The Magic Behind

The Black-Scholes Option Pricing Model

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A Thesis presented to Lafayette College for

Honors in Mathematics

on May 8th, 2001.

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I. Acknowledgments

I would first like to express my gratitude to my thesis advisor, Qin Lu. Her motivation and her love for mathematics are truly an inspiration. Working with Qin for this past year has been a pleasure, to say the least. She will forever hold a place in my life as a mentor and a friend.

Next, I would like to thank Tom Hill. I was very fortunate to have him as a reader of my thesis and as my academic advisor here at Lafayette. He provided me with invaluable guidance throughout my academic career and encouraged my love and pursuit of mathematics.

I would also like to thank my second reader, Don Chambers. His clear thinking and love of finance have given him a spirit to which I will always aspire. He has taught me to think with a financial mind—a quality that I will always cherish.

Lastly, I would like to thank my friends and family. Without them, I would be lost. They have supported my every endeavor. They have supported my choices in life, and for this, I am forever indebted. My family and friends make life both meaningful and fun.

II. Introduction

In 1973, Fischer Black, Myron Scholes and Robert Merton formulated a financial model that would forever change the financial world. This model, the Black-Scholes Option Pricing Model, quickly became the "Ford Model T" of derivative pricing and triggered a financial engineering revolution that is still strong and growing.

Myron Scholes and Robert Merton were recognized for their innovation in 1977, when they were awarded the Nobel Prize for economics. Sadly, Fischer Black passed away in 1995—surely he would have also been a recipient of this honor.

III. Background

The Black-Scholes Option Pricing Model has revolutionized financial engineering through the use of *derivatives*. A *derivative* is a financial instrument that derives its price from an underlying asset. An *option* is a derivative that affords the owner the privilege to buy or sell the underlying asset at a determined price, sometime in the future. Usually, the owner of an option pays a premium (the option price) for the right to *exercise* (or buy/sell the underlying asset of) that option. The Black-Scholes model finds a fair price for these options, thus allowing them to be efficiently traded.

When an investor purchases an option, the investor is said to have taken a *long position* in that option. Conversely, when an investor sells an option, the investor is said to have *written* the option, or taken a *short position* in that option. These positions are important, because an investor may create a portfolio where many different positions in options and their underlying assets are held, as part of a *hedge*, or risk-reducing strategy. Hedging strategies are where derivatives are used the most in today's markets. It is important to note also that the payoff from options, and derivatives in general, is a zero-sum game. When an option is exercised, a transfer of wealth occurs between the investor in the long position and the investor in the short position. Because of this, two parties must enter into the contract, covering both positions.

There are two types of options, *calls* and *puts*. A *call option* gives the owner the right to buy an asset for a predetermined price, at an agreed upon date. A *put option* gives the owner the right to sell an asset at a determined price, at an agreed upon date. The price at which the owner of an option has the right to buy or sell the underlying asset is called the *strike price*, or *exercise price* of the option, and the date at which the option may be exercised is the *maturity*, or *expiration date*. If the option can be exercised only on the maturity date, then it is referred to as a *European option*, whereas if the option can be exercised anytime before the maturity date, then the option is called an *American option*.

Example #1

An investor purchases a European call option on a stock with a strike price of $100, and a *spot price* (or current market price) of $90. The option has a price of $4 and an expiration date in 3 months.

Assume that the price of the stock after three months has gone up to $110/share. Under this scenario, it is in the best interest of the investor to exercise this option, and buy the stock for $100. This investor could then sell the same stock for $110, for a net cash flow of $10. Therefore, disregarding taxes and transaction costs, the investor made $6 (110 - 100 - 4) from this transaction.

Now assume that the price of the stock stayed at $90. Under this scenario, it would be foolish for the investor to exercise the option. The option would allow the stock to be purchased for $100, but the investor could purchase it in the spot market for $90. Therefore the investor will not exercise the option, and the option is said to expire worthless. The investor will have lost only the price paid for the option, $4.

Hence, the buyer of a call option hopes that the market price will rise, so that the option may be exercised. On the other hand, the writer of the call option hopes that the price of the underlying stock will stay below the strike price, so that he/she can collect the call price up-front, without obligation to fulfill the buyer's exercise.

We can generalize this example into a payoff scheme for all European call options. The payoff from an option of this type at the expiration date, excluding transaction costs, is given by

$$Y = Max(S_T - X, 0) ,$$

where S_T is the price of the stock at the expiration date, X is the strike price, and Y is the payoff of the call option. The net gain from the transaction is $Y - c$, where c is the price of the call.

The Black-Scholes Model only prices European call options. It is enough to consider only call options because of a principle known as *put-call parity*. Put-call parity states that in an efficient market, the price of a European put option can be deduced from the price of the corresponding call option, given that both options have the same underlying asset and expiration date. To demonstrate this parity, construct the following portfolios.

Portfolio A: 1 European call option + an amount of cash equal to Xe^{-rT}

Portfolio B: 1 European put option + 1 share of the underlying stock

(Note that Xe^{-rT} is the present value of the strike price, X, where r is the risk-free interest rate, and T is the time, in years, to maturity.) At the expiration date, both of these portfolios are worth $Max(S_T, X)$. Because the options are European, they cannot be exercised early. Therefore, the portfolios must have the same value today. So

$$c + Xe^{-rT} = p + S,$$

where c is the value of the call option, p is the value of the put option, X is the strike price, and S is the spot price of the underlying stock. Put-call parity is exactly this relationship between p and c. It suffices, therefore, to consider only call options when deriving the Black-Scholes model.

The Black-Scholes model finds a fair-market price for a European call option on a stock that does not pay dividends. It uses five parameters:

S_0 = the current stock price,

X = the strike price of the option,

r = the risk-free interest rate,

T = the time to expiration (in years),

σ^2 = the volatility of the stock.

The Black-Scholes model states that, if c is the unknown price of the call, then

$$c = S_0 N(d_1) - X e^{-rT} N(d_2),$$

where
$$d_1 = \frac{\ln\left(\frac{S_0}{X}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}},$$

and
$$d_2 = \frac{\ln\left(\frac{S_0}{X}\right) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}.$$

IV. Assumptions

The Black-Scholes model is based on a few simplifying assumptions.

1. **Stocks pay no dividends during the life of the option.** Because many companies pay dividends to their shareholders, this can be a limiting assumption. This assumption can be relaxed fairly easily but will not be discussed in this paper.

2. **The Black-Scholes model only prices European call options.** Due to put-call parity, it is sufficient to consider only European call options.

3. **No commissions are charged.** Options are used most often by large businesses, where trading costs are minimal.

4. **Interest rates remain constant and known.** The B-S model uses the risk-free interest rate, as seen in 3-month treasury bills. This rate does fluctuate under certain market conditions, thus violating an assumption of the model.

5. **Stock prices are lognormally distributed.** This assumption can be validated using empirical research. Although it is generally accepted that stock prices are lognormally distributed, they are not perfectly lognormal. In fact, the distribution of stock prices has a fatter tail than that of the lognormal distribution.

6. **Markets are efficient.** The efficient market hypothesis implies that there are no arbitrage opportunities in the market. Because no arbitrage opportunities exist, then in non-overlapping time intervals, the change in a stock price can be assumed to be independent. The efficient market hypothesis assumes that people prefer more wealth to less, trading is continuous, and that short-selling

is permitted. It also implies that money can be borrowed and lent at the same rate.

7. **Shares of a stock can be divided.** This assumption can easily be relaxed by finding the least common multiple between a stock and its derivative.

Many different variations of the Black-Scholes model exist which address these assumptions. For the purposes of this paper, the simplest version of the model is derived.

We will use only the knowledge of calculus and probability to derive the Black-Scholes model. The more traditional way is to solve the Black-Scholes-Merton differential equation that will be discussed later in the paper. Also, Ito's Lemma will be proved using a non-measure theory based method.

IV. Derivation of the Black-Scholes Option Pricing Model

1. Modeling Stock Price Movements

To begin deriving the Black-Scholes formula, we must first model the behavior of stock prices. When determining the relevant factors in the change of stock prices, ΔS, over a small time period, Δt, it is natural to assume that ΔS will reflect the expected return of the stock (μ), and Δt. If we may assume for the moment that stock prices have no volatility, we are left with the equation

$$\Delta S = \mu S \Delta t \ ,$$

where S is the current stock price, Δt is a small time period, and μ is the annual expected return of S, with continuous compounding. (Note that μ is a constant that is determined by the market and is independent of S. In other words, investors will seek a set annual return from a stock, regardless of the stock's current price.) This assumption can be verified as follows. When Δt is small, the stock price at Δt, $S_{\Delta t,}$ is equal to the initial price, S, times $e^{\mu\Delta t}$. That is,

$$S_{\Delta t} = Se^{\mu\Delta t}.$$

Therefore,

$$\begin{aligned} \Delta S &= S_{\Delta t} - S \\ &= Se^{\mu\Delta t} - S \\ &= S(e^{\mu\Delta t} - 1) \\ &\approx S\mu\Delta t \ , \end{aligned}$$

for small values of Δt.

The stock price will most certainly not increase at a constant rate, so it seems practical to include some randomness in our model. To do this, we will add the following term

$$\sigma S \sqrt{\Delta t}\varepsilon ,$$

where σ is the volatility per annum, i.e. the standard deviation of the return $\frac{\Delta S}{S}$ when annualized, and ε is a random number drawn from a standard normal distribution, N(0,1). This normality of ε is a consequence of the assumption that stock prices follow stochastic Brownian motion, which implies that in a very small time period, the price of a stock will not change by a large amount[1]. These deviations are then assumed to be normally distributed. The rationale behind the square root of Δt will be explored later in this paper. It is important to note that because Δt is a very small time period, the assumption of normality is very weak. Assuming that normality holds, then

$$\sigma S \sqrt{\Delta t}\varepsilon \sim N\left(0, \sigma S \sqrt{\Delta t}\right).$$

Therefore, our model of the behavior of S is

$$\Delta S = \mu S \Delta t + \sigma S \sqrt{\Delta t}\varepsilon . \tag{1}$$

According to our model, the change in a stock price depends on ε, which is random.

It is worthwhile to stress that this model is only *locally* true (namely if Δt is very small). Over larger time intervals, we will see that S is not normally distributed, but rather lognormally distributed

Our equation for ΔS is dependent on two variables, S and t, so we can write the equation in the form

$$\Delta S = a(S,t)\Delta t + b(S,t)\varepsilon\sqrt{\Delta t}\ .$$

Ito's Lemma: If we assume that equation 1 is true for very small values of Δt, and that f is a function of variables S and t with partials of the second order, then

$$\Delta f = (\tfrac{\partial f}{\partial S}\mu S + \tfrac{\partial f}{\partial t} + \tfrac{1}{2}\tfrac{\partial^2 f}{\partial S^2}\sigma^2 S^2)\Delta t + (\tfrac{\partial f}{\partial S}\sigma S\varepsilon)\sqrt{\Delta t} + o(\Delta t),$$

when Δt is small.

Proof: Regard f as a function of S and t, with second partials.

Using the Taylor Polynomial to approximate Δf, we are left with

$$\Delta f = \tfrac{\partial f}{\partial S}\Delta S + \tfrac{\partial f}{\partial t}\Delta t + \tfrac{1}{2}\tfrac{\partial^2 f}{\partial S^2}\Delta S^2 + \tfrac{1}{2}\tfrac{\partial^2 f}{\partial t^2}\Delta t^2 + \tfrac{\partial^2 f}{\partial f\partial S}\Delta S\Delta t + o(\Delta t)\ .$$

Using equation 1,

$$\begin{aligned}\Delta f &= \tfrac{\partial f}{\partial S}(\mu S\Delta t + \sigma S\sqrt{\Delta t}\varepsilon) + \tfrac{\partial f}{\partial t}\Delta t + \tfrac{1}{2}\tfrac{\partial^2 f}{\partial S^2}(\mu S\Delta t + \sigma S\sqrt{\Delta t}\varepsilon)^2 + \tfrac{1}{2}\tfrac{\partial^2 f}{\partial t^2}\Delta t^2 \\ &+ \tfrac{\partial^2 f}{\partial f\partial S}(\mu S\Delta t + \sigma S\sqrt{\Delta t}\varepsilon)\Delta t + o(\Delta t).\end{aligned}$$

So

$$\begin{aligned}\Delta f &= \tfrac{\partial f}{\partial S}\mu S\Delta t + \tfrac{\partial f}{\partial S}\sigma S\sqrt{\Delta t}\varepsilon + \tfrac{\partial f}{\partial t}\Delta t + \tfrac{1}{2}\tfrac{\partial^2 f}{\partial S^2}\mu^2 S^2\Delta t^2 + \tfrac{\partial^2 f}{\partial S^2}\mu\sigma S^2\Delta t^{\frac{3}{2}}\varepsilon + \tfrac{1}{2}\tfrac{\partial^2 f}{\partial S^2}\sigma^2 S^2\Delta t\varepsilon^2 \\ &+ \tfrac{1}{2}\tfrac{\partial^2 f}{\partial t^2}\Delta t^2 + \tfrac{\partial^2 f}{\partial t\partial S}\mu S\Delta t^2 + \tfrac{\partial^2 f}{\partial t\partial S}\sigma S\varepsilon\Delta t^{\frac{3}{2}} + o(\Delta t).\end{aligned}$$

Now, Δt can be thought of as a very small value (less that 1 day out of 250 trading days in a year). Because Δt is extremely small, anything multiplied by Δt to a power greater than 1 can be ignored, therefore simplifying our equation to

$$\Delta f = \tfrac{\partial f}{\partial S}\mu S\Delta t + \tfrac{\partial f}{\partial S}\sigma S\sqrt{\Delta t}\varepsilon + \tfrac{\partial f}{\partial t}\Delta t + \tfrac{1}{2}\tfrac{\partial^2 f}{\partial S^2}\sigma^2 S^2\Delta t\varepsilon^2 + o(\Delta t)\ .$$

So

[1] Brownian motion is the generally accepted model for stock price movements. It assumes that the changes in a stock price over small, non-overlapping time intervals are independent and normally distributed, which

$$\Delta f = (\tfrac{\partial f}{\partial S}\mu S + \tfrac{\partial f}{\partial t} + \tfrac{1}{2}\tfrac{\partial^2 f}{\partial S^2}\sigma^2 S^2 \varepsilon^2)\Delta t + (\tfrac{\partial f}{\partial S}\sigma S\varepsilon)\sqrt{\Delta t} + o(\Delta t)\,. \qquad (2)$$

Notice that the only difference between this formula and Ito's Lemma is an ε^2 in the first term. To remedy this,

Claim: $\varepsilon^2\Delta t = \Delta t + o(\Delta t)$.

Observe that
$$\begin{aligned} VAR(\varepsilon^2\Delta t) &= E(\varepsilon^4\Delta t^2)\text{-}[E(\varepsilon^2\Delta t)]^2 \\ &= [E(\varepsilon^4)\text{-}E(\varepsilon^2)^2]\,\Delta t^2 \\ &= o(\Delta t)\,. \end{aligned}$$

Notice further that
$$\begin{aligned} E(\varepsilon^2\Delta t) &= \Delta t[E(\varepsilon^2)] \\ &= \Delta t \qquad \text{(because } \varepsilon \sim N(0,1).) \end{aligned}$$

Therefore, if the variance of $\varepsilon^2\Delta t$ is almost 0, and the expected value of $\varepsilon^2\Delta t$ is Δt, then

$$\varepsilon^2\Delta t = \Delta t + o(\Delta t).$$

Equation 2 then simplifies to

$$\Delta f = (\tfrac{\partial f}{\partial S}\mu S + \tfrac{\partial f}{\partial t} + \tfrac{1}{2}\tfrac{\partial^2 f}{\partial S^2}\sigma^2 S^2)\Delta t + (\tfrac{\partial f}{\partial S}\sigma S\varepsilon)\sqrt{\Delta t} + o(\Delta t)\,,$$

thus satisfying the lemma. ■

For the purposes of this paper, the term $o(\Delta t)$ is so small that it will be ignored when Ito's lemma is used.

Lemma 2: If stock prices follow the motion described in equation 1, then stock prices are lognormally distributed when Δt is small.

$$\ln S_{\Delta t} \sim N\left(\ln S_0 + (\mu - \tfrac{1}{2}\sigma^2)\Delta t, \sigma\sqrt{\Delta t}\right).$$

Proof: From Ito's Lemma, we know that

$$\Delta f \approx (\tfrac{\partial f}{\partial S}\mu S + \tfrac{\partial f}{\partial t} + \tfrac{1}{2}\tfrac{\partial^2 f}{\partial S^2}\sigma^2 S^2)\Delta t + (\tfrac{\partial f}{\partial S}\sigma S\varepsilon)\sqrt{\Delta t}\,.$$

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is consistent with the efficient market hypothesis.

Let $$f = \ln(S).$$

So, $$\Delta f \approx (\tfrac{1}{S}\mu S + 0 + \tfrac{1}{2}\tfrac{-1}{S^2}\sigma^2 S^2)\Delta t + (\tfrac{1}{S}\sigma S\varepsilon)\sqrt{\Delta t}$$

$$= (\mu - \tfrac{1}{2}\sigma^2)\Delta t + (\sigma\varepsilon)\sqrt{\Delta t}$$

$$= (\mu - \tfrac{1}{2}\sigma^2)\Delta t + (\sigma\sqrt{\Delta t})\varepsilon.$$

Notice that, due to Ito's Lemma, our formula is now expressed in terms of constants and one random variable, ε. Numerical values for μ, σ, and Δt can be deduced from historical data. Therefore,

$$\Delta f \sim N\left((\mu - \tfrac{1}{2}\sigma^2)\Delta t, \sigma\sqrt{\Delta t}\right).$$

Now, $$\Delta f = \ln(S_{\Delta t}) - \ln(S_0).$$

Therefore, $$\ln S_{\Delta t} \sim N\left(\ln S_0 + (\mu - \tfrac{1}{2}\sigma^2)\Delta t, \sigma\sqrt{\Delta t}\right).$$ ■

Corollary 1: Stock prices follow a lognormal distribution when Δt is arbitrarily large (denoted T).

$$\ln S_T \sim N\left(\ln S_0 + (\mu - \tfrac{1}{2}\sigma^2)T, \sigma\sqrt{T}\right)$$

Proof: Notice that T can be divided into n, non-overlapping intervals

$$\{\Delta t_1, \Delta t_2, \Delta t_3, \ldots, \Delta t_n\},$$

such that $$\Delta t_1 + \Delta t_2 + \Delta t_3 + \ldots + \Delta t_n = T.$$

If we let S_T be the spot price of the stock at time T, then, from our assumption that stocks follow a Brownian motion and from our efficient market assumption, in non-overlapping time intervals, stock prices are independent, and $\ln S_{i+1} - \ln S_i$ is normally distributed. Therefore, their sum is normally distributed. So,

$$\ln S_T - \ln S_0 = \ln S_T - \ln S_{n-1}$$

$$+ \ln S_{n-1} - \ln S_{n-2}$$

$$\vdots$$

$$+ \ln S_2 - \ln S_1$$

$$+ \ln S_1 - \ln S_0 \,.$$

Also,

$$E(\ln S_T - \ln S_0) = E(\ln S_T - \ln S_{n-1}) + E(\ln S_{n-1} - \ln S_{n-2}) + \ldots + E(\ln S_1 - \ln S_0)$$

$$= \left(\mu - \tfrac{1}{2}\sigma^2\right)\Delta t_n + \left(\mu - \tfrac{1}{2}\sigma^2\right)\Delta t_{n-1} + \ldots + \left(\mu - \tfrac{1}{2}\sigma^2\right)\Delta t_1$$

$$= \left(\mu - \tfrac{1}{2}\sigma^2\right)T \,,$$

and

$$Var(\ln S_T - \ln S_0) = Var(\ln S_T - \ln S_{n-1}) + Var(\ln S_{n-1} - \ln S_{n-2}) + \ldots + Var(\ln S_1 - \ln S_0)$$

$$= \left(\sigma\sqrt{\Delta t_n}\right)^2 + \left(\sigma\sqrt{\Delta t_{n-1}}\right)^2 + \ldots + \left(\sigma\sqrt{\Delta t_1}\right)^2$$

$$= \sigma^2 T \,.$$

(Here you'll note the importance of $\sqrt{T}$ in our assumption that stock prices follow a Brownian motion.)

Therefore,

$$\ln S_T - \ln S_0 \sim N\left((\mu - \tfrac{1}{2}\sigma^2)T, \sigma\sqrt{T}\right)$$

or equivalently, $$\ln S_T \sim N\left(\ln S_0 + (\mu - \tfrac{1}{2}\sigma^2)T, \sigma\sqrt{T}\right).$$

■

This concludes our modeling of the behavior of stock prices.

2. The Black-Scholes-Merton Differential Equation

To begin to derive the Black-Scholes-Merton (BSM) differential equation, it is important to discuss the complications with μ, the expected annual return of a stock. It would greatly simplify our model if μ could be replaced with some constant, because μ is specific to a firm. The expected return of a stock is risk-adjusted. It reflects a higher premium that the market demands for stocks with greater volatility. Therefore if we diversified the risk associated with μ, we would be left with a constant, r, that represents the risk-free interest rate, as found in the 3-month treasury bills. The BMS differential equation does exactly that; it shows that μ can be replaced with r.

We begin by creating a riskless portfolio; we take a long position in a call option and short $\frac{\partial c}{\partial S}$ shares of its underlying stock. We chose to formulate the portfolio in this manner, because both assets are perfectly negatively correlated (subject to the same risk of stock price fluctuations). If our portfolio is riskless, then it must grow at the risk-free rate.

Let $$\Pi = c - \tfrac{\partial c}{\partial S} S,$$

where Π is the value of our portfolio, c is the call price, and S is the current stock price. If we hold $\frac{\partial c}{\partial S}$ constant, then at the next moment in time,

$$\Delta\Pi = \Delta c - \tfrac{\partial c}{\partial S}\Delta S .$$

Now, note that c is a function of S and t; therefore Ito's lemma says that

$$\Delta c = (\tfrac{\partial c}{\partial S}\mu S + \tfrac{\partial c}{\partial t} + \tfrac{1}{2}\tfrac{\partial^2 c}{\partial S^2}\sigma^2 S^2)\Delta t + (\tfrac{\partial c}{\partial S}\sigma S \varepsilon)\sqrt{\Delta t} .$$

Recall from equation 1, that

$$\Delta S = \mu S \Delta t + \sigma S \sqrt{\Delta t}\,\varepsilon .$$

A linear combination of these two equations yields

$$\Delta\Pi = (\tfrac{\partial c}{\partial t} + \tfrac{1}{2}\tfrac{\partial^2 c}{\partial S^2}\sigma^2 S^2)\Delta t . \qquad (3)$$

Notice that there is no ε in equation 3. This implies that the value of our portfolio, after a short time period, is predictable (namely, that no volatility exists). Due to there being no volatility, we can now consider ourselves in a risk-neutral world, so our portfolio must grow at the risk-free rate, *r*. Therefore,

$$\Delta\Pi = r\left(c - \tfrac{\partial c}{\partial S} S\right)\Delta t \,. \tag{4}$$

Combining equations 3 and 4 yields

$$r\left(c - \tfrac{\partial c}{\partial S} S\right)\Delta t = \left(\tfrac{\partial c}{\partial t} + \tfrac{1}{2}\tfrac{\partial^2 c}{\partial S^2}\sigma^2 S^2\right)\Delta t \,,$$

or equivalently

$$rc = \tfrac{\partial c}{\partial t} + \tfrac{1}{2}\tfrac{\partial^2 c}{\partial S^2}\sigma^2 S^2 + \tfrac{\partial c}{\partial S} Sr \,. \tag{5}$$

Equation 5 is the Black-Scholes-Merton differential equation. We also have the initial conditions

$$c(S_0,0) = Max(S_0 - X, 0),$$

$$c(0, T) = 0,$$

and

$$\frac{\partial c(0,T)}{\partial S_0} = 0 \,.$$

Therefore, as long as we can continuously hedge our portfolio (adjust the number of shares of *S*), we can consider ourselves in a risk-neutral world for an arbitrarily long period of time. This is significant, because it allows us to use a risk-neutral argument to prove the Black-Scholes model. It is important to note, however, that the portfolio is only risk-neutral for a very short time period; it must be adjusted continuously.

Corollary 2: In a risk-neutral world,

$$\ln S_T \sim N\left(\ln S_0 + (r - \tfrac{1}{2}\sigma^2)T, \sigma\sqrt{T}\right) ,$$

where r is the risk-free interest rate.

This follows directly from the BSM differential equation, and Corollary 1.

3. Derivation of the Black-Scholes Option Pricing Model

The Black-Scholes option pricing formula finds a price for a European call option.

Theorem (Black-Scholes Option Pricing Model): If S_0 is the current stock price, X is the option's strike price, r is the continuously compounded risk-free interest rate, T is the time to maturity in years, σ is the standard deviation of the price of the underlying stock, and $N(d)$ is the cumulative standard normal distribution function, then

$$c = S_0 N(d_1) - Xe^{-rT} N(d_2), \text{ where}$$

$$d_1 = \frac{\ln(S_0 / X) + (r + \sigma^2 / 2)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(S_0 / X) + (r - \sigma^2 / 2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

Proof: Since the BSM differential equation does not contain μ, we can consider ourselves in a risk-neutral world. Therefore, the price of an option c, is given by

$$c = E\left(Max\left((S_T - X)e^{-rT}, 0\right)\right),$$

where S_T is the stock price at the maturity date, T. That is, the price, c, is the payoff discounted at the risk-free rate. Because $\ln S_T \sim N\left(\ln S_0 + (r - \tfrac{1}{2}\sigma^2)T, \sigma\sqrt{T}\right)$, S_T will follow a lognormal distribution with density function

$$f(S) = \frac{1}{S}\frac{1}{\sigma\sqrt{2\pi T}}e^{\frac{\frac{-1}{2}\left[\ln S-\left(\ln S_0+(r-\frac{1}{2}\sigma^2)T\right)\right]^2}{\sigma^2 T}} .$$

Therefore,

$$c = E\left(Max\left((S_T - X)e^{-rT},0\right)\right) = \int_X^\infty e^{-rT}(S-X)\left(\frac{1}{S}\right)\left(\frac{1}{\sigma\sqrt{2\pi T}}\right)e^{\frac{\frac{-1}{2}\left[\ln S-\left(\ln S_0+(r-\frac{1}{2}\sigma^2)T\right)\right]^2}{\sigma^2 T}}dS .$$

Now, let $y = \ln S$.

Then,

$$c = \int_{\ln X}^\infty e^{-rT}\left(e^y - X\right)\left(\frac{1}{\sigma\sqrt{2\pi T}}\right)e^{\frac{\frac{-1}{2}\left[y-\left(\ln S_0+(r-\frac{1}{2}\sigma^2)T\right)\right]^2}{\sigma^2 T}}dy$$

$$= \int_{\ln X}^\infty e^{-rT}e^y\left(\frac{1}{\sigma\sqrt{2\pi}T}\right)e^{\frac{\frac{-1}{2}\left[y-\left(\ln S_0+(r-\frac{1}{2}\sigma^2)T\right)\right]^2}{\sigma^2 T}}dy$$

$$- \int_{\ln X}^\infty Xe^{-rT}\left(\frac{1}{\sigma\sqrt{2\pi T}}\right)e^{\frac{\frac{-1}{2}\left[y-\left(\ln S_0+(r-\frac{1}{2}\sigma^2)T\right)\right]^2}{\sigma^2 T}}dy .$$

Let $a = \ln S_0 + \left(r - \frac{1}{2}\sigma^2\right)T$,

and $b^2 = \sigma^2 T$.

Substituting a and b^2 into c, we are left with

$$c = \int_{\ln X}^\infty e^{-rT}e^y\left(\frac{1}{b\sqrt{2\pi}}\right)e^{\frac{\frac{-1}{2}(y-a)^2}{b^2}}dy - \int_{\ln X}^\infty Xe^{-rT}\left(\frac{1}{b\sqrt{2\pi}}\right)e^{\frac{\frac{-1}{2}(y-a)^2}{b^2}}dy .$$

Consider the first integral, $\int_{\ln X}^\infty e^{-rT}e^y\left(\frac{1}{b\sqrt{2\pi}}\right)e^{\frac{\frac{-1}{2}(y-a)^2}{b^2}}dy$

$$= e^{-rT}\int_{\ln X}^{\infty}\left(\frac{1}{b\sqrt{2\pi}}\right)e^{y-\frac{\frac{1}{2}(y-a)^2}{b^2}}dy$$

$$= e^{-rT}\int_{\ln X}^{\infty}\left(\frac{1}{b\sqrt{2\pi}}\right)e^{-\frac{1}{2}\left[-2y+\frac{(y-a)^2}{b^2}\right]}dy\,.$$

Now, define w by $$w = -2y + \frac{(y-a)^2}{b^2}$$

$$= \frac{-2yb^2}{b^2} + \frac{y^2 - 2ay + a^2}{b^2}$$

$$= \frac{y^2 - 2yb^2 - 2ay + a^2}{b^2}$$

$$= \frac{y^2}{b^2} - \frac{2y\left(b^2 + a\right)}{b^2} + \frac{a^2}{b^2}\,.$$

Completing the square, $$\frac{y^2}{b^2} - \frac{2y\left(b^2 + a\right)}{b^2} + \frac{a^2}{b^2} = \frac{\left[y - \left(b^2 + a\right)\right]^2}{b^2} - b^2 + 2a\,.$$

Substituting this into our integral, yields

$$e^{-rT}\int_{\ln X}^{\infty}\left(\frac{1}{b\sqrt{2\pi}}\right)e^{\frac{-\frac{1}{2}\left[y-\left(b^2+a\right)\right]^2}{b^2}+\frac{1}{2}b^2+a}dy$$

$$= e^{-rT+\frac{1}{2}b^2+a}\int_{\ln X}^{\infty}\left(\frac{1}{b\sqrt{2\pi}}\right)e^{\frac{-\frac{1}{2}\left[y-\left(b^2+a\right)\right]^2}{b^2}}dy\,.$$

Now define $$z = \frac{y - b^2 - a}{b}\,.$$

Then $$dz = \frac{1}{b}dy\,,$$

and $$bdz = dy\,.$$

Substituting back into the integral, we are left with

$$e^{-rT+\frac{1}{2}b^2+a}\int_{\frac{\ln X-(b^2+a)}{b}}^{\infty}\left(\frac{1}{\sqrt{2\pi}}\right)e^{-\frac{1}{2}z^2}\,dz$$

$$= e^{-rT+\frac{1}{2}b^2+a}N\left(\frac{-\ln X+b^2+a}{b}\right)$$

$$= e^{-rT+\frac{1}{2}\sigma^2T+\ln S_0+(r-\frac{1}{2}\sigma^2)T}N\left(\frac{-\ln X+\sigma^2T+\ln S_0+(r-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)$$

$$= e^{\ln S_0}N\left(\frac{\ln\frac{S_0}{X}+(r+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)= e^{\ln S_0}N(d_1)=S_0N(d_1).$$

Thus completes the first integral.

Now, consider the second integral,

$$\int_{\ln X}^{\infty}Xe^{-rT}\left(\frac{1}{b\sqrt{2\pi}}\right)e^{\frac{\frac{-1}{2}(y-a)^2}{b^2}}\,dy\,.$$

Define z by $$z=\frac{(y-a)}{b}\,.$$

Then $$dz=\frac{1}{b}dy\,,$$

and $$bdz = dy\,.$$

Substituting into our integral, yields:

$$\int_{\frac{\ln X-a}{b}}^{\infty}Xe^{-rT}\left(\frac{1}{b\sqrt{2\pi}}\right)e^{-\frac{1}{2}z^2}\,bdz$$

$$= Xe^{-rT}\int_{\frac{\ln X-a}{b}}^{\infty}\left(\frac{1}{b\sqrt{2\pi}}\right)e^{-\frac{1}{2}z^2}\,bdz$$

$$= Xe^{-rT}N\left(-\frac{\ln X-a}{b}\right)$$

$$= Xe^{-rT} N\left(-\frac{\ln X - \ln S_0 - (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)$$

$$= Xe^{-rT} N\left(\frac{\ln\frac{S_0}{X} + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) = Xe^{-rT} N(d_2).$$

Therefore, we are left with

$$c = S_0 N(d_1) - Xe^{-rT} N(d_2), \qquad (6)$$

where $$d_1 = \frac{\ln(S_0 / X) + (r + \sigma^2 / 2)T}{\sigma\sqrt{T}},$$

and $$d_2 = \frac{\ln(S_0 / X) + (r - \sigma^2 / 2)T}{\sigma\sqrt{T}}.$$

Equation 6 is call the Black-Scholes option pricing formula. Now we will verify that the Black-Scholes formula satisfies the Black-Scholes-Merton differential equation and the initial conditions.

Consider the first condition, $c(S_0,0) = Max(S_0 - X, 0)$. Observe that

$$c(S_0, 0) = \lim_{T\to 0} S_0 N(d_1) - Xe^{-rT} N(d_2).$$

Now, $$\lim_{T\to 0} d_1 = \lim_{T\to 0} \frac{\ln(S_0 / X)}{\sigma\sqrt{T}} + \lim_{T\to 0} \frac{(r + \sigma^2 / 2)T}{\sigma\sqrt{T}}$$

$$= \pm\infty + 0 \qquad \text{(depending on } S_0 \geq X \text{ or } S_0 < X\text{)}.$$

Therefore, $$\lim_{T\to 0} S_0 N(d_1) = \begin{cases} 0, \ if\ S_0 < X \\ S_0, \ if\ S_0 \geq X \end{cases}.$$

Similarly, $$\lim_{T\to 0} - Xe^{-rT} N(d_2) == \begin{cases} 0, \ if\ S_0 < X \\ -X, \ if\ S_0 \geq X \end{cases}.$$

So, $$\lim_{T\to 0} c = \lim_{T\to 0} S_0 N(d_1) - Xe^{-rT} N(d_2)$$

$$= \begin{cases} 0 \quad , \textit{if } S_0 < X \\ S_0 - X, \textit{if } S_0 \ge X \end{cases}$$

$$= Max(S_0 - X, 0).$$

To satisfy the second condition, we must show that $c(0,T) = 0$. Consider

$$\lim_{S_0 \to 0} c = \lim_{S_0 \to 0} S_0 N(d_1) - Xe^{-rT} N(d_2).$$

Now $$\lim_{S_0 \to 0} S_0 N(d_1) = 0,$$

because $N(d_1)$ is bounded by 1.

Also, $$\lim_{S_0 \to 0} d_2 = \lim_{S_0 \to 0} \frac{\ln(S_0 / X) + (r - \sigma^2 / 2)T}{\sigma\sqrt{T}} = -\infty.$$

Therefore, $$N(d_2) = 0,$$

and $$\lim_{S_0 \to 0} c = 0.$$

To satisfy the final condition, we must show that $\dfrac{\partial c(0,T)}{\partial S_0} = 0$.

It is an important result that $\dfrac{\partial c}{\partial S_0} = N(d_1)$ (Hull, 311). Observe that

$$\lim_{S_0 \to 0} \frac{\partial c}{\partial S_0} = \lim_{S_0 \to 0} N(d_1) = 0 \ ,$$

because $$\lim_{S_0 \to 0} \frac{\ln(S_0 / X) + (r + \sigma^2 / 2)T}{\sigma\sqrt{T}} = -\infty.$$

Therefore, the Black-Scholes formula satisfies all three initial conditions of the Black-Scholes-Merton differential equation. Although not presented in this paper, it can be shown that the Black-Scholes formula is indeed a solution of the Black-Scholes-Merton differential equation (Hull Solutions, 69).

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VI. Remarks on the Model

1. Intuition Behind the Model

As previously mentioned, it is an important result that

$$N(d_1) = \frac{\partial c}{\partial S_0}. \qquad \text{(Hull,311)}$$

This, coupled with the fact that, in a risk-neutral world,

$$N(d_2) = P(S_T > X),$$

provides some intuition behind the Black-Scholes Model.

Intuitively, the price of a call option should be the present value of the expected payoff from a European call option, namely

$$c = E\left(Max\left((S_T - X)e^{-rT}, 0\right)\right).$$

Therefore, the price given by the Black-Scholes Model,

$$c = S_0 N(d_1) - Xe^{-rT}N(d_2),$$

should be the present value of the expected payoff, if we use the risk-free rate as the discount rate.

Consider a portfolio consisting of a long position in $N(d_1)$ shares of stock and a short position in a zero-coupon bond with a face value of X, with an $N(d_2)$ probability of default (note that $N(d_1)$ is exactly the number of shares needed to make this portfolio riskless). Then the value of the portfolio is the difference between the values of these two assets. Black and Scholes proved that the value of a call option relies only on r and T, and not on μ. In other words, in order to avoid arbitrage opportunities, options in a risk-adverse world must be priced the same as if the option were in a risk-neutral world. Recall that in a risk-neutral world, $N(d_2)$ is the probability that the option is exercised. Therefore, the second term of the Black-Scholes model would be the expected value of the strike price, discounted to the present time. So the model is the present value of the expected payoff from a European call option, exactly what we intuitively expected it to be.

2. The Greek Letters

Nearly as important as the model itself, are the partial derivatives of the model. These partials measure specific types of risk associated with the model. Investors use "the Greek letters" to manage the risks involved with options. The Greek letters discussed in this paper are Delta, Gamma and Vega.

Delta is the partial of c with respect to the underlying stock, given by

$$\Delta = \frac{\partial c}{\partial S_0} = N(d_1)\,.$$

In a portfolio consisting of a short position in a call option and a long position in its underlying stock, Delta is exactly the number of shares of the underlying stock needed to make the portfolio risk-less, or Delta-neutral.

Delta can be thought of as similar to the duration of a bond. If the Delta of a long position in a call option equals 0.6, then for every dollar increase in the underlying stock, the option price will increase by about 60 cents. (It is not exactly $.60, because Delta is a linear approximation.) So to make a riskless portfolio including a long position in this call option, a short position in 0.6 shares must be taken to offset the risk of the option.

Recall, though, that the portfolio is only risk-less for a very small time period. In fact, Delta needs to be recalculated very often. That's why Gamma is important.

Gamma is the second partial of c with respect to the underlying stock.

$$\Gamma = \frac{\partial^2 c}{\partial S_0^2} = \frac{N'(d_1)}{S_0 \sigma \sqrt{T}}, \qquad \text{(Hull, 324)}$$

where $N'(d_1)$ is the probability density function for the standard normal distribution. Gamma gives the rate at which Delta is changing. It is used to figure out how often Delta needs to be changed to keep the portfolio riskless.

Vega is the partial of c with respect to the volatility of the underlying stock, given by

$$\mathcal{V} = \frac{\partial c}{\partial \sigma} = S_0 \sqrt{T} N'(d_1). \qquad \text{(Hull, 327)}$$

Vega represents the sensitivity of a call option to movements in the volatility of the underlying stock. A single option usually has a non-zero Vega, and a stock has a Vega of zero. A portfolio's Vega is the share-weighted sum of the Vegas of the individual assets. Therefore, a portfolio of only stocks has a Vega of zero. However, the Vega for a portfolio can be changed by adding a position in an option. If we let $\mathcal{V}$ be the Vega of a portfolio and $\mathcal{V}_T$ be the Vega for an option, then adding a position of $-\mathcal{V}/\mathcal{V}_T$ units of the option would make the portfolio instantaneously Vega-neutral ($\mathcal{V} + \mathcal{V}_T(-\mathcal{V}/\mathcal{V}_T) = 0$).

The Greek letters are tools that investors use to manage the risk associated with an option. They are all usually taken into consideration when the risk of an option is being analyzed. The Greek Letters are very important tools in financial engineering.

3. Option Theory and the Black-Scholes Model

It was mentioned in the beginning of this paper that the Black-Scholes option model revolutionized financial engineering. Perhaps some popularity of the Black-Scholes model is derived from the ability of investors to look at the payoff schemes of financial situations as if they were options. In doing so, the investor sees the potential benefit from the situation, with limited downside risk.

An example applying option theory lies in the fee that an investor, who can perfectly time the market, will charge for his or her services. A perfect market timer is someone who can determine when the price of a stock, or group of stocks, will increase or decrease. Now, assume that our investor is a perfect market timer and takes on a long position in a stock when the stock price is going to increase, and gets out of this position right before the market goes down. Then the portfolio of this investor has unlimited upside potential, with no downside risk—just like a call option for upward stock price movements and a put option for downward stock price movements. Put differently, this investor will never lose money, but has the potential to make money. Using the known volatility of the portfolio and the current risk-free interest rate, the value of the investor's ability can be calculated using the Black-Scholes model. In essence, the value of market-timing is the price that one would pay for a call option and a put option, given the same criteria.

4. The Magic Behind the Black-Scholes Option Pricing Model

Many different versions of the Black-Scholes model exist today. It has been tailored to meet the characteristics of many different types of options, and derivatives in general. Although the model has been extended to a variety of financial assets, the overall idea behind the model remains the same. The heart of the Black-Scholes model lies in the BSM differential equation. This equation allows the model to use a risk-neutral world in which to price options that live in a risk-adverse world. This risk-neutral valuation is what separated the Black-Scholes option pricing model from all other previous models. All subsequent pricing models have followed the precedents set by the Black-Scholes model.

This model has revolutionized modern finance by providing accurate pricing of derivatives. These derivatives are used to create synthetic assets with specific payoffs and risk levels. The Black-Scholes model affords investors the capability to tightly control the risk associated with a portfolio. The opportunities for risk control that the Black-Scholes model provides has been credited with substantially enhancing the efficiency of the productive sector. The numerous advances in financial engineering that the Black-Scholes model has brought about has earned itself a place among the most influential models in finance.

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