frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}\, S} e^{-\frac{d_2^2}{2}}$$

$$\frac{\partial d_2}{\partial S} = \frac{\partial}{\partial S} \frac{\log S - \log E + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} =$$

$$= \frac{1}{\sigma\sqrt{T-t}} \cdot \frac{\partial}{\partial S}(\log S) = \frac{1}{\sigma\sqrt{T-t}} \cdot \frac{1}{S} = \frac{1}{\sigma\sqrt{T-t}\, S}$$

6) $\Gamma = \frac{\partial^2 B}{\partial S^2} = \frac{\partial \Delta}{\partial S}$

$$\Delta = \frac{e^{-r(T-t)}}{\sigma \sqrt{2\pi (T-t)}\, S} e^{-\frac{d_2^2}{2}} = \frac{e^{-r(T-t)}}{\sigma \sqrt{T-t}\, S} \cdot \frac{e^{-\frac{d_2^2}{2}}}{\sqrt{2\pi}} =$$

$$\Delta = \frac{e^{-r(T-t)}}{\sigma \sqrt{T-t}\, S} \cdot N'(d_2)$$

$$\Gamma = \frac{\partial \Delta}{\partial S} = \frac{e^{-r(T-t)}}{\sigma \sqrt{T-t}\, S} N''(d_2) \frac{\partial d_2}{\partial S}$$

$$N''(d_2) = -\frac{d_1 e^{-\frac{d_2^2}{2}}}{\sqrt{2\pi}} \quad \text{. where } d_1 = d_2$$

$$\frac{\partial d_2}{\partial S} = + \frac{1}{\sigma \sqrt{T-t}} \frac{1}{S}$$

$$\Gamma = \frac{\partial \Delta}{\partial S} = \frac{e^{-r(T-t)}}{\sigma \sqrt{T-t}\, S} \cdot -\frac{d_1 e^{-d_2^2}}{\sqrt{2\pi}} \cdot \frac{1}{\sigma\sqrt{T-t}} \frac{1}{S} =$$

$$\Gamma = -\frac{e^{-r(T-t)}}{\sigma^2 (T-t) \sqrt{2\pi}\, S^2} d_1 e^{-\frac{d_2^2}{2}}$$

c) $\nu = \dfrac{\partial B}{\partial \sigma}$, $B = e^{-r(T-t)} N(d_2)$

$$N'(d_2) = \frac{1}{\sqrt{2\pi}} e^{-\frac{d_2^2}{2}}$$

$$\nu = \frac{\partial B}{\partial \sigma} = e^{-r(T-t)} N'(d_2) \frac{\partial d_2}{\partial \sigma}$$

$$\frac{\partial d_1}{\partial \sigma} = -\frac{d_2}{\sigma} \quad , \quad d_2 = d_1 - \sigma\sqrt{T-t}$$

$$\frac{\partial d_2}{\partial \sigma} = \frac{\partial}{\partial \sigma}(d_1 - \sigma\sqrt{T-t}) = \frac{\partial d_1}{\partial \sigma} - \sqrt{T-t} =$$

$$= -\frac{d_2}{\sigma} - \sqrt{T-t} = -\frac{d_2 + \sigma\sqrt{T-t}}{\sigma} = -\frac{d_1}{\sigma}$$

$$\nu = e^{-r(T-t)} \frac{1}{\sqrt{2\pi}} e^{-\frac{d_2^2}{2}} - \frac{d_1}{\sigma} =$$

$$= -\frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi}} d_1 e^{-\frac{d_2^2}{2}}$$

2)

$$P(S,t) = E\,e^{-r(T-t)}\,N(-d_2) - S\,N(-d_1)$$

a) $t \to T$, $\sigma > 0$, $e^{-r(T-t)} \div e^{-(T-t)} \to 1$

$$d_{1/2} \to \frac{\log(S/E)}{\sigma\sqrt{T-t}} + O(\sqrt{T-t}) \to \begin{cases} \infty & S > E \\ 0 & S = E \\ -\infty & S < E \end{cases} \quad \text{so}$$

$$p \to \begin{cases} 0 & S > E \\ 0 & S = E \\ E-S & S < E \end{cases}$$

b) $\sigma \to 0$, $t < T$

$$d_{1/2} \to \frac{\log\frac{S}{E} + r(T-t)}{\sigma\sqrt{T-t}} + O(\sigma) =$$

$$= \frac{\log \frac{S\,e^{(T-t)}}{E\,e^{-r(T-t)}}}{\sigma\sqrt{T-t}} + O(\sigma) \longrightarrow$$

$$\begin{cases} \infty & S e^{(T-t)} > E e^{-r(T-t)} \\ 0 & S e^{(T-t)} = E e^{-r(T-t)} \\ -\infty & S e^{(T-t)} < E e^{-r(T-t)} \end{cases} \quad \text{so}$$

$$p \to \max\left[E e^{-r(T-t)} - S e^{-(T-t)}\right]$$

c) $\sigma \to \infty$, $t < T$

$$d_{1/2} \to \pm \frac{1}{2}\sigma\sqrt{T-t} + O\left(\frac{1}{\sigma}\right) \longrightarrow \pm\infty$$

$$p \to E e^{-r(T-t)} N(\infty) - S e^{-(T-t)} N(-\infty) = E e^{-r(T-t)}$$

3)

The value of the option V satisfies the following Black-Scholes equation.

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-D)S\frac{\partial V}{\partial S} - rV = -C(S,t)$$

$$V(S,T) = S$$

where $C(S,t)$ and $V(S,t)$ has the form

$$C(S,t) = f(t)S \quad \text{and} \quad V = \phi(t)S$$

Solution:

writing $C(S,t) = f(t)S$ gives

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-D)S\frac{\partial V}{\partial S} - rV = -f(t)S$$

and we now use the transformation $V = \phi(t)S$ to convert to an ODE which is a function of t alone.

$$\frac{\partial V}{\partial t} = \phi'(t)S \; ; \quad \frac{\partial V}{\partial S} = \phi(t) \; ; \quad \frac{\partial^2 V}{\partial S^2} = 0$$

For the final condition we know

$$V(S,T) = S \equiv \phi(T)S$$

$$\Rightarrow \phi(T) = 1$$

So the original problem reduces to

$$\frac{d\phi}{dt} + (r - D)\phi - r\phi = -f$$

which is a first order linear equation (i.e. integrating factor method). I.F is

$$e^{-Dt}$$

so the ODE becomes

$$e^{-D}\frac{d\phi}{dt} - D\phi e^{-Dt} = -fe^{-Dt}$$

$$\frac{d}{dt}(e^{-Dt}\phi) = -fe^{-Dt}$$

$$\int_t^T d(e^{-Dt}\phi(\tau)) = -\int_t^T f(\tau)e^{-Dt}dt$$

$$(e^{-Dt}\phi(\tau))\Big|_t^T = -\int_t^T f(\tau)e^{-Dt}dt$$

$$e^{-DT}\phi(T) - e^{-Dt}\phi(t) = -\int_t^T f(\tau)e^{-Dt}dt$$

and we know $\phi(T) = 1$ then

$$e^{-DT} - e^{-Dt}\phi(t) = -\int_t^T f(\tau)e^{-D\tau}d\tau$$

$$e^{-Dt}\phi(t) = e^{-Dt} + \int_t^T f(\tau)e^{-D\tau}d\tau$$

$$\phi(t) = e^{-D(T-t)} + \int_t^T f(\tau)\, e^{-D(\tau - t)}\, d\tau$$

So the option price $V(S,t) = \phi(t) S$

AND $\Delta(S,t) = \frac{\partial V}{\partial S} = \phi(t) =$

$$e^{-D(T-t)} + \int_t^T f(\tau)\, e^{-D(\tau-t)}\, d\tau$$

4)

$$C(S,t) = S\,N(d_1) - E\,e^{-r(T-t)}\,N(d_2)$$

$$\frac{\partial C(S,t)}{\partial S} = \frac{\partial}{\partial S}\left[S\,N(d_1) - E\,e^{-r(T-t)}\,N(d_2)\right] =$$

$$= S\,N'(d_1) - E\,e^{-r(T-t)}\,N'(d_2)$$

$$N'(d_1) = \frac{1}{\sqrt{2\pi}}\,e^{-\frac{d_1^2}{2}}$$

$$N'(d_2) = N'(d_1)\,\frac{S}{E}\,e^{r(T-t)}$$

$$S\,N'(d_1) = E\,e^{-r(T-t)}\,N'(d_2)$$

$$\frac{1}{\sqrt{2\pi}}\left(S\,e^{-\frac{d_1^2}{2}}\right) = E\,e^{-r(T-t)} \cdot \frac{1}{\sqrt{2\pi}}\,e^{-\frac{d_1^2}{2}} \cdot \frac{S}{E}\,e^{r(T-t)}$$

$$\frac{1}{\sqrt{2\pi}}\left(S\,e^{-\frac{d_1^2}{2}}\right) = \frac{1}{\sqrt{2\pi}}\left(S\,e^{-\frac{d_1^2}{2}}\right)\underbrace{\frac{E}{E}}_{''1}\;\underbrace{e^{-r(T-t)+r(T-t)}}_{''1}$$

$$\frac{1}{\sqrt{2\pi}}\,S\,e^{-\frac{d_1^2}{2}} = \frac{1}{\sqrt{2\pi}}\left(S\,e^{-\frac{d_1^2}{2}}\right) \quad \Big|\cdot \sqrt{2\pi}$$

$$S\,e^{-\frac{d_1^2}{2}} = S\,e^{-\frac{d_1^2}{2}}$$

$$N'(d_2) = \frac{1}{\sqrt{2\pi}} e^{-\frac{d_2^2}{2}} =$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}} e^{\sigma\sqrt{T-t}\, d_1 - \frac{\sigma^2 (T-t)}{2}} =$$

$$= N'(d_1)\, e^{\log\frac{S}{E} + (r + \frac{1}{2}\sigma^2)(T-t) - \sigma^2 \frac{(T-t)}{2}} =$$

$$= N'(d_1)\, \frac{S}{E}\, e^{r(T-t)}$$

5)

$$C^m_{n-1} = C(s-\delta s, t) = C_{s,t} - \delta s\frac{\partial C}{\partial s} + \frac{1}{2}\delta s^2\frac{\partial^2 C}{\partial s^2} - \frac{1}{3}\delta s^3\frac{\partial^3 C}{\partial s^3} + O(\delta s^4)$$

$$C^m_n = C_{s,t}$$

$$C^m_{n+1} = C(s+\delta s, t) = C_{s,t} + \delta s\frac{\partial C}{\partial s} + \frac{1}{2}\delta s^2\frac{\partial^2 C}{\partial s^2} + \frac{1}{3}\delta s^3\frac{\partial^3 C}{\partial s^3} + O(\delta s^4)$$

$$C^m_{n+2} = C(s+2\delta s, t) = C_{s,t} + 2\delta s\frac{\partial C}{\partial s} + 2\delta s^2\frac{\partial^2 C}{\partial s^2} + \overset{!}{\approx}\frac{5}{3}\delta s^3\frac{\partial^3 C}{\partial s^3} + O(\delta^4)$$

FORWARD DIFFERENCE FORMULA - 3ᴿᴰ DERIVATIVE

$$f'''(x_0) = \frac{f(x_2) - 3f(x_1) + 3f(x_0) - f(x_{-1})}{\delta x}$$

$$C^m_{n+2} - 3C^m_{n+1} + 3C^m_n - C^m_{n-1} =$$

$$= C^m_n + 2\delta s\frac{\partial C}{\partial s} + 2\delta s^2\frac{\partial^2 C}{\partial s^2} + \frac{5}{3}\delta s^3\frac{\partial^3 C}{\partial s^3} +$$

$$- 3C^m_n - 3\delta s\frac{\partial C}{\partial s} - \frac{3}{2}\delta s^2\frac{\partial^2 C}{\partial s^2} - \frac{3}{3}\delta s^3\frac{\partial^3 C}{\partial s^3} +$$

$$3C^m_n +$$

$$- C^m_n + \delta s\frac{\partial C}{\partial s} - \frac{1}{2}\delta s^2\frac{\partial^2 C}{\partial s^2} + \frac{1}{3}\delta s^3\frac{\partial^3 C}{\partial s^3} =$$

$$= \approx \delta s^3\frac{\partial^3 C}{\partial s^3}$$

$$\text{Speed} = \frac{\partial^3 C}{\partial s^3} \approx \frac{C^m_{n+2} - 3C^m_{n+1} + 3^m_n - C^m_{n-1}}{\delta s^3}$$

6)

1) The S_t process follows dynamics of

$$dS_t = \mu(t, S_t)\, dt + \sigma(t, S_t)\, dX_{(t)},$$ where

$X(t)$ is a Brownian motion

S_t written as an integral is

$$S_t = S_0\, e^{\mu t - \frac{1}{2}\sigma^2 t + \sigma X_t}$$

Applying ITÔ formula to get differential form

$$V(t, S_t) = S_0\, e^{\sigma S_t + (\mu - \frac{1}{2}\sigma^2)t}$$

$$V(t + \Delta t, S_t + \Delta S_t) = V_t(t, S_t)\,\Delta t + V_S(t, S_t)\,\Delta S_t + \frac{1}{2} V_{SS}(t, S_t)\,\Delta S_t^2$$

ITÔ's formula

$$dV(t, S(t)) = V_t(t, S(t))\,dt + V_S(t, S(t))\,dS + \frac{1}{2} V_{SS}(t, S(t))\,dS^2$$

where $(dS)^2 = dt$

$$dV(t, S(t)) = \left[V_t(t, S(t)) + \frac{1}{2} V_{SS}(t, S(t))\right] dt + V_S(t, S(t))\, dS$$

$$V(t,S_t) = S_0\, e^{\sigma S_t + (\mu - \frac{1}{2}\sigma^2)t}$$

$$S_t = V(t, S(t))$$

$$V_t = \frac{\partial V}{\partial t} = (\mu - \tfrac{1}{2}\sigma^2)\, V(t,S_t)$$

$$V_S = \frac{\partial V}{\partial S} = \sigma\, V(t,S_t) \quad , \quad V_{SS} = \frac{\partial^2 V}{\partial S} = \sigma^2\, V(t,S_t)$$

$$dV(t,S_t) = \left(V_t(t,S(t)) + \tfrac{1}{2} V_{SS}(t,S(t))\right) dt + V_S(t,S(t))\, dS$$

$$dV(t,S_t) = \left((\mu - \tfrac{1}{2}\sigma^2) S_t + \tfrac{1}{2}\sigma^2 S_t\right) dt + \sigma S_t\, dS$$

6

2)

$$V^*_{(t,S_t)} = e^{-zt} V_{(t,S_t)} = \frac{V(t,S_t)}{e^{zt}} = \frac{V(t,S_t)}{B_t}$$

$$B_{(0)} = 1$$

$$B_t = B_{(0)} e^{zt} = e^{zt}$$

$$dB_t = z B_t \, dt \quad \Rightarrow dB_t^{-1} = -z B_t^{-1} dt$$

$$d V^*_{(t,S_t)} = \frac{d V_{(t,S_t)}}{d B_t} = d\left(V_{(t,S_t)} B_t^{-1}\right) =$$

$$= V_{(t,S_t)} \, d B_t^{-1} + B^{-1} d V_{(t,S_t)} =$$

$$= \left(\phi_t^S S_t + \phi_t^B B_t\right) dB^{-1} + B^{-1}\left(\phi_t^S dS_t + \phi_t^B dB_t\right)$$

$$= \phi_t^S \left(B_t^{-1} dS_t + S_t \, dB_t^{-1}\right)$$

$$= \phi_t^S \, dS_t^*$$

$$S^* = S_t e^{-zt} \quad \Rightarrow d S_t^* = d\left(S_t^* B_t^{-1}\right)$$

6

3/

CONSIDER function $V(T,S) = G(S)$

RELATION BETWEEN $V(t,S_t)$ AND $V(T,S)$

$$V(t,S_t) = e^{-r(T-t)} E[G(S_T) | F_t]$$

APPLY ITÔ'S lemma TO THE function

$V(T,S)$ AND INTEGRATE OVER $[t, T]$

$$V(T,S) = V_T(0) + \int_t^T \left(\frac{\partial V_T}{\partial t} + \frac{\partial V_T}{\partial S_t} + \frac{1}{2} \frac{\partial^2 V_T}{\partial S_t^2} \right) dt + \int_t^T \frac{\partial V}{\partial S_t} \, dS_T$$

$$V_T(0) = 0 \quad , \quad V_T = e^{-rt} V(t,S_t)$$

$$V(T,S) = e^{-rt} \int_t^T \left[\frac{\partial}{\partial t} V(t,S_t) + \frac{\partial}{\partial S_t} V(t,S_t) + \frac{1}{2} \frac{\partial^2}{\partial S^2} V(t,S_t) \right] dt +$$

$$+ \quad e^{-rt} \int_t^T \frac{\partial}{\partial t} V(t,S_t) \, dS_T$$

4) Fix $T > 0$, and consider the stochastic process

$$V(t, S_t) = e^{-r(T-t)} E[G(S_T) | F_t]$$

where

$$V(T, s) = G(s) \text{ is the terminal condition}$$

Because $V(t, S_t)$ is a solution of the PDE, it is continuosly differentiable once in t and twice in S. Moreover, since V is bounded, so is process Y_t. By Itô's theorem

$$\begin{aligned} dY_t = &- e^{-r(T-t)} V(T-t, S_t)\, dt \\ &- V_t(T-t, S_t)\, e^{-r(T-t)}\, dt \\ &+ V_S(T-t, S_t)\, e^{-r(T-t)}\, dS_T \\ &+ \frac{1}{2} V_{SS}(T-t, S_t)\, e^{-r(T-t)}\, dt \end{aligned}$$

Since V satisfies the partial differential equation

$$\frac{\partial V}{\partial t}(t,s) + \mu(t,s)\frac{\partial V}{\partial S}(t,s) + \frac{1}{2}\sigma^2(t,s)\frac{\partial^2 V}{\partial S^2}(t,S) - rV(t,s) = 0,$$

the dt terms in the last expression sum to zero, leaving

$$dY_t = V(T-t, S_t)\, e^{-r(T-t)}\, dS_T$$

Thus Y_t is a martingale up to time T. By the "Conservation of Expectation" law for martingales, is follows that

$$Y_0 = V(t, S_t) = E^S Y_t = E^S e^{-r(T-t)} V(0, S_T) =$$

$$= e^{-r(T-t)} E^S V(T, S_t)$$

$$V(T, S_t) = G(S_T)$$

$$= e^{-r(T-t)} E\left[G(S_T) \mid F_t\right]$$

7)

1) FOR EUROPEAN BINARY CALL THAT PAYS OFF \$1 $G(S_T)=1$ AT EXPIRATION T IF THE UNDERLYING S IS OVER THE STRIKE K, THE PAYOFF $G(S_T)=1_{\{S>K\}}$ CAN BE SUMMARIZED AS,

$$c(T) = \begin{cases} 1, & S_T >= K \\ 0, & S_T < K \end{cases}$$

, where S_T is underlying at option expiration

In risk neutral world, the price of such an option can be determined by integrating over the range of possible price outcomes

if $C_{(t)} = e^{-r(T-t)} E^Q[G(S_T)|\mathcal{F}_t]$ and

$$dS_t = r\,dt + \sigma\,dX_t^Q, \quad S(0) = 0$$

$$B_T^{-1} = e^{-r(T-t)}, \quad B_0 = 1$$

using the basic definition of expectation can be written in terms of random variable S

$$C_{(t)} = e^{-r(T-t)} \int_K^\infty G(S_T)\,p(S)\,dS$$

where r is the risk free rate and $p(S)$ is PDF probability density function, with lognormally distributed prices, $\ln(S)$ has a normal distribution with mean $\ln(S_0) + (r - \frac{\sigma^2}{2})(T-t)$ and standard deviation $\sigma\sqrt{T-t}$

EXPRESSED IN TERMS OF CUMULATIVE PROBABILITY CDF, THE BINARY CALL PRICE IS

$$C_{(t)} = e^{-r(T-t)} P[S_T >= K] \quad \text{OR}$$

$$C_{(t)} = e^{-r(T-t)} P[\ln(S_0) >= \ln(K)]$$

USING THE FACT THAT $\ln(S_0)$ IS NORMALLY DISTRIBUTED, THIS EQUATES TO

$$C_{(t)} = e^{-r(T-t)} \left[1 - N\left(\frac{\ln(S_0) + (r - \frac{\sigma^2}{2})(T-t) - \ln(K)}{\sigma\sqrt{T-t}}\right)\right]$$

OR

$$C_{(t)} = e^{-r(T-t)} [1 - N(-d_2)] \Rightarrow C_{(t)} = e^{-r(T-t)} N(d_2)$$

II) FOR THE EUROPEAN BINARY PUT THAT PAYS \$1 AT EXPERATION T IF STRIKE K IS OVER THE UNDERLYING S, THE EXPIRATION PAYOFF $G(S)_{\{K>S\}} = 1$

CAN BE SUMMARIZED AS

$$p(T) = \begin{cases} 1, & S_T <= K \\ 0, & S_T > K \end{cases}, \text{ where } S_T \text{ IS THE UNDERLYING AT OPTION EXPIRATION}$$

$$P_{(t)} = e^{-r(T-t)} [1 - N(d_2)]$$

$$d_2 = \frac{\ln(\frac{S_t}{K}) + (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$