1 Exercise 2.2 Solutions

1. We will proceed using the Lagrangian to obtain a parametrization of the boundary of opportunity sets based on the return objective, m.

   Define the vector of returns, μ as

$$\mu = \begin{pmatrix} 0.08 \\ 0.10 \\ 0.10 \\ 0.14 \end{pmatrix}$$

   We will solve one general problem for an arbitrary correlation matrix R and then, once it is solved, we will substitute for the three correlations matrices given in the exercise.

   To solve the general problem, we are going to need the little covariance matrix decomposition outlined in class. Define the standard deviation matrix, S as

$$S = \begin{pmatrix} 0.12 & 0 & 0 & 0 \\ 0 & 0.12 & 0 & 0 \\ 0 & 0 & 0.15 & 0 \\ 0 & 0 & 0 & 0.20 \end{pmatrix}$$

   then the covariance matrix Σ is given by:

$$\Sigma = SRS$$

   Finally define the weight vector w as

$$w = \begin{pmatrix} w_A \\ w_B \\ w_C \\ w_D \end{pmatrix}$$

Our optimization problem can be formulated as

$$\min_{w} \frac{1}{2} w^T SRSw$$

Subject to:

$$\begin{array}{rcl} \mu^T w & = & m \\ \mathbf{1}^T w & = & 1 \end{array}$$

where the two constraints are respectively the return constraint and the budget equation.

Next, we form the Lagrange function: with two lagrange multipliers λ and γ:

$$L(w, \lambda, \gamma) = \frac{1}{2} w^T SRSw + \lambda(m - \mu^T w) + \gamma(1 - \mathbf{1}^T w)$$

and solve for the first order condition:

$$\begin{array}{rcl} \frac{\partial L}{\partial w}(w, \lambda, \gamma) & = & w^T \Sigma - \lambda \mu^T - \gamma \mathbf{1}^T = 0 \\ \frac{\partial L}{\partial \lambda}(w, \lambda, \gamma) & = & m - \mu^T w = 0 \\ \frac{\partial L}{\partial \gamma}(w, \lambda, \gamma) & = & (1 - \mathbf{1}^T w) = 0 \end{array}$$

We then get the optimal weight vector v^*

$$w^* = (SRS)^{-1}(\lambda \mu + \gamma \mathbf{1}) \tag{1}$$

where

$$\begin{cases} \lambda = \frac{Am-B}{AC-B^2} \\ \gamma = \frac{C-Bm}{AC-B^2} \end{cases}$$

and

$$\begin{cases} A = \mathbf{1}^T (SRS)^{-1} \mathbf{1} \\ B = \mu^T (SRS)^{-1} \mathbf{1} \\ C = \mu^T (SRS)^{-1} \mu \end{cases}$$

that's essentially as far as we can go without putting some values into the correlation matrix R. Let's now examine each of the three cases.

Case 1:

$$R = \begin{pmatrix} 1 & 0.2 & 0.5 & 0.3 \\ 0.2 & 1 & 0.7 & 0.4 \\ 0.5 & 0.7 & 1 & 0.9 \\ 0.3 & 0.4 & 0.9 & 1 \end{pmatrix}$$

then

$$\begin{cases} A \approx 456.5217391 \\ B \approx 63.81642512 \\ C \approx 9.389694042 \end{cases}$$

so, for example, for a 10% return constraint, we would have

$$\begin{cases} \lambda \approx -0.0848546 \\ \gamma \approx 0.0140522 \end{cases}$$

and substituting into w^*, we would get:

$$w* = \begin{pmatrix} 76.23\% \\ 84.42\% \\ -98.76\% \\ 38.11\% \end{pmatrix}$$

Quite aggressive! Now what would happen as we the correlation structure changes?

Case 2: When assets are uncorrelated,

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

then

$$\begin{cases} A \approx 208.3333333 \\ B \approx 20.44444444 \\ C \approx 2.073333333 \end{cases}$$

so, for example, for a 10% return constraint, we would have

$$\begin{cases} \lambda \approx 0.0278392 \\ \gamma \approx 0.0020681 \end{cases}$$

and substituting into w^*, we would get:

$$w* = \begin{pmatrix} 29.83\% \\ 33.70\% \\ 21.56\% \\ 14.91\% \end{pmatrix}$$

Quite a change! No short position, and a very reasonable allocation.

Case 3: When assets are perfectly correlated,

$$R = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

and we have a problem! Indeed, the correlation matrix and hence the covariance matrix cannot be inverted. What to do?

An important point to see is that when asset returns are perfectly correlated, asset returns and volatilities are different and you are allowed to short with no restriction, you can **always** design a portfolio producing any arbitrary level of return m with 0 volatility.

How do you find this portfolio? You solve a system of 3 equations with n unknown variables, the weights. The three equations you need to solve are:

$$\begin{aligned} \sigma^T w &= 0 \\ \mu^T w &= m \\ \mathbf{1}^T w &= 1 \end{aligned} \tag{2}$$

where σ is the column **vector** of standard deviation.

Since you have more unknown variables than constraints, you will have several possible portfolios satisfying this system.

Of course this cannot happen in practice since this is a clear case of an **arbitrage trade**.

So a few conclusions with respect to this exercise:

- An optimizer will always take advantage of small discrepancies between the input parameters in order to get the best possible result. So, beware of the "Garbage-in-garbage-out" syndrome! a small changes in input data could produce very large changes in the decision variables.
- Always try to add constraints to ground your model in operational realities and increase the stability of your allocation model.
- The third case is an example of an arbitrage situation and of the optimizer finding ways to exploit this arbitrage. Optimizers can be a good tool to hedge or find complex arbitrage techniques... as long as you have good parameters.

- Hedging is a special case of diversification which occurs when two securities are perfectly positively or negatively correlated. Diversification has much wider applications than hedging and is generally much cheaper to implement.

2. A symmetric matrix A is positive definite if for all $\underline{x} \neq \underline{0}$, $\underline{x}^{\mathrm{T}} A \underline{x} > 0$. Therefore using the covariance matrix given

$$(x,\, y,\, z) \begin{pmatrix} 9 & 3 & 0 \\ 3 & 16 & 5 \\ 0 & 5 & 25 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = (x,\, y,\, z) \begin{pmatrix} 9x + 3y \\ 3x + 16y + 5z \\ 5y + 25z \end{pmatrix}$$

$= 9x^2 + 6xy + y^2 + 25z^2 + 10yz + y^2 + 14y^2$
$= (3x + y)^2 + (5z + y)^2 + 14y^2$
$\Rightarrow$ for $(x,\, y,\, z) \neq \underline{0}$, the quadratic form above is always positive, hence the covariance matrix is strictly positive definite.

The inverse of the covariance matrix A is

$$A^{-1} = \frac{1}{\det(A)} \cdot \operatorname{adj}(A)$$

$\det(A) = (9 \times 16 \times 25) - (9 \times 5 \times 5) - (25 \times 3 \times 3) = 15 \times 210$

$$\operatorname{adj}(A) = 15 \begin{pmatrix} 25 & -5 & 1 \\ -5 & 15 & -3 \\ 1 & -3 & 9 \end{pmatrix}$$

$$\Rightarrow \frac{1}{210} \begin{pmatrix} 25 & -5 & 1 \\ -5 & 15 & -3 \\ 1 & -3 & 9 \end{pmatrix}$$

3. Where should we be on the efficient frontier in question 1 if we wish to minimise the chance of a return less than 0.05?

If $C(\cdot)$ is the cumulative distribution function for the standardised Normal distribution then

$$C\left(\frac{\mu_\Pi - r^*}{\sigma_\Pi}\right)$$

is the probability that the return exceeds r^*. If we want to minimise the chance of a return less than r^* we should choose the portfolio from the efficient frontier set Π_{eff} with the largest value of the slope

$$\frac{\mu_{\Pi_{\text{eff}}} - r^*}{\sigma_{\Pi_{\text{eff}}}}.$$

To minimise the chance of a return less than 0.05, we must therefore maximise $\dfrac{\mu_\Pi - 0.05}{\sigma_\Pi}$, There is one relatively easy way to do this without having to maximize this fairly complex looking criterion. From the lecture, recall that we already know how to obtain the portfolio maximizing the slope of a line: the **tangency portfolio** maximizes the slope of the CML. All we need to do is adapt this result to our question.

Our question can be reinterpreted as finding the tangency portfolio between a line going through the point $(0, 0.05)$ and the efficient frontier. The equation of the non-investable line can be obtained by varying m and solving the following optimization:

$$\min_w \frac{1}{2} w^T \Sigma w$$

Subject to:

$$r^* + (\mu - r^* \mathbf{1})^T w = m$$

where $r^* = 0.05$ is the floor rate.

We form the Lagrange function:

$$L(x, \lambda) = \frac{1}{2} w^T \Sigma w + \lambda (m - r^* - (\mu - r^* \mathbf{1})^T w)$$

We now solve for the first order condition by taking the derivative with respect to the vector w:

$$\frac{\partial L}{\partial w} = w^T \Sigma - \lambda (\mu - r^* \mathbf{1})^T = 0$$

Checking the second order condition, the Hessian of the objective function is still the covariance matrix, which is positive definite. Therefore, we have reached the optimal weight vector w^*:

$$w^* = \lambda \Sigma^{-1} (\mu - r^* \mathbf{1})$$

Substituting in the constraint equation, we get:

$$\lambda = \frac{m - r^*}{(\mu - r^* \mathbf{1})^T \Sigma^{-1} (\mu - r^* \mathbf{1})}$$

Finally, we get w^*:

$$w^* = \frac{(m - r^*) \Sigma^{-1} (\mu - r^* \mathbf{1})}{(\mu - r^* \mathbf{1})^T \Sigma^{-1} (\mu - r^* \mathbf{1})}$$

Now, to find the tangency portfolio, simply recall that the tangency portfolio is 100% invested in the risky assets, i.e. $\mathbf{1}^T w = 1$ to finally obtain [1]:

$$w_t = \frac{\Sigma^{-1} (\mu - r^* \mathbf{1})}{B - A r^*}$$

Plugging in the values,

[1] To do so, substitute the definition of w^* into the budget equation in order to get m_t, the tangency portfolio's return. Next, substitute the equation for m_t back into the equation for w^* in order to deduce the tangency portfolio weights, w_t.

$$w_\Pi = w_t = \begin{pmatrix} 93.89\% \\ 166.47\% \\ -335.38\% \\ 175.01\% \end{pmatrix}$$

Moreover,

$$\mu_\Pi = \mu_t = \mu^T w_t = 15.12\%$$
$$\sigma_\Pi = \sigma_t = \sqrt{w_t^T \Sigma w_t} = 4.97\%$$

and

$$\mathbb{P}\left[R_\Pi > r^*\right] = C\left(\frac{\mu_\Pi - r^*}{\sigma_\Pi}\right) = 97.92\%$$

An important note: the problem we just solved shares a similar structure with the investment problem with a risk-free rate. However, it is NOT the investment problem with a risk-free rate. Indeed, the floor rate of 0.05% is NOT an asset and therefore CANNOT be invested in. Hence, while w_t makes sense from an investment perspective, none of the other w^* represent an investable portfolio.

4.

Risk is measured by standard deviations of returns $\sigma_i{}^2$ and covariance of returns σ_{ij}, reward by expected returns μ_i. Investing W_i of wealth in risky asset with $\sum W_i = 1$ gives

$$\mu_\sqcap = \sum_{i=1}^{N} W_i \mu_i, \quad \sigma_\sqcap{}^2 = \sum_{i=1}^{N}\sum_{j=1}^{N} W_i W_j \rho_{ij} \sigma_i \sigma_j \qquad ((**))$$

where ρ_{ij} is the correlation coefficient between the i^{th} and j^{th} asset. As we vary the weights W_i the quantities given in (**) also vary and fill a convex set of all possible risky-portfolios - the opportunity set. The efficient frontier is the top of the boundary set - it maximises return for a feasible level of risk.

Using the definitions for risk and reward given by (**) again we have

$$\mu_\Pi = W\mu_A + (1 - W)\,\mu_B,$$

and

$$\sigma_\Pi^2 = W^{\ 2}\sigma_A^2 + 2W\,(1 - W)\,\rho\sigma_A\sigma_B + (1 - W)^2\,\sigma_B^2,$$

Asset	μ_i	σ_i
A	0.1	0.2
B	0.2	0.3

and $\rho = 0.5$. Upon substituting the relevant values, we obtain

$$\begin{aligned}\mu_\Pi &= 0.1W + 0.2\,(1-W) = 0.2 - 0.1W \\ &\Rightarrow W = 2 - 10\mu_\Pi\end{aligned}$$

and

$$\begin{aligned}\sigma_\Pi^2 &= 0.2^2W^{\,2} + 2W\,(1-W)\,(0.5)\,(0.2)\,(0.3) + (1-W)^2\left(0.3^2\right) \\ &= 0.04W^{\,2} + 0.06W\,(1-W) + 0.09\,(1-W)^2 \\ \sigma_\Pi &= \sqrt{0.07W^{\,2} - 0.12W + 0.09} = \sqrt{0.07\,(2-10\mu_\Pi)^{\,2} - 0.12\,(2-10\mu_\Pi) + 0.09}\end{aligned}$$

To find the boundary of the opportunity set (and hence efficient frontier), we vary W (or μ_Π) on a $(\sigma_\Pi,\ \mu_\Pi)$ plot.

Introduce a risk-free rate $r_f = 5\%$. Now, any investment can be expressed as an allocation to some risky portfolio Π plus an allocation to the risk-free rate. Hence, the new efficient frontier will be a line joining the risk-free rate and some "special" risky portfolio to be determined. In fact, the new "efficient line" is the line determined by the risk-free rate and some "special" risky portfolio with maximum slope: it is the tangent to the old efficient frontier and the "special" risky portfolio is therefore called the tangency portfolio. This new efficient frontier is called the CML and its equation is:

$$\mu_P = r_f + \theta_t \times \sigma_P \tag{3}$$

where the Sharpe ratio of the tangency portfolio, θ_t, is defined as

$$\frac{\mu_t - r_f}{\sigma_t} \tag{4}$$

which tells us that now the return of any efficient portfolio P is the risk-free rate plus the Sharpe Ratio θ of the tangency (or market) portfolio times the risk of portfolio P.

So, one way or another, writing the CML requires that we find the tangency portfolio. There are basically two ways to proceed:

- either we recall that the tangency portfolio is the risky portfolio with the highest Sharpe ratio, and we therefore have to solve the following scalar unconstrained optimization problem

  $$\max_W \theta(W) = \frac{\mu_\Pi - r_f}{\sigma_\Pi} = \frac{0.15 - 0.1W}{\sqrt{0.07W^{\,2} - 0.12W + 0.09}}$$

  This method has the benefit of being direct, although it can only really be solved analytically if we have 2 assets;

- or we do what we already did in question 3. This method is longer to write, but works all the time and is computationally easier.

In the following, we will use the second method. The equation of the CML can be obtained by varying m and solving the following optimization:

$$\min_{w} \frac{1}{2} w^T \Sigma w$$

Subject to:

$$r_f + (\mu - r_f \mathbf{1})^T w = m$$

where $r_f = 0.05$ is the risk-free rate,

$$\mu = \begin{pmatrix} 0.1 \\ 0.2 \end{pmatrix}$$

$$\Sigma = SRS$$

$$R = \begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix}$$

$$S = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.3 \end{pmatrix}$$

We form the Lagrange function:

$$L(x, \lambda) = \frac{1}{2} w^T \Sigma w + \lambda(m - r_f - (\mu - r_f \mathbf{1})^T w)$$

We now solve for the first order condition by taking the derivative with respect to the <u>vector</u> w:

$$\frac{\partial L}{\partial w} = w^T \Sigma - \lambda(\mu - r_f \mathbf{1})^T = 0$$

Checking the second order condition, the Hessian of the objective function is still the covariance matrix, which is positive definite. Therefore, we have reached the optimal weight vector w^*:

$$w^* = \lambda \Sigma^{-1}(\mu - r_f \mathbf{1})$$

Substituting in the constraint equation, we get:

$$\lambda = \frac{m - r^*}{(\mu - r_f \mathbf{1})^T \Sigma^{-1} (\mu - r_f \mathbf{1})}$$

Finally, we get w^*:

$$w^* = \frac{(m - r_f) \Sigma^{-1} (\mu - r_f \mathbf{1})}{(\mu - r_f \mathbf{1})^T \Sigma^{-1} (\mu - r_f \mathbf{1})}$$

Now, to find the tangency portfolio, simply recall that the tangency portfolio is 100% invested in the risky assets, i.e. $\mathbf{1}^T w = 1$ to finally obtain:

$$w_t = \frac{\Sigma^{-1}(\mu - r_f \mathbf{1})}{B - Ar_f}$$

Plugging in the values,

$$w_t = \begin{pmatrix} 0\% \\ 100\% \end{pmatrix}$$

It turns out that in this case the tangency portfolio is asset B!!! Therefore

$$\begin{aligned} \mu_t &= \mu^T w_t = 20\% \\ \sigma_t &= \sqrt{w_t^T \Sigma w_t} = 30\% \\ \theta_t &= \frac{\mu_t - r_f}{\sigma_t} = 0.5 \end{aligned}$$

and therefore the equation of the CML is

$$\mu_P = 0.05 + 0.5 \times \sigma_P \qquad (5)$$