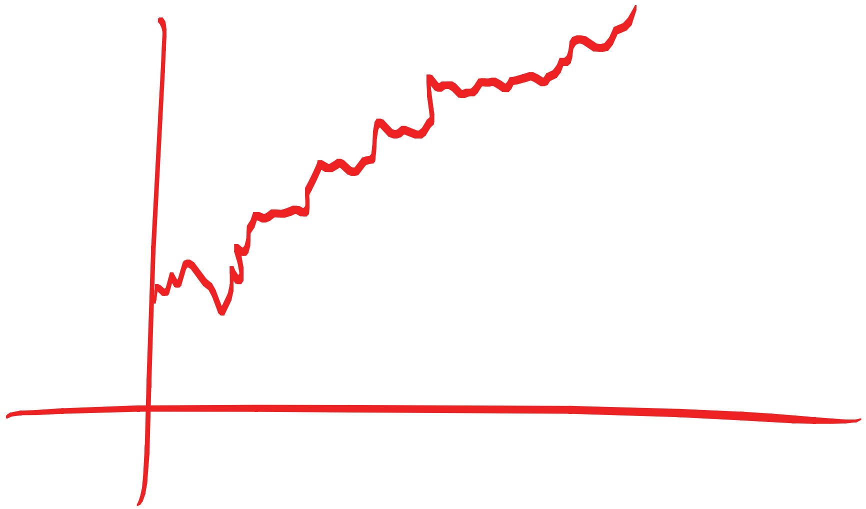
$$dS = \mu S \, dt + \sigma S \, dW$$ cts

$$\delta S = \mu S \, \delta t + \sigma S \, \phi \sqrt{\delta t}$$ discrete



$$d\boxed{S} = \mu S\,dt + \sigma S\,dW$$

stochastic process

state variable

$$dr = \ldots$$

$\boxed{a}\,dt$

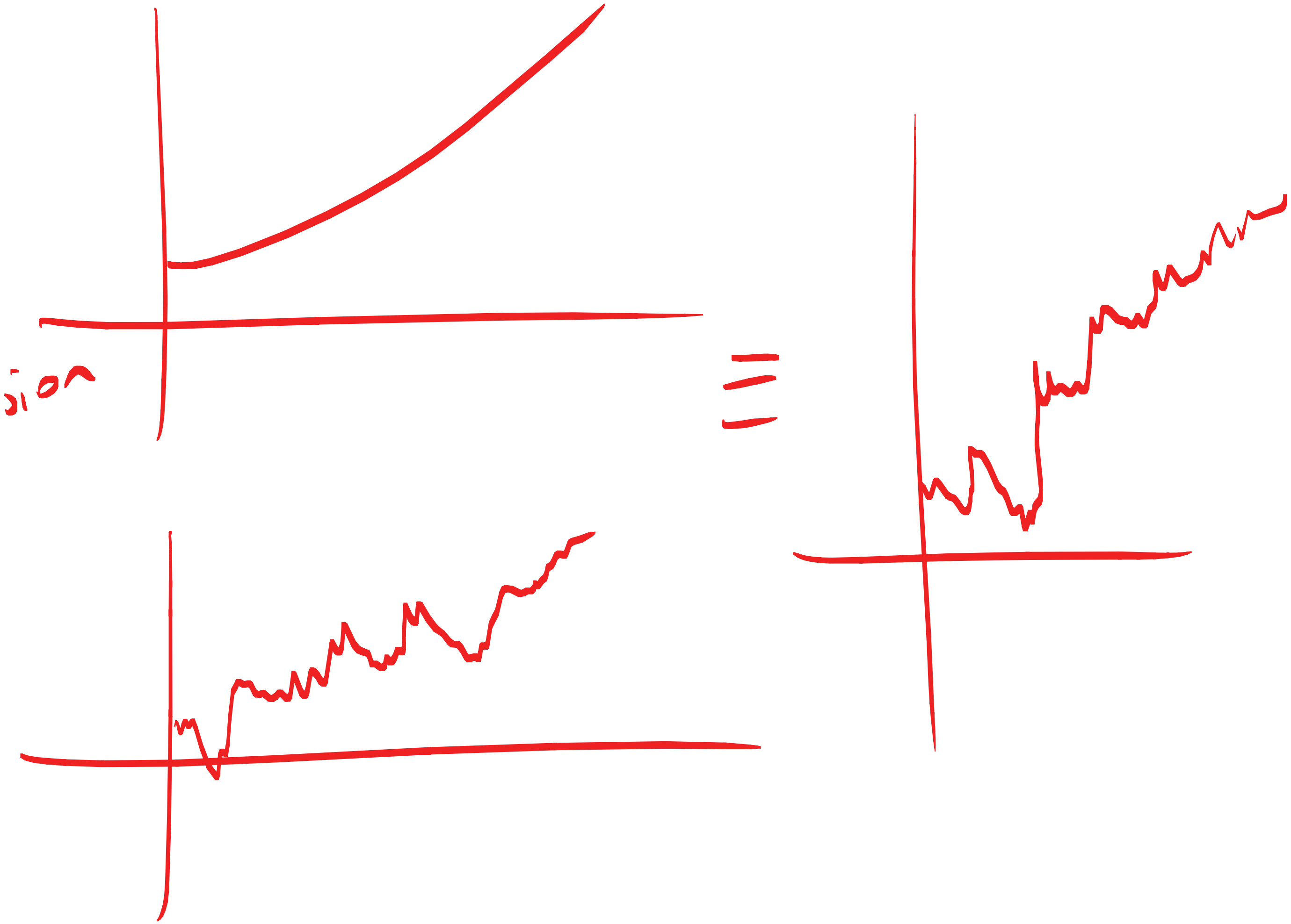
determ.

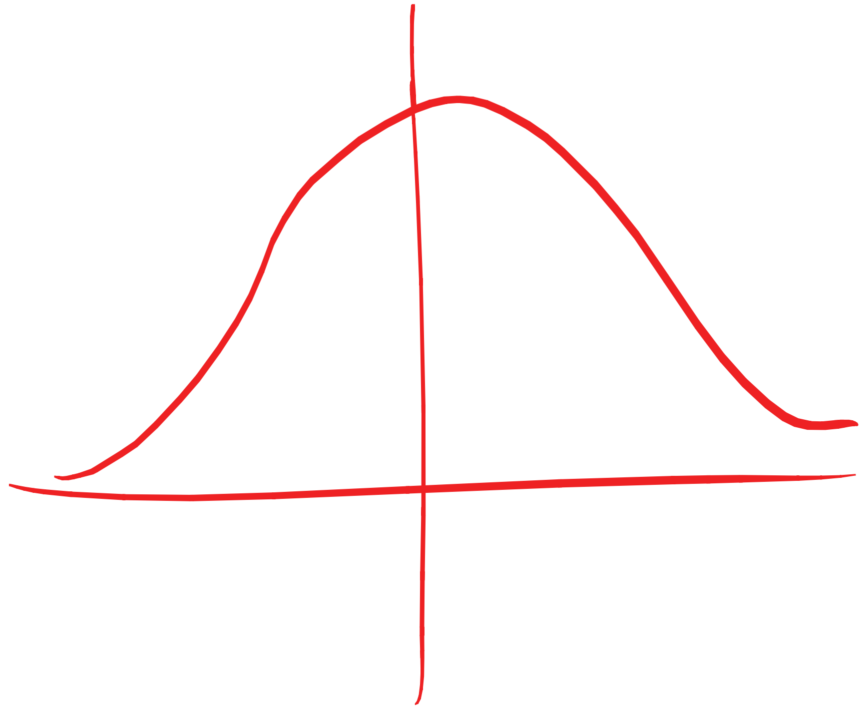
drift

diffusion

$\boxed{b}\,dX$

random part.





$S(t)$

diffusion $\longrightarrow$ $B(s,t)$ $\underline{\underline{dX}}$

SDE in integral form

$$① \quad dG = A(G,t)\,dt + B(G,t)\,dX_t$$

Integ. over $(0,t)$

$$\int_0^t dG = \int_0^t A\,d\tau + \int_0^t B\,dX_\tau$$

$$G_t - G_0 = \int_0^t A\,d\tau + \int_0^t B\,dX_\tau$$

$① \equiv ②$

$$② \quad G_t = G_0 + \int_0^t A\,d\tau + \underbrace{\int_0^t B\,dX_\tau}_{\text{Itô Integral}}$$

Put $\sigma = 0$ in GBM

$$dS = \mu S \, dt$$

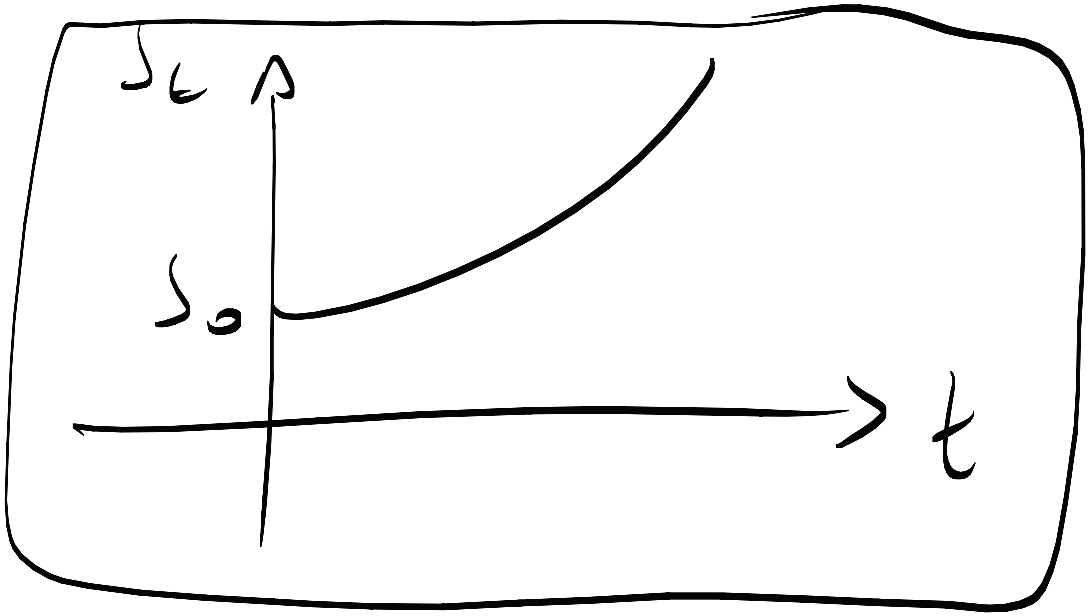
$$\int \frac{dS}{S} = \mu \int dt \qquad \ln S = \mu t + c$$

$$S_t = A e^{\mu t} \qquad \mu \to r$$

at $t = 0$ $S = S_0$

$$S_t = A e^{rt} \qquad A = S_0$$

$$S_t = S_0 e^{rt}$$



S_t
S_0
t

$$dS = \mu S\,dt + \sigma S\,dX$$

$$dS^2 = \mu^2 S^2 \cancel{dt^2} + \underline{\underline{\sigma^2 S^2 dt}} + 2\mu\sigma S^2 \cancel{dt\,dX}$$

$$\to 0 \qquad \to O(dt^{3/2})$$

$$\boxed{dS^2 = \sigma^2 S^2 dt}$$

Subst in the r.h.s of dV

Ex: $V = S^2$ S is G.B.M

$V' = 2S$ $V'' = 2$

Ex: ① $V = S^n$

② $V = e^S$

Subst in (3)

$$dV = \left(\mu S \cdot 2S + \frac{1}{2}\sigma^2 S^2 \cdot 2\right)dt + \sigma S \, 2S \, dX$$

$$dV = \underbrace{\left(2\mu S^2 + \sigma^2 S^2\right)}_{\text{drift}} dt \underbrace{+ 2\sigma S^2}_{\text{diffusion}} dX$$

$$dG = A(G,t)\,dt + B(G,t)\,dX$$

$$\boxed{dG^2 = B^2\,dt}$$

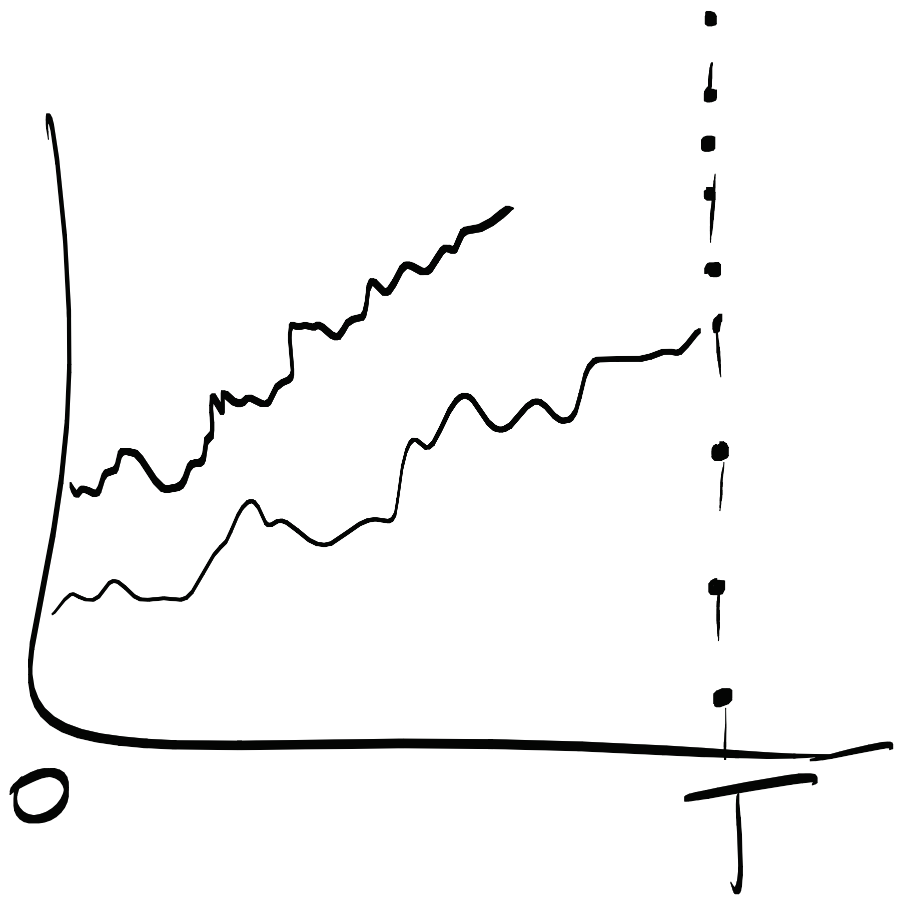
$$\mathbb{E}[dG] = \mathbb{E}[A(G,t)\,dt] + \mathbb{E}[B\,dX]$$

$$= A\,dt$$

$$\mathbb{V}[dG] = \underbrace{\mathbb{V}[\cancel{A\,dt}]}_{\to 0} + \underbrace{\mathbb{V}[B\,dX]}_{B^2\,dt}$$

$$dV = \left[\mu \cancel{S}\left(\frac{1}{\cancel{S}}\right) + \frac{1}{2}\sigma^2 \cancel{S^2}\left(-\frac{1}{\cancel{S^2}}\right)\right] dt + \sigma \cancel{S} \frac{1}{\cancel{S}}\, dX$$

$$d(\log S) = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma\, dX$$


0
T

$$dr = -\gamma \underbrace{(r - \bar{r})}_{u} dt + \sigma dX$$

$$u = r - \bar{r}$$

$$du = dr$$

$$\boxed{du = -\gamma u \, dt + \sigma dX}$$

Ornstein Uhlenbeck process

$$d(ue^{\gamma t}) = e^{\gamma t}\,du + u\gamma e^{\gamma t}\,dt$$

$$= e^{\gamma t}\left[-\cancel{\gamma u\,dt} + \sigma\,dX\right] + \cancel{\gamma u e^{\gamma t}}\,dt$$

$$\int_0^t d(ue^{\gamma s}) = \int_0^t \sigma e^{\gamma s}\,dX$$

$$\frac{d}{dt}(e^{\gamma t}) = \gamma e^{\gamma t}$$

$$de^{\gamma t} = \gamma e^{\gamma t}\,dt$$

$$u_s e^{\gamma s}\Big|_0^t = \sigma \int_0^t e^{\gamma s}\,dX_s$$

$$u_t e^{\gamma t} - u_0 = \sigma \int_0^t e^{\gamma s}\,dX_s$$

$$U_t = U_0 e^{-rt} + \sigma \int_0^t e^{r(s-t)}\, dX_s$$

$$\int_0^t e^{r(s-t)}\, dX_s \equiv \int v\, d\bar{u}$$

$$v = e^{r(s-t)} \qquad dX_s \equiv d\bar{u}$$

$$dv = r e^{r(s-t)}\, ds \qquad \bar{u} = X_s$$

$$X_s e^{\gamma(s-t)}\Big|_0^t - \gamma \int_0^t X_s e^{\gamma(s-t)}\, ds$$

$$X_t - X_0 e^{\gamma(\ )} \quad s=0$$

$$\sigma\, \boxed{X_t - \gamma \int_0^t e^{\gamma(s-t)} X_s \, ds}$$

Unsteady $\longrightarrow$ t - dep

Steady $\longrightarrow$ t - indep.

Steady state sol^{n} : as $t \to \infty$

sol^{n} is time indep.

$$\frac{1}{2}\sigma^2\frac{d^2}{dr^2}(p)-\gamma\frac{d}{dr}(\bar{r}-r)p=0$$

Integ. throughout

$$\frac{1}{2}\sigma^2\int\frac{d^2}{dr^2}(p)=\gamma\int\frac{d}{dr}(\bar{r}-r)$$

$$\frac{1}{2}\sigma^2\frac{d}{dr}p=-\gamma(r-\bar{r})p+C$$

as $r\to\infty$ $\begin{cases} p\to 0 \\ p'\to 0 \end{cases}$

$\therefore$ $C=0$

$$\frac{1}{2}\sigma^2 \frac{dp}{dr} = -\gamma(r-\bar{r})p$$

$$\frac{dp}{dr} = -\frac{2\gamma}{\sigma^2}(r-\bar{r})\,p \int \frac{d}{dr}\frac{1}{2}(r-\bar{r})^2$$

$$\frac{dp}{p} = -\frac{2\gamma}{\sigma^2}\int (r-\bar{r})\,dr$$

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$$

$$dS = S_{i+1} - S_i = \mu S_i \, \delta t + \sigma S_i \, dX$$

next time step

now

$$S(t) = S_i$$

$$S_{i+1} = S_i + \mu S_i \, \delta t + \sigma S_i \, \phi \sqrt{\delta t}$$

$$S_{i+1} = S_i \left(1 + \mu \, \delta t + \sigma \, \phi \sqrt{\delta t} \right)$$

Normal Distⁿ

pdf

$$p(x) = \frac{dF}{dx} \quad \text{CDF}$$

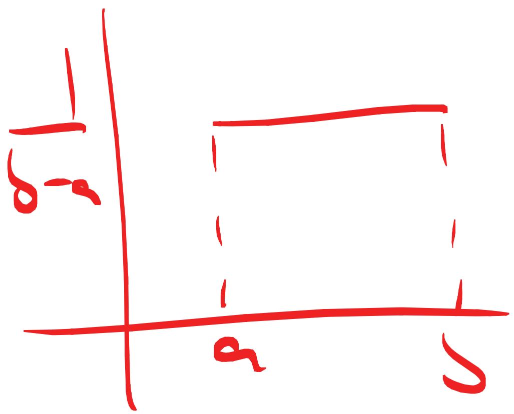
$$F(x) = \int_{-\infty}^{x} p(x)\, dx$$

Standard Normal.

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\boxed{x}} e^{-\frac{1}{2}\phi^2}\, d\phi \equiv \mathbb{P}[X \leq x]$$

$$\mathbb{E}\left[\sum_{i=1}^{12} R_i\right] = ?\sum_{i=1}^{12} \mathbb{E}(R_i) = 12 \cdot \frac{1}{2} = 6$$

$$p(x) = \begin{cases} \frac{1}{b-a} & a \le x \le b \\ 0 & \text{elsewhere} \end{cases}$$



$$\mathbb{E}[\mathrm{Unif}] = \frac{a+b}{2} \qquad \mathbb{E}(\mathrm{RAND}) = \frac{1}{2}$$

$$\mathbb{V}(\mathrm{Unif}) = (b-a)^2/12 \qquad \mathbb{V}(\mathrm{RAND}) = 1/12$$

$$p = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

$$\mathbb{E}(\mathrm{RAND}()) = \int_0^1 x \, dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$$

$$\mathbb{V}(\mathrm{RAND}()) = \int_0^1 x^2 p(x) \, dx - \frac{1}{4}$$

$$= \frac{x^3}{3}\Big|_0^1 - \frac{1}{4} = \frac{1}{12}$$

X_i i.i.d

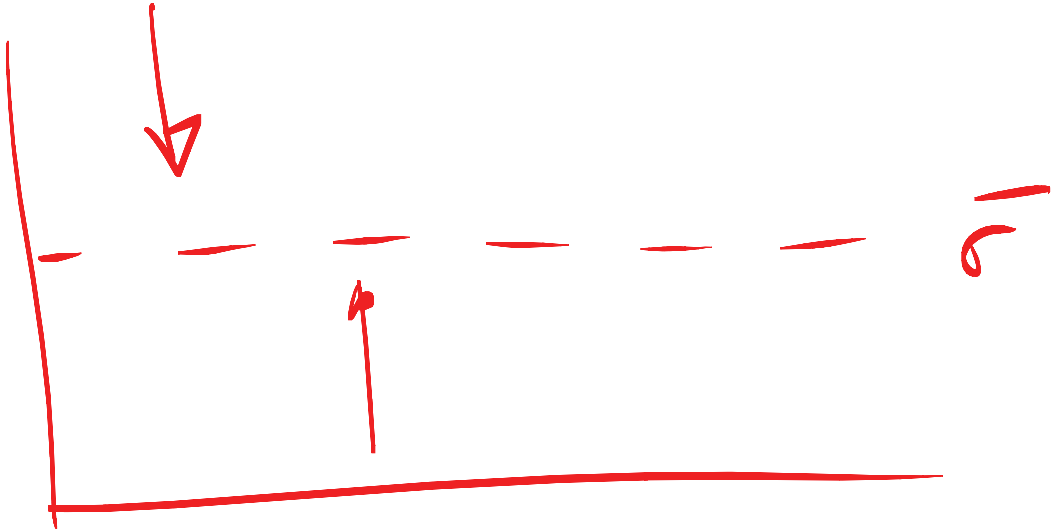
$$\frac{\sum_{i=1}^{N} X_i - N\mu}{\sigma\sqrt{N}} \longrightarrow \phi \sim N(0,1)$$

$$\frac{\sum_{i=1}^{12} RAND - \cancel{N}\,12\cdot\frac{1}{2}}{\sqrt{\frac{1}{12}}\,\sqrt{12}}$$

Ex: Extend to N RAND()

$$\sqrt{\frac{12}{N}}\left[\sum_{i=1}^{N} \mathrm{RAND}() - \frac{N}{2}\right]$$

γ high



γ low

$-\gamma(r - \bar{r})$

$r > \bar{r}$ $\quad -\gamma(+ve)$ $-ve$ trend

$r < \bar{r}$ $\quad -\gamma(-ve)$ $+ve$ trend

In Vasicek put $\sigma = 0$

$$dr = -\gamma(r - \bar{r})\,dt$$

$$\int \frac{dr}{r-\bar{r}} = -\gamma \int dt$$

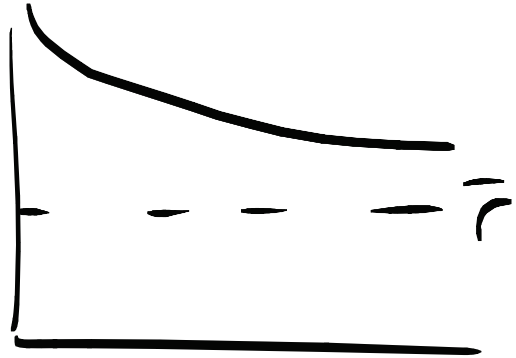
$$\ln(r-\bar{r}) = -\gamma t + A$$

$$r-\bar{r} = Ce^{-\gamma t}$$

$$r(t) = \bar{r} + Ce^{-\gamma t}$$

γ high

γ low



$$\varepsilon_1, \varepsilon_2 \sim N(0,1)$$

$$\mathbb{E}[\varepsilon_1] = 0 = \mathbb{E}[\varepsilon_2]$$

$$\mathbb{E}[\varepsilon_1^2] = 1 = \mathbb{E}[\varepsilon_2^2]$$

$$\mathbb{E}[\varepsilon_1 \varepsilon_2] = 0$$

We want produce ϕ_1, ϕ_2 such that

$$\phi_1, \phi_2 \sim N(0,1)$$

$$\mathbb{E}[\phi_1 \phi_2] = \rho$$

$$\phi_1 = \varepsilon_1$$

$$\phi_2 = \alpha \varepsilon_1 + \beta \varepsilon_2$$

what values of α, β gives

$$\mathbb{E}(\phi_1 \phi_2) = \rho$$

$$\mathbb{E}(\phi_1 \phi_2) = \rho = \mathbb{E}\left[\varepsilon_1(\alpha\varepsilon_1 + \beta\varepsilon_2)\right]$$

$$= \alpha \underbrace{\mathbb{E}[\varepsilon_1^2]}_{=1} + \beta \underbrace{\mathbb{E}[\varepsilon_1 \varepsilon_2]}_{=0} \Rightarrow \boxed{\alpha = \rho}$$

$$\mathbb{E}(\phi_2^2) = 1 = \mathbb{E}\left((\alpha\varepsilon_1 + \beta\varepsilon_2)^2\right)$$

$$= \mathbb{E}\left(\alpha^2\varepsilon_1^2 + \beta^2\varepsilon_2^2 + 2\alpha\beta\varepsilon_1\varepsilon_2\right) = 1$$

$$\underbrace{\alpha^2 \mathbb{E}[\varepsilon_1^2]}_{=1} + \underbrace{\beta^2 \mathbb{E}[\varepsilon_i^2]}_{=1} + 2\alpha\beta\, \underbrace{\mathbb{E}(\varepsilon_1 \varepsilon_2)}_{=0} = 1$$

$$\alpha^2 + \beta^2 = 1 \Longrightarrow \beta = \sqrt{1-\alpha^2}$$

$$\therefore \boxed{\begin{aligned} \alpha &= \rho \\ \beta &= \sqrt{1-\rho^2} \end{aligned}}$$

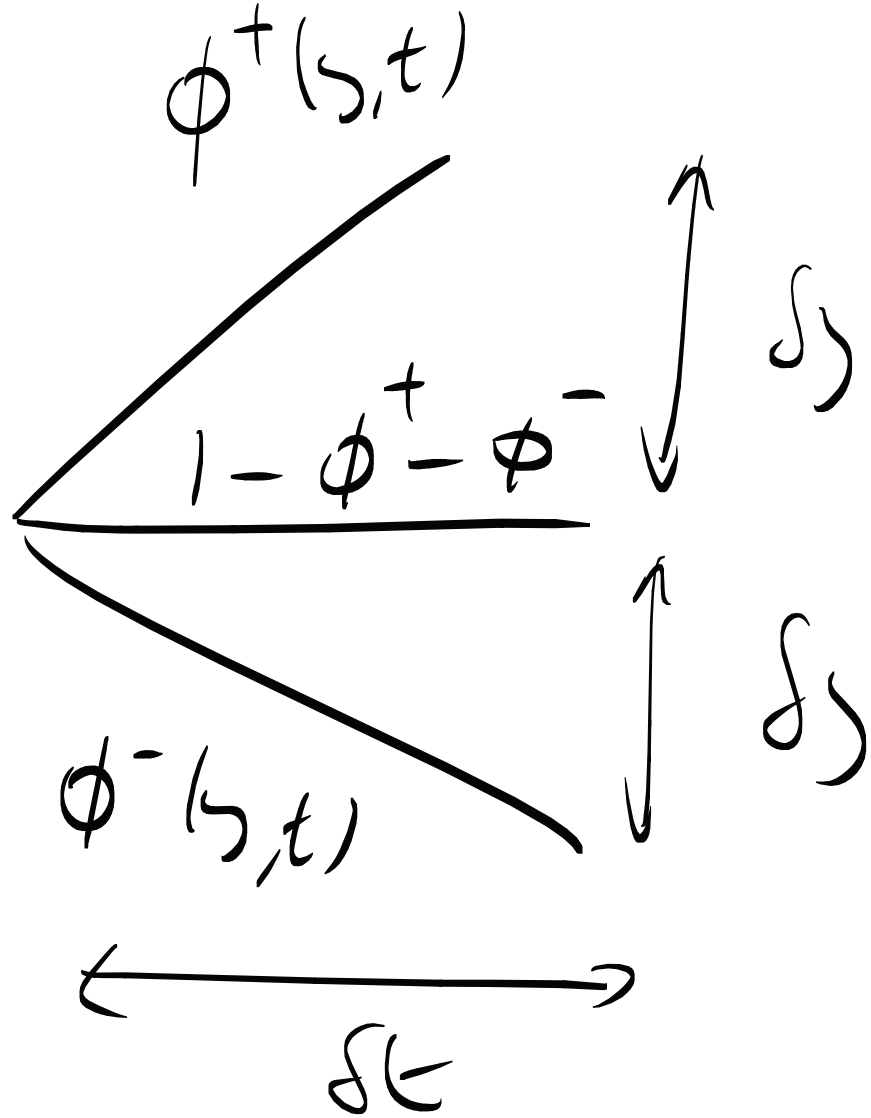
$$S_{1,i+1} = S_{1,i}\left[1 + \mu_1 \delta t + \sigma_1 \sqrt{\delta t}\, \varepsilon_1\right]$$

$$S_{2,i+1} = S_{2,i}\left[1 + \mu_2\, dt + \sigma_2 \sqrt{dt}\left(\rho \varepsilon_1 + \sqrt{1-\rho^2}\, \varepsilon_2\right)\right]$$

$$\mathbb{E}\left[dX_1, dX_2\right] = \rho\, dt$$

$\phi \sqrt{\delta t}$

$$S_{i+1} = S_i \left[1 + \mu\, \delta t + \sigma \sqrt{\delta t} * \text{NORMSINV}(\text{RAND}()) \right]$$



$\phi^+(s,t)$
$1-\phi^+-\phi^-$
$\phi^-(s,t)$
δs
δs
δt