CQF Solutions 4.1 Fixed Income

1. A coupon bond pays out 3% every year, with a principal of \$1 and a maturity of five years. Decompose the coupon bond into a set of zero coupon bonds.

$\boxed{\text{Solution}}$ The coupon bond can be expressed as
so

$$0.03\sum_{i=1}^{5} Z\left(t;\ i\right) + Z\left(t;\ 5\right),$$

2. Construct a spreadsheet to examine how \$1 grows when it is invested at a continuously-compounded rate of 7%. Redo the calculation for a discretely compounded rate of 7%, paid once per annum. Which rate is more profitable?

$\boxed{\text{Solution}}$ After T years, \$1 invested at a continuously-compounded rate of 7% is worth $\exp\left(0.07T\right)$.
\$1 invested at a discretely-compounded rate of 7% is worth $\left(1+0.07\right)^T$.
The continuously-compounded rate is clearly more profitable.

3. A zero-coupon bond (ZCB) has a principal of \$100 and matures in 4 years. The market price for the bond is \$72. Calculate the yield to maturity, duration and convexity for the bond.

$\boxed{\text{Solution}}$ For a ZCB,

$$V = P\exp\left(-y\left(T-t\right)\right)$$

and so the yield to maturity is

$$y = -\frac{\log\left(V/P\right)}{T-t} = -\frac{\log\left(72/100\right)}{4} = -\frac{1}{4}\log\left(0.72\right) = 0.082$$

Then

$$\frac{dV}{dy} = -\left(T-t\right)P\exp\left(-y\left(T-t\right)\right)$$

and the duration is

$$-\frac{1}{V}\frac{dV}{dy} = \frac{1}{V}(T-t)P\exp(-y(T-t)) = \frac{1}{72}\times 4\times 100\times e^{-4y} = 4.$$

Finally the convexity is

$$\frac{1}{V}\frac{d^2V}{dy^2} = \frac{1}{V}(T-t)^2 P\exp(-y(T-t)) = \frac{1}{72}\times 4^2\times 100\times e^{-4y} = 16.$$

4. A coupon bond pays out 2% every year on a principal of \$100. The bond matures in 6 years and has a market value \$92. Calculate the yield to maturity, duration and convexity for the bond.

$\boxed{\text{Solution}}$

$$V = P\exp(-y(T-t)) + \sum_{i=1}^{N} C_i \exp(-y(t_i - t)),$$

and so

$$92 = 100e^{-6y} + \sum_{i=1}^{6} 2e^{-y(i)}.$$

We must solve this equation for y to find the yield to maturity of the coupon bond. This can be done on Excel using solver, and we find

$$y = 0.034$$

The duration is given by

$$-\frac{1}{V}\frac{dV}{dy}$$

where

$$\frac{dV}{dy} = -(T-t)P\exp(-y(T-t)) - \sum_{i=1}^{N} C_i(t_i - t)\exp(-y(t_i - t)).$$

The duration is therefore

$$\frac{1}{92}\left(6\times 100e^{-6y} + \sum_{i=1}^{6} 2ie^{-y(i)}\right) = 5.699$$

The convexity is defined by

$$\begin{aligned}
\frac{1}{V}\frac{d^2V}{dy^2} &= \frac{1}{V}\left((T-t)^2 P\exp(-y(T-t)) + \sum_{i=1}^{N} C_i(t_i - t)^2\exp(-y(t_i - t))\right) \\
&= \frac{1}{92}\left(6^2\times 100e^{-6y} + \sum_{i=1}^{6} 2i^2 e^{-y(i)}\right) = 33.506
\end{aligned}$$

5. Zero-coupon bonds are available with principal of \$1 and the following maturities:

1 year (market price \$0.93)
2 years (market price \$0.82)
3 years (market price \$0.74)

Calculate the yield to maturities for the three bonds. Use a bootstrapping method to obtain the forward rates that apply between 1-2 years and 2-3 years.

$\boxed{\text{Solution}}$ The yield to maturity for the 1 year, 2 year and 3 year bond in turn, is

$$\begin{aligned} y_1 &= -\frac{\log(0.93)}{1} = 0.073 \\ y_2 &= -\frac{\log(0.82)}{2} = 0.099 \\ y_3 &= -\frac{\log(0.74)}{3} = 0.100 \end{aligned}$$

The 1-2 year forward rate satisfies

$$2y_2 = y_1 + F_{1-2}$$

which after rearranging gives

$$F_{1-2} = 2y_2 - y_1 = 0.126$$

The 2-3 year forward rate satisfies

$$3y_3 = 2y_2 + F_{2-3}$$

therefore

$$F_{2-3} = 3y_3 - 2y_2 = 0.103$$

6. Consider the following problem

$$\begin{cases} \dfrac{dV}{dt} + K(t) = r(t)V \\ V(T) = 1 \end{cases}$$

where $V = V(t)$ is the value of a coupon bond and the interest rate $r(t)$ is known. $K(t)$ represents a coupon payment, and T is maturity. By assuming a solution of the form

$$V = f\ \ (t)\, e^{-\int_t^T r(\tau)d\tau},$$

for the non-homogeneous part of the equation, obtain a particular solution.

$\boxed{\text{Solution}}$

Two parts to solving here. First the homogeneous equation, i.e.

$$\frac{dW}{dt} = r\,(t)\, W,$$

which is first order variable separable and gives

$$W = Ae^{-\int_t^T r(\tau)d\tau}.$$

Now solve the inhomogeneous equation by assuming existence of solution of the form

$$V = f\ \ (t)\, e^{-\int_t^T r(\tau)d\tau}$$

So substituting $V = f\ \ (t)\, e^{-\int_t^T r(\tau)d\tau}$ into $\dfrac{dV}{dt} + K\,(t) = r\,(t)\, V$ to find the form of $f\ \ (t)$. Using the product rule we get

$$\frac{df}{dt} e^{-\int_t^T r(\tau)d\tau} + r\,(t)\, fe^{-\int_t^T r(\tau)d\tau} + K\,(t) = r\,(t)\, fe^{-\int_t^T r(\tau)d\tau}$$

which upon simplifying becomes an equation of type variable separable

$$\frac{df}{dt} + K\,(t)\, e^{\int_s^T r(\tau)d\tau} = 0$$

Integrating this gives

$$f = A - \int_t^T K\,(s)\, e^{\int_s^T r(\tau)d\tau} ds.$$

So the general solution becomes

$$V = e^{-\int_t^T r(\tau)d\tau} \left(A - \int_t^T K\,(s)\, e^{\int_s^T r(\tau)d\tau} ds \right).$$

The final condition upon maturity $V\,(T) = 1$ gives $A = 1$ to give a particular solution

$$V = e^{-\int_t^T r(\tau)d\tau} \left(1 - \int_t^T K\,(s)\, e^{\int_s^T r(\tau)d\tau} ds \right).$$