CQF Examination 3

January 2009 Cohort

Instructions

All questions must be attempted. Books and lecture notes may be referred to. Spreadsheets and VBA may be used. Help from other people is not permitted.

Throughout this examination you may assume (where appropriate) the following:

$$\begin{aligned}
d_1 &= \frac{\log(S/E) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}, \\
d_2 &= \frac{\log(S/E) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \quad \text{and} \\
N(x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} \exp(-\phi^2/2)d\phi, \quad N'(x) = \frac{1}{\sqrt{2\pi}}\exp(-x^2/2)
\end{aligned}$$

where $S \geq 0$ is the spot price, $t \leq T$ is the time, $E > 0$ is the strike, $T > 0$ the expiry date, r the interest rate, and σ is the volatility of S.

1. The Black–Scholes formula for the value of a binary call option $B(S,t)$ is given by

$$B(S,t) = \exp(-r(T-t))N(d_2).$$

Show that the delta, gamma and vega for this option are given by

$$\begin{aligned}
\Delta &= \frac{\partial B}{\partial S} = \frac{\exp(-r(T-t))}{\sigma\sqrt{2\pi\,(T-t)}S}\exp(-{d_2}^2/2) \\
\Gamma &= \frac{\partial^2 B}{\partial S^2} = -\frac{\exp(-r(T-t))}{\sigma^2\sqrt{2\pi}\,(T-t)\,S^2}d_1\exp(-{d_2}^2/2) \\
\upsilon\text{ega} &= \frac{\partial B}{\partial \sigma} = \frac{-\exp(-r(T-t))}{\sigma\sqrt{2\pi}}d_1\exp(-{d_2}^2/2)
\end{aligned}$$

2. The Black–Scholes formula for the value of a put option $P(S,t)$ is

$$P(S,t) = E\exp(-r(T-t))N(-d_2) - SN(-d_1)$$

From this expression, find the Black–Scholes value of the put option in the following limits:

(a) (time tends to expiry) $t \to T$, $\sigma > 0$;
(b) (volatility tends to zero) $\sigma \to 0$, $t < T$;
(c) (volatility tends to infinity) $\sigma \to \infty$, $t < T$.

3. Consider the following Black-Scholes problem

$$\begin{aligned}\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+(r-D)\,S\frac{\partial V}{\partial S}-rV &= -C\,(S,t)\,,\\ V\,(S,T) &= S\end{aligned}$$

for a European option $V\,(S,t)$, where the volatility σ, interest rate r, and dividend yield D are constants. T is the expiry.

Suppose that the term $C\,(S,t)$ has the form $C\,(S,t) = f\,(t)\,S$. By writing $V\,(S,t) = \Psi\,(t)\,S$ show that the options delta Δ is

$$e^{-D(T-t)} + \int_t^T f\,(\tau)\,e^{-D(\tau-t)}d\tau.$$

4. The Black-Scholes pricing formula for a European call option is given by

$$C\,(S,t) = SN(d_1) - E\exp(-r(T-t))N(d_2).$$

Show that

$$S\exp(-d_1^2/2) = E\exp(-r(T-t))\exp(-d_2^2/2).$$

5. The "Speed" of an option $C(S,t)$ is given by

$$\text{Speed} = \frac{\partial^3 C}{\partial S^3}$$

If $S = n\delta S$ and $t = m\delta t$, by obtaining 3 <u>suitable</u> Taylor expansions for the option price $C(S,t)$ **<u>derive</u>** a Finite Difference Approximation for the Speed which is given by

$$\text{Speed} \approx \frac{1}{\delta S^3}\left(C_{n+2}^m - 3C_{n+1}^m + 3C_n^m - C_{n-1}^m\right)$$

Hint: Use Taylor expansions for $C(S+\delta S,t)$; $C(S-\delta S,t)$ and $C(S+2\delta S,t)$ as a starting point to **derive** the third derivative. No credit will be given for merely verifying that the left and right hand expressions are similar.

6. In this question, we will use the Itô formula to derive the Feynman-Kač formula. The Feynman-Kač formula state that if the function $V(t,s)$ solves the boundary value problem

$$\begin{aligned} \frac{\partial V}{\partial t}(t,s) + \mu(t,s)\frac{\partial V}{\partial s}(t,s) + \frac{1}{2}\sigma^2(t,s)\frac{\partial^2 V}{\partial s^2}(t,s) - rV(t,s) &= 0 \\ V(T,s) &= G(s) \end{aligned} \tag{6.1}$$

with r constant and that the process $S(t)$ follows the dynamics

$$dS_t = \mu(t,S_t)dt + \sigma(t,S_t)dX(t)$$

where $X(t)$ is a Brownian motion, then, the function V can be represented by the expectation

$$V(t,S_t) = e^{-r(T-t)}\mathbf{E}\left[G(S_T)|\mathcal{F}_t\right]$$

where $\mathcal{F}_t$ is the filtration up to time t and is such that $S(t)$ and $V(t,S_t)$ are $\mathcal{F}_t$-adapted (i.e. "known" at time t).

Questions:

1. Apply the Itô formula to derive an SDE for $V(t,S_t)$.
2. Apply the Itô product rule to get an integral equation for $e^{-rt}V(t,S_t)$.
3. Integrate over $[t,T]$ to obtain an integral equation for $e^{-rt}V(t,S_t)$.
4. Take the conditional expectation of this integral equation. The conditional expectation is with respect to the filtration up to time t, $\mathcal{F}_t$. Deduce the Feynman-Kač formula:

$$V(t,S_t) = e^{-r(T-t)}\mathbf{E}\left[G(S_T)|\mathcal{F}_t\right]$$

Hint: you may need to use the PDE and the terminal condition (introduced in equation (6.1)) in the latter part of the derivation.

7. Consider a Black-Scholes world with a stock $S(t)$ modelled as a Geometric Brownian Motion under the $\mathbb{P}$-measure so that

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dX(t), \qquad S(0) = s_0$$

where $X(t)$ is a Brownian motion, and a bank account $B(t)$ satisfying the ODE

$$\frac{dB(t)}{B(t)} = rdt, \qquad B(0) = 1$$

where r is the risk-free rate, we know that the price $V(t)$ of a European-style derivative on the stock $S(t)$ with expiry date T and payoff function $G(T)$ is given by the *fundamental asset pricing formula* as:

$$V(t) = e^{-r(T-t)}\mathbf{E}^{\mathbb{Q}}\left[G(S(T))|\mathcal{F}_t\right] \tag{7.1}$$

where $\mathbf{E}^{\mathbb{Q}}[\cdot]$ denotes the expectation taken with respect to the equivalent martingale (a.k.a. "risk-neutral") measure.

(i) Using formula (7.1) as your starting point, **show** that the value of a (European) **Binary call** on the stock $S(t)$ with expiry date T, strike E and payoff $G(S_T) = \mathbf{1}_{\{S_T \geq E\}}$ is given by

$$C(t) = e^{-r(T-t)}N(d_2)$$

with

$$d_2 = \frac{\ln\left(\frac{S_t}{E}\right) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$

(ii) What is the value of a **Binary put** with same characteristics as the call priced above? (***Hint****: this question should* ***not*** *take more than a few lines to answer*)