Pricing Inflation-Indexed Derivatives Using the Extended Vasicek Model of Hull and White

Alan Stewart

Exeter College

University of Oxford

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Contents

Chapter 1

Introduction

The purpose of this thesis is to review the framework for pricing inflation-indexed derivatives using the two currency Heath-Jarrow-Morton approach introduced by Jarrow and Yildirim [11] and to derive prices for the most commonly traded inflation-indexed derivatives using the Hull White model.

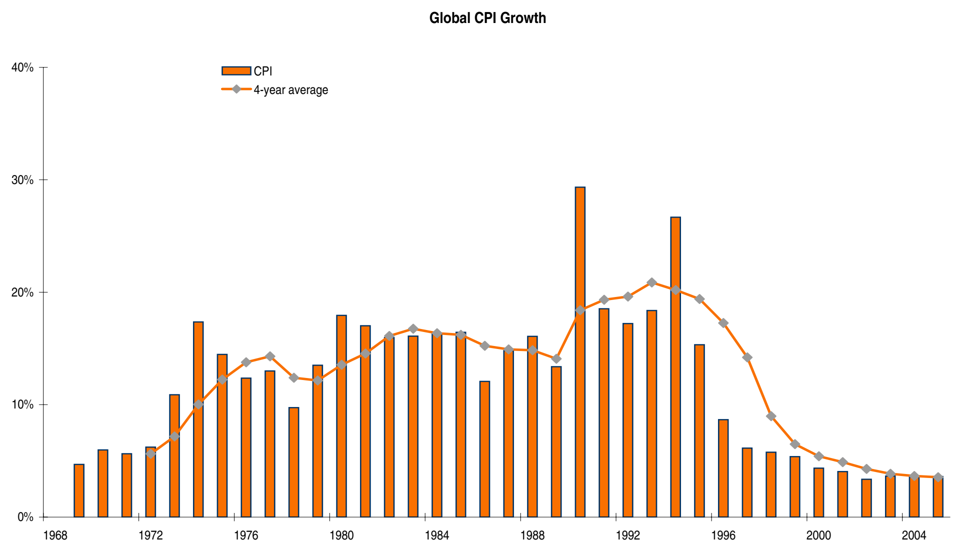
The first chapter gives an overview of the inflation markets and gives a brief description of the securities that are traded and their liquidity in the major markets. Some of the considerations peculiar to the inflation-indexed markets such as seasonality and indexation are then reviewed. The second chapter provides the mathematical background for the model. Gaussian Markov short-rate models are described in the HJM framework and the restrictions on the HJM volatility structure that allow the dynamics to be represented by a Markovian short rate model are described. The dynamics of the zero-coupon bond in the single currency setting is then derived in terms of the short rate parameters. The real economy is then introduced in terms of the economy of the foreign currency and the martingale measure is constructed for the extended set of nominal tradables. The third chapter describes the most popular inflation-indexed derivative securities in more detail and in particular derives prices for year-on-year inflation-indexed swaps and inflation-indexed caps and floors using the model. The fourth chapter describes calibration considerations. In particular it reviews how the most popular nominal derivatives such as swaptions and caps/floors can be expressed in terms of options on zero-coupon bonds and hence how they can be priced using the dynamics of the zero-coupon bond that were derived in the second chapter. A simple approach to calibrating the inflation model is also described.

1.1 Inflation

Economists generally define inflation as a sustained increase in the level of prices in an economy. The price level may be defined either in terms of the GDP deflator or by the consumer price index. The GDP deflator is a measure of the average price of goods produced by an economy whereas the consumer price index is concerned with the price of goods consumed. The two alternative measures are in general not the same but are not significantly different. We are only concerned with consumer prices here as almost all price indexed securities are linked to this index. The consumer price index is defined in terms of a basket of goods and services.

We might ask why economists are concerned with inflation and why controlling the rate of inflation is such a significant element of central bank policy. If the inflation rate was constant and all prices and wages rose at the same rate, inflation would not really be a significant problem. The concern is with uncertainty and the relative changes in prices of different items and wages. Changes in prices relative to wages has an obvious effect on the standard of living and the lack of certainty is also of concern to businesses making investment decisions and negotiating wages.

Over the past thousand years, periods of sustained increases in prices have actually been in the minority and price stability has been the norm. However, during the second half of the last century inflation took off and has only been contained again in recent years.



1.2 State of the market

Although inflation-indexed securities have been around in some form for hundreds of years, significant liquid markets only really started developing in the early 1980s. The first products were government bonds, originally issued by Canada and Australia and then by the UK in 1981. France followed and then in 1997 the US Federal Reserve issued its first inflation indexed treasury bonds, which are known as Treasury Inflation Protected Securities (TIPS). The Inflation-indexed derivative market didn't really exist prior to 2002, but has developed rapidly since then. In Europe there is now a developed market in vanilla inflation indexed derivatives - primarily in zero coupon inflation indexed swaps.

From a government's perspective issuing inflation indexed bonds sends a strong statement to the markets about its inflation fighting intentions. If market inflation expectations are higher than those of the government, it also provides a relatively cheap source of government funding as the market is willing to pay more for the higher expected nominal cash flows. For both of these reasons, the proportion of government debt issued in inflation-linked form has been increasing in most developed countries in recent years.

Inflation indexed derivatives have all of the normal benefits associated with derivative products. They can be traded over the counter and can therefore be tailored to meet specific needs, they are off-balance sheet and in theory have limitless supply. Inflation swap curves are now well defined out to about 30 years in Europe and for the first time the swap market is starting to drive the bond markets. However, zero-coupon inflation-indexed swaps are the only really liquid inflation-indexed derivatives. Year-on-year swaps and inflation-indexed caps and floors are becoming increasingly common, but are still considered to be exotic products in most respects. The pricing formulae for zero-coupon swaps, year-on-year swaps and inflation-indexed caps and floors are derived using the selected model in chapter 3.

1.3 Main users

The economy contains some entities that are natural payers of inflation and others that are natural receivers. Driven to some extent by government accounting legislation, there is currently a structural surplus of demand for inflation protection over supply in most developed economies. The excess demand has caused real yield levels to fall and real yield curves to flatten driven by demand for long dated inflation protection and guaranteed real yields. In the United States and the United Kingdom, the real yield curves are currently inverted for longer maturities.

1.4 Characteristics of the market

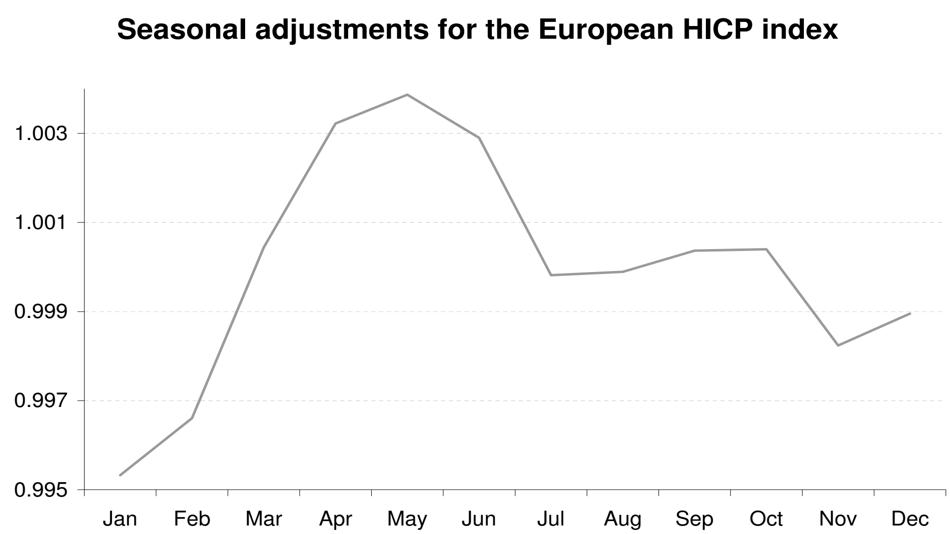
1.4.1 Choice of Index

There are many alternatives to choose from when deciding which index to link a security to. In theory securities could pay cashflows linked to the level of wages, core inflation (excluding energy) or the more common headline consumer prices. In the United Kingdom most index linked securities are linked to the RPI index, but the government's official inflation target is now expressed in terms of the CPI index. The CPI index is more consistent with the European harmonised measure of inflation, but someone hedging CPI exposure with inflation derivatives is exposed to basis risk as the two measures are not perfectly correlated. In a similar way, exposure to increases in wages cannot be hedged perfectly using RPI linked instuments. Because the indices are generally based on headline CPI (including food and energy), inflation linked securities may be sensitive to short term fluctuations caused by volatile factors such as the price of oil. Central banks generally target core inflation as they are more concerned with the longer term trend than with short term fluctuations, but this means that the two measures can deviate significantly for short periods and this introduces hedging risk for those hedging core inflation exposure with inflation indexed derivatives.

1.4.2 Seasonality

Most inflation indexed securities are linked to unrevised indices that are not seasonally adjusted. Consumer prices generally exhibit seasonal behaviour and the calculation of seasonal adjustments is an important aspect of pricing inflation indexed products, particularly for vanilla products when cash flows occur at different times of year.

Until recently seasonal adjustments have been calculated from historical index data using statistical techniques such as those reviewed by Belgrade and Benhamou [2]. They discuss a parametric least squares technique and a non-parametric (X11) approach. The following seasonal adjustments were calculated using the parametric approach fitting a log-linear model to historical index values.



In the European inflation markets prices are now quoted for inflation-indexed bonds in both the cash and asset swap markets. The relationship between the two prices is dependent on the seasonal adjustments used and in general adjustments derived from historic data will result in prices that are inconsistent between the two markets.

It is possible to infer seasonal adjustments from the bond prices, asset swap spreads and inflation swap rates that are quoted in the market. Swap spreads for inflation indexed bonds are most commonly quoted in terms of the Z-spread, which is the constant spread to the interbank curve that would make the discounted future cash flows equal to the market price of the bond. For the inflation indexed bonds, the future cashflows are determined using inflation swap rates. If a bond has annual coupons, the Z-spread will contain information about the relative seasonality of the reference index for today, and that of the coupon payment month. Given a selection of bonds with coupon payments in different months, seasonal adjustments may be fitted to market prices using a least squares approach.

Recent developments in the asset swap market has made it possible to infer seasonal adjustments from market inflation swap rates and asset swap rates on a range of bonds with cash flows at different times of year.

1.4.3 Indexation

In most developed economies, inflation indices are released monthly and in general the index level for each month is released around the middle of the following month. But inflation indexed securities are traded continuously and in order to calculate the real cash flows it is necessary to define a reference index which can be applied at any point in time. The reference index must be lagged so as to cope with the delayed release in the publication of the index. There are two alternative methods of defining the reference index for days during the month. The original UK IL Gilts and European inflation swaps use a constant reference index during the month which resets discretely at the end of each month. For most other instruments the reference index is calculated by linear interpolation between two lagged month-end index settings. Indexation can have quite a distorting effect on changes in real yields observed in the markets as the reference index is highly seasonal and the real yield at any point is defined in the context of a reference index at that time.

Chapter 2

Pricing inflation-indexed derivatives in the Heath-Jarrow-Morton framework

This chapter introduces the model for inflation and derives the no-arbitrage dynamics that are used in the following chapter to derive pricing formulae for inflation indexed derivatives. The model is based on the foreign currency analogy approach introduced by Jarrow & Yildirim in 2003 [11]. In the two currency model, the term structure of the domestic and foreign economies are defined as Heath Jarrow & Morton (HJM) models and the spot FX rate between the two currencies is modelled as a lognormal process. The model supports correlations between the domestic rates, foreign rates and the exchange rate between the two economies. This approach to modelling FX derivatives was introduced by Amin & Jarrow in 1991 [9]. The inflation model considers the nominal term structure to be the domestic term structure, the real term structure to be the foreign term structure and the spot inflation index to be the spot exchange rate.

The two currency model is an extension of the single currency model and so the first section of this chapter reviews the derivation of the HJM risk neutral dynamics in the single currency setting. Although the model is derived in the full generality of the HJM framework, in order to derive analytic price formulations for the most commonly traded inflation products it is necessary to restrict the HJM volatility structure so as to result in Gaussian forward rates and lognormal bond prices. The next section describes how the extended Vasicek model of Hull and White can be defined in the HJM framework and the restrictions on the volatility structure that allow an HJM model to be expressed as a Hull White Markov process. This section is also discussed in the single currency setting.

The next section introduces the real term structure and inflation index into the model. The risk neutral measure is defined by considering assets tradable in the nominal economy and the dynamics of the zero coupon bonds and inflation index are derived under this measure. The dynamics under the forward measures are also reviewed as these are of use when deriving the price formulae in the following chapter.

2.1 HJM no-arbitrage dynamics in a single currency setting

The Heath, Jarrow & Morton approach models the instantaneous forward interest rate as

$$f_{tT} = f_{0T} + \int_0^t \alpha_{uT} du + \int_0^t \sigma_{uT} dW_u \tag{2.1}$$

where f_{tT} is the instantaneous forward rate observed at time t for borrowing at time T and W_t is a multi-dimensional Brownian motion in the risk neutral measure. This section shows that in a complete market which is arbitrage free, the drift α_{tT} of the forward rate is uniquely determined by the volatility structure as

$$\alpha_{tT} = \sigma_{tT} \int_t^T \sigma_{tu} du \tag{2.2}$$

Let P_{tT} denote the price at time t of the zero-coupon bond maturing at time T. P_{tT} can be expressed in terms of the forward rate as

$$\begin{aligned}
P_{tT} &= \exp\left(-\int_t^T f_{tu} du\right) \\
&= \exp\left(-\int_t^T \left(f_{0s} + \int_0^t \sigma_{us} dW_u + \int_0^t \alpha_{us} du\right) ds\right) \\
&= \frac{P_{0T}}{P_{0t}} \exp\left(-\int_0^t \left(\int_t^T \sigma_{us} ds\right) dW_u - \int_0^t \left(\int_t^T \alpha_{us} ds\right) du\right) \\
&= \frac{P_{0T}}{P_{0t}} \exp\left(\int_0^t (\Sigma_{uT} - \Sigma_{ut}) dW_u - \int_0^t (A_{uT} - A_{ut}) du\right)
\end{aligned}$$

where $A_{tT} = \int_t^T \alpha_{ts} ds$ and $\Sigma_{tT} = -\int_t^T \sigma_{tu} du$. The money market account B_t has the dynamics $dB_t = r_t B_t dt$ where r_t is the spot exchange rate and can be written as $r_t = f_{tt}$. B_t can therefore be written as

$$\begin{aligned}
B_t &= \exp\left(\int_0^t f_{ss} ds\right) \\
&= \exp\left(\int_0^t \left(f_{0s} + \int_0^s \sigma_{us} dW_u + \int_0^s \alpha_{us} du\right) ds\right) \\
&= \frac{1}{P_{0t}} \exp\left(\int_0^t \left(\int_u^t \sigma_{us} ds\right) dW_u + \int_0^t \left(\int_u^t \alpha_{us} ds\right) du\right) \\
&= \frac{1}{P_{0t}} \exp\left(-\int_0^t \Sigma_{ut} dW_u + \int_0^t A_{ut} du\right)
\end{aligned}$$

the discounted bond $\frac{P_{tT}}{B_t}$ given by

$$\frac{P_{tT}}{B_t} = P_{0T} \exp\left(\int_0^t \Sigma_{uT} dW_u - \int_0^t A_{uT} du\right) \tag{2.3}$$

must be a martingale and therefore implies

$$\int_0^t A_{uT} du = \frac{1}{2} \int_0^t \Sigma_{uT} \Sigma_{uT}^* du \tag{2.4}$$

where $*$ denotes the adjoint. This implies that $A_{tT} = \frac{1}{2} \Sigma_{tT} \Sigma_{tT}^*$ for $t \leq T$ and hence the no-arbitrage condition 2.2.

2.2 The extended Vasicek model of Hull and White in the HJM framework

Although the no-arbitrage dynamics for the inflation model will be derived in the general HJM framework, in order to derive explicit prices for inflation indexed derivatives it is necessary to restrict the HJM volatility so that the dynamics can be represented by a Gaussian Markov process for the short rate. The extended Vasicek model of Hull & White is the most general formulation of such a process. This section reviews the extended Vasicek model and derives the restrictions on the HJM volatility that allow it to be represented in this form.

The stochastic differential equation for the extended Vasicek process is written as

$$dr_t = (a(t) - b(t)r_t)dt + \sigma(t)dW_t \tag{2.5}$$

where W_t is a Brownian motion in the martingale measure. The solution to this equation is given by

$$r_t = e^{-\beta(t)}\left(r_0 + \int_0^t e^{\beta(u)}a(u)du + \int_0^t e^{\beta(u)}\sigma(u)dW_u\right) \tag{2.6}$$

where $\beta(t) = \int_0^t b(u)du$ and hence

$$\int_0^T r_s ds = \int_0^T e^{-\beta(s)}\left(r_0 + \int_0^s e^{\beta(u)}a(u)du\right)ds + \int_0^T \left(\int_u^T e^{-\beta(s)}ds\right)e^{\beta(u)}\sigma(u)dW_u \tag{2.7}$$

Because W_t is a Brownian motion in the martingale measure the process $\frac{P_{tT}}{B_t}$ must be a martingale under this measure. P_{tT} can be written as $P_{tT} = \mathbb{E}\left[\exp(\int_0^T r_s ds)|\mathcal{F}_t\right]$ and B_t can be written $B_t = \exp\left(\int_0^t r_s ds\right)$ and so

$$\begin{aligned}\frac{P_{tT}}{B_t} &= \mathbb{E}\left[\exp\left(-\int_0^T r_s ds\right)\Big|\mathcal{F}_t\right]\\ &= P_{0T}\exp\left(-\int_0^t(\phi_T-\phi_u)g_u dW_u - \frac{1}{2}\int_0^t(\phi_T-\phi_u)^2 g_u^2 du\right)\end{aligned}$$

where $\phi_t = \int_0^t e^{-\beta(u)}du$ and $g_t = e^{\beta(t)}\sigma(t)$ taking logarithms and differentiating with respect to T this gives an expression for the instantaneous forward rate in terms of the extended Vasicek parameters

$$\begin{aligned}f_{tT} &= -\partial_T \log P_{tT}\\ &= -\partial_T \log \frac{P_{tT}}{B_t}\\ &= f_{0T} + \frac{\partial\phi_T}{\partial T}\int_0^t g_u dW_u + \frac{\partial\phi_T}{\partial T}\int_0^t(\phi_T-\phi_u)^2 g_u^2 du\end{aligned}$$

and so under the martingale measure, the dynamics of the forward rate are given by

$$df_{tT} = g_t\frac{\partial\phi_T}{\partial T}dW_t - g_t^2\frac{\partial\phi_T}{\partial T}(\phi_t-\phi_T)^2 dt \tag{2.8}$$

This shows that the extended Vasicek model is an HJM model with the volatility of the instantaneous forward rate given by $\sigma_{tT} = g_t \frac{\partial \phi_T}{\partial T} = \sigma(t)e^{\beta(t)-\beta(T)}$.

The derivation of the prices of many interest rate derivative products can be expressed in terms of the dynamics of the zero coupon bond price under an appropriate martingale measure. Using the above formulation of the extended Vasicek parameters we now derive an expression for the dynamics of the zero coupon bond in terms of the extended Vasicek parameters, firstly under the martingale measure and then under the T-forward measure when discounted by the zero coupon bond P_{tT}.

$$
\begin{aligned}
P_{tT} &= \exp\left(-\int_t^T f_{ts}ds\right) \\
&= \exp\left(-\int_0^t \left(\int_t^T \frac{\partial \phi_s}{\partial s}\right) g_u dW_u - \int_0^t \left(\int_t^T \frac{\partial \phi_s}{\partial s}(\phi_s - \phi_u)^2 ds\right) g_u^2 du\right) \\
&= \frac{P_{0T}}{P_{0t}} \exp\left(-(\phi_T - \phi_u)\left(\int_0^t g_u dW_u + \frac{1}{2}\int_0^t (\phi_T + \phi_t - 2\phi_u) g_u^2 du\right)\right)
\end{aligned}
$$

It turns out that it is easier to price most of the products of interest in the T-forward measure instead of the risk neutral measure. Under the T-forward measure $\mathbb{P}_T$ the expression $\frac{P_{tS}}{P_{tT}}$ is a martingale with dynamics given by

$$
\frac{P_{tS}}{P_{tT}} = \frac{P_{0S}}{P_{0T}} \exp\left(-(\phi_S - \phi_T)\int_0^t g_u dW_u^T - \frac{1}{2}(\phi_S - \phi_T)^2 \int_0^t g_u^2 du\right) \tag{2.9}
$$

where W_t^T is a Brownian motion under the $\mathbb{P}_T$ measure. 2.9 expresses the variance of the zero coupon bond in terms of the original Hull-White parameters and is useful when calibrating this model to market prices.

At this point the notation $\mathcal{E}$ is introduced to represent the Doléans-Dade exponential of a continuous semi-martingale X_t which is

$$
\mathcal{E}(X_t) = \exp\left(X_t - \frac{1}{2}\langle X_t, X_t\rangle\right) \tag{2.10}
$$

Using this notation 2.9 can be re-written as

$$
\frac{P_{tS}}{P_{tT}} = \frac{P_{0S}}{P_{0T}} \mathcal{E}\left(-(\phi_S - \phi_T)\int_0^t g_u dW_u^T\right) \tag{2.11}
$$

2.3 Introducing the real economy

In this section the model is extended to include the dynamics of the real term structure and that of the inflation index. Using the foreign currency analogy the nominal interest rate is considered to be the interest rate in the domestic economy, the real interest rate is the interest rate in the foreign economy and the inflation index is the exchange rate between the two economies. The analysis presented so far applies equally to the nominal and real economies if they are considered in isolation. There exists a unique martingale measure $\mathbb{P}^r$ in the real economy under which the real instantaneous forward rate follows the arbitrage free dynamics

$$df^r_{tT} = \sigma^r_{tT} dW^r_t - \sigma^r_{tT}(\Sigma^r_{tT})^* dt \tag{2.12}$$

where the superscript r has been introduced to denote the real economy and from now on n will be used to denote the nominal economy. W^r_t is a Brownian motion under $\mathbb{P}^r$, the risk neutral measure in the real economy.

The two economies are related by the inflation index I_t, which is the price of a unit of real currency in units of nominal currency. Every asset in the real economy can be converted into a tradable asset in the nominal economy via the inflation index. We wish to extend the nominal risk neutral measure $\mathbb{P}^n$ to include this additional set of nominal tradable assets. Every nominal tradable asset must be a martingale under the $\mathbb{P}^n$ measure when discounted by the nominal money market account. In particular this is true for the real money market account B^r_t, so there must exist a pre-visible process σ^I_t such that

$$\frac{I_t B^r_t}{B^n_t} = I_0 \mathcal{E}\left(\int_0^t \sigma^I_s d\tilde{W}^I_s\right) \tag{2.13}$$

where $\tilde{W}^I_t$ is a Brownian motion in the nominal risk neutral measure $\mathbb{P}^n$ and $B^r_0 = B^n_0 = 1$. When normalised by I_0, this is the Radon-Nikodym density of the $\mathbb{P}^r$ measure with respect to the $\mathbb{P}^n$ measure and is given by

$$Z_t = \frac{I_t B^r_t}{I_0 B^n_t} \tag{2.14}$$

$$= \mathcal{E}\left(\int_0^t \sigma^I_s d\tilde{W}^I_s\right) \tag{2.15}$$

The dynamics of the real instantaneous forward rate under the $\mathbb{P}^r$ measure were given by 2.12 and using the Radon-Nikodym density 2.15 we can see that f^r_{tT} has the following dynamics under the extended nominal measure $\mathbb{P}^n$

$$f^r_{tT} = f^r_{0T} + \int_0^t \sigma^r_{sT} d\tilde{W}^r_s - \int_0^t \sigma^r_{sT}((\Sigma^r_{sT})^* + \rho^{rI}_s \sigma^I_s) ds \tag{2.16}$$

where ρ^{rI}_t is the instantaneous correlation between the $\mathbb{P}^n$-Brownian motions $\tilde{W}^I_t$ and $\tilde{W}^r_t$. From the definition of the martingale measure $\mathbb{P}^n$, the product $I_t P^r_{tT}$ is a martingale when discounted by the nominal money market account B^n_t with dynamics given by

$$\frac{I_t P^r_{tT}}{B^n_t} = I_0 P^r_{0T} \mathcal{E}\left(\int_0^t \sigma^I_s d\tilde{W}^I_s + \int_0^t \Sigma^r_{sT} d\tilde{W}^r_s\right)$$

or, in terms of the Hull-White parameterisation introduced earlier

$$\frac{I_t P^r_{tT}}{B^n_t} = I_0 P^r_{0T} \mathcal{E}\left(-\int_0^t (\phi^r_T - \phi^r_u) g^r_u d\tilde{W}^r_u + \int_0^t \sigma^I_u d\tilde{W}^I_u\right)$$

2.3.1 Dynamics in the forward measure

It turns out that it is easier to derive prices for the most popular inflation indexed products in the appropriate forward measure. In this section the dynamics under the forward measure are reviewed and in particular expressions for the forward inflation index and the Radon-Nikodym density of the real forward measure $\mathbb{P}_T^r$ with respect to the nominal forward measure $\mathbb{P}_T^n$ are derived. These expressions will be useful in the derivation of prices for year-on-year swaps and inflation-indexed caps and floors that are derived in the next chapter.

As in the single currency case, the ratio $\frac{P_{tS}^r}{P_{tT}^r}$ is a martingale in the real T-forward measure $\mathbb{P}_T^r$.

$$\frac{P_{tS}^r}{P_{tT}^r} = \frac{P_{0S}^r}{P_{0T}^r}\mathcal{E}\left(\int_0^t (\Sigma_{sS}^r - \Sigma_{sT}^r)dW_s^{r,T}\right) \tag{2.17}$$

where $W_t^{r,T}$ is a Brownian motion in the real T-forward measure $\mathbb{P}_T^r$. With the appropriate choice of T and with $t = T$, the price at time T of a bond paying 1 unit of real currency at time S is therefore given by

$$P_{TS}^r = \frac{P_{0S}^r}{P_{0T}^r}\mathcal{E}\left(\int_0^T (\Sigma_{sS}^r - \Sigma_{sT}^r)dW_s^{r,T}\right) \tag{2.18}$$

which is a martingale under $\mathbb{P}_T^r$ with expected price equal to the current forward price

$$\mathbb{E}_T^r\left[P_{TS}^r \middle| \mathcal{F}_t\right] = \frac{P_{tS}^r}{P_{tT}^r} \tag{2.19}$$

Using the Hull-White extended Vasicek parameterisation introduced earlier, 2.17 and 2.18 are given by

$$\frac{P_{tS}^r}{P_{tT}^r} = \frac{P_{0S}^r}{P_{0T}^r}\mathcal{E}\left(-(\phi_S^r - \phi_T^r)\int_0^t g_u^r dW_u^{r,T}\right) \tag{2.20}$$

and

$$P_{TS}^r = \frac{P_{0S}^r}{P_{0T}^r}\mathcal{E}\left(-(\phi_S^r - \phi_T^r)\int_0^T g_u^r dW_u^{r,T}\right) \tag{2.21}$$

When discounted by P_{tT}^n, the product $I_t P_{tT}^r$ is a martingale under the nominal T-forward measure $\mathbb{P}_T^n$ and has dynamics

$$\frac{I_t P_{tS}^r}{P_{tT}^n} = \frac{I_0 P_{0S}^r}{P_{0T}^n}\mathcal{E}\left(\int_0^t \sigma_s^I d\tilde{W}_s^{I,T} + \int_0^t \Sigma_{sS}^r d\tilde{W}_s^{r,T} - \int_0^t \Sigma_{sT}^n dW_s^{n,T}\right) \tag{2.22}$$

where $\tilde{W}_t^{I,T}$, $\tilde{W}_t^{r,T}$ and $W_t^{n,T}$ are correlated $\mathbb{P}_T^n$ Brownian motions. Under the nominal T-forward measure $\mathbb{P}_T^n$ we obtain

$$\mathbb{E}_T^n[I_T|\mathcal{F}_t] = \frac{I_t P_{tT}^r}{P_{tT}^n} \tag{2.23}$$

and

$$I_T \quad = \quad \frac{I_0 P^r_{0T}}{P^n_{0T}} \mathcal{E}\left(\int_0^T \sigma^I_s d\tilde{W}^{I,T}_s + \int_0^T \Sigma^r_{sT} d\tilde{W}^{r,T}_s - \int_0^T \Sigma^n_{sT} dW^{n,T}_s\right) \tag{2.24}$$

The dynamics of the inflation index are used in the derivation of pricing formulae for inflation-indexed caps/floors which depend on the distribution of the index at T.

The Radon-Nikodym density of the real T-forward measure $\mathbb{P}^r_T$ with respect to the nominal T-forward measure $\mathbb{P}^n_T$ is given by

$$Z_{tT} \quad = \quad \mathcal{E}\left(\int_0^t \sigma^I_s d\tilde{W}^{I,T}_s + \int_0^t \Sigma^r_{sT} d\tilde{W}^{r,T}_s - \int_0^t \Sigma^n_{sT} dW^{n,T}_s\right) \tag{2.25}$$

Chapter 3

Derivation of prices for inflation-indexed derivatives

This chapter describes the most liquid instruments in the inflation-indexed derivative market and derives prices for them using the Hull-White extended Vasicek model. The inflation-indexed derivative markets are relatively new and there are only a small number of products that are actively traded. Zero-coupon inflation-indexed swaps are the most liquid product and have model independent prices. Year-on-year inflation swaps are probably the next most common followed by inflation-indexed caps/floors and forward starting zero coupon swaps.

3.1 Zero-coupon inflation-indexed swaps

Zero coupon inflation-indexed swaps are actively traded in the European, UK and US markets. They are also the most simple inflation-indexed instruments and because they have model independent prices, the term structure of real rates can be easily derived from the nominal term-structure and market inflation swap rates.

A zero-coupon inflation swap is defined by its maturity T, its fixed rate r and its base index I_0 which is the reference index when the swap is initially traded. At time T Party A pays the floating inflation leg $\frac{I_T}{I_0} - 1$ and Party B pays the fixed amount $(1+r)^k - 1$, where k is the number of years to maturity when the swap is initially traded and I_T is the reference index at maturity. The fixed rate r is chosen so as to make the value of the fixed leg equal to that of the floating leg when the swap is initially traded and inflation swaps are quoted in terms of the fixed rate. The value of the inflation leg is denoted by $ZCIIS(t, T, I_0)$ and is given by

$$\frac{ZCIIS(t,T,I_0)}{P^n_{tT}} = \mathbb{E}^n_T \left[\frac{I_T}{I_0} - 1 \middle| \mathcal{F}_t \right] \tag{3.1}$$

where $\mathbb{E}^n_T$ denotes the expectation with respect to the nominal T-forward measure $\mathbb{P}^n_T$. Because $\frac{I_t P^r_{tT}}{P^n_{tT}}$ is a martingale under this measure

$$\mathbb{E}^n_T[I_T|\mathcal{F}_t] = \frac{I_t P^r_{tT}}{P^n_{tT}} \tag{3.2}$$

under $\mathbb{P}^n_T$ the expected index value at time T is the current forward value. The price of the inflation leg at time t is then given by

$$ZCIIS(t,T,I_0) = \frac{I_t P^r_{tT}}{I_0} - P^n_{tT} \tag{3.3}$$

and at $t=0$, when the swap is initially traded, $I_t = I_0$ and the price of the floating leg simplifies to

$$ZCIIS(0,T,I_0) = P^r_{0T} - P^n_{0T} \tag{3.4}$$

The value of the fixed side is simply the value of the fixed cash flow at time T discounted by the nominal discount factor and is given by

$$P^n_{0T}((1+r)^k - 1) \tag{3.5}$$

Equating this to the value of the inflation leg, it is clear that the price of the real zero coupon bond P^r_{0T} is given by

$$P^r_{0T} = P^n_{0T}(1+r)^k \tag{3.6}$$

This expression provides a simple, model independent, solution for the real discount factors given the nominal discount factors and the market fixed rates for zero coupon inflation-indexed swaps and proves to be very useful in pricing the more complicated products, all of which depend on the initial term structure of real rates

3.2 Year-on-year inflation-indexed swaps

A year-on-year inflation-indexed swap consists of a fixed and a floating leg with annual payments at $T_1, ..., T_N$. At the end of each period Party A pays the floating inflation leg which is based on the inflation rate over the previous period. The floating payment on a swap reset at time T_{i-1} and paid at time T_i would be $\frac{I_{T_i}}{I_{T_{i-1}}} - 1$. It is similar to the floating payment on a zero coupon inflation swap, and in fact the first payment is the same as a zero coupon swap because $I_{T_{i-1}}$ is known at the trade date. However for subsequent payments $I_{T_{i-1}}$ is not known until the end of the previous period and because both I_{T_i} and $I_{T_{i-1}}$ cannot both be martingales under the same measure, the price of each payment on the floating leg of a year-on-year swap contains a convexity correction term which accounts for the required measure change.

As with zero coupon swaps, year-on-year swaps are quoted in terms of the fixed rate that would make the value of the floating inflation leg equal to that of the fixed leg at trade date $t=0$. At the end of each period, Party B pays the fixed leg r and so the price at $t=0$ of the fixed payment paid at time T_i is $P^n_{0T_i} r$.

As discussed above, each floating payment on a year-on-year swap can be considered as the floating payment on a forward starting zero coupon swap. The value at time t of the floating payment reset at T_1 and paid at T_2 is therefore given by

$$\frac{YYIIS(t, T_1, T_2)}{P^n_{T_1}} = \mathbb{E}^n_{T_1}[ZCIIS(T_1, T_2, I_{T_1})] \tag{3.7}$$

where $\mathbb{E}^n_{T_1}$ denotes the expectation with respect to the nominal T_1-forward measure. At T_1 this is a zero coupon swap traded at market rates with base index given by the reference index at that time and 3.4 can therefore be applied to give

$$\frac{YYIIS(t, T_1, T_2)}{P^n_{tT_1}} = \mathbb{E}^n_{T_1}[P^r_{T_1T_2} - P^n_{T_1T_2}|\mathcal{F}_t] \tag{3.8}$$

Noting that $P^n_{tT_1}\mathbb{E}^n_{T_1}[P^n_{T_1T_2}] = P^n_{tT_2}$, the price of the floating inflation payment at time t is given by

$$YYIIS(t, T_1, T_2) = P^n_{tT_1}\mathbb{E}^n_{T_1}[P^r_{T_1T_2}|\mathcal{F}_t] - P^n_{tT_2} \tag{3.9}$$

This expectation is model dependent and is evaluated most easily by noting that $P^r_{T_1T_2}$ is a martingale under the $\mathbb{P}^r_{T_1}$ measure and then using the Radon-Nikodym density for the $\mathbb{P}^r_{T_1}$ measure with respect to the $\mathbb{P}^n_{T_1}$ measure that was developed in the previous chapter to derive the expectation for $P^r_{T_1T_2}$ in the nominal T_1-forward measure. It is this measure change that introduces the convexity adjustment in the price of year-on-year swaps.

As shown in the previous chapter $\frac{P^r_{tT_2}}{P^r_{tT_1}}$ is a martingale in the real T_1-forward measure with dynamics given by

$$\frac{P^r_{tT_2}}{P^r_{tT_1}} = \frac{P^r_{0T_2}}{P^r_{0T_1}}\mathcal{E}\left(\int_0^t (\Sigma^r_{sT_2} - \Sigma^r_{sT_1})dW^{r,T_1}_s\right) \tag{3.10}$$

where W^{r,T_1}_t is a Brownian motion under the $\mathbb{P}^r_{T_1}$ measure. Choosing $t = T_1$, the price at T_1 of a real zero coupon bond maturing at T_2 is therefore given by

$$P^r_{T_1T_2} = \frac{P^r_{0T_2}}{P^r_{0T_1}}\mathcal{E}\left(\int_0^{T_1} (\Sigma^r_{sT_2} - \Sigma^r_{sT_1})dW^{r,T_1}_s\right) \tag{3.11}$$

and under this measure has an expected price equal to the current forward price.

Written in terms of the three correlated $\mathbb{P}^n_{T_1}$ Brownian motions $\tilde{W}^{r,T_1}_t$, $\tilde{W}^{I,T_1}_t$ and W^{n,T_1}_t, the Radon-Nikodym density 2.25 of the real T_1-forward measure $\mathbb{P}^r_{T_1}$ with respect to the nominal T_1-forward measure $\mathbb{P}^n_{T_1}$ is given by

$$Z_t = \mathcal{E}\left(\int_0^t \sigma^I_s d\tilde{W}^{I,T_1}_s + \int_0^t \Sigma^r_{sT_1} d\tilde{W}^{r,T_1}_s - \int_0^t \Sigma^n_{sT_1} dW^{n,T_1}_s\right) \tag{3.12}$$

and so the required dynamics for $P^r_{T_1T_2}$ under the nominal T_1-forward measure $\mathbb{P}^n_{T_1}$ are given by

$$
\begin{aligned}
P^r_{T_1T_2} &= \frac{P^r_{0T_2}}{P^r_{0T_1}}\mathcal{E}\left(\int_0^{T_1}(\Sigma^r_{sT_2}-\Sigma^r_{sT_1})d\tilde{W}^{r,T_1}_s\right)\times \\
&\quad \exp\left(\int_0^{T_1}(\Sigma^r_{sT_2}-\Sigma^r_{sT_1})(\Sigma^n_{sT_1}\rho^{nr}_s-\sigma^I_s\rho^{Ir}_s-\Sigma^r_{sT_1})ds\right)
\end{aligned}
$$

It is therefore clear that

$$
\mathbb{E}^n_{T_1}[P^r_{T_1T_2}|\mathcal{F}_t] = \frac{P^r_{tT_2}}{P^r_{tT_1}}e^{C(t,T_1,T_2)} \tag{3.13}
$$

where

$$
C(t,T_1,T_2) = \int_t^{T_1}(\Sigma^r_{sT_2}-\Sigma^r_{sT_1})(\Sigma^n_{sT_1}\rho^{nr}_s-\sigma_s\rho^{Ir}_s-\Sigma^r_{sT_1})ds \tag{3.14}
$$

and the floating inflation leg is given by

$$
YYIIS(t,T_1,T_2) = P^n_{tT_1}\left(\frac{P^r_{tT_2}}{P^r_{tT_1}}e^{C(t,T_1,T_2)}-\frac{P^n_{tT_2}}{P^n_{tT_1}}\right) \tag{3.15}
$$

The convexity adjustment C is dependent on the correlation between the nominal and real rates and between the inflation index and real rates.

3.2.1 Year-on-year inflation-indexed swap for Hull White model with constant volatility parameters

When the volatility parameters and correlation coefficients are constant it is possible to derive an explicit formula for the year-on-year inflation indexed swap in 3.15. As shown in the previous chapter, constant volatility parameters b and σ result in a HJM forward volatility of the form

$$
\sigma_{tT} = \sigma e^{-b(T-t)} \tag{3.16}
$$

with the zero-coupon bond volatility Σ_{tT} given by

$$
\begin{aligned}
\Sigma_{tT} &= -\int_t^T \sigma e^{-b(u-t)}du = \frac{\sigma}{b}[e^{-b(u-t)}]_t^T \\
&= \frac{\sigma}{b}[e^{-b(T-t)}-1]
\end{aligned}
$$

The convexity correction $C(t,T_1,T_2)$ can then be written as

$$\begin{aligned}
C(t,T_1,T_2) &= \int_t^{T_1} \frac{\sigma_r}{b_r}\left(e^{-b_r(T_2-s)} - e^{-b_r(T_1-s)}\right) \\
&\quad \left(\frac{\sigma_n\rho^{nr}}{b_n}(e^{-b_n(T_1-s)}-1) - \sigma_I\rho^{Ir} - \frac{\sigma_r}{b_r}(e^{-b_r(T_1-s)}-1)\right)ds \\
&= \frac{\sigma_r\sigma_n\rho^{nr}}{b_rb_n}\int_t^{T_1} e^{-b_r(T_2-s)-b_n(T_1-s)} - e^{-b_r(T_2-s)} - e^{-b_r(T_1-s)-b_n(T_1-s)} + e^{-b_r(T_1-s)}ds \\
&\quad -\frac{\sigma_r\sigma_I\rho^{Ir}}{b_r}\int_t^{T_1} e^{-b_r(T_2-s)} - e^{-b_r(T_1-s)}ds \\
&\quad -\frac{\sigma_r^2}{b_r^2}\int_t^{T_1} e^{-b_r(T_1+T_2-2s)} - e^{-2b_r(T_1-s)} - e^{-b_r(T_2-s)} + e^{-b_r(T_1-s)}ds
\end{aligned}$$

$$\begin{aligned}
&= \frac{\sigma_r\sigma_n\rho^{nr}}{b_rb_n}\Bigg[\frac{1}{b_r+b_n}e^{-b_r(T_2-s)-b_n(T_1-s)} - \frac{1}{b_r}e^{-b_r(T_2-s)} \\
&\quad -\frac{1}{b_r+b_n}e^{-b_r(T_1-s)-b_n(T_1-s)} + \frac{1}{b_r}e^{-b_r(T_1-s)}\Bigg]_t^{T_1} \\
&\quad -\frac{\sigma_r\sigma_I\rho^{Ir}}{b_r}\left[\frac{1}{b_r}e^{-b_r(T_2-s)} - \frac{1}{b_r}e^{-b_r(T_1-s)}\right]_t^{T_1} \\
&\quad -\frac{\sigma_r^2}{b_r^2}\left[\frac{1}{2b_r}e^{-b_r(T_1+T_2-2s)} - \frac{1}{2b_r}e^{-2b_r(T_1-s)} - \frac{1}{b_r}e^{-b_r(T_2-s)} + \frac{1}{b_r}e^{-b_r(T_1-s)}\right]_t^{T_1}
\end{aligned}$$

$$\begin{aligned}
&= \frac{\sigma_r\sigma_n\rho^{nr}}{b_rb_n}\Bigg[\frac{1}{b_r+b_n}e^{-b_r(T_2-T_1)} - \frac{1}{b_r}e^{-b_r(T_2-T_1)} - \frac{1}{b_r+b_n} + \frac{1}{b_r} \\
&\quad -\frac{1}{b_r+b_n}e^{-b_r(T_2-t)-b_n(T_1-t)} + \frac{1}{b_r}e^{-b_r(T_2-t)} + \frac{1}{b_r+b_n}e^{-b_r(T_1-t)-b_n(T_1-t)} - \frac{1}{b_r}e^{-b_r(T_1-t)}\Bigg] \\
&\quad -\frac{\sigma_r\sigma_I\rho^{Ir}}{b_r^2}\left[e^{-b_r(T_2-T_1)} - 1 - e^{-b_r(T_2-t)} + e^{-b_r(T_1-t)}\right] \\
&\quad -\frac{\sigma_r^2}{b_r^3}\Bigg[\frac{1}{2}e^{-b_r(T_2-T_1)} - \frac{1}{2} - e^{-b_r(T_2-T_1)} + 1 \\
&\quad -\frac{1}{2}e^{-b_r(T_1+T_2-2t)} + \frac{1}{2}e^{-2b_r(T_1-t)} + e^{-b_r(T_2-t)} - e^{-b_r(T_1-t)}\Bigg]
\end{aligned}$$

and finally the convexity adjustment C is given by

$$\begin{aligned}
C(t,T_1,T_2) &= \sigma_rB^r_{T_1T_2}B^r_{tT_1}\frac{\rho_{nr}\sigma_n}{b_n+b_r}(1+b_rB^n_{tT_1}) + \sigma_rB^r_{T_1T_2}\frac{\rho_{nr}\sigma_n}{b_n+b_r}B^n_{tT_1} \\
&\quad +\sigma_r\sigma_I\rho_{rI}B^r_{T_1T_2}B^r_{tT_1} \\
&\quad -\frac{1}{2}\sigma_r^2(B^r_{tT_1})^2B^r_{T_1T_2}
\end{aligned} \tag{3.17}$$

where $B_{tT} = \frac{1}{b}(1-e^{-b(T-t)})$

3.3 Inflation indexed Caps and Floors

An inflation indexed caplet/floorlet is a call/put option on the inflation rate with payoff at time T_2 defined by

$$N\psi\left[\omega\left(\frac{I_{T_2}}{I_{T_1}}-1-\kappa\right)\right]^+ \tag{3.18}$$

where κ is the strike, ψ is the year fraction for the interval $[T_1, T_2]$, N is the notional and $\omega = 1$ for a caplet and $\omega = -1$ for a floorlet. Defining $K = 1 + \kappa$, the price of the option at $t < T_1$ is given by

$$\frac{IICF(t,\psi,N,T_1,T_2,\kappa,\omega)}{P^n_{tT_2}} = N\psi\mathbb{E}^n_{T_2}\left\{\left[\omega\left(\frac{I_{T_2}}{I_{T_1}}-K\right)\right]^+\middle|\mathcal{F}_t\right\} \tag{3.19}$$

where $\mathbb{E}^n_{T_2}$ denotes the expectation under the nominal T_2-forward measure $\mathbb{P}^n_{T_2}$. As shown in the previous chapter, under the nominal T_2-forward measure $\mathbb{P}^n_{T_2}$

$$I_{T_2} = \frac{I_0P^r_{0T_2}}{P^n_{0T_2}}\mathcal{E}\left(\int_0^{T_2}\sigma^I_s d\tilde{W}^{I,T_2}_s + \int_0^{T_2}\Sigma^r_{sT_2}d\tilde{W}^{r,T_2}_s - \int_0^{T_2}\Sigma^n_{sT_2}dW^{n,T_2}_s\right) \tag{3.20}$$

Under the nominal T_1-forward measure

$$I_{T_1} = \frac{I_0P^r_{0T_1}}{P^n_{0T_1}}\mathcal{E}\left(\int_0^{T_1}\sigma^I_s d\tilde{W}^{I,T_1}_s + \int_0^{T_1}\Sigma^r_{sT_1}d\tilde{W}^{r,T_1}_s - \int_0^{T_1}\Sigma^n_{sT_1}dW^{n,T_1}_s\right) \tag{3.21}$$

and under the nominal T_2-forward measure

$$I_{T_1} = \frac{I_0P^r_{0T_1}}{P^n_{0T_1}}\mathcal{E}\left(\int_0^{T_1}\sigma^I_s d\tilde{W}^{I,T_2}_s + \int_0^{T_1}\Sigma^r_{sT_1}d\tilde{W}^{r,T_2}_s - \int_0^{T_1}\Sigma^n_{sT_1}dW^{n,T_2}_s\right)e^{D(0,T_1,T_2)} \tag{3.22}$$

where $e^{D(0,T_1,T_2)}$ is the change in drift due to the measure change. The inflation rate $\frac{I_{T_2}}{I_{T_1}}$ is therefore lognormally distributed under the nominal T_2-forward measure with dynamics given by

$$\begin{aligned}\frac{I_{T_2}}{I_{T_1}} \quad = \quad & \frac{P^r_{0T_2}P^n_{0T_1}}{P^n_{0T_2}P^r_{0T_1}}\mathcal{E}\bigg(\int_{T_1}^{T_2}\sigma^I_s d\tilde{W}^{I,T_2}_s + \int_0^{T_2}\Sigma^r_{sT_2}d\tilde{W}^{r,T_2}_s - \int_0^{T_2}\Sigma^n_{sT_2}dW^{n,T_2}_s \\ & -\int_0^{T_1}\Sigma^r_{sT_1}d\tilde{W}^{r,T_2}_s + \int_0^{T_1}\Sigma^n_{sT_1}dW^{n,T_2}_s\bigg)e^{-D(0,T_1,T_2)}\end{aligned} \tag{3.23}$$

because the inflation rate is lognormally distributed the option prices can be derived from the following property of the lognormal distribution. If $\ln(X)$ is normally distributed with $\mathbb{E}[X] = m$ and the variance of $\ln(X) = v^2$ then

$$E[\omega(X-K)]^+ = \omega m\Phi\left(\omega\frac{\ln\frac{m}{K}+\frac{1}{2}v^2}{v}\right) - \omega K\Phi\left(\omega\frac{\ln\frac{m}{K}-\frac{1}{2}v^2}{v}\right) \tag{3.24}$$

The expected value of $\frac{I_{T_2}}{I_{T_1}}$ can be derived by considering the analysis in the previous section. The YYIIS price was given both by

$$\frac{YYIIS(t,T_1,T_2)}{P^n_{tT_2}} = \mathbb{E}^n_{T_2}\left[\frac{I_{T_2}}{I_{T_1}} - 1|\mathcal{F}_t\right] \tag{3.25}$$

or alternatively in terms of a forward starting zero coupon inflation swap

$$\frac{YYIIS(t,T_1,T_2)}{P^n_{tT_1}} = \mathbb{E}^n_{T_1}\left[P^r_{T_1T_2} - P^n_{T_1T_2}|\mathcal{F}_t\right] \tag{3.26}$$

So the required expectation can be given in terms of T_1-forward expectations as

$$\mathbb{E}^n_{T_2}\left[\frac{I_{T_2}}{I_{T_1}} - 1|\mathcal{F}_t\right] = \frac{P^n_{T_1}}{P^n_{tT_2}}\mathbb{E}^n_{T_1}[P^r_{T_1T_2} - P^n_{T_1T_2}|\mathcal{F}_t] \tag{3.27}$$

and so, following the working from the year-on-year swap

$$\mathbb{E}^n_{T_2}\left[\frac{I_{T_2}}{I_{T_1}} - 1|\mathcal{F}_t\right] = \frac{P^n_{tT_1}}{P^n_{tT_2}}\frac{P^r_{tT_2}}{P^r_{tT_1}}e^{C(t,T_1,T_2)} \tag{3.28}$$

with $C(t,T_1,T_2)$ given by 3.14. The value of the option is therefore given by

$$\begin{aligned} IICF(t,\psi,N,T_1,T_2,\kappa,\omega) \quad = \quad & \omega N\psi P^n_{tT_2}\Bigg[\frac{P^n_{tT_1}}{P^n_{tT_2}}\frac{P^r_{tT_2}}{P^r_{t,T_1}} \\ & e^{C(t,T_1,T_2)}\Phi\left(\omega\frac{\ln\frac{P^n_{tT_1}P^r_{tT_2}}{KP^n_{tT_2}P^r_{tT_1}} + C(t,T_1,T_2) + \frac{1}{2}V^2(t,T_1,T_2)}{V(t,T_1,T_2)}\right) \\ & -K\Phi\left(\omega\frac{\ln\frac{P^n_{tT_1}P^r_{tT_2}}{KP^n_{tT_2}P^r_{tT_1}} + C(t,T_1,T_2) - \frac{1}{2}V^2(t,T_1,T_2)}{V(t,T_1,T_2)}\right)\Bigg] \end{aligned}$$

where $V^2(t,T_1,T_2)$ is the variance of $\frac{I_{T_2}}{I_{T_1}}$ and can be derived from 3.23

$$\begin{aligned} V^2(t,T_1,T_2) \quad = \quad & \int_{T_1}^{T_2}(\sigma^I_s)^2 ds + \int_t^{T_2}\Sigma^r_{sT_2}(\Sigma^r_{sT_2})^* ds \\ & + \int_t^{T_1}\Sigma^r_{sT_1}(\Sigma^r_{sT_1})^* ds + \int_t^{T_2}\Sigma^n_{sT_2}(\Sigma^n_{sT_2})^* ds \\ & + \int_t^{T_1}\Sigma^n_{sT_1}(\Sigma^n_{sT_1})^* ds - 2\int_t^{T_1}\Sigma^r_{sT_2}(\Sigma^r_{sT_1})^* ds - 2\int_t^{T_1}\Sigma^n_{sT_1}(\Sigma^n_{sT_2})^* ds \\ & +2\rho^{Ir}\int_{T_1}^{T_2}\sigma^I_s\Sigma^r_{sT_2} ds - 2\rho^{In}\int_{T_1}^{T_2}\sigma^I_s\Sigma^n_{sT_2} ds \\ & +\rho^{nr}\Bigg(-2\int_t^{T_2}\Sigma^r_{sT_2}(\Sigma^n_{sT_2})^* ds + 2\int_t^{T_1}\Sigma^r_{sT_2}(\Sigma^n_{sT_1})^* ds \\ & +2\int_t^{T_1}\Sigma^r_{sT_1}(\Sigma^n_{sT_2})^* ds - 2\int_t^{T_1}\Sigma^r_{sT_1}(\Sigma^n_{sT_1})^* ds\Bigg) \end{aligned} \tag{3.29}$$

For constant parameters this expands to

$$
\begin{aligned}
V^2(t,T_1,T_2) &= \frac{\sigma_n^2}{2b_n^3}(1-e^{-b_n(T_2-T_1)})^2[1-e^{-2b_n(T_1-t)}] + \sigma_I^2(T_2-T_1) \\
&+\frac{\sigma_r^2}{2b_r^3}(1-e^{-b_r(T_2-T_1)})^2[1-e^{-2b_r(T_1-t)}] - 2\rho_{nr}\frac{\sigma_n\sigma_r}{b_nb_r(b_n+b_r)} \\
&(1-e^{-b_n(T_2-T_1)})(1-e^{-b_r(T_2-T_1)})[1-e^{-(b_n+b_r)(T_1-t)}] \\
&+\frac{\sigma_n^2}{b_n^2}\left[T_2-T_1+\frac{2}{b_n}e^{-b_n(T_2-T_1)}-\frac{1}{2b_n}e^{-2b_n(T_2-T_1)}-\frac{3}{2b_n}\right] \\
&+\frac{\sigma_r^2}{b_r^2}\left[T_2-T_1+\frac{2}{b_r}e^{-b_r(T_2-T_1)}-\frac{1}{2b_r}e^{-2b_r(T_2-T_1)}-\frac{3}{2b_r}\right] \\
&-2\rho_{nr}\frac{\sigma_n\sigma_r}{b_nb_r}\left[T_2-T_1-\frac{1-e^{-b_n(T_2-T_1)}}{b_n}\right. \\
&\left.-\frac{1-e^{-b_r(T_2-T_1)}}{b_r}+\frac{1-e^{-(b_n+b_r)(T_2-T_1)}}{b_n+b_r}\right] \\
&+2\rho_{nI}\frac{\sigma_n\sigma_I}{b_n}\left[T_2-T_1-\frac{1-e^{-b_n(T_2-T_1)}}{b_n}\right] \\
&-2\rho_{rI}\frac{\sigma_r\sigma_I}{b_r}\left[T_2-T_1-\frac{1-e^{-b_r(T_2-T_1)}}{b_r}\right]
\end{aligned}
$$

The working for this is given in Appendix A.

Chapter 4

Calibration

The objective of calibration is to choose the model parameters in such a way that the model prices are consistent with the market prices of simple instruments. In the case of nominal interest rates, the most common products in the market are swaptions and caps/floors.

Starting with the Hull-White extended Vasicek model for the spot rate, we derive an expression for the price of a zero coupon bond at a future time in terms of the Hull-White parameters. The bond price is lognormally distributed, and we are therefore able to use the properties of the lognormal distribution, combined with the first fundamental theorem of finance to derive the price of an option on a zero-coupon bond. We then show that both swaptions and caps/floors can be expressed in terms of options on zero-coupon bonds and consequently we are able to price these instruments in terms of the Hull-White volatility parameters. The calibration process is then a matter of choosing a particular form of the volatility parameters and fitting them so as to match the prices of selected market instruments.

In what follows we will use the following result which is a property of the lognormal distribution and allows us to price European options when the asset price is lognormally distributed under the martingale measure. If V is lognormally distributed with the variance of $\ln(V)$ given by w, then

$$E[(V-K)^+] = E(V)N(d_1) - KN(d_2) \tag{4.1}$$

$$E[(K-V)^+] = KN(-d_2) - E(V)N(-d_1) \tag{4.2}$$

where

$$d_1 = \frac{\ln[E(V)/K] + w^2/2}{w}$$

$$d_2 = \frac{\ln[E(V)/K] - w^2/2}{w}$$

4.1 Hull-White zero coupon bond dynamics

As shown previously, when $a(t)$ is chosen so as to fit the initial term structure, the risk-neutral Hull-White spot rate dynamics are given by

$$dr_t = (a(t) - b(t)r_t)dt + \sigma(t)dW \tag{4.3}$$

and are equivalent to a one factor Heath-Jarrow-Morton model with forward volatility structure given by

$$\sigma_{tT} = \sigma(t)e^{\beta(t)-\beta(T)} \tag{4.4}$$

where $\beta(t) = \int_0^t b(u)du$. The dynamics of the zero coupon bond P_{tS} in the T-forward measure is given by 2.11

$$\frac{P_{tS}}{P_{tT}} = \frac{P_{0S}}{P_{0T}}\mathcal{E}\left(-(\phi_S - \phi_T)\int_0^t g_u dW_u^T\right) \tag{4.5}$$

where $\phi_t = \int_0^t e^{-\beta(u)}du$ and $g_t = e^{\beta(t)}\sigma(t)$.

4.2 Hull-White zero coupon bond option

We now derive the price at time t of a call option $V(t, K, T, S)$ expiring at time T and struck at K on a zero-coupon bond with unit value at time S where $S > T$. Using the first fundamental theorem of finance, we can see that under the T-forward measure

$$\frac{V(t, K, T, S)}{P(t, T)} = \mathbb{E}_T[(P(T, S) - K)^+|\mathcal{F}_t] \tag{4.6}$$

In section 4.1 we showed that under the T-forward measure the zero-coupon bond is lognormally distributed with expected price equal to its forward price and with the variance of the logarithm given by $(\phi_S - \phi_T)^2 G_t$. We can therefore use result 4.2 to derive the option price in terms of the Hull-White volatility parameters as follows

$$V(t, K, T, S) = P(t, T)(FN(d_1) - KN(d_2)) \tag{4.7}$$

where

$$d_1 = \frac{\ln[F/K] + w^2/2}{w}$$
$$d_2 = \frac{\ln[F/K] - w^2/2}{w}$$

and F is the expected price of the bond in the T-forward measure i.e. the forward price $P(t, S)/P(t, T)$ and w is the variance of the logarithm of the bond, which is given above as

$$w^2 = (\phi_S - \phi_T)^2 G_t \tag{4.8}$$

In the Hull-White case with constant volatility parameters, this expression reduces to the form

$$w^2 = \frac{\sigma}{a}(1 - e^{-a(S-T)})^2 \left(\frac{1 - e^{-2aT}}{2a}\right) \tag{4.9}$$

4.3 Swaptions

A swaption is an option on a swap with the fixed rate on the swap given by the strike. A *receiver swaption* gives the holder the right to enter into a swap receiving the fixed strike rate and paying the floating rate and a *payer swaption* gives the holder the right to pay the fixed strike rate and receive the floating rate.

4.3.1 Swaption market quotes

Market prices of swaptions are quoted as lognormal volatilities where the price of the swaption is implied using Black's swaption formula. Swaption volatilities are generally quoted as a matrix of volatilities for a combination of option expiries and underlying swap terms. We now give a brief derivation of Black's swaption formula.

If we define the forward annuity $A(t, T_1, T_N)$ as the price at time t of an annuity that pays regular cash flows at times $T_1, ..., T_N$, we can write the forward swap rate starting at T and maturing at T_N as

$$S(t, T, T_N) = \frac{P(t, T) - P(t, T_N)}{A(t, T_1, T_N)} \tag{4.10}$$

It is clear that the forward swap rate is a martingale in the forward annuity measure, and so

$$\mathbb{E}_A[S(T, T, T_N)|\mathcal{F}_t] = S(t, T, T_N) \tag{4.11}$$

Black's formula assumes a lognormal distribution for the forward swap rate and the option price is calculated in the forward annuity measure. Defining $\mathrm{rec}(S_k, S_0, t, T, T_1, T_N)$ as the price of a receiver swaption struck at S_k and expiring at T on a swap starting at T with payment dates $T_1, ..., T_N$ and writing S_t for the forward swap rate at time t, i.e. $S(t, T, T_N)$, we have the relationship:

$$\frac{\mathrm{rec}(S_k, S_t, t, T, T_1, T_N)}{A(t, T_1, T_N)} = \mathbb{E}_A\left[\frac{\left((S_T - S_k)A(T, T_1, T_N)\right)^+}{A(T, T_1, T_N)}\right] \tag{4.12}$$

which simplifies to

$$\mathrm{rec}(S_k, S_t, t, T, T_1, T_N) = A(t, T_1, T_N)\mathbb{E}_A[(S_T - S_k)^+] \tag{4.13}$$

Using result 4.2 we have Black's swaption formula for a receiver swaption

$$\mathrm{rec}(S_k, S_t, t, T, T_1, T_N) = A(t, T_1, T_N)\Big(S_t N(d_1) - S_k N(d_2)\Big) \tag{4.14}$$

where

$$d_1 = \frac{\ln\left(\frac{S_t}{S_k}\right) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$$
$$d_2 = \frac{\ln\left(\frac{S_t}{S_k}\right) - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$$

with the corresponding payer swaption formula given by

$$\mathrm{pay}(S_k, S_t, t, T, T_1, T_N) = A(t, T_1, T_N)\Big(S_k N(-d_2) - S_t N(d_1)\Big) \tag{4.15}$$

4.3.2 Hull-White swaption

The payoff at time T of a receiver swaption which expires at time T with a fixed rate strike of S_k on a swap with cashflows at times $T_1, ..., T_N$ is given by

$$\left(S_k \sum_{i=1}^{N} P(T, T_i) - 1 + P(T, T_N)\right)^+ \tag{4.16}$$

This is identical to the payoff of a call option struck at 1 on a bond paying a coupon S_k at times $T_1 ... T_N$. To price an option on a coupon paying bond we use a technique proposed by Jamshidian. The technique is based on the observation that in a one-factor model, the price of each cash flow decreases monotonically with the spot interest rate. It is therefore possible to price an option on a coupon bond as a portfolio of options on the individual cash flows, with each option struck at the respective zero coupon bond rate at option expiry when the spot rate is the critical rate r^*. Here r^* is the spot rate that makes the price of the coupon bond at option expiry equal to the strike on the coupon bond option.

The approach is therefore as follows:

1. Using the Hull-White zero-coupon bond formula $P(T, T_i) = A(T, T_i)\exp(-rB(T, T_i))$ solve for r^*, the value of r that makes the price at time T of a bond paying a coupon S_k at times $T_1, ..., T_N$ equal to 1.

2. The price of the swaption is then the price of a portfolio of zero coupon bond options. Each option expires at T, with the underlying zero coupon bond paying 1 at T_i where $1 \leq i \leq N$. The strike of each option is the price at time T of a zero coupon bond maturing at time T_i if the spot rate were r^* and the notional of the option is size of the cash flow. Each option can be priced using 4.7.

4.4 Caps and Floors

A cap is a portfolio of call options on a Libor rate. Each option is known as a caplet and has the payoff $\delta(L_i - K)^+$ at time T_i where L_i is the Libor rate reset at time T_{i-1} and paid at time T_i and δ is the accrual fraction for the period. In the same way, a floor is a portfolio of floorlets with payoff $\delta(K - L_i)^+$. In what follows we will only discuss caps, but the logic applies equally to floors and both are connected by the put/call parity relation:

Value of cap = value of floor + value of swap

4.4.1 Cap/Floor market quotes

Caps and floors are quoted in the market as at-the-money Black lognormal flat volatities. That is to say each caplet is struck at the swap rate with the same tenor and cashflow frequency as the cap and the price of the cap is the sum of the prices of the individual caplets implied via the quoted lognormal volatility using Black's caplet formula.

Black's caplet formula is based on the Libor rate $L(t, T_{i-1}, T_i)$ which defines the value at time t of the interest rate that resets at time T_{i-1} and pays at time T_i. It can be written in terms of zero coupon bonds as follows

$$L(t, T_{i-1}, T_i) = \frac{1}{\delta}\left(\frac{P(t, T_{i-1})}{P(t, T_i)} - 1\right) \tag{4.17}$$

which can be written as

$$L(t, T_{i-1}, T_i) = \frac{1}{\delta}\left(\frac{P(t, T_{i-1}) - P(t, T_i)}{P(t, T_i)}\right) \tag{4.18}$$

Clearly this is a martingale under the T_i-forward measure so

$$\mathbb{E}_{T_i}[L(T_{i-1}, T_{i-1}, T_i)|\mathcal{F}_t] = L(t, T_{i-1}, T_i) \tag{4.19}$$

By the first fundamental theorem of finance we have the value of a caplet with strike K in the T_i forward measure as

$$\frac{\text{caplet}}{P(0, T_i)} = \delta\mathbb{E}_{T_i}[(L(T_{i-1}, T_{i-1}, T_i) - K)^+] \tag{4.20}$$

and using the result 4.2 we get Black's caplet model

$$\delta P(0, T_i)[L_t N(d_1) - KN(d_2)] \tag{4.21}$$

where

$$d_1 = \frac{\ln(L_t/K) + \frac{1}{2}\sigma^2 T_{i-1}}{\sigma\sqrt{T_{i-1}}}$$
$$d_2 = \frac{\ln(L_t/K) - \frac{1}{2}\sigma^2 T_{i-1}}{\sigma\sqrt{T_{i-1}}}$$

Here L_t is defined as the forward Libor given by $L(t, T_{i-1}, T_i)$.

4.4.2 Hull-White caps and floors

Here we show that each caplet can be considered as an option on a zero coupon bond and can therefore be priced using the Hull-White zero coupon bond option formula discussed in section 4.2

The payoff from a caplet at time T_i struck at K is

$$\delta\left(L_i - K\right)^+ \tag{4.22}$$

Where L_i is the rate reset at time T_{i-1} and paid at time T_i. This is equivalent to the payment

$$\frac{\delta}{1 + L_i\delta}\left(L_i - K\right)^+ \tag{4.23}$$

at time T_{i-1} and can be simplified to give

$$\left(1 - \frac{1 + K\delta}{1 + L_i\delta}\right)^+ \tag{4.24}$$

the expression

$$\frac{1 + K\delta}{1 + L_i\delta}$$

is the value at time T_{i-1} of a zero-coupon bond that pays off $1 + K\delta$ at time T_i and the caplet is therefore a put option with maturity T_{i-1} on a zero-coupon bond with maturity T_i with a strike of 1 and the nominal value of the bond equal to $1 + K\delta$. The caplet and floorlet can then be priced using the bond option formula 4.7

$$\begin{aligned}
\text{floorlet} &= (1 + K\delta)P_{tT_i}N(d_1) - P_{tT_{i-1}}N(d_2) \\
\text{caplet} &= P_{tT_{i-1}}N(-d_2) - (1 + K\delta)P_{tT_i}N(-d_1) \\
d_1 &= \frac{\ln\left(\frac{(1+K\delta)P_{tT_i}}{P_{tT_{i-1}}}\right) + \frac{w^2}{2}}{w} \\
d_2 &= \frac{\ln\left(\frac{(1+K\delta)P_{tT_i}}{P_{tT_{i-1}}}\right) - \frac{w^2}{2}}{w}
\end{aligned}$$

where w^2 is the variance at time T_{i-1} of the logarithm of the zero-coupon bond maturing at time T_i.

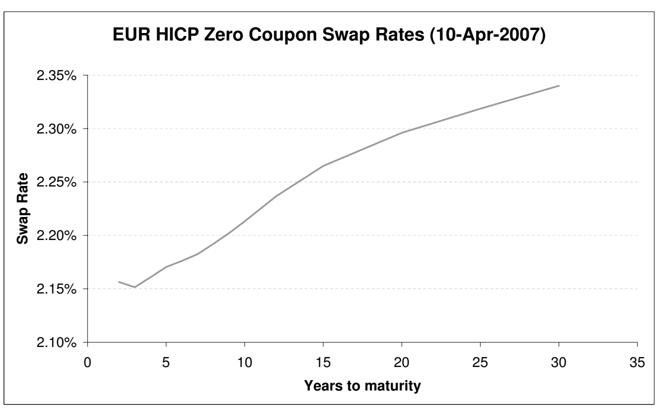
4.5 Implementation of the inflation model calibration

In this section we show how the Hull White inflation model with constant volatility parameters can be calibrated to market data. The initial term structures are derived from the nominal zero coupon curve and market inflation swap rates. The nominal volatility parameters are then fitted to at-the-money caps and the other parameters are then fitted to the inflation instruments. The most liquid instruments containing information on the inflation volatility and correlation are year-on-year inflation swaps. Although inflation-indexed caps and floors are becoming increasing popular, they are not yet sufficiently liquid to permit satisfactory calibration and so here we concentrate on fitting the model to year-on-year inflation swaps.

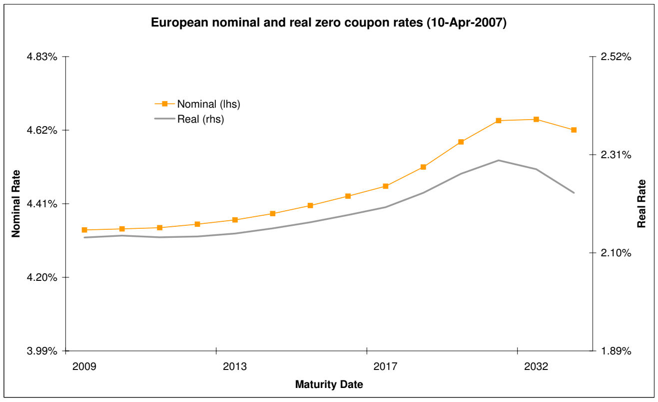
The first stage in the calibration process is to derive the initial term structures. The nominal term structure is derived from traded instruments in the cash, futures and swap markets. We do not discuss the details of yield curve interpolation here, but assume a continuous set of nominal discount factors which are derived from the traded market instruments using a proprietary model. The real discount factors are derived from the nominal discount factors and the zero coupon inflation swap rates using the model independent expression 3.6

$$P_{0T}^r = P_{0T}^n (1+r)^k$$

There are liquid zero-coupon inflation swap markets for European CPI, French domestic CPI, US CPI and UK RPI. We choose to calibrate to European CPI because of the more liquid year-on-year swap markets. In the European market, zero coupon swaps are liquid out to a maturity of about thirty years. The inflation swap term structure on the European HICPXT index on 10 April is shown here. The term structures for the other markets mentioned above are shown in appendix A.



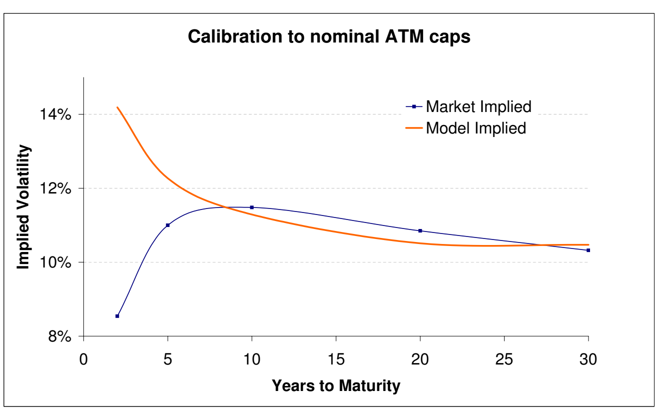
The one year inflation swap rate is not very liquid and is highly dependent on the reference index and economists' forecasts and so we do not include it in this analysis. On 10 April, the nominal and real zero coupon curves for the European HICPXT index were as follows. (The real curves for the other indices are also shown in appendix A)



The year-on-year inflation swap pricing formula developed in chapter 3 contains the seven parameters: b_n, σ_n, b_r, σ_r, σ_I, ρ_{nr} and ρ_{Ir}. We first fit the nominal volatility parameters b_n and σ_n to at-the-money nominal caps and then fit the remaining five parameters to year-on-year inflation swaps. The calibration to nominal caps is done by choosing b_n and σ_n so as to minimise the sum of the square difference between market and model cap prices using the goodness-of-fit measure

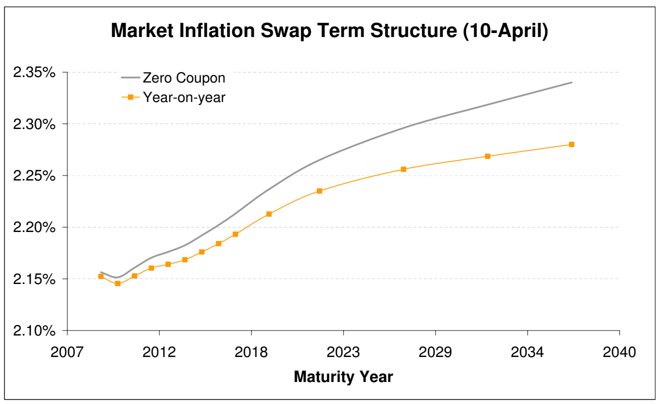
$$\sum_{i=1}^{n} (\text{MarketCap}_i - \text{ModelCap}_i)^2$$

where there are n calibrating instruments. The Hull White cap price was introduced earlier in the chapter and is dependent on the variance of the logarithm of the zero coupon bond for each option. The model implied volatility (the implied Black volatility backed out from the calibrated model cap prices) is shown here with the market Black implied volatility structure

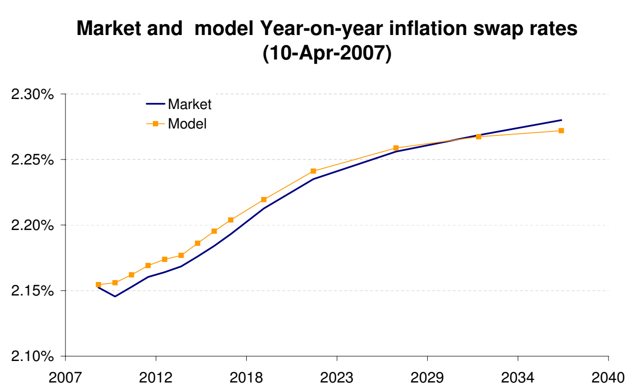


Although the fit is not bad for longer maturities, the model does not support the humped structure typically observed in the cap market. The behaviour is to be expected from the constant parameter model with the forward volatility decaying exponentially with maturity. This is the volatility structure proposed by Jarrow and Yildirim but, as described in chapter 2, the model can be extended to include deterministic time dependent parameters and thus provide a much closer fit to the initial market volatility structure. In this case either σ or b or both parameters may be made time dependent, but this extension is outside the scope of this thesis.

Year-on-year inflation swap rates trade below the corresponding zero-coupon swap rates due to the convexity adjustment. The following chart shows market zero-coupon and year-on-year swap rates on 10 April 2007



The model was also fitted to market year-on-year swaps using a least-squares approach, with the remaining five parameters chosen so as to minimise the square of the difference between the market and model year-on-year swap rates. The calibrated model rates and market rates for year-on-year swaps are shown here



The fit seems quite satisfactory, particularly given that bid/offer spreads are typically of the order of five basis points. However, given the rather poor fit to the nominal volatility structure, it seems unlikely that the constant parameter model should be used to price more exotic products. In this case extending the model to include deterministic time dependent volatility parameters would allow a much better initial fit and result in more reliable prices for more complex inflation-linked products.

Appendix A

Inflation Indexed Caplet formula for constant Hull-White parameters

In this appendix the variance $V^2(t, T_1, T_2)$ of the logarithm of the inflation rate $\frac{I_{T_2}}{I_{T_1}}$ is derived for the extended Vasicek model of Hull and White with constant volatility parameters. As shown in 3.29 the variance $V^2(t, T_1, T_2)$ is given by

$$
\begin{aligned}
V^2(t, T_1, T_2) \quad = \quad & \int_{T_1}^{T_2} (\sigma_s^I)^2 ds + \int_t^{T_2} \Sigma_{sT_2}^r (\Sigma_{sT_2}^r)^* ds + \int_t^{T_1} \Sigma_{sT_1}^r (\Sigma_{sT_1}^r)^* ds + \int_t^{T_2} \Sigma_{sT_2}^n (\Sigma_{sT_2}^n)^* ds \\
& + \int_t^{T_1} \Sigma_{sT_1}^n (\Sigma_{sT_1}^n)^* ds - 2 \int_t^{T_1} \Sigma_{sT_2}^r (\Sigma_{sT_1}^r)^* ds - 2 \int_t^{T_1} \Sigma_{sT_1}^n (\Sigma_{sT_2}^n)^* ds \\
& + 2\rho^{Ir} \int_{T_1}^{T_2} \sigma_s^I \Sigma_{sT_2}^r ds \\
& - 2\rho^{In} \int_{T_1}^{T_2} \sigma_s^I \Sigma_{sT_2}^n ds \\
& + \rho^{nr} \Big(- 2 \int_t^{T_2} \Sigma_{sT_2}^r (\Sigma_{sT_2}^n)^* ds + 2 \int_t^{T_1} \Sigma_{sT_2}^r (\Sigma_{sT_1}^n)^* ds \\
& + 2 \int_t^{T_1} \Sigma_{sT_1}^r (\Sigma_{sT_2}^n)^* ds - 2 \int_t^{T_1} \Sigma_{sT_1}^r (\Sigma_{sT_1}^n)^* ds \Big)
\end{aligned}
$$

The volatility of the zero coupon bond P_{tT} under this model is given by

$$
\Sigma_{tT} = \frac{\sigma}{b} [e^{-b(T-t)} - 1] \tag{A.1}
$$

The ρ_{Ir} coefficient comes from

$$
\begin{aligned}
& \frac{2\sigma_I\rho_{Ir}\sigma_r}{b_r}\int_{T_1}^{T_2}(e^{-b_r(T_2-s)}-1)ds \\
= \quad & \frac{2\sigma_I\rho_{Ir}\sigma_r}{b_r}\left[\frac{e^{-b_r(T_2-s)}}{b_r}-s\right]_{T_1}^{T_2} \\
= \quad & \frac{2\sigma_I\rho_{Ir}\sigma_r}{b_r}\left[\frac{1-e^{-b_r(T_2-T_1)}}{b_r}+T_1-T_2\right]
\end{aligned}
$$

and the ρ_{In} coefficient comes from

$$
\begin{aligned}
& -\frac{2\sigma_I\rho_{In}\sigma_n}{b_n}\int_{T_1}^{T_2}(e^{-b_n(T_2-s)}-1)ds \\
= \quad & -\frac{2\sigma_I\rho_{In}\sigma_n}{b_n}\left[\frac{e^{-b_n(T_2-s)}}{b_n}-s\right]_{T_1}^{T_2} \\
= \quad & -\frac{2\sigma_I\rho_{In}\sigma_n}{b_n}\left[\frac{1-e^{-b_n(T_2-T_1)}}{b_n}-T_2+T_1\right]
\end{aligned}
$$

The ρ_{nr} coefficients come from

$$
\begin{aligned}
- \quad & \frac{2\rho_{nr}\sigma_r\sigma_n}{b_rb_n}\int_t^{T_2}(e^{-b_r(T_2-s)}-1)(e^{-b_n(T_2-s)}-1)ds \\
+ \quad & \frac{2\rho_{nr}\sigma_r\sigma_n}{b_rb_n}\int_t^{T_1}(e^{-b_r(T_2-s)}-1)(e^{-b_n(T_1-s)}-1)ds \\
+ \quad & \frac{2\rho_{nr}\sigma_r\sigma_n}{b_rb_n}\int_t^{T_1}(e^{-b_r(T_1-s)}-1)(e^{-b_n(T_2-s)}-1)ds \\
- \quad & \frac{2\rho_{nr}\sigma_r\sigma_n}{b_rb_n}\int_t^{T_1}(e^{-b_r(T_1-s)}-1)(e^{-b_n(T_1-s)}-1)ds
\end{aligned}
$$

which would expand to give

$$
\begin{aligned}
\frac{2\rho_{nr}\sigma_r\sigma_n}{b_rb_n}\Bigg[\quad - \quad & \int_t^{T_2}(e^{-b_r(T_2-s)-b_n(T_2-s)}-e^{-b_r(T_2-s)}-e^{-b_n(T_2-s)}+1)ds \\
+ \quad & \int_t^{T_1}(e^{-b_r(T_2-s)-b_n(T_1-s)}-e^{-b_r(T_2-s)}-e^{b_n(T_1-s)}+1)ds \\
+ \quad & \int_t^{T_1}(e^{-b_r(T_1-s)-b_n(T_2-s)}-e^{-b_r(T_1-s)}-e^{-b_n(T_2-s)}+1)ds \\
- \quad & \int_t^{T_1}(e^{-b_r(T_1-s)-b_n(T_1-s)}-e^{-b_r(T_1-s)}-e^{-b_n(T_1-s)}+1)ds\Bigg]
\end{aligned}
$$

and in turn this gives

$$
\begin{aligned}
= \frac{2\rho_{nr}\sigma_r\sigma_n}{b_r b_n}\Bigg(& - \left[\frac{e^{-b_r(T_2-s)-b_n(T_2-s)}}{b_r+b_n} - \frac{e^{-b_r(T_2-s)}}{b_r} - \frac{e^{-b_n(T_2-s)}}{b_n} + s\right]_t^{T_2} \\
& + \left[\frac{e^{-b_r(T_2-s)-b_n(T_1-s)}}{b_r+b_n} - \frac{e^{-b_r(T_2-s)}}{b_r} - \frac{e^{-b_n(T_1-s)}}{b_n} + s\right]_t^{T_1} \\
& + \left[\frac{e^{-b_r(T_1-s)-b_n(T_2-s)}}{b_r+b_n} - \frac{e^{-b_r(T_1-s)}}{b_r} - \frac{e^{-b_n(T_2-s)}}{b_n} + s\right]_t^{T_1} \\
& - \left[\frac{e^{-b_r(T_1-s)-b_n(T_1-s)}}{b_r+b_n} - \frac{e^{-b_r(T_1-s)}}{b_r} - \frac{e^{-b_n(T_1-s)}}{b_n} + s\right]_t^{T_1}
\end{aligned}
$$

Finally for the ρ_{nr} coeffs we get

$$
\begin{aligned}
- & \frac{2\rho_{nr}\sigma_n\sigma_r}{b_n b_r}\left(T_2 - T_1 - \frac{1-e^{-b_n(T_2-T_1)}}{b_n} - \frac{1-e^{-b_r(T_2-T_1)}}{b_r} + \frac{1-e^{-(b_n+b_r)(T_2-T_1)}}{b_n+b_r}\right) \\
- & \frac{2\rho_{nr}\sigma_n\sigma_r}{b_n b_r(b_n+b_r)}(1-e^{-b_n(T_2-T_1)})(1-e^{-b_r(T_2-T_1)})(1-e^{-(b_n+b_r)(T_1-t)})
\end{aligned}
$$

Now we move on to those terms with no correlation coefficents

$$
\begin{aligned}
= & \quad \sigma_I^2(T_2-T_1) + \frac{\sigma_r^2}{b_r^2}\int_t^{T_2}(e^{-b_r(T_2-s)}-1)^2 ds + \frac{\sigma_r^2}{b_r^2}\int_t^{T_1}(e^{-b_r(T_1-s)}-1)^2 ds \\
+ & \quad \frac{\sigma_n^2}{b_n^2}\int_t^{T_2}(e^{-b_n(T_2-s)}-1)^2 ds + \frac{\sigma_n^2}{b_n^2}\int_t^{T_1}(e^{-b_n(T_1-s)}-1)^2 ds \\
- & \quad \frac{2\sigma_r^2}{b_r^2}\int_t^{T_1}(e^{-b_r(T_2-s)}-1)(e^{-b_r(T_1-s)}-1) ds \\
- & \quad \frac{2\sigma_n^2}{b_n^2}\int_t^{T_1}(e^{-b_n(T_2-s)}-1)(e^{-b_n(T_1-s)}-1) ds
\end{aligned}
$$

and doing the integrations we get

$$
\begin{aligned}
= & \quad \sigma_I^2(T_2-T_1) \\
+ & \quad \frac{\sigma_r^2}{b_r^2}\left[\frac{e^{-2b_r(T_2-s)}}{2b_r} - \frac{2e^{-b_r(T_2-s)}}{b_r} + s\right]_t^{T_2} + \frac{\sigma_n^2}{b_n^2}\left[\frac{e^{-2b_n(T_2-s)}}{2b_n} - \frac{2e^{-b_n(T_2-s)}}{b_n} + s\right]_t^{T_2} \\
+ & \quad \frac{\sigma_r^2}{b_r^2}\left[\frac{e^{-2b_r(T_1-s)}}{2b_r} - \frac{2e^{-b_r(T_1-s)}}{b_r} + s\right]_t^{T_1} + \frac{\sigma_n^2}{b_n^2}\left[\frac{e^{-2b_n(T_1-s)}}{2b_n} - \frac{2e^{-b_n(T_1-s)}}{b_n} + s\right]_t^{T_1} \\
- & \quad \frac{2\sigma_r^2}{b_r^2}\left[\frac{e^{-b_r(T_2+T_2-2s)}}{2b_r} - \frac{e^{-b_r(T_2-s)}}{b_r} - \frac{e^{-b_r(T_1-s)}}{b_r} + s\right]_t^{T_1} \\
- & \quad \frac{2\sigma_n^2}{b_n^2}\left[\frac{e^{-b_n(T_2+T_1-2s)}}{2b_n} - \frac{e^{-b_n(T_2-s)}}{b_n} - \frac{e^{-b_n(T_1-s)}}{b_n} + s\right]_t^{T_1}
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \sigma_I^2(T_2-T_1) \\
& +\frac{\sigma_n^2}{b_n^2}\left[\frac{1}{2b_n}-\frac{2}{b_n}-\frac{e^{-2b_n(T_2-t)}}{2b_n}+\frac{2e^{-b_n(T_2-t)}}{b_n}+\frac{1}{2b_n}-\frac{2}{b_n}-\frac{e^{-2b_n(T_1-t)}}{2b_n}+\frac{2e^{-b_n(T_1-t)}}{b_n}+T_2-t+T_1-t\right] \\
& -\frac{2\sigma_n^2}{b_n^2}\left[\frac{e^{-b_n(T_2-T_1)}}{2b_n}-\frac{e^{-b_n(T_2-T_1)}}{b_n}-\frac{1}{b_n}+T_1-\frac{e^{-b_n(T_1+T_2-2t)}}{2b_n}+\frac{e^{-b_n(T_2-t)}}{b_n}+\frac{e^{-b_n(T_1-t)}}{b_n}-t\right] \\
& +\frac{\sigma_r^2}{b_r^2}\left[\frac{1}{2b_r}-\frac{2}{b_r}-\frac{e^{-2b_r(T_2-t)}}{2b_r}+\frac{2e^{-b_r(T_2-t)}}{b_r}+\frac{1}{2b_r}-\frac{2}{b_r}-\frac{e^{-2b_r(T_1-t)}}{2b_r}+\frac{2e^{-b_r(T_1-t)}}{b_r}+T_2-t+T_1-t\right] \\
& -\frac{2\sigma_r^2}{b_r^2}\left[\frac{e^{-b_r(T_2-T_1)}}{2b_r}-\frac{e^{-b_r(T_2-T_1)}}{b_r}-\frac{1}{b_r}+T_1-\frac{e^{-b_r(T_1+T_2-2t)}}{2b_r}+\frac{e^{-b_r(T_2-t)}}{b_r}+\frac{e^{-b_r(T_1-t)}}{b_r}-t\right]
\end{aligned}
$$

Which finally gives

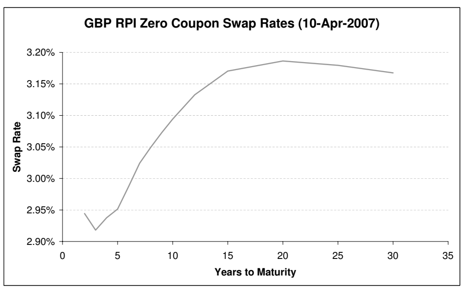
$$
\begin{aligned}
= \quad & \frac{\sigma_n^2}{2b_n^3}(1-e^{-b_n(T_2-T_1)})^2(1-e^{-2b_n(T_1-t)})+\sigma_I^2(T_2-T_1) \\
+ \quad & \frac{\sigma_r^2}{2b_r^3}(1-e^{-b_r(T_2-T_1)})^2(1-e^{-2b_r(T_1-t)}) \\
+ \quad & \frac{\sigma_n^2}{b_n^2}\left[T_2-T_1+\frac{2}{b_n}e^{-b_n(T_2-T_1)}-\frac{1}{2b_n}e^{-2b_n(T_2-T_1)}-\frac{3}{2b_n}\right] \\
+ \quad & \frac{\sigma_r^2}{b_r^2}\left[T_2-T_1+\frac{2}{b_r}e^{-b_r(T_2-T_1)}-\frac{1}{2b_r}e^{-2b_r(T_2-T_1)}-\frac{3}{2b_r}\right]
\end{aligned}
$$

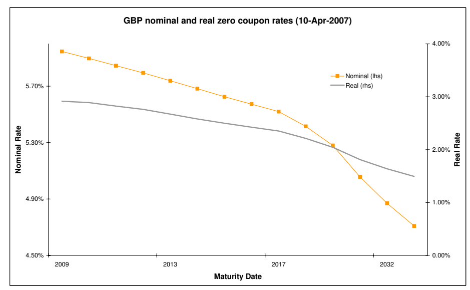
so the variance of the logarithm of $\frac{I_{T_2}}{I_{T_1}}$ is given by

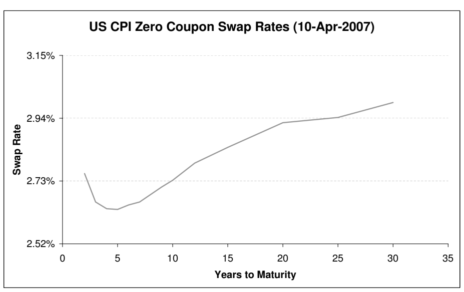
$$
\begin{aligned}
V^2(t,T_1,T_2) = {} & \frac{\sigma_n^2}{2b_n^3}(1-e^{-b_n(T_2-T_1)})^2[1-e^{-2b_n(T_1-t)}]+\sigma_I^2(T_2-T_1) \\
& +\frac{\sigma_r^2}{2b_r^3}(1-e^{-b_r(T_2-T_1)})^2[1-e^{-2b_r(T_1-t)}]-2\rho_{nr}\frac{\sigma_n\sigma_r}{b_nb_r(b_n+b_r)} \\
& (1-e^{-b_n(T_2-T_1)})(1-e^{-b_r(T_2-T_1)})[1-e^{-(b_n+b_r)(T_1-t)}] \\
& +\frac{\sigma_n^2}{b_n^2}\left[T_2-T_1+\frac{2}{b_n}e^{-b_n(T_2-T_1)}-\frac{1}{2b_n}e^{-2b_n(T_2-T_1)}-\frac{3}{2b_n}\right] \\
& +\frac{\sigma_r^2}{b_r^2}\left[T_2-T_1+\frac{2}{b_r}e^{-b_r(T_2-T_1)}-\frac{1}{2b_r}e^{-2b_r(T_2-T_1)}-\frac{3}{2b_r}\right] \\
& -2\rho_{nr}\frac{\sigma_n\sigma_r}{b_nb_r}\left[T_2-T_1-\frac{1-e^{-b_n(T_2-T_1)}}{b_n}-\frac{1-e^{-b_r(T_2-T_1)}}{b_r}+\frac{1-e^{-(b_n+b_r)(T_2-T_1)}}{b_n+b_r}\right] \\
& +2\rho_{nI}\frac{\sigma_n\sigma_I}{b_n}\left[T_2-T_1-\frac{1-e^{-b_n(T_2-T_1)}}{b_n}\right] \\
& -2\rho_{rI}\frac{\sigma_r\sigma_I}{b_r}\left[T_2-T_1-\frac{1-e^{-b_r(T_2-T_1)}}{b_r}\right]
\end{aligned}
$$

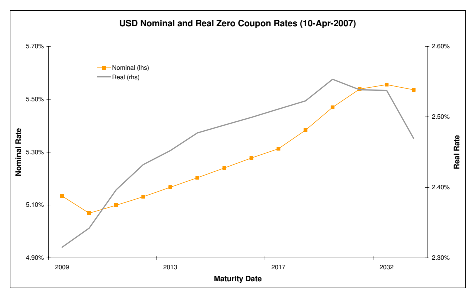
Appendix B

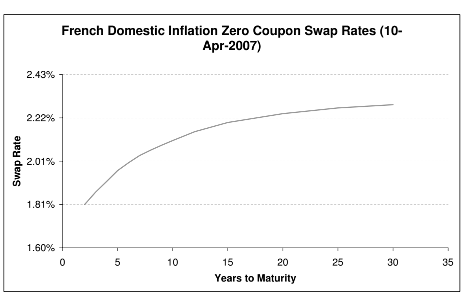
Market Data

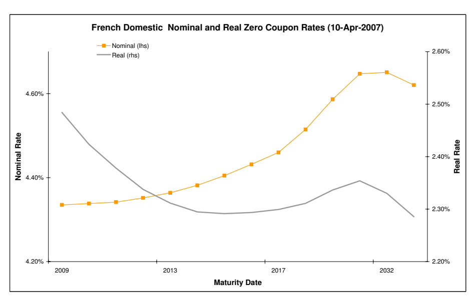












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