CQF Module 2 Examination Solutions

Acknowledgement:

The diagrams in question 10 are courtesy of Emerson Bedford (June 2008 Cohort)

- The investment universe is composed of a set of 4 assets:

Asset	$\boldsymbol{\mu}$	$\boldsymbol{\sigma}$
A	0.04	0.07
B	0.08	0.12
C	0.12	0.18
D	0.15	0.26

with the following correlation structure

$$R = \begin{pmatrix} 1 & 0.2 & 0.5 & 0.3 \\ 0.2 & 1 & 0.7 & 0.4 \\ 0.5 & 0.7 & 1 & 0.9 \\ 0.3 & 0.4 & 0.9 & 1 \end{pmatrix}$$

Denote the column vector of asset weights by $\mathbf{w}$, the column vector of asset returns by $\boldsymbol{\mu}$ and the covariance matrix by Σ

a. *Compute the covariance matrix* Σ*.*

Answer: We will use the covariance matrix decomposition described in class. Define $\mathbb{S}$ as the diagonal matrix with standard deviation on its diagonal:

$$S = \begin{pmatrix} 0.07 & 0 & 0 & 0 \\ 0 & 0.12 & 0 & 0 \\ 0 & 0 & 0.18 & 0 \\ 0 & 0 & 0 & 0.26 \end{pmatrix}$$

Then, by the covariance matrix decomposition, the covariance matrix Σ is given by $\Sigma = SRS$, i.e.

$$\Sigma = \begin{pmatrix} 0.0049 & 0.00168 & 0.0063 & 0.00546 \\ 0.00168 & 0.0144 & 0.01512 & 0.01248 \\ 0.0063 & 0.01512 & 0.0324 & 0.04212 \\ 0.00546 & 0.01248 & 0.04212 & 0.0676 \end{pmatrix}$$

b. *Consider the following optimization:*

$$\min_{\mathbf{w}} \frac{1}{2}\mathbf{w}^T\Sigma\mathbf{w}$$

Subject to

$$\begin{aligned} \mathbf{w}^T\mathbf{1} &= 1 \\ \mathbf{w}^T\boldsymbol{\mu} &= 0.1 \end{aligned}$$

- *Explain in plain English what this optimization does.*
- *Solve this optimization using the Lagrangian method.*
- *Compute the standard deviation of the optimal portfolio.*
- *On a graph of expected returns plotted against standard deviation, identify the optimal portfolio.*

Answer:

- This optimizations is used to select an optimal asset weight vector $\mathbf{w}^*$ with the objective to minimize half of the portfolio variance, subject to two constraints: the budget equation, and a return constraint specifying the portfolio must provide a 10% return over the period. In short, this optimization determines the minimum-variance portfolio for a given level of return of 10%. As we vary the return constraint from $-\infty$ to $+\infty$, this optimization will enable us to parametrize the **boundary** of the opportunity set .

- In order to solve the optimization problem, we start by forming the Lagrange function: with two lagrange multipliers λ and γ:

  $$L(\mathbf{w}, \lambda, \gamma) = \frac{1}{2}\mathbf{w}^T\Sigma\mathbf{w} + \lambda(0.1 - \mu^T\mathbf{w}) + \gamma(1 - \mathbf{1}^T\mathbf{w})$$

  and solve for the first order condition:

  $$\begin{aligned} \frac{\partial L}{\partial \mathbf{w}}(\mathbf{w}, \lambda, \gamma) &= \mathbf{w}^T\Sigma - \lambda\mu^T - \gamma\mathbf{1}^T = 0 \\ \frac{\partial L}{\partial \lambda}(\mathbf{w}, \lambda, \gamma) &= 0.1 - \mu^T w = 0 \\ \frac{\partial L}{\partial \gamma}(\mathbf{w}, \lambda, \gamma) &= (1 - \mathbf{1}^T\mathbf{w}) = 0 \end{aligned}$$

  We then get the optimal weight vector $\mathbf{w}^*$

  $$\mathbf{w}^* = (\Sigma)^{-1}(\lambda\mu + \gamma\mathbf{1})$$

where

$$\begin{cases} \lambda & \approx & 0.2542605 \\ \gamma & \approx & -0.0078808 \end{cases}$$

and therefore

$$\mathbf{w}^* = \begin{pmatrix} 5.87\% \\ 75.90\% \\ -31.95\% \\ 50.18\% \end{pmatrix}$$

- The standard deviation of the portfolio is equal to

$$\sigma_\Pi = \sqrt{\mathbf{w}^{*T}\Sigma\mathbf{w}^*} =\approx 13.25\%$$

- The optimal portfolio is located on the efficient frontier, at the coordinates $(0.1, 0.1325)$.

c. Consider the following optimization:

$$\min_{\mathbf{w}} \frac{1}{2}\mathbf{w}^T\Sigma\mathbf{w}$$

Subject to

$$\mathbf{w}^T\mathbf{1} \quad = \quad 1$$

- Explain in plain English what this optimization does.
- Solve this optimization using the Lagrangian method.
- Compute the standard deviation of the optimal portfolio.
- On a graph of expected returns plotted against standard deviation, identify and name the optimal portfolio.

Answer:

- This optimizations is used to select an optimal asset weight vector $\mathbf{w}^*$ with the objective to minimize half of the portfolio variance, subject to the sole budget constraint. In short, this optimization problem gives us the **global minimum variance portfolio**.
- In order to solve the optimization problem, we start by forming the Lagrange function: with two lagrange multipliers λ and γ:

$$L(\mathbf{w}, \lambda) = \frac{1}{2}\mathbf{w}^T\Sigma\mathbf{w} + \lambda(1 - \mathbf{1}^T\mathbf{w})$$

and solve for the first order condition:

$$\begin{aligned} \frac{\partial L}{\partial \mathbf{w}}(\mathbf{w}, \lambda) &= \mathbf{w}^T\Sigma - \lambda\mathbf{1}^T = 0 \\ \frac{\partial L}{\partial \lambda}(\mathbf{w}, \lambda) &= (1 - \mathbf{1}^T\mathbf{w}) = 0 \end{aligned}$$

From the first equation, we get the optimal weight vector $\mathbf{w}^*$

$$\mathbf{w}^* = (\Sigma)^{-1}(\lambda \mathbf{1})$$

and substituting into the second, we obtain

$$\lambda = \frac{1}{\mathbf{1}^T \Sigma^{-1} \mathbf{1}} \approx 0.000664337$$

and therefore

$$\mathbf{w}^* = \begin{pmatrix} 90.54\% \\ 82.91\% \\ -137.46\% \\ 64.01\% \end{pmatrix}$$

- The return of the portfolio is equal to

$$\sigma_\Pi = \mathbf{w}^{*T}\mu = \approx 3.3608\%$$

- The standard deviation of the portfolio is equal to

$$\sigma_\Pi = \sqrt{\mathbf{w}^{*T}\Sigma\mathbf{w}^*} = \approx 2.5775\%$$

- The optimal portfolio is the **global minimum variance portfolio**. It is located at the tip of the efficient frontier, at the coordinates $(0.033608, 0.025775)$.

2. A butterfly spread can be created by buying call options with strike prices of £15 and £20, and selling two call options with strike prices of £17.5. The initial investment is therefore

$$4 + 0.5 - 2 \times 2 = 0.5$$

The table shows the variation of profit with the final stock price:

Stock Price $S(T)$	Profit
$S(T) < 15$	-0.5
$15 < S(T) < 17.5$	$S(T) - 15.5$
$17.5 < S(T) < 20$	$19.5 - S(T)$
$S(T) > 20$	-0.5

3. Fill in the table

Position	Strategy	Max loss	Max gain	Breakeven
Long Call	Bullish	Premium	Unlimited	Strike + Premium
Short Call	Bearish/neutral	Unlimited	Premium	Strike + Premium
Long Put	Bearish	Premium	Strike − Premium	Strike − Premium
Short Put	Bullish/neutral	Strike − Premium	Premium	Strike − Premium

4. **(i):**

$$\begin{array}{ccc} & & 15 \\ & V_1 & \\ V & & 0 \\ & V_{-1} & \\ & & 0 \end{array}$$

To find V_1 from portfolio: $\Pi = V - \Delta S$. Then from T_1 to T we have

$$\Pi \longrightarrow \left\{ \begin{array}{c} 15 - \Delta\left(\alpha + 20\right) \\ 0 - \Delta\alpha \end{array} \right.$$

so for risk free portfolio$\Rightarrow$ choose $15 - \Delta\left(\alpha + 20\right) = -\Delta\alpha \longrightarrow \Delta = 3/4$. For no arbitrage we want

$$V_1 - \Delta\left(\alpha + 10\right) = -\Delta\alpha$$

since $r = 0$. Solving gives $V_1 = 7.5$
For V_{-1} :

$$\Pi \longrightarrow \left\{ \begin{array}{c} 0 - \Delta\alpha \\ 0 - \Delta\left(\alpha - 20\right) \end{array} \right. \Rightarrow \Delta = 0$$

Therefore $V_{-1} = 0$.
For V :

$$V - \Delta\alpha = \left\{ \begin{array}{ll} V_1 - \Delta\left(\alpha + 10\right) & \equiv \frac{15}{2} - \Delta\left(\alpha + 10\right) \\ V_{-1} - \Delta\left(\alpha - 10\right) & \equiv 0 - \Delta\left(\alpha - 10\right) \end{array} \right.$$

so $\frac{15}{2} - 20\Delta = 0 \longrightarrow \Delta = 3/8$.
Finally $V - \Delta\alpha = -\left(\alpha - 10\right) \longrightarrow V = 10\Delta = 15/4$. Hence

$$\begin{array}{ccc} & & 15 \\ & 15/2 & \\ V = 15/4 & & 0 \\ & 0 & \\ & & 0 \end{array}$$

(ii)

$$\begin{array}{ccc} & & 0 \\ & V_1 & \\ V & & 0 \\ & V_{-1} & \\ & & 15 \end{array}$$

same method gives:-

$$\begin{array}{ccc} & & 0 \\ & 0 & \\ V = 15/4 & & 0 \\ & 15/2 & \\ & & 15 \end{array}$$

(iii)

$$V(S,T)=\begin{cases} S-\alpha-5 & S\geq\alpha+5\\ 0 & \alpha-5\leq S\leq\alpha+5\\ \alpha-5-S & S\leq\alpha-5\end{cases}$$

hence

$$\begin{array}{ccc} & & 15\\ & V_1 & \\ V & & 0\\ & V_{-1} & \\ & & 15\end{array}$$

add **(i)** and **(ii)**

$$\text{Payoff } = \max(S-\alpha-5,0)+\max(\alpha-5-S,0)\equiv\textbf{(iii)}$$

$$\begin{array}{ccc} & & 15\\ & 15/2 & \\ V=15/2 & & 0\\ & 15/2 & \\ & & 15\end{array}$$

5. See spreadsheet

6. **a)** We define the return R_i on an asset S_i as

$$R_i=\frac{S_i-S_{i-1}}{S_{i-1}}$$

In the EWMA model,the variance rate (i.e. square of volatility) calculated for day n is a weighted average of the R_{n-i}^2's $(i=1,\ 2,\ 3,\ ..)$.
For some constant $\lambda\ (0<\lambda<1)$ the weight given to R_{n-i-1}^2 is λ times the weight given to R_{n-i}^2. The volatility estimated for day n, σ_n is related to the volatility estimated for day $(n-1)$, σ_{n-1}, by

$$\sigma_n{}^2=\lambda\sigma_{n-1}^2+\frac{(1-\lambda)}{\delta t}R_{n-1}^2.$$

We note a very attractive property of the EWMA model. To calculate the volatility estimate for day n, it is sufficient to know the volatility estimate for $(n-1)$ and σ_{n-1}.

b) In reducing λ from 0.95 to 0.85, we are putting more weight on recent observations of $R_i{}^2$ and less weight is given to older observations. Volatilities calculated with $\lambda=0.85$ will react more quickly to new information and will move around much more than volatilities calculated with $\lambda=0.95$.

7. Find the Fourier transform $\widehat{f}(\omega)$ of

$$f(x) = \begin{cases} 1/2\epsilon & |x| \le \epsilon \\ 0 & |x| > \epsilon \end{cases}$$

and show that

$$\lim_{\epsilon \longrightarrow 0^+} \widehat{f}(\omega) \longrightarrow \frac{1}{\sqrt{2\pi}}$$

$$\begin{aligned}
\widehat{f}(\omega) &= \mathcal{F}(f) = \tfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{ix\omega} dx \quad (\text{ignore } 1/\sqrt{2\pi} \text{ for time being}) \\
&= \tfrac{1}{2\epsilon} \int_{-\epsilon}^{\epsilon} e^{ix\omega} dx = \tfrac{1}{2\epsilon} \left. \frac{e^{ix\omega}}{i\omega} \right|_{-\epsilon}^{\epsilon} = \tfrac{1}{2\epsilon} \frac{\left(e^{i\epsilon\omega} - e^{-i\epsilon\omega}\right)}{i\omega} \\
&= \tfrac{1}{2\epsilon} . \frac{2}{\omega} . \frac{\left(e^{i\epsilon\omega} - e^{-i\epsilon\omega}\right)}{2i} = \tfrac{1}{\omega\epsilon} . \sin(\epsilon\omega) \\
\widehat{f}(\omega) &= \tfrac{1}{\sqrt{2\pi}} \tfrac{1}{\omega\epsilon} . \sin(\epsilon\omega)
\end{aligned}$$

$$\lim_{\epsilon \longrightarrow 0^+} \tfrac{1}{\sqrt{2\pi}} \frac{\sin(\epsilon\omega)}{\omega\epsilon}$$

using L'Hospital's rule

$$\begin{aligned}
\lim_{\epsilon \longrightarrow 0^+} \tfrac{1}{\sqrt{2\pi}} \frac{\omega \cos(\epsilon\omega)}{\omega} &= \lim_{\epsilon \longrightarrow 0^+} \tfrac{1}{\sqrt{2\pi}} \cos(\epsilon\omega) \\
&\longrightarrow \tfrac{1}{\sqrt{2\pi}}
\end{aligned}$$

8. Evaluate $\int_1^4 (x-1)^2 (4-x)^3 \, dx$ using the Beta function. **Hint:** consider a change of variable.

$$I = \int_1^4 (x-1)^2 (4-x)^3 \, dx$$

let

$$x = 3u + 1 \longrightarrow dx = 3du$$

the limits $x = 1$ and 4 become in turn $u = 0$ and 1.

$$\begin{aligned}
I &= \int_0^1 (3u)^2 (3-3u)^3 . 3du \\
&= 3^6 \int_0^1 u^2 (1-u)^3 \, du = 3^6 B(3,4) \\
&= 3^6 \frac{\Gamma(3)\Gamma(4)}{\Gamma(7)} = 3^6 \frac{2!3!}{6!} = \frac{3^5}{20} \\
&= \frac{243}{20}
\end{aligned}$$

9. Identify any singular points and classify them (you are not expected to solve any equation). Start by writing each in standard form $y''+p(x)\,y'+q(x)\,y=0$.

   (i) $x^2y''+2xy'+y=0$

$$y''+\frac{2}{x}y'+\frac{1}{x^2}y=0$$

   $x=0$ is a singular point. Checking $x.\frac{2}{x}$ and $x^2.\frac{1}{x^2}$ tells us that the singularity is a regular singular point

   (ii) $x^2y''+xy'+\left(x^2-4\right)y=0$

$$y''+\frac{1}{x}y'+\frac{\left(x^2-4\right)}{x^2}y=0$$

   $x=0$ is a singular point. $x.\frac{1}{x}$ and $x^2.\frac{\left(x^2-4\right)}{x^2}$ removes the singularity hence we have a regular singular point.

   (iii) $xy''+x^2y'+y=0$

$$y''+xy'+\frac{1}{x}y=0$$

   $x=0$ is a singular point. $x^2.\frac{1}{x}$ has a Taylor series expansion (TSE), so $x=0$ is a regular singular point.

   (iv) $\left(1-x^2\right)^2y''+xy'+y=0$

$$y''+\frac{x}{\left(1-x^2\right)^2}y'+\frac{1}{\left(1-x^2\right)^2}y=0$$

   $x=\pm1$ is a singular point. Recall that if x_0 is a singularity then we classify according to the existence of TSE of

$$(x-x_0)\,p(x)\ \text{ and }\ (x-x_0)^2\,q(x)$$

   Rewriting the the equation as

$$y''+\frac{x}{(1-x)^2\,(1+x)^2}y'+\frac{1}{(1-x)^2\,(1+x)^2}y=0$$

   So at $x_0=1$

$$(x-1)\,p(x)=(x-1)\,\frac{x}{(1+x)^2\,(1-x)^2}$$

   which does not have a TSE at $x=1$ hence it is irregular. At $x_0=-1$

$$(x+1)\,p(x)=(x+1)\,\frac{x}{(1+x)^2\,(1-x)^2}$$

   which is singular at -1. Hence it is an irregular singular point. We could have checked $(x-x_0)^2\,q(x)$, but the fact that the initial test failed confirms the nature of the singular point.

(v) $(x+5)\,y'' + x^4 y = 0$

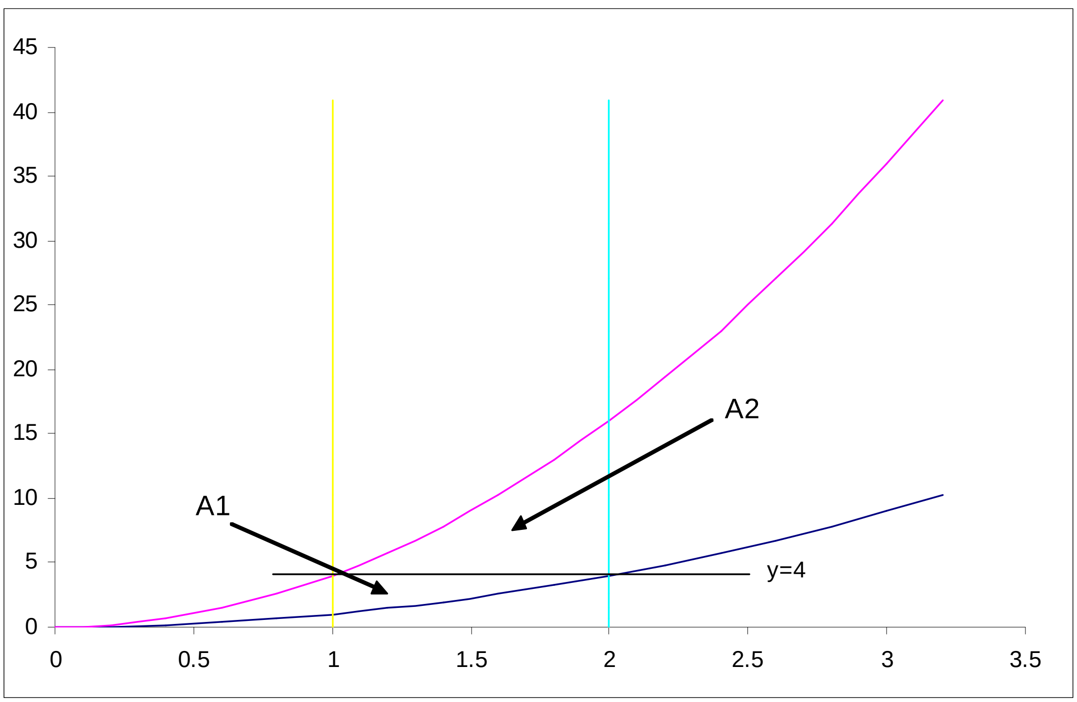
$$y'' + \frac{x^4}{(x+5)}y = 0$$

singular point at $x = -5$. Here $p\,(x)$ is zero. Since $(x+5)^2 \cdot \dfrac{x^4}{(x+5)}$ is $(x+5)\,x^4$ which is analytic and has a TSE about the point $x = 5$, it is a regular singular point.

10. (i) Consider the double integral

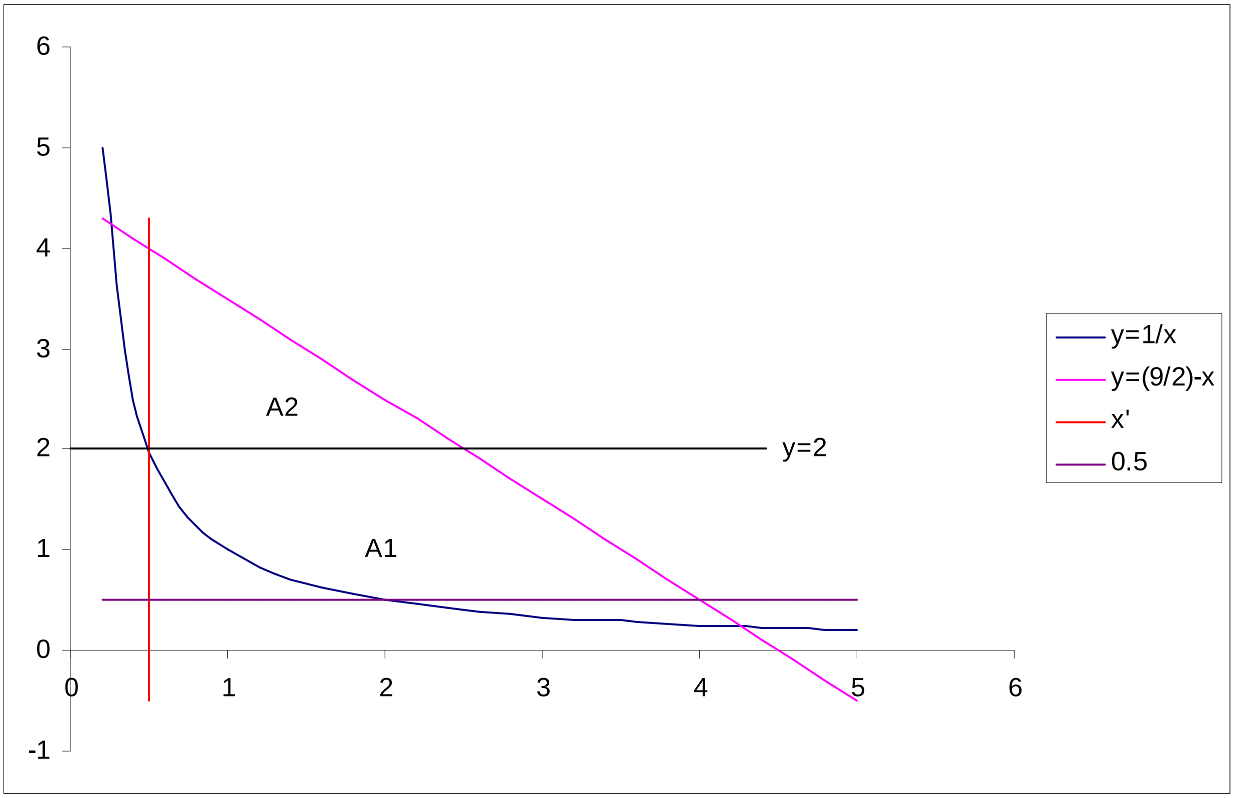
$$\int_1^2 \int_{x^2}^{4x^2} f(x,y)\, dy dx$$

where $f(x,y) = x+y$. By changing the order of integration, evaluate the integral. The region is drawn below. This is more of a mathematical exercise - normally you would never reverse the integral in this case as it introduces a lot more unnecessary work. Call the complete area bounded by the four lines $A = A_1 + A_2$. The line $y = 4$ is only drawn in to split the region in to two:



$$\iint_A = \iint_{A_1} + \iint_{A_2}$$

$$
\begin{aligned}
\int_1^2 \int_{x^2}^{4x^2} f(x,y)\, dydx &= \iint_{A_1} f(x,y)\, dxdy + \iint_{A_2} f(x,y)\, dxdy \\
&= \int_1^4 \int_1^{\sqrt{y}} (x+y)\, dxdy + \int_4^{16} \int_{\sqrt{y}/2}^{2} (x+y)\, dxdy \\
&= \int_1^4 \tfrac{1}{2}x^2 + xy\Big|_1^{\sqrt{y}} dy + \int_4^{16} \tfrac{1}{2}x^2 + xy\Big|_{\sqrt{y}/2}^{2} dy \\
&= \int_1^4 \left(y^{3/2} - \tfrac{1}{2}y - \tfrac{1}{2}\right) dy + \int_4^{16} \left(2 - \tfrac{1}{2}y^{3/2} + \tfrac{15}{8}y\right) dy \\
&= 231/4
\end{aligned}
$$



(ii) Using double integration, calculate the area bounded by

$$y = \frac{1}{x},\ y = \frac{9}{2} - x,\ x = \frac{1}{2},\ y = \frac{1}{2}.$$

There are a number of ways to tackle this. Here we draw the line $y = 2$ in, to split the bounded region into A_1 and A_2

$$
\begin{aligned}
\iint_A &= \iint_{A_1} dxdy + \iint_{A_2} dxdy \\
&= \int_{1/2}^{2}\int_{1/y}^{9/2-y} dxdy + \int_{2}^{4}\int_{1/2}^{9/2-y} dxdy \\
&= \int_{1/2}^{2} x\big|_{1/y}^{9/2-y}\, dy + \int_{2}^{4} x\big|_{1/2}^{9/2-y}\, dy \\
&= \int_{1/2}^{2} \left(\tfrac{9}{2} - y - 1/y\right) dy + \\
&\quad \int_{2}^{4} (4-y)\, dy \\
&= \tfrac{9}{2}y - \tfrac{1}{2}y^2 - \ln y\big|_{1/2}^{2} + 4y - \tfrac{1}{2}y^2\big|_{2}^{4} \\
&= \tfrac{55}{8} - 2\ln 2
\end{aligned}
$$

(iii) Calculate the area of the circle $x^2 + y^2 = 4y$ between $\theta = \pi/3$ and $\theta = \pi/4$ **Hint:** use plane polars in (iii)

Completing the square on $x^2+y^2 = 4y$ gives $x^2+(y-2)^2 = 2^2$ which is a circle centre $(0, 2)$ radius 2. This is the region defined by A

$$\iint_A dxdy = \iint_A rdrd\theta$$

where $x = r\cos\theta$, $y = r\sin\theta$ which gives $r^2\cos^2\theta + r^2\sin^2\theta = 4r\sin\theta$, i.e. $r = 0$ to $r = 4\sin\theta$. The θ limits go from $\pi/4$ to $\pi/3$.

$$\begin{aligned}
\int_{\pi/4}^{\pi/3}\int_0^{4\sin\theta} rdrd\theta &= \int_{\pi/4}^{\pi/3} \left.\frac{r^2}{2}\right|_0^{4\sin\theta} d\theta = \int_{\pi/4}^{\pi/3} 8\sin^2\theta d\theta \\
&= 4\int_{\pi/4}^{\pi/3} (1-\cos 2\theta)\, d\theta \\
&= 4\left[\theta - \frac{1}{2}\sin 2\theta\right]_{\pi/4}^{\pi/3} \\
&= 2 - \sqrt{3} + \frac{\pi}{3}
\end{aligned}$$

11. Consider the stochastic process $Y(t)$ satisfying the SDE

$$dY(t) = f(t)dt + g(t)dX(t), \qquad Y(0) = Y_0 \tag{1}$$

where $f(t)$ and $g(t)$ are two time-dependent functions and $X(t)$ is a standard Brownian motion.

How should we choose $f(t)$ if we want the process $Z(t) = e^{Y(t)}$ to be an exponential martingale?

Consider the process $Z(t) = e^{Y(t)}$. Applying Itô to the function $F(y) = e^y$ and the process $Y(t)$ given in (1), we obtain:

$$\begin{aligned} dZ(t) &= de^{Y(t)} \\ &= \frac{dF}{dy}\left(f(t)dt + g(t)dX(t)\right) + \frac{1}{2}\frac{d^2F}{dy^2}g^2(t)dt \\ &= e^{Y(t)}\left(f(t) + \frac{1}{2}g^2(t)\right)dt + e^{Y(t)}g(t)dX(t) \end{aligned}$$

$Z(t)$ is a martingale iff it is a driftless process, and therefore for $Z(t)$ to be a martingale we must have

$$e^{Y(t)}\left(f(t) + \frac{1}{2}g^2(t)\right)$$

This is only possible if

$$f(t) = -\frac{1}{2}g^2(t).$$

12. Consider the function $m_n(t)$ defined as

$$m_n(t) = \mathbb{E}[X^n(t)], \qquad n = 1, 2, \ldots \tag{2}$$

where $X(t)$ is a standard Brownian motion.

Applying Itô's lemma, show that:

$$m_n(t) = \frac{1}{2}n(n-1)\int_0^t m_{n-2}(t), \quad n = 2, 3, \ldots \tag{*}$$

Deduce from $(*)$ that

$$m_4(t) = 3t^2$$

and compute $m_6(t)$.
Because of the expectation, we cannot tackle expression (2) up-front.

Consider instead the auxiliary function $g_n(t,x) = x^n$ for $n \geq 2$. Note the relation between $g_n(t,x)$ and $m_n(t)$:

$$m_n(t) = \mathbf{E}[g_n(t, X(t))]$$

Applying Itô's lemma to the function g_n and the standard Brownian motion, we get

$$\begin{aligned}
g_n(t) &= g_n(0) + \int_0^t \frac{\partial g_n}{\partial s} ds + \int_0^t \frac{\partial g_n}{\partial x} dX(s) + \frac{1}{2}\int_0^t \frac{\partial^2 g_n}{\partial x^2} ds \\
&= n\int_0^t X^{n-1}(s)dX(s) + \frac{1}{2}n(n-1)\int_0^t X^{n-2}(s)ds
\end{aligned}$$

since $g_n(0) = 0$

Take expectation on both sides to get:

$$\begin{aligned}
m_n(t) &= \mathbf{E}\left[g_n(t, X(t))\right] \\
&= n\mathbf{E}\left[\int_0^t X^{n-1}(s)dX(s) + \frac{1}{2}n(n-1)\int_0^t X^{n-2}(s)ds\right]
\end{aligned}$$

By linearity of expectation,

$$m_n(t) = n\mathbf{E}\left[\int_0^t X^{n-1}(s)dX(s)\right] + \frac{1}{2}n(n-1)\mathbf{E}\left[\int_0^t X^{n-2}(s)ds\right]$$

Recall that $\int_0^t X^{n-1}(s)dX(s)$ is an Itô integral and it is therefore a martingale, so $\mathbf{E}[\int_0^t X^{n-1}(s)dX(s)] = 0$.

Interchanging the order of integration to take the expectation inside the integral (that's Fubini's theorem), we finally get

$$\begin{aligned}
m_n(t) &= \frac{1}{2}n(n-1)\int_0^t \mathbf{E}\left[X^{n-2}(s)\right] ds \\
&= \frac{1}{2}n(n-1)\int_0^t m_{n-2}(s)ds
\end{aligned}$$

Now let's apply this formula for $n = 4$:

$$\begin{aligned} m_4(t) &= 6\int_0^t \mathbf{E}\left[X^2(s)\right] ds \\ &= 6\int_0^t s ds \\ &= 3t^2 \end{aligned}$$

What about $n = 6$?

$$\begin{aligned} m_6(t) &= 15\int_0^t \mathbf{E}\left[X^4(s)\right] ds \\ &= 45\int_0^t s^2 ds \\ &= 15t^3 \end{aligned}$$