A comprehensive Analysis of Advanced Pricing Models for Collateralised Debt Obligations

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Abstract

The subject of this paper is the single tranche portfolio credit default swap or synthetic single tranche CDO, which has received a great deal of interest in recent years. Unlike single name CDS, tranche portfolio products depend on the joint default behavior of the underlying credits or in other words their default correlation.The Gaussian copula has emerged as a market standard for modeling the dependence structure and pricing CDOs. The introduction of credit indices has made it possible to compute a market implied correlation, which for the Gaussian copula results in a smile similar to the volatility smile for the Black and Scholes model in the equity options market. The smile is inconsistent with the model and causes problems for pricing bespoke CDOs. Therefore a lot of research has been devoted to develop extensions to the Gaussian copula and/or alternative pricing models that are better able to explain the correlation smile. In this thesis we will review and compare the performance of a number of advanced pricing models for collateralised debt obligations which were recently suggested, including the Base Correlation framework, the double t copula, the NIG copula and two extensions to the normal copula comprising random recovery rates and random factor loadings. The models are assessed in terms of how well they are able to replicate/explain the correlation skew in the index market. The performance of the models is tested over time for both the iTraxx and CDX 5 year index. Unlike previous research which focused on particular points in time we tested the models on a monthly basis between January 2005 and May 2006.

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Chapter 1

Introduction

Credit Derivatives emerged in the early 90s and have experienced tremendous growth over the last couple of years. The most common credit derivative is a credit default swap (CDS) which essentially is an insurance contract against credit risk losses. The two counterparty in a CDS contract are referred to as protection buyer and seller respectively. The protection seller agrees to make payment to the protection buyer following a credit event in an amount to cover credit losses net of recoveries. The protection buyer in return for the default protection agrees to pay a regular premium until the contract matures or a credit event occurs.
Depending on the number of underlying credits CDS can be classified into single Name CDS where the underlying exposure is a single obligor or reference name such as for example General Motors or Ford and portfolio credit derivatives such as nth to default basket swaps and collateralised debt obligations (CDOs).

Other credit derivatives include options on CDS including the index and index tranches, forward starting CDS and CDO squareds among others. Most recently hybrid products have also emerged, which include Constant Proportion Portfolio Insurance (CPPI) and Constant Proportion Debt Obligations (CPDO).

1.1 Single Tranche Portfolio CDS (Synthetic CDOs)

There are many types of different CDOs including cash flow arbitrage, balance sheet and collaterlised loan obligations or CLOs. They all have in common that they transfer the credit risk of a portfolio of assets, which is divided into tranches of different seniority. Credit Losses in the portfolio will be allocated in reverse sequential order starting with the equity or first loss tranche, followed by the mezzanine tranches and finally the senior and/or super senior tranches. Thereby tranches that rank lower in the capital structure provide credit enhancement to the senior tranches in the form of subordination.

The underlying portfolio in traditional CDOs is usually made up of cash assets, which tend to be relatively illiquid and can only be sourced by established asset managers. They are also complicated structures, which are costly and time consuming to execute. This has lead to the development of synthetic CDOs which use single name credit defaults swaps rather than cash bonds to build the reference portfolio. Originally synthetic CDOs like more traditional CDOs would issue the entire capital structure ranging from the first loss position up to the super senior. This did not require explicit correlation models, since the arranging bank was fully hedged and tranches like in traditional CDOs were priced by investors on a relative value basis. This however required to find investors for the equity, mezzanine and senior positions, which at the least is a long process and would often require the arranging bank to retain individual tranches on their balance sheet.

As the single name CDS market for corporate , financial institution and sovereign risk became more liquid[1] arrangers started to issue single tranche CDOs, employing dynamic delta hedging rather than buying the entire reference portfolio. This created a need for correlation models to price and hedge these correlation products dynamically. The following diagram shows the structure of a typical single tranche synthetic CDO.

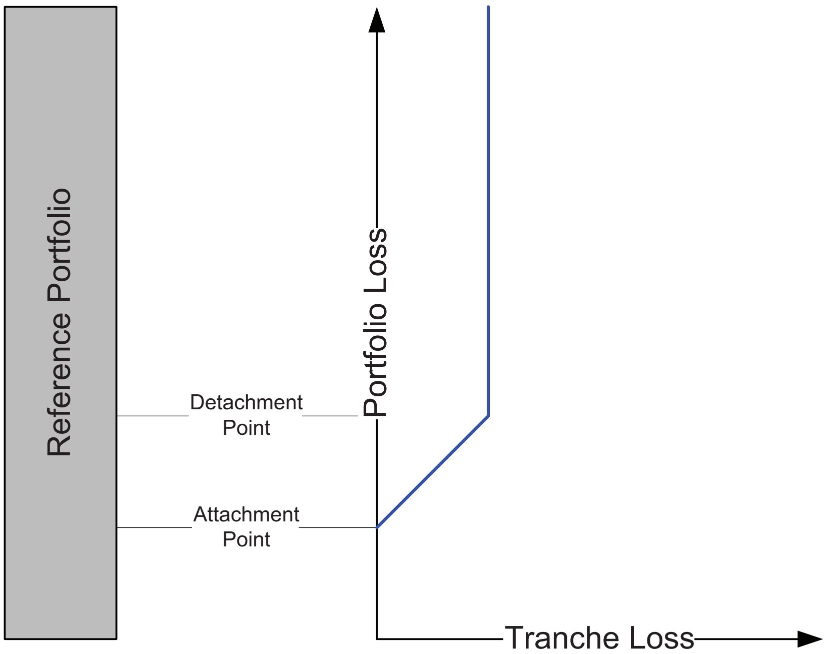


Figure 1.1: Structural Diagram for Portfolio CDS

The payout profile of a single tranche CDO is determined by the attachment point and detachment point of the tranche. The protection seller makes a payment if cumulative portfolio losses exceed the attachment point. The maximum exposure for the protection seller is capped at the detachment point. Unlike single name CDS where a default terminates the swap, portfolio CDS continue until either maturity or losses exceed the detachment point. Therefore the protection seller may have to make several payments over the life of the swap. In return for the protection payments the buyer agrees to pay a premium (usually fixed) on a regular basis on the remaining notional amount of the swap. Portfolio CDS apply the same definition for credit events (defaults) and cash settlement payments (losses) as the single name CDS.

[1]Less liquid Asset classes such as ABS, Leveraged Loans and Loans to small and medium enterprises are still being securitised using traditional cash or synthetic full capital structure CDOS.

iTraxx EUR IG		
	5y Spread(upfront)	10y Spread (upfront)
0-3%	500 (24.1%)	500 (51.5%)
3%-6%	68.6	553.4
6%-9%	20.8	131.2
9%-12%	8.9	61.4
12%-22%	4.2	23.7

Table 1.1: Mid Spread for iTraxx EUR IG and CDX NA IG on the 24th May 2006

CDX NA IG		
	5y Spread(upfront)	10y Spread (upfront)
0-3%	500 (34.5%)	500 (57.6%)
3%-7%	105.1	625.0
7%-10%	24.0	131.3
10%-15%	10.0	57.5
15%-30%	6.0	16.0

Table 1.2: Mid Spread for iTraxx EUR IG and CDX NA IG on the 24th May 2006

1.2 Credit Indices

Potential investors who are interest in synthetic CDOs can sell protection on bespoke portfolios offered by credit desks of all major investment banks or standardised index portfolios. Over the past three years several credit indices were created including the Dow Jones CDX NA IG (125 names), Dow Jones CDX HY(100 names), iTraxx EUR IG(125 names), iTraxx Asia (50 names) and iTraxx Japan (50 names). In addition to the index itself the market trades several tranches on each of these indices, which are offered with standard maturities of five, seven and ten years. The index portfolios are selected to include the most liquid names in the single named credit default swap market. All names in the index are equally weighted and the indices are 'rolled' every six months, whereby non eligible names are replaced.

For example, the iTraxx index family includes standard tranches with APs and DPs of 0-3%, 3-6%, 6-9%, 9-12% and 12-22%. The DJ CDX index market differs from the iTraxx family and the traded tranches have APs and DPs of 0-3%,3-7%, 7-10%, 10-15% and 15-30%. Under the terms and conditions of the tranche swaps the protection buyer pays a fixed spread on a quarterly basis. The notable exception are the equity tranches which pay an upfront amount (usually expressed as percentage of the tranche notional) in addition to a fixed spread of 500bps. This is in order to reduce the cash flow volatility to the protection seller caused by the timing of defaults.

Tables 1.1 and 1.2 shows the price quotes for the two major indices on the 24 May 2005.

Index tranches are also referred to as correlation products and provide the basis for correlation trading. The spread on the standard index tranches allows to infer market expectations on the joint default behavior of the underlying names in the index portfolio. The market convention is to quote prices in terms of base correlations (see section 3.1).

1.3 Review of Default Correlation Methodologies

Credit Risk Models for default co-dependence is a relatively new area in Quantitative finance and there are a multitude of modeling approaches. One approach is based on the structural form model which was initially developed by Merton in 1974 [11]. The structural form framework takes a micro economic view, whereby default occurs if the firm value drops below the value of its liabilities. The asset value process if often modeled as a geometric brownian motion, which gives closed form pricing formulas for the companies debt. Modeling co-dependence in this framework involves correlating the asset process for different firms. This was done for example by the widely used KMV model developed by Zeng and Zhang in 2001 [17]. As the number of obligors increases this approach rapidly becomes computationally intensive and as a result the models tend to be slow.

Another branch of research which was also developed initially to price risky debt is based on a compound poison process for the default arrival time. Such models are typically referred to as reduced from models. The hazard rate of default intensity is modeled as a stochastic process. Co-dependence between defaults is introduced by correlating the stochastic process of the default intensity/hazard rate. This approach was for example suggested by Duffie and Singleton in 1999 [3]. Similar to structural form models this approach is also very computationally intensive and slow.

A third approach which is the subject of this thesis models the co-dependence between defaults by using so-called copual functions. These models were specifically developed to model portfolio credit derivatives. The most popular model is the Gaussian copula which has become a market standard for pricing CDO tranches. This model was initially suggested by Li in 2000 [8]. A good survey of copula models can be found in Schonbucher (2003) [13] and Embrechts (2005) [10]. The advantage of copula models over structural or reduced form models is their fast semi-analytic implementations, which provide an efficient way to calibrate the models to market prices. However copulas are static models for the loss distribution and say nothing about the evolution of the loss distribution over time. Therefore copula models can not be used to price credit derivatives with optional features such as tranche options. In contrast the structural form and the intensity approach can also be used to price option structures.

Other approaches that were also suggested include stochastic correlation by Burtschell, Gregory and Laurent [16], dynamic copula processes by Totouom and Armstrong [2] and a model that is altogether different to the copula approach is the Intensity Gamma Model by Joshi and Stacey [9].

1.4 Outline and Research Proposal

The subject of this paper is the single tranche portfolio credit default swap or synthetic single tranche CDO, which has received a great deal of interest in recent years. Unlike single name CDS, tranche portfolio products depend on the joint default behavior of the underlying credits or in other words their default correlation.The Gaussian copula has emerged as a market standard for modeling the dependence structure and pricing CDOs. The introduction of credit indices has made it possible to compute a market implied correlation, which for the Gaussian copula results in a smile similar to the volatility smile for the Black and Scholes model in the equity options market. The smile is inconsistent with the model and causes problems for pricing bespoke CDOs. Therefore a lot of research has been devoted to develop alternative pricing models that are better able to explain the correlation smile. In this thesis we will review and compare the performance of a number of advanced pricing models for collateralised debt obligations which were recently suggested. The models are assessed in terms of how well they are able to replicate/explain the correlation skew in the index market. We will focus on copula model only.

The first chapter reviews the general approach for pricing credit default swaps and in particular portfolio CDS. Chapter two reviews the mathematical properties of a number of pricing models including the Base Correlation, the double t copula, the NIG copula and two extensions to the normal copula comprising random recovery rates and random factor loadings. In the final chapter we provide the results of the models and show how well they are able to match the correlation smile for the five year iTraxx EUR IG and the CDX NA IG indices. Unlike previous research which focused on particular points in time we tested the models on a monthly basis between January 2005 and May 2006.

Chapter 2

Pricing CDOs

This chapter outlines the general theory for pricing credit default swaps and in particular portfolio credit default swaps or CDOs.

2.1 Pricing and the Balance Equation

Under the general terms and conditions of a credit default swap the protection seller receives a regular spread payment from the protection buyer (the Fixed Leg). In the case of default of the underlying reference entity the protection seller has to make a compensation payment to the protection buyer in an amount equal to the resulting loss (the Loss Leg) (see 1.4). Here we consider only the most general terms. As we have seen in chapter one there are many different types of credit derivatives with numerous variations in their specific terms and conditions.

The price of such contracts is given as the expected present value difference between both swap legs. This is expressed in the so called balance equation. As always expectations for pricing are evaluated using the risk neutral measure. In the next chapter we will see how this is done for credit default swaps. From the perspective of the protection buyer the expected present value of the credit default swap is given by;

$$E(PV_{SWAP}) = E(PV_{Fixed}) - E(PV_{Loss}) \tag{2.1}$$

The terms of a CDS typically stipulate a fixed spread s for the term T of the swap to be paid on the remaining swap notional, which is given by the initial notional of the swap N less the amount of any loss payments up to that date. The payments are made on regular discreet payment dates $t_0 \leq t_j \leq T$ (for example quarterly). Assuming independence between interest rates and defaults simplifies the evaluation of the expection to computing the expected losses. The balance equation becomes:

$$PV_{SWAP} = \left(\sum_{j=1} D(t_0, t_j) s \Delta_{t_{j-1},t_j} (N - EL(t_j)) \right) - \left(\int_{t_0}^{T} D(t_0, s) dEL(s) \right) \tag{2.2}$$

Here Δ_{t_{j-1},t_j} is the day count fraction between payment dates, $D(t_0, t_j)$ is the discount factor and $EL(t_j)$ is the expected cumulative loss up to t_j.
The integral equation for the expected loss leg can be approximated by discreet sum as follows

$$\left(\int_{t_0}^{T} D(t_0,s)dEL(s)\right) \approx \left(\sum_{j=1} D(t_0,t_j)(EL(t_j) - EL(t_{j-1}))\right) \tag{2.3}$$

The approach so far is the same for all credit default swaps regardless of the underlying credit risk factors. Before moving on to CDOs we will briefly review the valuation of single name credit default swaps, which is important for determining risk neutral default probabilities which are needed to price CDO tranches.

2.2 Hazard Rates from Single Name Credit Default Swap Prices

Single name credit default swaps are the most common and most actively traded credit derivative. SCDS are typically traded with terms of 1,3,5,7 and 10 years. The bid offer spread varies depending on the underlying reference name and changes over time as some names become more liquid, especially when their credit quality changes. Under the term of such a swap the protection seller agrees to pay to the protection buyer the loss incurred following the default of the specified reference name. The default of the reference name terminates the contract and no further premium payments are made.
Assuming a constant recovery rate RR such a contract can be easily valued in the above framework. For a single name the expected loss at t_j is given by;

$$EL(t_j) = N(1 - RR)\mathbb{P}(\tau \leq t_j) \tag{2.4}$$

where τ is the default time for the reference name and $\mathbb{P}(\tau \leq t_j)$ is the probability that the name default prior to t_j. Substituting the expected loss in the balance equation results in the pricing formula for a single name credit default swap. The recovery rate for the premium leg is assumed to be zero, as default terminates the contract.

$$\begin{aligned} PV_{SWAP} \quad = \quad & \left(sN\sum_{j=1} D(t_0,t_j)\Delta_{t_{j-1},t_j}(1 - \mathbb{P}(\tau \leq t_j))\right) \\ & - \left(N(1-RR)\sum_{j=1} D(t_0,t_j)(\mathbb{P}(\tau \leq t_j) - \mathbb{P}(\tau \leq t_{j-1}))\right) \end{aligned} \tag{2.5}$$

The value of a swap at initiation is zero, which means neither party has to make a payment to enter the swap. The spread that makes the swap value zero is referred to as the fair market spread. As these are traded instruments the fair spread is given by the market and we can determine the market implied or risk neutral default probabilities $\mathbb{P}(\tau \leq t_j)$ that value the swap to zero by rearranging the balance equation.

In practice it is difficult to extract default probabilities directly from quoted SCDS spreads, as there is not a quoted spread for every payment period to back out a corresponding default rate. Therefore hazard rates are calibrated instead. The default time τ is assumed to have an exponential distribution with hazard rate function $\lambda(t)$. The cumulative probability of default up to time t is given by

$$\mathbb{P}(\tau \leq t) = 1 - e^{-\int \lambda(t)dt} \approx 1 - e^{-\sum \lambda(t)\Delta t} \tag{2.6}$$

The integral can be approximated by a discreet sum assuming a piecewise constant hazard rate function.

The conditional default probability between t and $t + \delta t$ is given by

$$\mathbb{P}(t \leq \tau \leq t + \delta t | \tau \geq t) = \frac{\mathbb{P}(\tau \geq t) - \mathbb{P}(\tau \geq t + \delta t)}{\mathbb{P}(\tau \geq t)} = -\frac{S(t + \delta t) - S(t)}{S(t)} \tag{2.7}$$

where $S(t)$ is the cumulative survival probability. The hazard rate gives the instantaneous default probability at time t, conditional on survival up to time t, normalised to one unit of time.

$$\lambda(t) = \lim_{\delta t \to 0} \frac{\mathbb{P}(\tau \geq t) - \mathbb{P}(\tau \geq t + \delta t)}{\delta t \mathbb{P}(\tau \geq t)} = -\frac{1}{S(t)} \frac{dS(t)}{dt} \tag{2.8}$$

Solving this ODE leads to equation (2.6).

Substituting this into the balance equation for a SCDS we get

$$\begin{aligned} PV_{SWAP} &= \left(sN \sum_{j=1} D(t_0, t_j) \Delta_{t_{j-1}, t_j} e^{-\sum_{k=1}^{j} \lambda_k \Delta_{t_{k-1}, t_k}} \right) \\ &\quad - \left(N(1 - RR) \sum_{j=1} D(t_0, t_j) e^{-\sum_{k=1}^{j-1} \lambda_k \Delta_{t_{k-1}, t_k}} (1 - e^{-\lambda_j \Delta_{t_{j-1}, t_j}}) \right) \end{aligned} \tag{2.9}$$

The hazard rate is assumed to be piecewise constant and can be bootstrapped from SCDS spread. The equation has to be solved numerically using for example Newton Raphson.

2.3 Pricing CDOs

Collateralised Debt Obligations reference debt portfolios that typically include corporates and may also include financial institutions or sovereigns. The iTraxx EUR Investment Grade index for example references a portfolio of 125 enitites. In chapter one we have introduced the general structure for CDO tranches (see 1.4). This section applies the general framework for pricing credit default swaps to CDO tranches. Recall that CDO tranches are defined by their attachment point K_A and detachment point K_D. The terms of a CDO tranche default swap require the protection seller to make a payment if the cumulative portfolio loss exceeds the attachment point. However the maximum possible payment is capped at the detachment point.

Let $L(t)$ be the cumulative loss in the reference portfolio due to individual credit events. Let n be the number for reference obligors in the CDO portfolio and τ_i the default time of the ith obligor. The cumulative portfolio loss is given by

$$L(t) = \sum_{i=1}^{n} N_i(1 - RR_i) 1_{(\tau_i \leq t)} \tag{2.10}$$

where $1_{(\tau_i \leq t)}$ is the indicator function which takes the value 1 if $\tau_i \leq t$ and zero otherwise. The loss to a CDO tranche $LT(t)$ is given by

$$LT(t) = \begin{cases} 0 & \text{if } L(t) \leq K_A; \\ L(t) - K_A & \text{if } K_A \leq L(t) \leq K_D; \\ K_D - K_A & \text{if } L(t) \geq K_D. \end{cases}$$

This can also be written in a payoff function as follows

$$LT(t) = min(K_D - K_A, (L(t) - K_A)^+) = (L(t) - K_A)^+ - (L(t) - K_D)^+ \tag{2.11}$$

The tranche payoff therefore is equivalent to a call option spread on the realised portfolio loss. The expected tranche loss $ELT_{A,D}(t)$ needed in the balance equation for the CDO tranches is given by

$$\begin{aligned} ELT(t) &= \int_0^{L_{max}} (L(t) - K_A)^+ - (L(t) - K_D)^+ dF(L(t)) \\ &= \int_{K_A}^{L_{max}} (L(t) - K_A) dF(L(t)) - \int_{K_D}^{L_{max}} (L(t) - K_D) dF(L(t)) \end{aligned} \tag{2.12}$$

The Balance Equation for valuing the CDO tranche swap is

$$\begin{aligned} PV_{SWAP} &= s\left(\sum_{j=1} D(t_0, t_j)\Delta_{t_{j-1},t_j}(K_D - K_A - ELT(t_j))\right) \\ &\quad - \left(\sum_{j=1} D(t_0, t_j)(ELT(t_j) - ELT(t_{j-1}))\right) \end{aligned} \tag{2.13}$$

In order to evaluate equation 2.12 for the expected tranche loss and hence to price a CDO tranche we need the probability distribution the portfolio loss $dF(L(t))$ or portfolio loss distribution as shown in figure 3.3.1.
While it is straight forward to compute the expected loss for the portfolio it is in no way a trivial task to determine the expected loss of a CDO tranche. The main problem is the dependence between the default times of individual obligors in the reference portfolio, which determines the shape of the portfolio loss distribution. The next section reviews copula functions, which have become popular for modelling the portfolio loss distribution.

2.4 Copulas Functions and the Portfolio Loss Distribution

Copula functions were popularised in credit risk modeling by Li in 2000 [8]. A copula is a function which maps individual marginal distributions into a joint distribution $[0, 1]^n \rightarrow [0, 1]$.

The probably most important property of copulae is given by Sklar's theorem, which states that for any n-dimensional distribution function $H(x_1, \dots, x_n)$ with marginal distribution functions $F_1(x_1), \dots, F_n(x_n)$ there exists an n-dimensional copula function C, such that

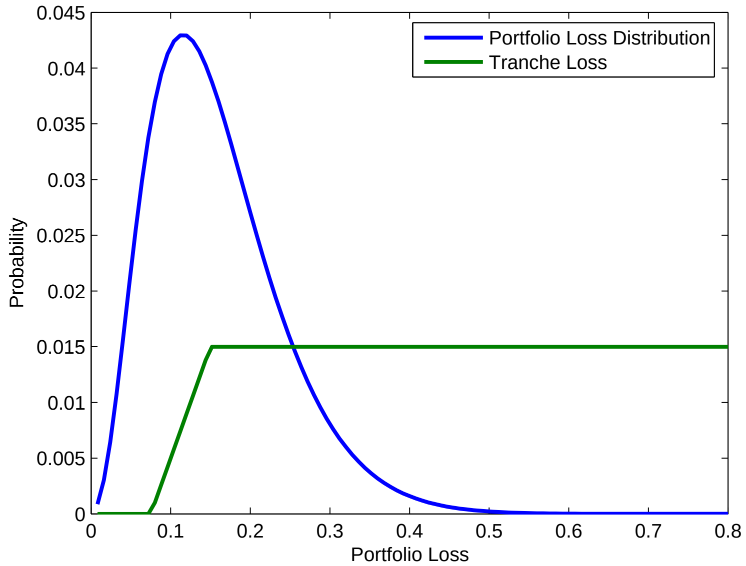


Figure 2.1: Portfolio Loss Distribution and Tranche Payoff Function.

$$H(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n)) \tag{2.14}$$

The joint distribution function H is depends on the choice of C but is independent of the marginal distribution functions $F_1(x_1), \ldots, F_n(x_n)$.

The simplest copula function is the product copula, which is given by

$$H(x_1, \ldots, x_n) = F_1(x_1)*, \ldots, *F_n(x_n) \tag{2.15}$$

In the context of credit risk modelling we are interested in the joint distribution of default times.

$$\mathbb{P}(\tau_1 \leq t, \ldots, \tau_n \leq t) = C(\mathbb{P}(\tau_1 \leq t), \ldots, \mathbb{P}(\tau_n \leq t)) \tag{2.16}$$

The individual marginal distributions of default times are derived from the single name credit default swaps. The fact that the joint distribution only depends on the copula and is independent of the marginal distribution functions makes is possible to separately calibrate the marginals and the dependency structure. For credit risk modelling this is important in order to match both the single name default swap prices as well as the index tranche prices for example.
In the following sections we will briefly review a number of popular copula functions used in credit risk modelling and their properties.

2.4.1 Gaussian Copula

The most popular copula function is the Gaussian normal copula. Here the marginal distributions are all normal Gaussian and the joint distribution is multi variate normal with correlation matrix Σ.

$$C_\Sigma(u_1, \ldots, u_n) = \frac{1}{(2\pi)^{\frac{n}{2}}\sqrt{det\Sigma}} \int_{-\infty}^{\Phi^{-1}(u_1)} \cdots \int_{-\infty}^{\Phi^{-1}(u_n)} exp\left(-0.5\overline{x}\Sigma^{-1}\overline{x}\right) dx_1, \ldots, dx_n \tag{2.17}$$

For credit risk modelling $u_i = \mathbb{P}(\tau_i \leq t)$. The Gaussian Copula function has become a market standard in credit risk modelling, because of its well defined properties and ease of implementation. The Gaussian copula is used both for pricing as well as credit risk management. The integral function can be evaluated by Monte Carlo integration, which is very efficient for higher dimensional problems.

Alternatively the Gaussian copula also has a factor representation, which gives very fast semi analytic solutions. In the one factor case the default times are modelled from a Gaussian vector $(X_1, \ldots, X_n)$.

$$X_i = \beta_i V + \sqrt{1-\beta_i^2}\epsilon_i \tag{2.18}$$

Here V and ϵ_i are independent Gaussian random variables. The factor representation only allows to model a reduced correlation structure which is defined by the exposure vector $(\beta_1, \ldots, \beta_n)$, with $-1 \leq \beta_i \leq 1$.

A more diverse correlations structure would require additional factors. The probability of default is given by $\mathbb{P}(\tau_i \leq t) = \Phi(X_i)$ and the default times are given by $\tau_i = F_i^{-1}(\Phi(X_i))$. Here F is the mapping function between default times and default rates. Given a realisation of the factor V one can compute the conditional default probabilities from the unconditional probabilities as follows

$$\mathbb{P}(\tau_i \leq t|V) = \mathbb{P}(X_i \leq \Phi^{-1}\left[\mathbb{P}(\tau_i \leq t)\right]|V) = \Phi\left(\frac{\Phi^{-1}\left[\mathbb{P}(\tau_i \leq t)\right] - \beta_i V}{\sqrt{1-\beta_i^2}}\right) \tag{2.19}$$

Conditional on the realisation of the factor V the default times for the individual obligors in the reference portfolio are independent. This property allows provides a very fast semi-analytic solution for the portfolio loss distribution as we will see in section 2.4.4. The Gaussian copula is often criticised to lack lower tail dependence, which leads to low probabilities int he tail of the portfolio loss distribution. This particular problem is expressed in the correlation smile in the credit market.

2.4.2 Student t Copula

The Student t Copula was suggested as possible alternative. It contains an additional parameter ν, which is referred to as the degrees of freedom, which allows to calibrate the lower tail dependence in the t copula.

The probability density function of a Student t distribution is given by

$$f_\nu(x) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\Gamma\left(\frac{\nu}{2}\right)}\left(1+\frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}} \tag{2.20}$$

where $\Gamma(x)$ is the Gamma Function. As $\nu \rightarrow \infty$ the Student t distribution converges to the normal distribution.

The multivariate Student t distribution with correlation matrix Σ is given by

$$C_{\nu,\Sigma}(u_1,\ldots,u_n) = \frac{\Gamma\left(\frac{\nu+n}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{|\Sigma|\left(\nu\pi\right)^n}} \int_{-\infty}^{t_\nu^{-1}(u_1)} \cdots \int_{-\infty}^{t_\nu^{-1}(u_n)} \left(1 + \frac{x\Sigma^{-1}x^T}{\nu}\right)^{-\frac{\nu+2}{2}} dx_1,\ldots,dx_n \quad (2.21)$$

The student t copula also has a factor representation. In the symmetric case the default times for the student t copula are determined by a vector t distributed random variables $(Y_1,\ldots,Y_n)$ with ν degrees of freedom.

$$Y_i = \sqrt{\frac{\nu}{W}}\left(\beta_i V + \sqrt{1-\beta_i^2}\epsilon_i\right) \quad (2.22)$$

Here V and ϵ_i are independent normally distributed random variables and W is a χ^2_ν distributed random variable independent of V and ϵ_i. Thus the simplest student t factor copula is already a two factor model. The covariance between Y_i and Y_j is given by $\frac{\nu}{\nu-2}\beta_i\beta_j$. The product of the factor exposures $\beta_i = \beta_j = 0$ does not represent independence in the case of the student t copula.

For given a realisation of the factor V and W one can calculate the conditional default probabilities from the unconditional probabilities as follows

$$\mathbb{P}(\tau_i \le t|V,W) = \mathbb{P}(X_i \le \Phi^{-1}\left[\mathbb{P}(\tau_i \le t)\right]|V,W) = \Phi\left(\frac{\sqrt{\frac{W}{\nu}}\Phi^{-1}\left[\mathbb{P}(\tau_i \le t)\right] - \beta_i V}{\sqrt{1-\beta_i^2}}\right) \quad (2.23)$$

In the case of the student t copula defaults are independent conditional on the realisation of the random variables V and W.

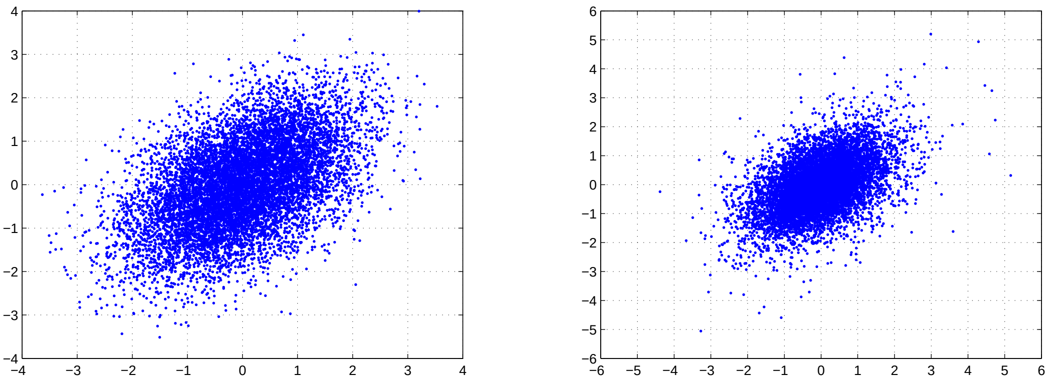


Figure 2.2: Bivariate Normal with 50% Corr and Bivariate Student t with 50% Corr and 10 dF

2.4.3 Clayton Copula

The Clayton copula is part of the family of Archimedean copulae, which are characterised by a common generator function $\varphi(u)$ and its inverse $\varphi^{-1}(u)$. For $u \in [0,1]$ the generator $\varphi(u)$ is a mapping function $[0,1] \to [0,\infty]$.

The generator for the Clayton copula is $\varphi(u) = u^{-\theta} - 1$ and its inverse is $\varphi^{-1}(u) = (1+u)^{-1/\theta}$.

$$C_\theta(u_1, \ldots, u_n) = \varphi^{-1}(\varphi(u_1), \ldots, \varphi(u_n)) = (u_1^{-\theta}, \ldots, u_n^{-\theta} - n + 1)^{-1/\theta} \tag{2.24}$$

Note that the inverse of the generator is equal to the Laplace transform of the Gamma distribution with shape parameter $1/\theta$ and probability density function $f(x) = \frac{1}{\Gamma(1/\theta)} e^{-x} x^{(1-\theta)/\theta}$.

$$\psi(u) = \int_0^\infty f(x) e^{-ux} dx = (1+u)^{-1/\theta} \tag{2.25}$$

For the factor representation of the Clayton copula we introduce a positive random variable V, which is called a frailty, following a standard Gamma distribution with shape parameter $1/\theta$ where $\theta > 0$. Next we introduce a vector of random variables $(Y_1, \ldots, Y_n)$.

$$Y_i = \psi\left(-\frac{lnU_i}{V}\right) \tag{2.26}$$

where $(U_1, \ldots, U_n)$ are independent uniform random variables also independent of V.
The conditional default probabilities for the Clayton copula are expressed as:

$$\mathbb{P}(\tau_i \leq t|V) = exp(V(1 - \mathbb{P}\left[(\tau_i \leq t)\right]^{-\theta})) \tag{2.27}$$

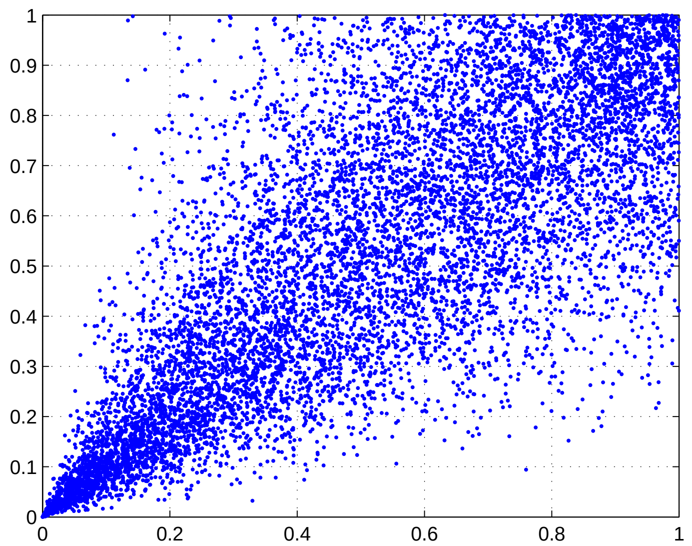


Figure 2.3: Bivariate Clayton Copula with Shape Parameter of one third

2.4.4 Loss Distribution via Semi Analytic Recursion Method

One way to implement copula functions is monte carlo integration. The approach would be to simulate default times directly and compute the expected tranche losses for each simulation trial. This is generally necessary for credit derivatives for which the payoff depends on the order of defaults in the underlying reference portfolio (for example CDO squared). Monte carlo simulation suffers from slow convergence especially when for high dimensionality. A typical CDO would include around 100 names and for reasonable convergence especially for senior tranches that depend on the tail of the loss distribution a large number (typically several hundred thousand) simulations would be required. However the valuation of plain vanilla CDO tranches only requires knowledge of the expected cumulative tranche loss on each payment date and not the identity of the defaulted names. The expected tranche loss in turn depends on the marginal loss. One approximation, which was suggested by Shelton [14] assumed that the conditional loss distribution is given by the normal distribution. Given the assumption of normality one can derive a closed form solution for the loss distribution. Another approximation is the large homogenous portfolio approximation suggested initially by Vasicek [15]. This approach has been adopted by many risk management departments and we will review it in the next chapter.

The approach we will use here was developed by Anderson, Sidenius and Basu [7] and provides a fast semi analytic method of computing the marginal loss distribution from the conditional default rates as given by the copula functions.

In general the loss distribution of a sum of independent random variables is given by the convolution of the loss distribution for each of the individual random variables. Assuming fixed recovery rates the loss distribution for each of the reference credit in the CDO portfolio is a two point distribution. The convolution can be performed using fourier transformation and the characteristic function of the loss distribution or alternatively using probability generating functions. However the convolution may also be calculated in a simple and very illustrative recursion approach.

First we introduce a simplified notation. Let $p_t^{i|\Omega} = \mathbb{P}(\tau_i \leq t|\Omega)$ be the risk neutral default probability for the ith reference entity conditional on the realisations of all systematic factors Ω. For the Gaussian Copula and the Clayton copula for example $\Omega = V$. In the case of the student t copula Ω would include both V and W. Let $\mathbb{P}(L^t = K|\Omega)$ be the conditional probability that the portfolio loss L^t at time t is equal to k (the conditional portfolio loss distribution). We introduce an arbitrary loss united,u, such that the loss given default l_i of each reference credit in the CDO portfolio can be well approximated by integer multiples of u $l_i = mlt_i^u u$. Let L_n be the portfolio loss as measured in loss units in the subportfolio consisting of the first n obligors. Since defaults are independent conditioned on Ω it is possible to compute the conditional marginal loss distribution at time t by the following simple recursion.

$$\mathbb{P}(L_{n+1}^t = K|\Omega) = p_t^{i|\Omega}\mathbb{P}(L_n^t = K - mlt_i^u|\Omega) + (1 - p_t^{i|\Omega})\mathbb{P}(L_n^t = K|\Omega) \tag{2.28}$$

The conditional portfolio loss distribution can be computed from the boundary case of the empty portfolio for which $\mathbb{P}(L_0^t = K|\Omega) = 1_{K=0}$. The recursive approach also allows a fast calculation of individual name sensitivities. For example to compute the dollar value change in the tranche PV for a one basis point increase in the spread of one of the underlying credits, it is not necessary to rerun the full pricing model with the risk neutral default probability for the bumped spread. One can simply 'remove' the obligor using the above recursive relationship and add it again with the bumped default probability.

The unconditional loss distribution can be obtained by numerically integrating across the common factor/factors as follows

$$\mathbb{P}(L_n^t = K) = \int_\Omega \mathbb{P}(L_n^t = K|\Omega)f(\Omega)d\Omega \tag{2.29}$$

When choosing the loss step u there is a general trade off between accuracy and runtime. In the case of the credit indices the reference notional of each of the credits is 8 million. Assuming fixed recovery rates of 40% (which is a common assumptions for the credit indices) the loss given default for each of the reference credits is 4.8 million. Therefore we could choose u equal to 4.8 million and the integer multiple for each credit in the reference portfolio would be one. Generally u should be choosen as large as possible in oder to reduce the number of required recursion steps. However in if either the notionals or the reocvery rates between the reference credits in the CDO portfolio are very heterogenous u can become very small, increasing the number of recursion steps.
The following chart shows the conditional loss distribution derived with Monte Carlo vs. the distribution derived using the above recursive approach. The portfolio consisted of 100 obligors with the same notional and recovery rate but varying default probabilities that ranged between 50bps and 5%.

2.4.5 Loss Distribution and the Large Homogeneous Portfolio Approximation

For large portfolios there exists an approximation that allows an analytical solution for the loss distribution. This model was proposed by Vasicek [15] and is known as the large homogeneous portfolio approximation or LHP. The approximation is based on the assumption that the portfolio consists of a very large number $n \to \infty$ of obligors, which are identical in size, recovery rates and unconditional default probabilities. Using this assumptions the law of large numbers causes the conditional loss distribution to be exactly the conditional mean given by $\mathbb{P}(L^t_{n\to\infty}|\Omega) = \mathbb{P}(\tau \leq t|\Omega)(1 - RR)N$. The law of large numbers hold that the sum of n independent random variables approaches the mean as n approaches to infinity. The LHP provides a surprisingly good approximation of the loss distribution even for portfolios of 125 obligors such as the index. Therefore many dealers still use the LHP instead of the exact recursion.

As before the unconditional loss distribution is obtained by integrating across the common factors as shown in equation 2.29.

The LHP for the Gaussian one factor copula can be derived as follows

$$\mathbb{P}(L^t \leq K) = \mathbb{E}\left[1_{p_t^V(1-RR)N \leq K}\right] \tag{2.30}$$

Since the conditional default probabilities for the Gaussian copula are given by

$$p_t^V = \Phi\left(\frac{\Phi^{-1}\left[\mathbb{P}(\tau \leq t)\right] - \beta V}{\sqrt{1-\beta^2}}\right) \tag{2.31}$$

Substitution and rearranging gives

$$\begin{aligned}\mathbb{P}(L^t \leq K) &= \int_{-\infty}^{\infty} 1_{V \leq A}\phi(V)dV \\ &= \int_{-\infty}^{A} \phi(V)dV \\ &= \Phi(A).\end{aligned} \tag{2.32}$$

where A is given by

$$A = \frac{1}{\beta}\left(\sqrt{1-\beta^2}\Phi^{-1}\left(\frac{K}{(1-RR)N}\right) - \Phi^{-1}\left[\mathbb{P}(\tau \leq t)\right]\right) \tag{2.33}$$

Similarly one can derive an LHP expression for the student t and the Clayton one factor copulas.

The LHP for the Gaussian copula also allows to derive closed form expressions for Expected tranche payoffs. The following derivation was provided by O'Kane and Livesey [12] As we have seen in section 2.3 the tranche payoff can be decomposed as a call option spread on the cumulative portfolio loss. This can also be written as follows

$$\begin{aligned}LT(t) &= min(K_D - K_A, (L(t) - K_A)^+) \\ &= min(L(t), K_D) - min(L(t), K_A).\end{aligned} \tag{2.34}$$

For pricing CDO tranches we need to compute the expected tranche payoff ELT(t). taking expectations we need to evaluate $\mathbb{E}(min(L(t), K))$, which can be rewritten as follows

$$\begin{aligned}\mathbb{E}(min(L(t), K) &= \mathbb{E}\left(K1_{L(t)>K} + L(t)1_{L(t)<K}\right) \\ &= K\Phi(-A) + \mathbb{E}\left(L(t)1_{L(t)<K}\right).\end{aligned} \tag{2.35}$$

The second term can be evaluated using equation 3.31 and iterated expectations as follows

$$\begin{aligned}\mathbb{E}(L(t)1_{L(t)<K}) &= \mathbb{E}\left(\mathbb{E}\left(L(t)1_{L(t)<K}|V\right)\right) \\ &= \mathbb{E}\left(\mathbb{E}\left(p_t^V(1-RR)N1_{p_t^V(1-RR)N\leq K}|V\right)\right) \\ &= (1-RR)N\mathbb{E}\left(\mathbb{E}\left(p_t^V 1_{V\leq A}|V\right)\right) \\ &= (1-RR)N\int_{\infty}^{A}\Phi\left(\frac{\Phi^{-1}\left[\mathbb{P}(\tau \leq t)\right] - \beta V}{\sqrt{1-\beta^2}}\right)\phi(V)dV \\ &= (1-RR)N\Phi_2\left(\Phi^{-1}\left[\mathbb{P}(\tau \leq t)\right], A, \beta\right).\end{aligned} \tag{2.36}$$

Hence the expected tranche loss is given by

$$\begin{aligned}ELT(t) &= \mathbb{E}\left(min(L(t), K_D)\right) - \mathbb{E}\left(min(L(t), K_A)\right) \\ &= K_D\Phi(-A_D) + (1-RR)N\Phi_2\left(\Phi^{-1}\left[\mathbb{P}(\tau \leq t)\right], A_D, \beta\right) \\ &\quad -K_A\Phi(-A_A) + (1-RR)N\Phi_2\left(\Phi^{-1}\left[\mathbb{P}(\tau \leq t)\right], A_A, \beta\right).\end{aligned} \tag{2.37}$$

Chapter 3

Modelling the Correlation Smile

The Gaussian Copula has become a market standard for modelling the dependence structure in credit derivatives. Before the introduction of credit indices such as the iTraxx and CDX the market generally applied historical correlations, which typically were derived from equity correlations. KMV for example marketed its asset correlation, which were computed from equity correlation using the merton structural form framework.

The introduction of credit indices and in particular active trading in standardised index tranches has made it possible to compute market implied correlations, by solving for the correlation input in the one factor Gaussian model for which the model price is equal to the tranche price. The resulting implied or as it is also called compound correlation turned out to be different for each tranche in the capital structure. Similar to the smile in the implied option volatility the credit market with the Gaussian copula also exhibited what is referred to as a 'correlation smile'. The following chart shows the correlation smile for the iTraxx EUR Investment Grade Index on the (DATE).

The correlation smile would suggests that the one factor flat gaussian copula is not a good model for the dependence structure between default times of different credits. If it was the correlation would be the same for all tranches of the same index. Nevertheless the market continued to use the one factor gaussian copula to price index tranches and other bespoke CDOs. However correlation trading required a model that can match the index tranche prices, which has lead to the development of the base correlation framework and a lot of research has been carried out to find alternative models or variations to the normal copula that produce a better fit and would allow the pricing of bespoke CDOs. In fact the Student t copula and the Clayton copula outlined above are two of the earlier models that were suggested in order to provide a better fit to for the correlation smile.

In this chapter we review the theory for a number of recently suggested models. The first section will review the base correlation methodology. Section two to four review a number of popular correlation models including the Double t copula, the NIG copula and an extension to the normal one factor copula in the form of random factor loadings. The focus of this thesis is on copula models only and we do not cover other approaches such as intensity models or structural form models.

3.1 Base Correlation

The simple way to derive the market-implied correlation is to find the correlation level that reproduces the price of a standard index tranches using the standard pricing model for single tier CDOs. This approach is referred to as compound correlation and is illustrated in the following chart, which shows the price of the equity, mezzanine and senior tranche as function of asset correlation.

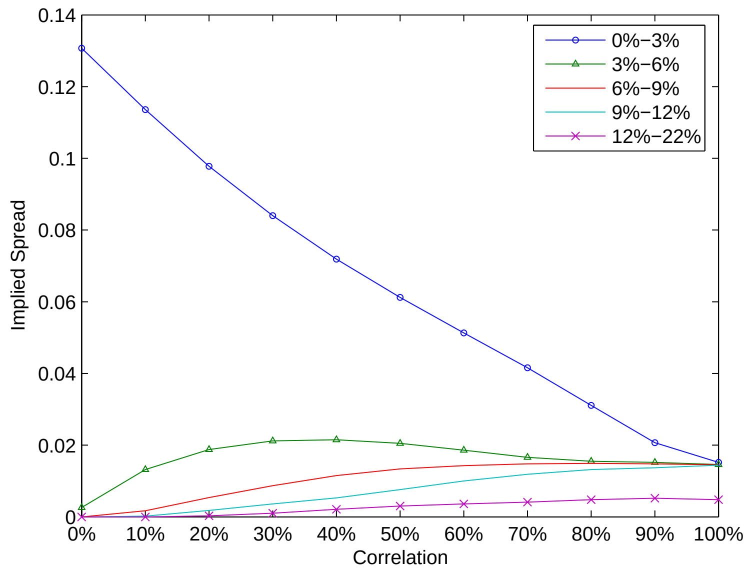


Figure 3.1: Implied CDS spread for different levels of Correlation.

The equity tranche is long correlation and the fair spread decreases as correlation increases. The opposite is true for the senior tranche, which are long correlation. The price-correlation relationship for the mezzanine tranche highlights one of the major drawbacks of compound correlation as a measure of market-implied correlation. The price of the mezzanine tranche increases at first with increasing correlation but decreases for high levels of asset correlation. As a result there are often two correlation levels that reproduce the market price in the standard copula model, with inherently very different risk profiles.

Compound correlation is not constant across the capital structure, but exhibits a smile like shape. The smile raises the problem of how to interpolate compound correlation for bespoke index tranches with non standard attachment points. For example a tranche with an AP of 0% and a DP of 6% comprises the equity and mezz tranche and should therefore have a PV that is the sum of the PVs of the two underlying tranches[1]. A simple interpolation of the compound correlation does not reproduce the PV as expected. Compound correlation is therefore problematic if used to value bespoke index tranches.

[1] Assuming both have the same spread

This has led to the development of the base correlation approach. The core idea in this approach is that each tranche can be valued as the difference between the value of two base (equity) tranches. For example the value of a long position in the 3% to 6% tranche is the difference between the value of long position in the 0% to 6% equity tranche and a short position in the 0% to 3% equity tranche.

$$PV(K_A, K_D, \rho_{A,D}) = PV(K_0, K_D, \rho_{0,D}) - PV(K_0, K_A, \rho_{0,A}) \tag{3.1}$$

The base correlations are the flat correlation values for the Gaussian copula model that correctly reprice the base (equity) tranches. For the 0%-3% equity tranche the base correlation is the same as the compound correlation. Equity tranche values are monotonic functions of correlation in the Gaussian copula model, which means each price has only one corresponding correlation value. This prevents the problem of multiple solutions encountered with compound correlations.
To find the base correlations for the mezzanine and senior detachment points we use a bootstrapping approach as follows;

1. Find the 0-3% implied correlation using the normal copula model.
2. Find the PV of the 0-3% tranche using the 3-6% premium and the 0%-3% base correlation calculated in step 1.
3. Find the 0-6% implied correlation such that PV is equal to the PV computed in step 2. The PV of the 3-6% tranche with the 3-6% fair spread is zero.
4. Find the PV of the 0-6% tranche using the 6-9% premium and the 0%-6% base correlation calculated in step 3.
5. Find the 0-9% implied correlation such that the PV is equal to the PV computed in step 4.
6. Repeat the procedure for the remaining tranches.

The follwing charts show the correlation smile and the base correlation skew for the iTraxx EUR.

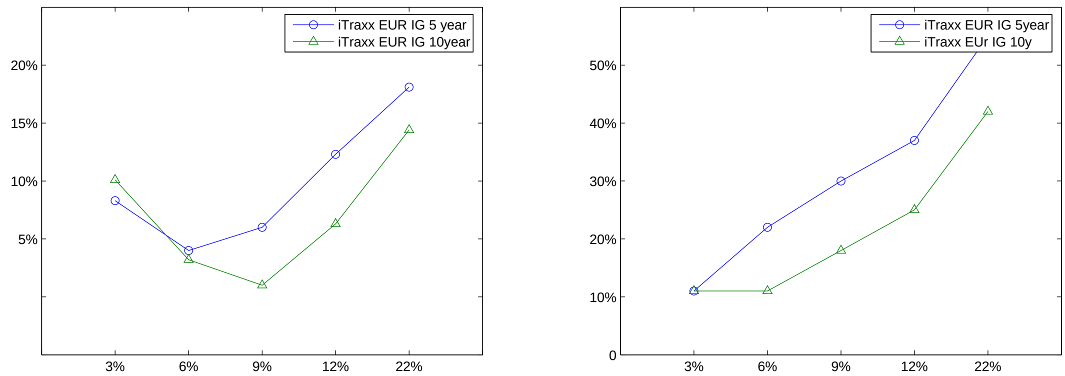


Figure 3.2: Implied Correlation and Base Correlation for the iTraxx EUR IG on the 24 May 2006

Unlike the compound correlation, which showed a smile, base correlation results in an upward sloping skew, whereby the values typically increase for higher detachment points.

The concept of base correlation allows to correctly reprice all index tranches and to some extend bespoke tranches. The base correlation approach can be compared to local volatility functions for equity options, which allow to fit the market but are inconsistent with the original model and do not explain the existence of a smile.

3.2 Double t Copula

The Double t Copula was suggested by Hull and White [4]. The default times are determined by a vector of random variables $(Y_1, \ldots, Y_n)$, given by.

$$Y_i = \beta_i \left(\frac{\nu - 2}{\nu}\right)^{1/2} V + \sqrt{1 - \beta_i^2} \left(\frac{\overline{\nu} - 2}{\overline{\nu}}\right)^{1/2} \epsilon_i \tag{3.2}$$

Here V and ϵ_i are independent student t distributed random variables with ν and $\overline{\nu}$ degrees of freedom. The student t distribution is not stable under convolution, which means the Y_i do not follow a student t distribution. Let $H(Y_i)$ be the cumulative distribution function of Y_i. For the double t distribution H has to be found numerically. Another difference between the double t distribution and the Student t distribution is that the default times are independent for $\beta_i = 0$.

For given a realisation of the factor V the conditional default probabilities are expressed as:

$$\mathbb{P}(\tau_i \leq t|V) = \mathbb{P}(Y_i \leq H^{-1}\left[\mathbb{P}(\tau_i \leq t)\right]|V) = t_{\overline{\nu}} \left(\left(\frac{\overline{\nu}}{\overline{\nu} - 2}\right)^{1/2} \frac{H^{-1}\left[\mathbb{P}(\tau_i \leq t)\right] - \beta_i \left(\frac{\nu - 2}{\nu}\right)^{1/2} V}{\sqrt{1 - \beta_i^2}} \right) \tag{3.3}$$

Conditional on the factor V defaults are again independent and we can use the recursive algorthim (see 2.4.4) to compute the portfolio loss distribution. The double t copula is a one factor model unlike its close relative the student t copula which requires two factors. Therefore the numerical integration for deriving the portfolio loss distribution is significantly faster for the double t copula. The double t copula has three parameters $\nu, \overline{\nu}$andβ (assuming flat correlation or $\beta_i = \beta$), which can be used to fit the model to the market prices. $\nu, \overline{\nu}$ have to be integer values greater than two.

3.3 Double NIG Copula

The double NIG copula model was suggested by Kalemanova, Schmidt and Werner [5]. The vector of random variables $(Y_1, \ldots, Y_n)$ is given by

$$Y_i = \beta_i V + \sqrt{1 - \beta_i^2} \epsilon_i \tag{3.4}$$

Which looks identical to the Normal Gaussian representation, except that V and ϵ_i are independent normal inverse Gaussian random variables with the following parameters;

$$\begin{aligned} V &\sim NIG\left(\alpha, \lambda, -\tfrac{\alpha\lambda}{\sqrt{\alpha^2 - \lambda^2}}, \alpha\right) \\ \epsilon_i &\sim NIG\left(\tfrac{\sqrt{1-\beta_i^2}}{\beta_i}\alpha, \tfrac{\sqrt{1-\beta_i^2}}{\beta_i}\lambda, -\tfrac{\sqrt{1-\beta_i^2}}{\beta_i}\tfrac{\alpha\lambda}{\sqrt{\alpha^2-\lambda^2}}, \tfrac{\sqrt{1-\beta_i^2}}{\beta_i}\alpha\right) \end{aligned}$$

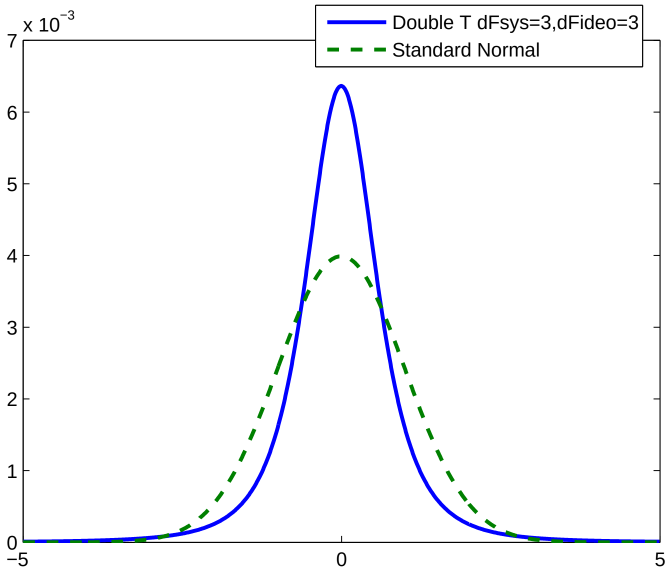


Figure 3.3: Double t Distribution versus Standard Normal Same mean and variance.

The parameters were chosen such that we get zero expected value and unit variance for the distribution of Y_i. The distribution will be skewed for non zero λ.

In the next section we will briefly review the main properties of the NIG distribution.

3.3.1 Definition and Properties of the NIG Distribution

The normal inverse Gaussian distribution is a mixture of normal and inverse Gaussian distributions.

A non-negative random variable Y has an inverse Gaussian distribution with parameters α and λ if its density function is of the form:

$$f_{IG}(y;\alpha,\lambda) = \begin{cases} \frac{\alpha}{\sqrt{2\pi\lambda}} y^{-3/2} exp\left(-\frac{(\alpha-\lambda y)^2}{2\lambda y}\right) & \text{if } y > 0; \\ 0 & \text{if } y \leq 0. \end{cases}$$

A random variable X follows a Normal Inverse Gaussian (NIG) distribution with parameters α,λ,μ and δ if:

$$\begin{aligned} X|Y = y &\sim N(\mu + \lambda y, y) \\ Y &\sim IG(\delta\gamma, \gamma^2) \text{ with } \gamma := \sqrt{\alpha^2 - \lambda^2} \end{aligned}$$

with parameters satisfying the following conditions: $0 \leq |\lambda| < \alpha$ and $\delta > 0$. We write $X \sim NIG(\alpha, \lambda, \mu, \delta)$ and denote the density and distribution function by $f_{NIG}(x;, \alpha, \lambda, \mu, \delta)$ and $F_{NIG}(x;, \alpha, \lambda, \mu, \delta)$. The density of a random variable $X \sim NIG(\alpha, \lambda, \mu, \delta)$ is:

$$f_{NIG}(x, \alpha, \lambda, \mu, \delta) = \frac{\delta\alpha exp(\delta\gamma + \lambda(x-\mu))}{\pi\sqrt{\delta^2 + (x-\mu)^2}} K_1\left(\alpha\sqrt{\delta^2 + (x-\mu)^2}\right), \tag{3.5}$$

where $K_1(\omega) := \frac{1}{2}\int_0^\infty exp\left(-0.5\omega(t+t^{-1})\right)dt$ is the modified Bessel function of the third kind. The density depends on four parameters:

- $\alpha > 0$ determines the shape,
- λ with $0 \leq |\lambda| < \alpha$ the skewness,
- μ the location, and
- $\delta >$ is a scaling parameter.

The following chart shows the density of the NIG distribution for different parameter values.

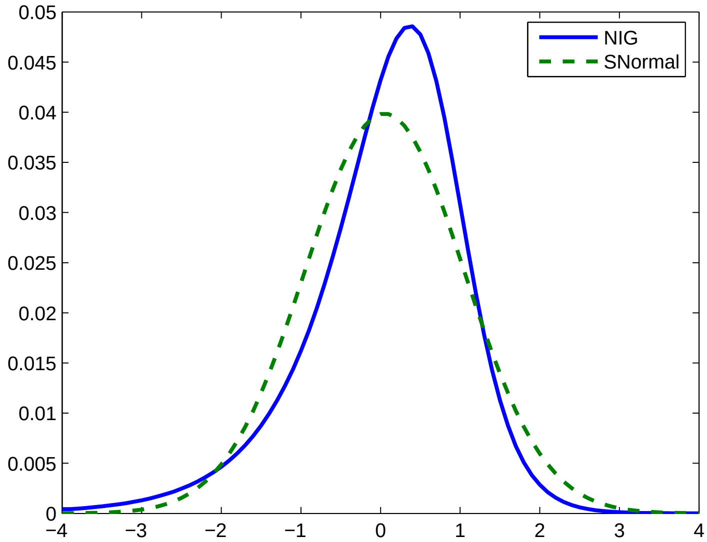


Figure 3.4: NIG versus Normal Distribution. Same mean and variance.

Mean and Variance
The mean and the variance of the NIG distribution $NIG(x, \alpha, \lambda, \mu, \delta)$are given by:

$$\mathbb{E}[X] = \mu + \delta\frac{\lambda}{\sqrt{\alpha^2-\lambda^2}} \qquad \mathbb{V}[X] = \delta\frac{\alpha^2}{\sqrt{\alpha^2-\lambda^2}} \tag{3.6}$$

Convolution Property
The NIG distribution is stable under convolution, meaning the sum of two independent NIG distributed random variables also follows the NIG distribution.

$$\begin{array}{rlcl} & X & \sim & NIG(\alpha, \lambda, \mu_1, \delta_1) \\ & Y & \sim & NIG(\alpha, \lambda, \mu_2, \delta_2) \\ \Rightarrow & X+Y & \sim & NIG(\alpha, \lambda, \mu_1+\mu_2, \delta_2+\delta_2) \end{array}$$

Scaling Property
Another main property of the NIg distribution is the scaling property

$$X \sim NIG(\alpha, \lambda, \mu, \delta) \Rightarrow cX \sim NIG(\frac{\alpha}{c}, \frac{\lambda}{c}, c\mu, c\delta)$$

An efficient implemenation of the cumulative NIG distribution function can be found for Matlab at **include URL**
For the calculation of the inverse cumulative NIG distribution it is computationally much more efficient to table the cumulative distribution function on a fine grid around the mean. This table can be used to quickly look up the inverse of the NIG distribution.

3.3.2 Conditional Default Probabilities under the NIG

Using the scaling property and stability under convolution from 3.3.1 the distribution of Y_i is also normal inverse gaussian and is given by;

$$Y_i \quad \sim \quad NIG\left(\frac{\alpha}{\beta_i}, \frac{\lambda}{\beta_i}, -\frac{1}{\beta_i}\frac{\alpha\lambda}{\sqrt{\alpha^2-\lambda^2}}, \frac{\alpha}{\beta_i}\right)$$

To simplify the notation we denote $F_{NIG}\left(x; s\alpha, s\lambda, -s\frac{\alpha\lambda}{\sqrt{\alpha^2-\lambda^2}}, s\alpha\right) = F_{NIG(s)}(x)$
For given a realisation of the factor V the conditional default probabilities under the NIG copula can be calculated as follows:

$$\mathbb{P}(\tau_i \leq t|V) = \mathbb{P}(Y_i \leq F^{-1}_{NIG(\frac{1}{\beta_i})}\left[\mathbb{P}(\tau_i \leq t)\right]|V) = F_{NIG(\frac{\sqrt{1-\beta_i}}{\beta_i})}\left(\frac{F^{-1}_{NIG(\frac{1}{\beta_i})}\left[\mathbb{P}(\tau_i \leq t)\right] - \beta_i V}{\sqrt{1-\beta_i^2}}\right) \tag{3.7}$$

3.4 Gaussian Copula with Random Factor Loadings

The extension of Random Factor Loadings was suggested by Anderson and Sidenius [6] to to better fit the correlation skew observed with the one factor Gaussian copula. In the random factor model the exposure to the systematic factor is made a function of the the systematic factor. The appeal of the model is an intuitively clear justification. Assuming the systematic factor represents the state of the economy, with low values in downturns and higher values in benign and boom phases. Research on Equity correlation found that the correlation increases during economic downturns relative to the correlation level in benign economic phases. The random factor model reflects these results by changing the correlation through the factor exposure as a function of the factor itself.
The vector of the default drivers in the Gaussian copula with random factor loadings is given by:

$$Y_i = \beta(V)V + \sqrt{1-\upsilon}\epsilon_i + m \tag{3.8}$$

As in the case of the Gaussian copula V and ϵ_i are independent standard normal random variables. The parameters υ and m are choosen to ensure Y_i has mean zero and standard deviation of one. The factor exposure β is now a function of the factor value V [2]. In this section we will derive the expressions for a three point (five parameter) factor loadings, which is more general, but contains both the two point and the flat correlation normal copula models.

$$\beta(V) = \begin{cases} \sqrt{\rho_l} & \text{if } V \leq \theta_l; \\ \sqrt{\rho_c} & \text{if } \theta_l < V \leq \theta_r; \\ \sqrt{\rho_r} & \text{if } V > \theta_r; \end{cases}$$

[2]For ease of notation we have dropped the reference asset ID subscript for β. Here $\beta_i(V) = \beta$.

Here ρ_l, ρ_c, ρ_r are positive constant values and $\theta_l, \theta_r \in (R)$. Hence we model regime switching in correlation. If $\rho_l > \rho_c > \rho_r$ the factor loadings decrease in Y. Conditional default probabilities, which could be viewed as indicator for the state of the economy also decrease in Y. Hence correlation increases as conditional default probabilities increase. If $\rho_l = \rho_c = \rho_r$ we regain the standard Gaussian copula function with flat correlation structure.

The conditional default probabilities for this model are given by

$$\mathbb{P}(\tau_i \leq t|V) = \mathbb{P}(Y_i \leq \Phi^{-1}\left[\mathbb{P}(\tau_i \leq t)\right]|V) = \Phi\left(\frac{F_Y^{-1}\left[\mathbb{P}(\tau_i \leq t)\right] - \beta(V)V - m}{\sqrt{1-\upsilon}}\right) \tag{3.9}$$

The cumulative distribution function F_Y is no longer normal due to the dependence of β on V. In the follwoing section we will derive a closed form solution for F_Y that allows to compute the thresholds with reasonable efficiency. The defaults are again conditionally independent and which allows us to use the semi analytic recursion again to compute the portfolio loss distribution.

3.4.1 Derivation of Parameters υ and m

The parameters υ and m are determined for the Y_i to have zero mean and unit variance.

Mean m
since $\mathbb{E}(\sqrt{1-\upsilon}\epsilon_i) = 0$.

$$\begin{aligned} m = \mathbb{E}(\beta(V)V) &= \mathbb{E}\left[\sqrt{\rho_l}1_{V\leq\theta_l}V + \sqrt{\rho_c}1_{\theta_l<V\leq\theta_r}V + \sqrt{\rho_r}1_{V>\theta_r}V\right] \\ &= -\sqrt{\rho_l}\varphi(\theta_l) + \sqrt{\rho_c}(\varphi(\theta_l) - \varphi(\theta_r)) + \sqrt{\rho_r}\varphi(\theta_r) \end{aligned} \tag{3.10}$$

Variance υ
$\mathbb{V}(Y_i) = \mathbb{V}(\beta(V)V) + (1-\upsilon)\mathbb{V}(\epsilon_i) = \mathbb{V}(\beta(V)V) + (1-\upsilon)$, since $\mathbb{V}(\epsilon_i) = 1$ by definition. Therefore

$$\upsilon = \mathbb{V}(\beta(V)V) = \mathbb{E}(\beta(V)^2V^2) - [\mathbb{E}(\beta(V)V)]^2 \tag{3.11}$$

$$\begin{aligned} &= \mathbb{E}\left[\rho_l 1_{V\leq\theta_l}V^2 + \rho_c 1_{\theta_l<V\leq\theta_r}V^2 + \rho_r 1_{V>\theta_r}V^2\right] - [\mathbb{E}(\beta(V)V)]^2 \\ &= \rho_l(\Phi(\theta_l) - \theta_l\varphi(\theta_l)) \\ &\quad + \rho_c[(\Phi(\theta_r) - \Phi(\theta_l)) + (\theta_l\varphi(\theta_l) - \theta_r\varphi(\theta_r))] \\ &\quad + \rho_r(1 - \Phi(\theta_r) + \theta_r\varphi(\theta_r)) \\ &\quad - (-\sqrt{\rho_l}\varphi(\theta_l) + \sqrt{\rho_c}(\varphi(\theta_l) - \varphi(\theta_r)) + \sqrt{\rho_r}\varphi(\theta_r))^2 \end{aligned} \tag{3.12}$$

Lemma
For a standarised Gaussian variable x and arbitrary constants c and d, we have

$$\begin{aligned}
\mathbb{E}\left[1_{x\leq d}x\right] &= -\varphi(d); && (3.13)\\
\mathbb{E}\left[1_{x>d}x\right] &= \varphi(d); && (3.14)\\
\mathbb{E}\left[1_{c<x\leq d}x\right] &= 1_{d\geq c}\left(\varphi(c)-\varphi(d)\right); && (3.15)
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}\left[1_{x\leq d}x^2\right] &= \Phi(d)-d\varphi(d); && (3.16)\\
\mathbb{E}\left[1_{x>d}x^2\right] &= c\varphi(d)+(1-\Phi(d)); && (3.17)\\
\mathbb{E}\left[1_{c<x\leq d}x^2\right] &= 1_{d\geq c}\left(\Phi(d)-\Phi(c)\right)+1_{d\geq c}\left(c\varphi(c)-d\varphi(d)\right). && (3.18)
\end{aligned}$$

Proof

Note that for the normal distribution, the density is given by $\varphi(x)=\frac{1}{\sqrt{2\pi}}exp(-\frac{1}{2}x^2)$ and the first and second derivative of the density functions are

$$\begin{aligned}
f'(x) &= \tfrac{-x}{\sqrt{2\pi}}exp(-\tfrac{1}{2}x^2) = -xf(x)\\
f''(x) &= \tfrac{-1}{\sqrt{2\pi}}exp(-\tfrac{1}{2}x^2)+\tfrac{-x^2}{\sqrt{2\pi}}exp(-\tfrac{1}{2}x^2) \quad = \quad -f(x)+x^2f(x)
\end{aligned}$$

Equations 3.13 and 3.14 follow from

$$\mathbb{E}\left[1_{x\leq d}x\right]=\int_{-\infty}^{d}x\varphi(x)dx=-\int_{-\infty}^{d}\varphi\prime(x)dx \tag{3.19}$$

Equation 3.16 follows from

$$\begin{aligned}
\mathbb{E}\left[1_{x\leq d}x^2\right] &= \int_{-\infty}^{d}x^2\varphi(x)dx && (3.20)\\
&= \int_{-\infty}^{d}\left(\varphi\prime\prime(x)+\varphi(x)\right)dx && (3.21)\\
&= \varphi\prime(d)+\Phi(d)=\Phi(d)-d\varphi(d) && (3.22)
\end{aligned}$$

Equation 3.17 follows from

$$\begin{aligned}
\mathbb{E}\left[1_{x\geq d}x^2\right] &= \int_{d}^{-\infty}x^2\varphi(x)dx && (3.23)\\
&= \int_{d}^{\infty}\left(\varphi\prime\prime(x)+\varphi(x)\right)dx && (3.24)\\
&= \int_{d}^{\infty}\left(\varphi\prime\prime(x)+\varphi(x)\right)dx && (3.25)\\
&= -\varphi\prime(d)+1-\Phi(d)=1-\Phi(d)+d\varphi(d) && (3.26)
\end{aligned}$$

The expectations in 3.15 and 3.18 are obviously zero if $d<c$. For $c<d$ we have

$$\mathbb{E}\left[1_{c<x\leq d}x\right]=\int_{c}^{d}x\varphi(x)dx=\varphi(c)-\varphi(d) \tag{3.27}$$

$$\mathbb{E}\left[1_{c<x\leq d}x^2\right]=\int_{c}^{d}x^2\varphi(x)dx=\int_{c}^{d}\left(\varphi\prime\prime(x)+\varphi(x)\right)dx=\Phi(d)-\Phi(c)+\left(c\varphi(c)-d\varphi(d)\right) \tag{3.28}$$

3.4.2 Derivation of Distribution of Y_i

If $\rho_l \neq \rho_c \neq \rho_r$ than Y_i is no longer normally distributed, which means the copula function for the random factor extension of the normal copula is no longer a Gaussian copula function. However for computing the conditional default rates it is necessary to compute the default threshold as the inverse of the cumulative distribution of the unconditional default rates $F_Y^{-1}(\mathbb{P}(\tau_i \leq t))$.

The Probability of default before time t is given by

$$\begin{aligned}
\mathbb{P}(\tau_i \leq t) &= \mathbb{E}[\mathbb{P}(m + \sqrt{\rho_l}1_{V\leq\theta_l}V + \sqrt{\rho_c}1_{\theta_l<V\leq\theta_r}V + \sqrt{\rho_r}1_{V>\theta_r}V + \sqrt{1-\upsilon}\epsilon_i \leq c_i|V)] \\
&= \mathbb{E}\left[\Phi\left(\frac{c_i - m - \sqrt{\rho_l}1_{V\leq\theta_l}V - \sqrt{\rho_c}1_{\theta_l<V\leq\theta_r}V - \sqrt{\rho_r}1_{V>\theta_r}V}{\sqrt{1-\upsilon}}\right)|V\right] \\
&= \int_{-\infty}^{\theta_l}\Phi\left(\frac{c_i - m - \sqrt{\rho_l}V}{\sqrt{1-\upsilon}}\right)\varphi(V)dV \\
&\quad + \int_{\theta_l}^{\theta_r}\Phi\left(\frac{c_i - m - \sqrt{\rho_c}V}{\sqrt{1-\upsilon}}\right)\varphi(V)dV \\
&\quad + \int_{\theta_r}^{\infty}\Phi\left(\frac{c_i - m - \sqrt{\rho_r}V}{\sqrt{1-\upsilon}}\right)\varphi(V)dV \\
\mathbb{P}(\tau_i \leq t) &= \Phi_2\left(\frac{c_i - m}{\sqrt{1-\upsilon+\rho_l}}, \theta_l; \frac{\sqrt{\rho_l}}{\sqrt{1-\upsilon+\rho_l}}\right) \\
&\quad + \Phi_2\left(\frac{c_i - m}{\sqrt{1-\upsilon+\rho_c}}, \theta_r; \frac{\sqrt{\rho_c}}{\sqrt{1-\upsilon+\rho_c}}\right) - \Phi_2\left(\frac{c_i - m}{\sqrt{1-\upsilon+\rho_c}}, \theta_l; \frac{\sqrt{\rho_c}}{\sqrt{1-\upsilon+\rho_c}}\right) \\
&\quad + \Phi\left(\frac{c_i - m}{\sqrt{1-\upsilon+\rho_r}}\right) - \Phi_2\left(\frac{c_i - m}{\sqrt{1-\upsilon+\rho_r}}, \theta_r; \frac{\sqrt{\rho_r}}{\sqrt{1-\upsilon+\rho_r}}\right) \qquad (3.29)
\end{aligned}$$

Where $\Phi_2(a, b; c)$ is the standard bivariate Gaussian cumulative distribution function.

Lemma
For arbitrary constants a,b and c

$$\int_{-\infty}^{\infty}\Phi(ax+b)\varphi(x)dx = \Phi\left(\frac{b}{\sqrt{1+a^2}}\right); \qquad (3.30)$$

$$\int_{-\infty}^{c}\Phi(ax+b)\varphi(x)dx = \Phi_2\left(\frac{b}{\sqrt{1+a^2}}, c; \frac{-a}{\sqrt{1+a^2}}\right); \qquad (3.31)$$

Proof
To proof 3.30 we note that $\mathbb{P}(\epsilon - ax \leq b) = \Phi\left(\frac{b}{\sqrt{1+a^2}}\right)$, where a is an arbitrary constant and $\epsilon, x \sim N(0, 1)$. Therefore $\mathbb{E}(\epsilon - ax) = 0$ and $\mathbb{V}(\epsilon - ax) = 1 - a^2$ and hence $(\epsilon - ax) \sim N(0, \sqrt{1 - a^2})$. Using the law of iterated expectations

$$\begin{aligned}\int_{-\infty}^{\infty} \Phi(b+ax)\varphi(x)dx &= \mathbb{E}(\Phi(b+ax)) \\ &= \mathbb{E}(\mathbb{P}(\epsilon \leq b+ax|x)) \\ &= \mathbb{E}(\mathbb{P}(\epsilon - ax \leq b|x)) \\ &= \mathbb{P}(\epsilon - ax \leq b) \\ &= \Phi\left(\frac{b}{\sqrt{1+a^2}}\right); \end{aligned} \tag{3.32}$$

To proof 3.31 note that the correlation ρ between the two random variables $-ax+\epsilon$ and x is given by

$$\rho(-ax+\epsilon; x) = \frac{\mathrm{COV}(-ax+\epsilon; x)}{\sqrt{\mathbb{V}(-ax+\epsilon)}\sqrt{\mathbb{V}(x)}} = \frac{-a}{\sqrt{1-a^2}} \tag{3.33}$$

$$\begin{aligned}\int_{-\infty}^{c} \Phi(b+ax)\varphi(x)dx &= \mathbb{E}(1_{x\leq c}\mathbb{P}(\epsilon \leq b+ax|x)) \\ &= \mathbb{E}(\mathbb{P}(-ax+\epsilon \leq b, x \leq c|x)) \\ &= \mathbb{P}(-ax+\epsilon \leq b, x \leq c) \\ &= \Phi_2\left(\frac{b}{\sqrt{1+a^2}}, c; \frac{-a}{\sqrt{1+a^2}}\right); \end{aligned} \tag{3.34}$$

3.5 Gaussian Copula with Random Recovery Rates

Anderson and Sidenius [6] also suggested another extension for the Gaussian copula by introducing a tractable model for random recovery rates with the Gaussian copula framework. So far all the models under consideration make the assumption of deterministic recovery rates. For senior unsecured the market typically assumes a recovery rate of 40%. The actual experience in the credit derivative market has shown that recovery rates can vary significantly between credit events **Cite Fitch Credit Event Study**. Also Altman, Bradi, Resti and Sironi [**?**] found strong evidence for correlation between default rates and recovery rates. Generally recovery rates were found to be lower during economic downturns. The model suggested by Anderson and Sidenius allows for random recovery rates that have an ideosyncratic as well as a systematic risk compoenent.

In the random recovery rate model the vector of latend variables for the default process is given by the one factor gaussian copula [3].

$$Y_i = \beta V + \sqrt{1-\beta^2}\epsilon_i \tag{3.35}$$

The conditional default rates are again given by

$$\mathbb{P}(\tau_i \leq t|V) = \mathbb{P}(Y_i \leq \Phi^{-1}\left[\mathbb{P}(\tau_i \leq t)\right]|V) = \Phi\left(\frac{\Phi^{-1}\left[\mathbb{P}(\tau_i \leq t)\right] - \beta V}{\sqrt{1-\beta^2}}\right) \tag{3.36}$$

[3] We assumed $\beta_i = \beta$

The recovery rates are specified as follows

$$R_i = \Phi(\mu_i + \gamma_i V + \sigma_{\xi_i}\xi_i) \tag{3.37}$$

where Φ is again the cumulative Gaussian Distribution. Recovery rates in this model are driven by the same systematic factor V, which also drives the default process and an idiosyncratic factor ξ_i. The random variable ξ_i is assumed to have a standard normal distribution. The factor exposure γ_i determines the co-dependence between defaults and recovery rates.

Conditional on the systematic factor V recovery rates between different reference entities are again independent. The conditional recovery rates distribution is given by

$$\mathbb{P}(R_i \leq x|V) = \Phi\left(\frac{\Phi^{-1}(x) - \mu_i - \gamma_i V}{\sigma_{\xi_i}}\right) \tag{3.38}$$

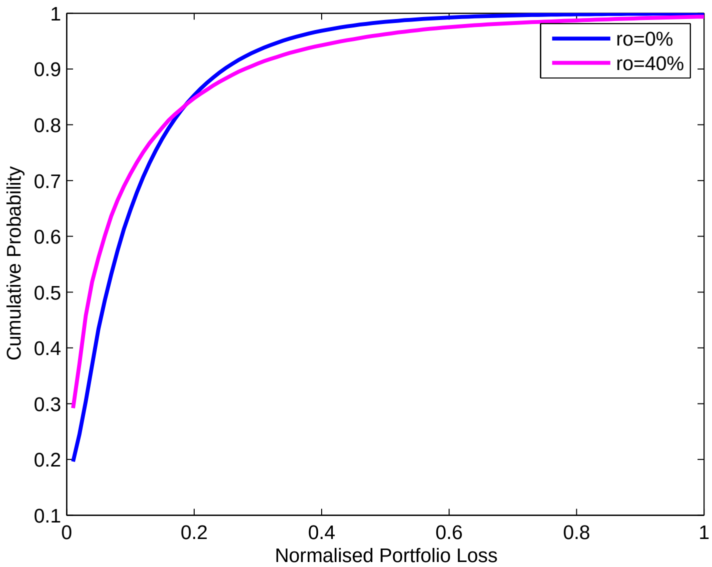


Figure 3.5: Portfolio Loss Distribution with Random Recovery Rates

3.5.1 Model Calibration

Mean, Variance and Correlation

The recovery rate model contains three parameters γ_i, μ_i and σ_{ξ_i}. These can be fitted to the correlation skew for example or alternatively calibrated to empirical data by matching the moments and rank corelation properties of the recovery rate distribution. In this section we derive expressions for the moments of the recovery rate distribution as functions of the model parameters.

Setting $X_i = \mu_i + \gamma_i V + \sigma_{\xi_i}\xi_i$.

X_i is normally distributed with mean and variance of $\mathbb{E}(X_i) = \mu_i$ and $\mathbb{V}(X_i) = \sigma_i^2 = \gamma_i\gamma_i + \sigma_{\xi_i}^2$.

Using the results in 3.30 and 3.31 we can derive the mean and variance of the recovery rate distribution as follows

$$
\begin{aligned}
\mathbb{E}(R_i) &= \mathbb{E}(\Phi(X_i)) \\
&= \int_{-\infty}^{\infty} \Phi(\mu_i + \sigma_i x)\varphi(x)dx \qquad \text{since } X_i \sim N(\mu_i, \sigma_i) \\
&= \Phi\left(\frac{\mu_i}{\sqrt{1+\sigma_i^2}}\right) \qquad (3.39)
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{V}(R_i) &= \mathbb{E}(\Phi(\mu_i + \gamma_i V + \sigma_{\xi_i}\xi_i)^2) - \mathbb{E}(\Phi(\mu_i + \gamma_i V + \sigma_{\xi_i}\xi_i))^2 \\
&= \int_{-\infty}^{\infty} \Phi(\mu_i + \sigma_i x)^2\varphi(x)dx - \Phi\left(\frac{\mu_i}{\sqrt{1+\sigma_i^2}}\right)^2 \\
&= \Phi_2\left(\frac{\mu_i}{\sqrt{1+\sigma_i^2}}, \frac{\mu_i}{\sqrt{1+\sigma_i^2}}; \frac{\sigma_i^2}{1+\sigma_i^2}\right) - \Phi\left(\frac{\mu_i}{\sqrt{1+\sigma_i^2}}\right)^2 \qquad (3.40)
\end{aligned}
$$

Proof of Step2

$$
\begin{aligned}
\int_{-\infty}^{\infty} \Phi(\mu_i + \sigma_i x)^2\varphi(x)dx &= \mathbb{E}(\mathbb{P}(\epsilon_1 \leq \mu_i + \sigma_i x|x)\mathbb{P}(\epsilon_2 \leq \mu_i + \sigma_i x|x)) \\
&= \mathbb{E}(\mathbb{P}(-\sigma_i x + \epsilon_1 \leq \mu_i, -\sigma_i x + \epsilon_2 \leq \mu_i|x)) \\
&= \mathbb{P}(-\sigma_i x + \epsilon_1 \leq \mu_i, -\sigma_i x + \epsilon_2 \leq \mu_i) \\
&= \Phi_2\left(\frac{\mu_i}{\sqrt{1+\sigma_i^2}}, \frac{\mu_i}{\sqrt{1+\sigma_i^2}}; \frac{\sigma_i^2}{1-\sigma_i^2}\right); \qquad (3.41)
\end{aligned}
$$

Since -the correlation ρ between the two random variables $-\sigma_i x + \epsilon_1$ and $-\sigma_i x + \epsilon_2$ is given by

$$
\rho(-\sigma_i x + \epsilon_1; -\sigma_i x + \epsilon_2) = \frac{\text{COV}(-\sigma_i x + \epsilon_1; \sigma_i x + \epsilon_1)}{\sqrt{\mathbb{V}(-\sigma_i x + \epsilon_1)}\sqrt{\mathbb{V}(-\sigma_i x + \epsilon_1)}} = \frac{\sigma_i^2}{1-\sigma_i^2} \qquad (3.42)
$$

Expected Recovery Conditional on Default prior to T

Importantly the calibration must ensure that the model is consistent with the credit default swap market. In order to value a credit default swap with random recovery it is necessary to compute the expected recovery rate conditional on default taking place prior to a certain time $\mathbb{E}(R_i|\tau_i \leq T) = \mathbb{E}(R_i|Y_i \leq \Phi^{-1}[\mathbb{P}(\tau_i \leq T)])$. Using Bayes Theorem and setting $\Phi^{-1}[\mathbb{P}(\tau_i \leq T)] = c_i$ this can be derived as follows

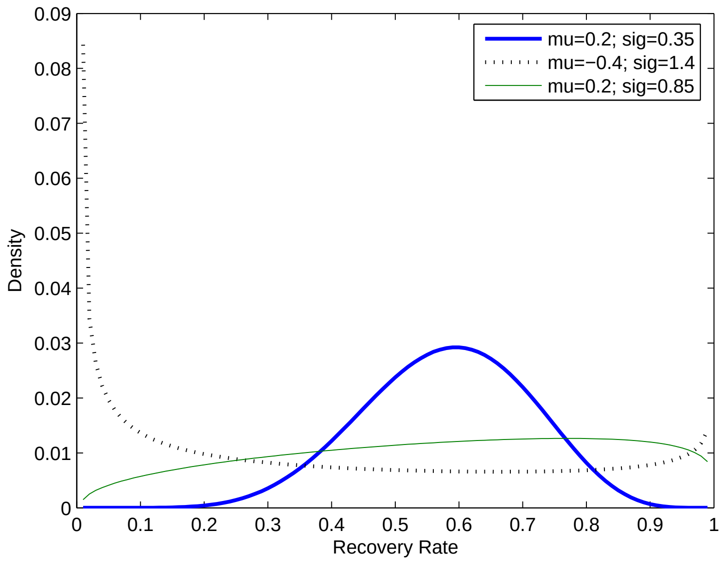


Figure 3.6: Recovery Rate Densities in cumulative Gaussian model

$$
\begin{aligned}
\mathbb{E}\left(R_i|Yi \leq c_i\right) &= \int_0^1 r\mathbb{P}\left(R_i = r|Y_i \leq c_i\right)dr \\
&= \int_0^1 r\frac{\mathbb{P}\left(R_i = r, Y_i \leq c_i\right)}{\mathbb{P}\left(Y_i \leq c_i\right)}dr \\
&= \int_0^1 r\frac{\int_{-\infty}^{c_i}\mathbb{P}\left(R_i = r, Y_i = y\right)dy}{\mathbb{P}(\tau_i \leq T)}dr \\
&= \mathbb{P}(\tau_i \leq T)^{-1}\int_{-\infty}^{c_i}\int_0^1 r\mathbb{P}\left(R_i = r, Y_i = y\right)drdy \\
&= \mathbb{P}(\tau_i \leq T)^{-1}\int_{-\infty}^{c_i}\int_0^1 r\mathbb{P}\left(R_i = r|Y_i = y\right)\varphi(y)drdy \\
&= \mathbb{P}(\tau_i \leq T)^{-1}\int_{-\infty}^{c_i}\mathbb{E}\left(R_i|Y_i = y\right)\varphi(y)dy \qquad (3.43)
\end{aligned}
$$

Anderson and Sidenius derived the following expression for the expected recovery rate conditional on Y_i.

$$
\mathbb{E}(R_i|Y_i = y) = \Phi\left(\frac{\mu_i + \rho_i\sigma_i y}{\sqrt{1+\sigma_i^2(1-\rho_i^2}}\right) \qquad (3.44)
$$

Here $\rho_i = \beta\gamma_i/\sigma_i$ is the correlation between X_i and V. Using (3.34) this can be solved to give

$$\mathbb{E}(R_i|\tau_i \le T) = \Phi_2\left(\frac{\mu_i}{\sqrt{1+\sigma_i^2}}, \Phi^{-1}\left[\mathbb{P}(\tau_i \le T)\right]; \frac{-\rho_i\sigma_i}{\sqrt{1+\sigma_i^2}}\right)/\mathbb{P}(\tau_i \le T) \tag{3.45}$$

The following relationship allows to derive an expression for the expected recovery rate at a particular time.

$$\begin{aligned} f_i(t) &= \mathbb{E}((1-R_i)|\tau_i \in [t, t+dt]) \\ F_i(t) &= \mathbb{E}((1-R_i)|\tau_i \le t) \end{aligned}$$

then, setting $p_i(t) = \mathbb{P}(\tau_i \le T)$ we use the following general relationship

$$\begin{aligned} F_i(T)p_i(T) &= \int_0^T f_i(t)p_i'(t)dt \\ f_i(t) &= F_i(t) + F_i'(t)\frac{p_i(t)}{p_i'(t)} \end{aligned}$$

where $p_i'(t)$ is the marginal default probability.

3.5.2 Portfolio Loss Distribution under Random Recovery Rates

Due to the idiosyncratic risk factor in the recovery rates the conditional loss distribution is no longer just a two point distribution. However the conditional recovery rates and losses are still independent, which makes it possible to derive the conditional portfolio loss distribution as the convolution product of all the individual loss distributions. The convolution product can be computed by fourier transform or using a slightly more involved recursion method. Here we will outline the recursion approach. In order to compute the convolution product it is necessary to approximate the loss $l_i \in [0, N_i]$ of each individual exposure in the portfolio by non negative integer multiples k_i of an arbitrary positive loss unit u such that $l_i = k_i u$.

To derive the portfolio loss distribution at time T we need the individual name loss distributions conditional on default of that name prior to time t, $l_i^c(t) = l_i(t)1_{\tau_i<t}$. Conditional on V, default and recovery are independent and we have

$$\mathbb{P}(l_i^c(t) < z) = \mathbb{P}(l_i(t) < z|V)\mathbb{P}(\tau_i < t|V) + \mathbb{P}(\tau_i > t|V) \tag{3.46}$$

The expression 3.38 can be used to compute $\mathbb{P}(l_i(t) < z|V)$.

$$\begin{aligned} \mathbb{P}(l_i(t) < z|V) &= \mathbb{P}(N_i(1-R_i) \le N_i x|V) \\ &= \mathbb{P}((1-R_i) \le x|V) \\ &= 1 - \mathbb{P}(R_i \le x|V) \\ &= 1 - \Phi\left(\frac{\Phi^{-1}(x) - \mu_i - \gamma_i V}{\sigma_{\xi_i}}\right) \end{aligned}$$

Note on Implementation. Computing the marginal probability in the implementation as

$$\mathbb{P}(z < l_i(t) < z + \Delta z|V) = \mathbb{P}(l_i(t) < z + \Delta z|V) - \mathbb{P}(l_i(t) < z|V). \tag{3.47}$$

will introduce a bias to the results for larger loss steps, although the bias would disappear asymptotically. It is possible to avoid the bias, while using larger loss units with the following discreetisation

$$\mathbb{P}(z < l_i(t) < z + \Delta z|V) = \mathbb{P}(l_i(t) < z + 0.5\Delta z|V) - \mathbb{P}(l_i(t) < z - 0.5\Delta z|V). \tag{3.48}$$

We achieved results that were very close to the monte carlo results even with loss united of 20% of the reference notional.

Let L_n^t be the loss (as measure in multiples of the loss unit u) in the sub-portfolio consisting of n individual credit exposures. The portfolio loss L_{n+1}^t can be computed by the following recursive relation

$$\mathbb{P}(L_{n+1}^t = K|V) \quad = \quad \sum_{k=0}^{k_{n+1}^{max}} \mathbb{P}(L_n^t = K - k|V)\mathbb{P}(l_{n+1}^c(t) = ku|V) \tag{3.49}$$

Here k_{n+1}^{max} is the maximum loss for the n+1 exposure expressed in multiples of u, $k_{n+1}^{max} = N_{n+1}/u$. The conditional portfolio loss distribution can be computed from the boundary case of the empty portfolio for which $\mathbb{P}(L_0^t = K|V) = 1_{K=0}$.

Chapter 4

Numerical Results

This final chapter gives the numerical results for each of the tested models. The first section briefly describes the data used for the model calibration. For this thesis the performance of each model during the period of increased volatility in May 2005 is of particular interest. Therefore the first section will also review the factors that caused the turmoil in the correlation markets during May 2005.

4.1 Description of Data

Each of the tested models requires the following data

- Single Name CDS spreads for each name in the index and maturities of 1,3,5,7 and 10 years.
- The index spread
- Index Tranche spreads
- Zero Rates corresponding to the quarterly payment dates of the index.

Index spreads and single name default swap spreads were provided by MarkIT Parterners and Index Tranche spreads by Lehman Brothers. The models were tested on a monthly basis between the 5th of January 2005 and the 10th of May 2006. Zero Rates were obtained from Bloomberg.

The following charts show the index spread for the iTraxx and CDX investment grade indices since 2004. Spread levels have been declining steadily since 2004.

During May of 2005 spreads widened dramatically by around 20bps for the iTraxx and 30bps for the CDX, peaking at 60.3bps in case of the iTraxx and 78bps for the CDX. The spread widening occurred as result of the downgrade of Ford and GM to sub investment grade and a general credit deterioration in the north american auto industry. For example the five year credit spread of GM on the 10th May 2005 was 695bps. The increase was only shortlived and index spreads quickly returned to their normal levels. For example GM traded much tighter on the 3rd of August at 363bps.

The sharp increase in spreads during May 2005 also lead to a widening of the bid-offer spread, which can be seen for the iTraxx in the chart below.

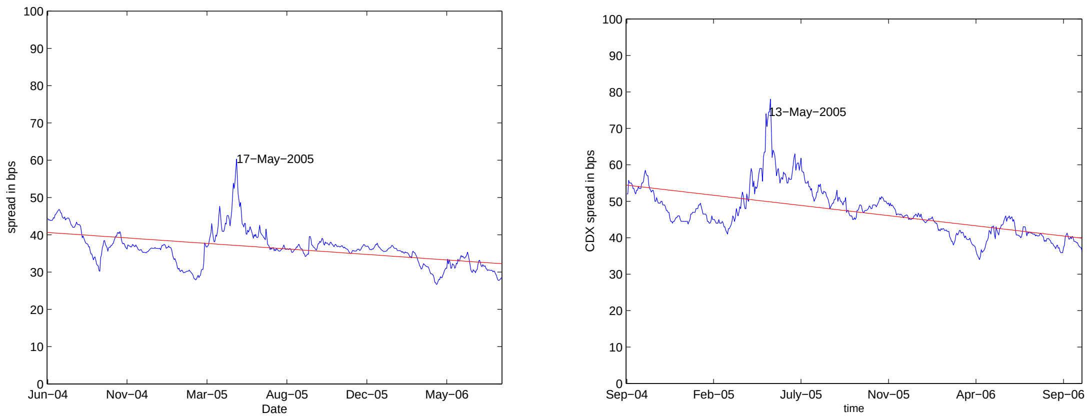


Figure 4.1: Five Year iTraxx and CDX Index Spread

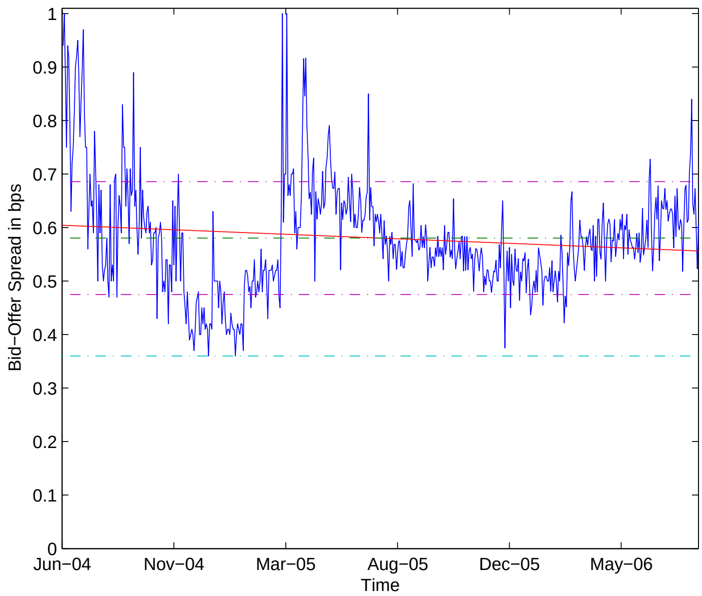


Figure 4.2: Five Year iTraxx Bid Offer Spread

Prior to the May period bid-offer spreads had been continuously decreasing as liquidity in the index market increased. However during may 2005 bid offer levels peaked up to 1bps. The overall average for the entire time period is around 0.6bps.

The increased volatility in the credit markets during May of 2005 also affected the tranche prices as can be seen in the following charts.

For both iTraxx and CDX the Equity upfront premium increased, while mezzanine and senior spreads remained the same or actually decreased. The market moves were significantly larger for the CDX

Figure 4.3: Five and 10 Year iTraxx and CDX Tranche Prices and Equity Upfront

IG index. The opposite market moves of mezzanine and equity indicated that the tranche market has shifted the view on implied correlation downwards. This shift in correlation also affected the iTraxx index but to a much lesser extend than the CDX, which we will see in the numerical results below.

The widening of the CDX index was a result of idiosyncratic spread moves of a number of names in the Auto industry, while the spread widening of the remaining index names was much less. At the time a popular trade among correlation traders and hedgefunds alike was to buy protection on the mezzanine position and hedge the exposure by selling protection on the equity position. The spread widening caused many traders to unwind their positions at the same time, by taking offsetting positions, which drove up the cost of equity and lowered the cost of mezzanine protection. Given the nature of the trade this meant mark to market losses on both legs, forcing other traders to also unwind or face large mark to market losses.

4.2 Description of Results

Each of the models described in chapter two and three was tested in terms of how well the model implied prices can be fitted to the index tranche prices shown above. The simplex downhill solver as well as a gradient method were used to minimise the absolute error sum between model implied spreads and market spreads across all tranches. In this section we present the best fit parameters as well as the absolute error over the sample period for each of the models. A table containing all detailed results can also be found in the appendix.

4.2.1 Clayton, One Factor Gauss Copula and Base Correlation

The following charts show the base correlations over time for the five year iTraxx and CDX index.

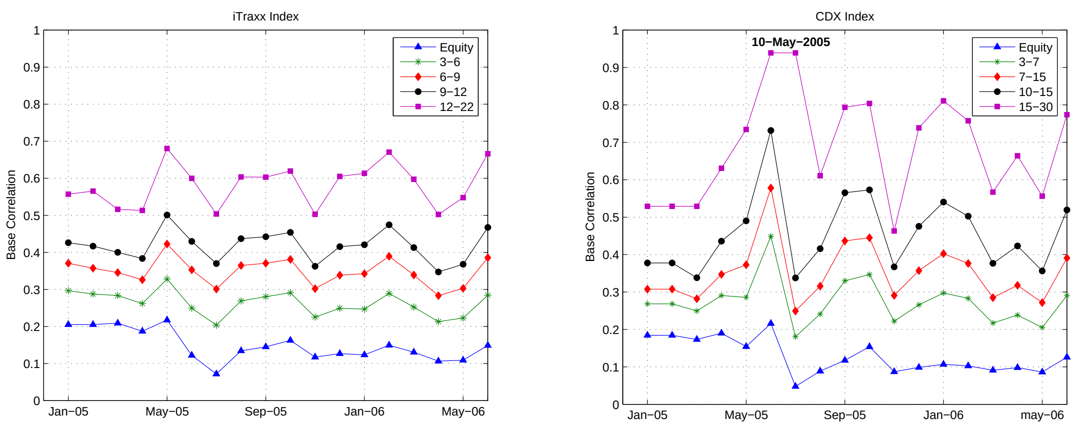
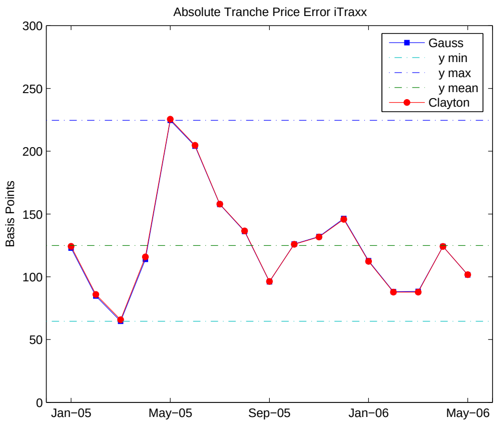


Figure 4.4: Base Correlation five year iTraxx and CDX

The data clearly shows a structural shift between the period prior to May 2005 and the period thereafter. The average level post May 2005 is clearly lower for all tranches both in the iTraxx and the CDX index. For the iTraxx the Equity correlation before May 2005 was approximately 20.5%, which compares to 12.8% thereafter. For the CDX the average before and after May 2005 was 17.2% and 11.2% respectively. At the same time the volatility in BC levels has increased significantly post May 2005.

Base correlations fit the data exactly and hence the model implied tranche spreads are equal to the market spread. However base correlation is not a model for the correlation skew but rather a fitting procedure to match the market. In order to assess the one factor Gaussian copula model, which only has one correlation parameter we fitted the model to the market prices by minimising the absolute tranche price error accross the capital structure.

The following charts show the best fit correlation levels and absolute model error for both the iTraxx and CDX indices over the sample period.



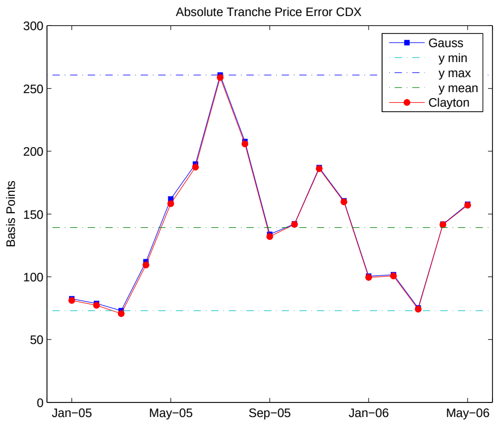


Figure 4.5: Absolute Model Error for One Factor Gauss and Clayton Copula

The best fit correlation level that minimises the tranche error is in all cases equal to the base correlation for the equity tranche, which has the largest spread (500bps) and hence the largest weight in the absolute tranche price error. Therefore the one factor Gauss copula shows the behaviour pre and post May 2005 that was described above for the base correlations. The minimum absolute model error varies over the sample period. For the iTraxx the error ranges between 64 and 224bps with average of 124bps. For the CDX the error is somewhat larger with a minimum of 72bps, a maximum of 260bps and an average of 139bps. The best fit model spreads implied by the Gauss copula consistently overestimate the spreads on the two mezzanine tranches and consistenly underestimate the spread on the most senior tranche.

An alternative one factor Copula which we introduced in Chaptor two is the Clayton copula which belongs to the family of Archimedean copulae. Interestingly the best fit model spreads for the Clayton copula although it has quite different properties to the Gaussian copula are nearly identical to the one factor Gaussian copula. Also the best fit value for theta, while it is different to the Gaussian

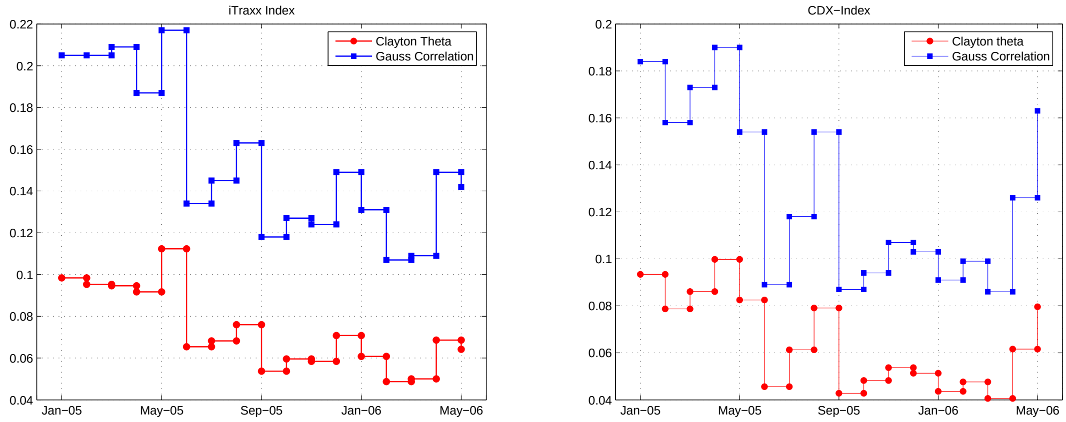


Figure 4.6: Parameter for One Factor Gauss and Clayton Copula

correlation shows a high degree of correlation over time with the best fit correlation level in the Gauss copula.

4.2.2 Random Recovery Rate

In order to obtain a better fit with the one factor Gaussian Copula Anderson and Sidenius [6] introduced stochastic recovery rates correlated with defaults. The next chart shows the absolute model error for the five year iTraxx and CDX.

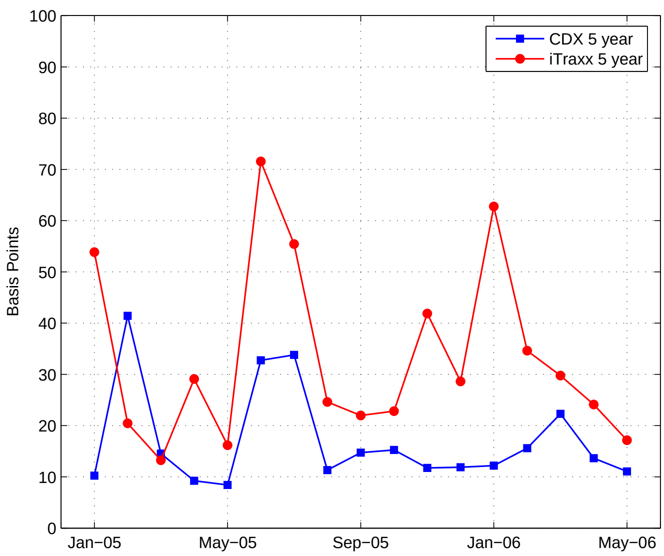


Figure 4.7: Absolute Model Error with Random Recovery Rates

The model shows a significant improvement compared to the standard one factor Gauss copula with deterministic recovery rates. The minimum absolute error is significantly lower for every datapoint in the sample period for both CDX and iTraxx. The average error for the iTraxx is 33bps with a minimum of 13bps and a maximum of 71bps. The fit is even better for the CDX for which the average is only 17bps with a minimum of 8.4bps and a maximum of 41bps.

The model is able to fit almost exactly both the equity and lowest mezzanine tranche. The fit of the second mezzanine tranche varies over time and is significantly better for the CDX compared to the iTraxx. The model still produces implied spreads that underestimates the two senior most tranches consistently. The absolute error increases between May and August 2005 for both iTraxx and CDX, which is caused by the increase in spread of the senior tranches relative to the mezzanine tranches

Table 4.1 shows the best fit parameters for the random recovery rate model. This random recovery rate model has three more parameters in addition to correlation, which include the mean and the variance of the recovery rate distribution and the correlation between recovery rates and the defaults driver.

	Best Fit iTraxx				Best Fit CDX			
	ρ_{Def}	$\mathbb{E}[R]$	$STDev$	ρ_{DP-RR}	ρ_{Def}	$\mathbb{E}[R]$	$STDev$	ρ_{DP-RR}
5-Jan-05	21.9%	58.8%	14.1%	24.6%	17.8%	57.3%	19.3%	38.9%
1-Feb-05	21.2%	58.5%	15.4%	40.6%	23.5%	59.6%	18.7%	47.2%
2-Mar-05	20.5%	58.3%	16.3%	38.1%	19.1%	57.9%	19.4%	39.3%
6-Apr-05	12.5%	55.0%	15.7%	31.3%	19.4%	57.9%	18.8%	40.0%
4-May-05	18.5%	57.4%	13.8%	37.6%	17.1%	56.9%	17.4%	37.5%
1-Jun-05	10.3%	54.1%	13.2%	17.1%	7.0%	52.9%	17.3%	26.3%
6-Jul-05	12.4%	55.0%	12.8%	19.2%	7.9%	53.2%	14.9%	27.9%
3-Aug-05	13.3%	55.4%	15.5%	35.6%	13.1%	55.3%	15.1%	35.8%
7-Sep-05	10.7%	54.3%	16.3%	31.0%	12.3%	55.0%	16.6%	35.1%
5-Oct-05	10.2%	54.1%	15.7%	28.9%	9.9%	54.0%	17.1%	31.4%
2-Nov-05	14.8%	55.9%	14.4%	38.4%	9.5%	53.8%	14.1%	30.7%
7-Dec-05	10.2%	54.1%	14.3%	21.7%	9.8%	54.0%	15.1%	31.3%
4-Jan-06	17.6%	57.0%	4.1%	21.9%	10.7%	54.3%	17.3%	30.9%
1-Feb-06	12.6%	55.1%	14.7%	20.1%	10.7%	54.3%	17.0%	31.6%
1-Mar-06	11.8%	54.7%	14.5%	25.3%	13.2%	55.3%	15.6%	34.7%
5-Apr-06	10.7%	54.3%	14.6%	7.7%	6.9%	52.8%	15.4%	23.8%
10-May-06	6.3%	52.5%	15.5%	22.8%	8.6%	53.5%	14.0%	27.6%

Table 4.1: Best Fit Parameters with Random Recovery Rates

The best fit mean and standard deviation for the specific recovery model are relatively stable over time. The mean for both indices ranges between 55% and 60% with an average standard deviation of 13.7% for the iTraxx and 15.4% for the CDX. The correlation between defaults and recovery rates shows more variability. The best is on average 27.2% for the CDX and 33.5% for the iTraxx. It is interesting that the average correaltion level between defauls and recoveries that provides the best fit to the data is very similar to results reported by Altman [1] based on historical data.

4.2.3 Random Factor Loadings

Anderson and Sidenius [6] in the same paper also suggested another extension of the one factor gaussian copula, by making the factor loadings a function of the factor itself. In their paper they outline the model with a two point loadings distribution. In chapter three we extended this model to a three point loadings setup. The following chart shows the best fit absolute model error for both versions of the random factor loadings model.

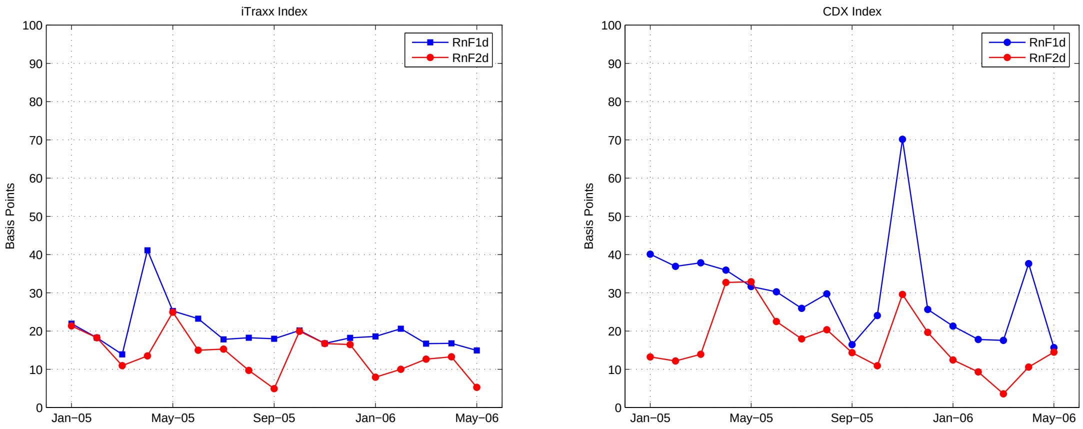


Figure 4.8: Absolute Model Error with Random Factor Loadings

As the three point model contains the two point version as a special case its minimum absolute error is always equal or below the error of the two point version.

For the iTraxx both models perform reasonably well. During the benign environment pre and post May 2005 the minimum absolute error is 20bps or below. Even during May 2005 the models perform well for the iTraxx index and the error only increases to 26bps. The only outlier occurred during April 2005 when the error was 41bps, which might be caused by the numerical optimisation routine, especially since the three point model performs well on this date. The three point model produces an minimum absolute error that is on average 5.6bps below the error of the two point model.

For the CDX index the model does not perform as well as for the iTraxx. The minimum absolute error of the two point model ranges between 15bps and 70bps with an average of 30bps. The three point model for the CDX provides a more significant improvement over the two point model. The minimum absolute error is always below 32bps with an average of 17bps, which includes the May 2005 period.

The random factor matches the spread on the equity and lower mezzanine tranches, but underestimates the spread of the second mezzanine tranches. Unlike the other models the random factor extensions tends to overestimate the spread on the two senior tranches.

Table 4.2 below shows the best fit parameters for each of the sample dates for the two point distribution model. The impact of the May 2005 period is mainly felt in the lower correlation, which drops dramatically from an average of 12% to an average of around 5% for both iTraxx and CDX.

The threshold level moves up during and after May 2005 but returns to the pre May 2005 levels for both indices. The senior correlation fluctuates but does not show a clear trend.

	Best Fit iTraxx			Best Fit CDX		
	ρ_1	ρ_2	θ	ρ_1	ρ_2	θ
5-Jan-05	51.4%	11.6%	-2.29	47.0%	12.8%	-2.31
1-Feb-05	57.8%	13.0%	-2.52	28.3%	13.4%	-2.35
2-Mar-05	46.7%	14.2%	-2.52	47.4%	12.9%	-2.48
6-Apr-05	63.3%	11.4%	-2.41	50.3%	11.8%	-2.17
4-May-05	67.7%	9.6%	-2.15	45.5%	7.9%	-2.05
1-Jun-05	49.5%	3.6%	-2.15	37.6%	2.7%	-2.05
6-Jul-05	34.5%	4.1%	-2.05	62.7%	1.6%	-2.09
3-Aug-05	61.4%	6.4%	-2.41	56.5%	4.1%	-2.09
7-Sep-05	52.7%	5.3%	-2.55	38.7%	3.5%	-2.26
5-Oct-05	54.3%	4.8%	-2.40	33.6%	2.9%	-2.05
2-Nov-05	76.5%	4.5%	-2.52	35.0%	4.4%	-2.05
7-Dec-05	67.1%	4.8%	-2.41	41.7%	2.2%	-2.07
4-Jan-06	48.4%	5.2%	-2.40	48.0%	4.6%	-2.49
1-Feb-06	53.3%	4.9%	-2.62	47.2%	4.5%	-2.41
1-Mar-06	51.1%	4.6%	-2.56	59.0%	4.6%	-2.69
5-Apr-06	71.6%	5.1%	-2.50	47.9%	3.9%	-2.19
10-May-06	46.2%	4.8%	-2.39	88.2%	4.6%	-2.39

Table 4.2: Best Fit Parameters with Random Factor Loadings (two point)

Overall the random factor model performs reasonably well in terms of its ability to reproduce the correlation skew and match the tranche prices. The three point model model provides some improvement mainly for the CDX index. However given the additional two parameter the gains are relatively small. The three point model also showed some numerical instability during the optimisation and a significant deterioration in runtime relative to the two point version.

4.2.4 NIG Copula

The normal inverse Gaussian Copula was suggested by Kalemanova, Schmid and Werner [5] as an alternative to the one factor Gaussian Copula. The model has three parameters that determine the correlation and the first four moments of the marginal distributions. The minimum absolute error for the NIG copula over the sample period is shown in the following chart.

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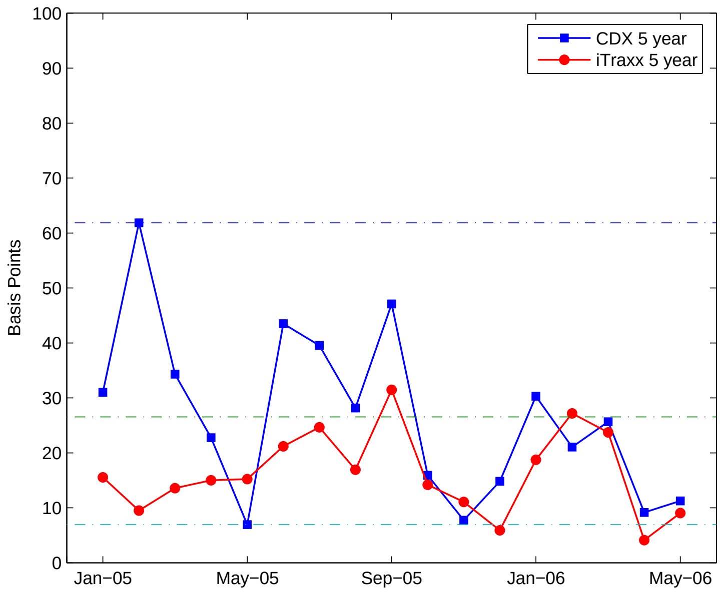

Figure 4.9: Absolute Model Error for NIG Copula

The minimum absolute error for the iTraxx ranges between 4bps and 31bps with an average of 16.3bps. Similar to the previous models the error is larger for the CDX, ranging between 7bps and 61bps with an average of 26.5bps. The model is able to match the market spread for the equity and junior mezzanine tranches almost exactly. The second and third mezzanine tranche spreads are matched to within 4bps for ten out of the 17 sample dates. The NIG as the one factor Gaussian copula tends to underestimate the senior tranche spreads or in other words can not produce a steep enough skew.

Table 4.3 shows the best fit parameters for the NIG copula. Prior to May 2005 the model shows a strong skew (high lambda), fat tails (high alpha) and high correlation. Following May 2005 all three parameters drop significantly, both for the iTraxx and the CDX index. The latter also shows higher volatility in the parameters over time. Although the NIG is not always able to match the correlation skew as well as the one factor Gaussian copula with random factor loadings, it produces a continuous smooth cumulative loss distribution.

	Best Fit iTraxx			Best Fit CDX		
	ρ	α	λ	ρ	α	λ
5-Jan-05	22.7%	0.477	0.323	21.0%	0.605	0.329
1-Feb-05	21.2%	0.592	0.331	16.8%	0.910	0.450
2-Mar-05	20.8%	0.669	0.378	18.7%	0.769	0.300
6-Apr-05	21.5%	0.500	0.330	23.5%	0.470	0.304
4-May-05	25.3%	0.691	-0.06	22.3%	0.310	0.212
1-Jun-05	19.8%	0.189	0.159	19.0%	0.139	0.123
6-Jul-05	18.0%	0.221	0.176	26.0%	0.122	0.111
3-Aug-05	18.5%	0.269	0.217	23.0%	0.127	0.101
7-Sep-05	12.8%	0.280	0.221	12.2%	0.234	0.166
5-Oct-05	15.5%	0.234	0.196	15.6%	0.146	0.123
2-Nov-05	15.6%	0.220	0.199	19.4%	0.128	0.045
7-Dec-05	18.5%	0.221	0.165	21.2%	0.130	0.076
4-Jan-06	15.0%	0.249	0.212	11.5%	0.269	0.213
1-Feb-06	11.3%	0.295	0.227	12.9%	0.247	0.189
1-Mar-06	11.7%	0.262	0.220	10.0%	0.303	0.250
5-Apr-06	16.6%	0.255	0.164	17.5%	0.238	0.132
10-May-06	14.8%	0.251	0.1815	21.9%	0.201	0.115

Table 4.3: Best Fit Parameters for the NIG Copula

4.2.5 Double-t Copula

The last model that was analysed was the double t copula suggested by Hull and White [4]. The minimum absolute error is shown in the chart below.

The model although it outperforms the one factor Gaussian copula on all sample dates is generally not able to reproduce the market skew as well as the other models above. The model performs poorly for the CDX index both in terms of maximum error and average error, which are well above those of the other models. The performance is somewhat better for the iTraxx but still the minimum absolute error for almost all sample dates is above 20bps.

On several sample dates the model fails to match the spread of the junior mezzanine tranche and significantly underestimates the spread of the senior tranches.

Table 4.4 shows the best fit model parameters for the double t copula, which appear volatile in comparison to other models, confirming that the double t copula is not able to match the correlation skew observed in the iTraxx and CDX market.

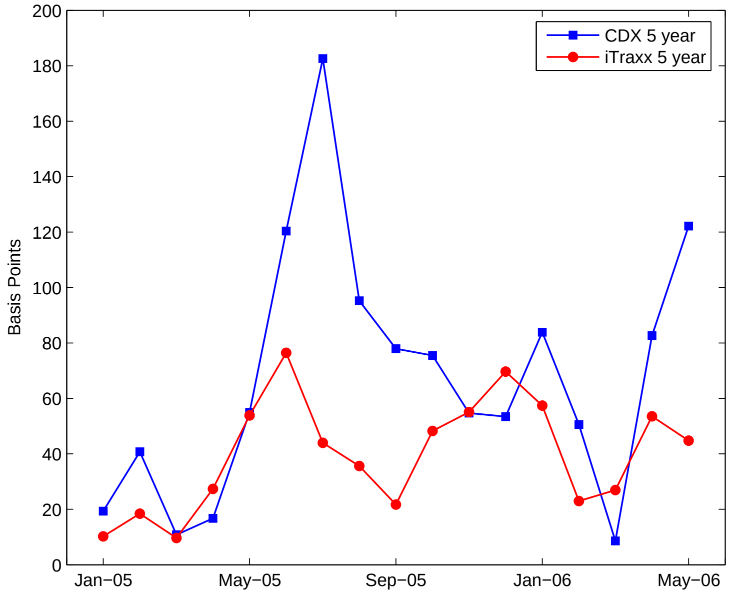

Figure 4.10: Absolute Model Error with Double-t Copula

4.3 Summary

We have so far reviewed the performance of each model individually. The following chart plots the minimum absolute error for all models.

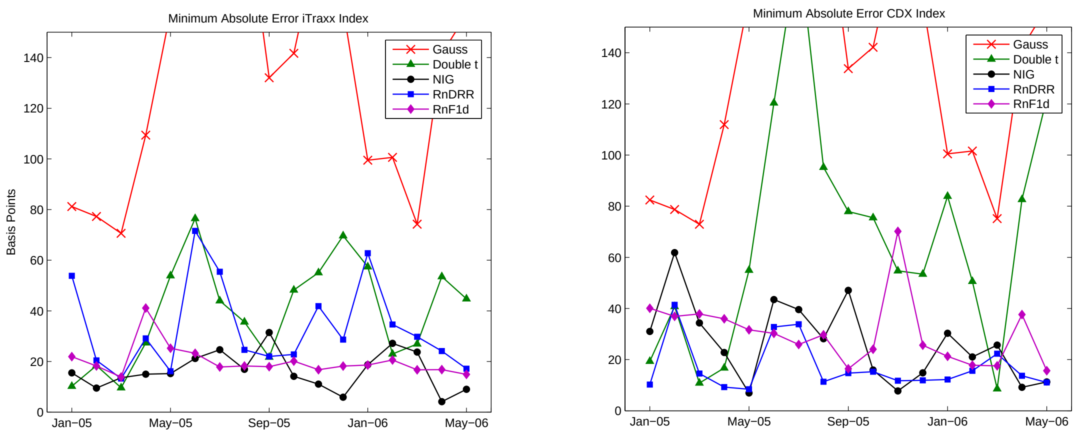

Figure 4.11: Minimum Absolute Model Error

There are a number of conclusions that we can draw from these results. Firstly, with the exception of the Clayton copula all models outperform the one factor Gaussian copula (without base correlation)

	Best Fit iTraxx			Best Fit CDX		
	ρ	ν	$\bar{\nu}$	ρ	ν	$\bar{\nu}$
5-Jan-05	22.7%	3	9	22.6%	5	7
1-Feb-05	22.0%	4	8	21.8%	9	5
2-Mar-05	24.2%	5	6	21.9%	5	6
6-Apr-05	25.8%	7	3	25.1%	4	5
4-May-05	29.6%	3	3	23.7%	3	4
1-Jun-05	20.0%	3	3	12.5%	4	6
6-Jul-05	19.8%	3	3	15.8%	4	7
3-Aug-05	21.2%	3	3	22.4%	3	3
7-Sep-05	14.9%	3	6	15.8%	4	7
5-Oct-05	18.0%	3	4	22.4%	3	3
2-Nov-05	17.8%	3	4	13.3%	3	3
7-Dec-05	20.6%	3	4	15.1%	3	3
4-Jan-06	19.2%	4	5	16.2%	3	3
1-Feb-06	13.5%	3	6	15.8%	3	3
1-Mar-06	13.2%	3	7	12.7%	3	3
5-Apr-06	19.6%	3	3	16.7%	3	8
10-May-06	18.4%	3	4	20.0%	5	7

Table 4.4: Best Fit Parameters for the Double-t Copula

in terms of the minimum absolute error. The one factor Clayton copula produces identical results as the Gaussian copula. Second, all models except the random recovery rate model perform better for the iTraxx index than the CDX. This also includes the one factor Gaussian model with Base correlations, where we encountered two sample dates for which the base correlation solver produced no result for the senior tranches. Thirdly, all models either over or underestimate the spread on the senior tranches. This may help to explain the better fit for the iTraxx, which has lower attachment and detachment points. Fourth, the NIG copula and the random factor loadings version of the Gaussian Copula (with two and three point distributions) consistently produced the lowest minimum absolute error, and were able to replicate the spreads on the equity and junior mezzanine tranches. Both models continued to performed well during May and June 2005. Fifth, the worst performance was gained with the double t copula especially for the CDX index.

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Chapter 5

Appendix

Results iTraxx 5 Year							
5-Jan-05	Best Fit Model Parameters	Model Spreads					Abs Err
BC	20.5%,29.7%,37.1%,42.6%,55.7%	500	131.5	42.0	25.8	14.0	-
Gauss	ρ=20.5%	500	217.6	64.4	21.8	3.6	122.9
Clayton	θ=0.0984	500	218.6	63.9	21.1	3.3	124.2
NIG	α=0.4771,λ=0.323,ρ=22.7%	500	131.5	41.6	19.7	4.9	15.5
Double-t	ρ=22.7%,ν=3,$\bar{\nu}$=10	500	133.7	48.2	27.6	14.0	10.2
RndRR	ρ=21.9%,μ=0.8,σ=0.135,γ=24.6%	501	136.2	27.0	6.3	0.6	53.8
RnF1d	ρ_h=51.4%,ρ_l=11.6%,θ=-2.29	500	131.5	35.2	28.5	26.4	21.9
RnF2d	53.8%,49.6%,11.6%,-4.08,-2.27	500	131.5	35.1	28.4	25.8	21.3
1-Feb-05							
BC	20.5%,28.8%,35.7%,41.7%,56.6%	500	109.0	33.0	15.7	8.4	-
Gauss	ρ=20.5%	500	172.4	48.2	15.6	2.4	84.7
Clayton	θ=0.0953	500	173.3	47.7	14.9	2.2	85.9
NIG	α=0.592,λ=0.3313,ρ=21.15%	500	109.0	32.4	13.4	2.2	9.5
Double-t	ρ=22.0%,ν=4,$\bar{\nu}$=8	500	112.9	39.9	21.8	10.0	18.4
RndRR	ρ=21.2%,μ=0.86,σ=0.162,γ=40.6%	496	103.1	32.6	10.8	2.9	20.5
RnF1d	ρ_h=57.8%,ρ_l=13.0%,θ=-2.52	500	109.0	21.5	12.3	11.7	18.2
RnF2d	59.4%,57.3%,13.04%,-3.46,-2.50	500	109.0	21.5	12.3	11.7	18.2
2-Mar-05							
BC	20.9%,28.4%,34.5%,40.0%,51.6%	500	92.5	28.8	13.4	8.1	-
Gauss	ρ=20.9%	500	140.3	37.9	12.0	1.85	64.5
Clayton	θ=0.0946	500	141.2	37.4	11.4	1.6	65.8
NIG	α=0.669,λ=0.378,ρ=20.8%	500	92.5	26.1	9.5	1.1	13.6
Double-t	ρ=24.2%,ν=5,$\bar{\nu}$=6	500	92.9	33.2	18.0	8.0	9.6
RndRR	ρ=20.5%,μ=0.87,σ=0.18,γ=38.1%	502	92.8	29.7	9.5	2.1	13.3
RnF1d	ρ_h=46.7%,ρ_l=14.2%,θ=-2.52	500	92.2	19.5	12.1	11.1	13.9
RnF2d	59.6%,33.1%,14.2%,-2.70,-2.50	500	92.5	19.4	11.8	8.1	10.9
6-Apr-05							
BC	18.7%,26.2%,32.6%,38.3%,51.3%	500	175.0	56.5	26.5	14.8	-
Gauss	ρ=18.7%	500	256.7	77.5	25.9	4.1	113.9
Clayton	θ=0.0917	500	258.2	76.7	25.1	3.8	115.9
NIG	α=0.50,λ=0.33,ρ=21.45%	500	176.0	53.1	24.5	6.4	15
Double-t	ρ=25.8%,ν=7,$\bar{\nu}$=3	500	182.2	66.6	35.7	15.7	27.4
RndRR	ρ=12.5%,μ=0.733,σ=0.165,γ=31.3%	497	176.2	55.2	16.2	1.3	29.1
RnF1d	ρ_h=63.3%,ρ_l=11.4%,θ=-2.41	500	176.0	33.5	12.9	11.2	41.1
RnF2d	74.88%,21.6%,10.9%,-2.80,-1.72	500	176.0	56.0	31.8	8.0	13.5

Table 5.1: Best Fit Parameters and Tranche Spreads for the five year iTraxx index

Results iTraxx 5 Year							
4-May-05	Best Fit Model Parameters	Model Spreads					Abs Err
BC	21.7%,32.8%,42.3%,50.1%,68.0%	500	176.5	53.0	25.5	16.8	-
Gauss	ρ=21.7%	500	316.0	111.8	43.7	8.7	224.6
Clayton	θ=0.1123	500	317.9	111.1	42.9	8.2	225.5
NIG	α=0.69.1,λ=-0.058,ρ=25.3%	500	176.5	56.5	25.5	5.04	15.2
Double-t	ρ=29.6%,ν=3,$\bar{\nu}$=3	500	182.4	72.9	45.1	25.2	53.9
RndRR	ρ=18.5%,μ=0.76.2,σ=0.127,γ=37.6%	501	176.1	50.3	25.5	4.5	16.2
RnF1d	ρ_h=67.7%,ρ_l=9.6%,θ=-2.15	500	176.5	39.1	27.4	26.2	25.3
RnF2d	67.9%,67.3%,9.6%,-2.814,-2.15	500	176.6	39.1	26.4	26.2	24.9
1-Jun-05							
BC	13.4%,26.9%,36.5%,43.7%,60.4%	500	118.3	39.5	25.0	14.5	-
Gauss	ρ=13.4%	500	275.1	65.2	16.4	1.7	204.1
Clayton	θ=0.0654	500	275.8	64.5	15.9	1.55	204.7
NIG	α=0.1877,λ=0.1588,ρ=19.8%	500	118.3	31.8	18.3	7.8	21.2
Double-t	ρ=20.0%,ν=3,$\bar{\nu}$=3	500	177.9	52.3	28.2	14.0	76.5
RndRR	ρ=10.3%,μ=0.75,σ=0.114,γ=17.06%	501	114.6	10.4	1.24	0.1	71.6
RnF1d	ρ_h=49.5%,ρ_l=3.6%,θ=-2.14	500	118.3	28.5	26.6	25.2	23.2
RnF2d	93.5%,26.7%,3.5%,-2.43,-2.32	500	118.3	27.8	25.8	17.0	15.0
6-Jul-05							
BC	14.5%,28.1%,37.1%,44.2%,60.3%	500	82.8	31.8	18.3	11.6	-
Gauss	ρ=14.5%	500	208.4	46.3	11.3	1.2	157.6
Clayton	θ=0.0682	500	208.7	45.7	10.9	1.1	157.9
NIG	α=0.2211,λ=0.176,ρ=18.0%	500	82.9	22.7	11.5	3.0	24.6
Double-t	ρ=19.8%,ν=3,$\bar{\nu}$=3	500	115.7	37.8	22.5	12.2	44.0
RndRR	ρ=12.4%,μ=0.773,σ=0.107,γ=19.2%	500	84.0	6.9	0.8	0.05	55.4
RnF1d	ρ_h=34.5%,ρ_l=4.1%,θ=-2.05	500	82.8	28.1	26.7	17.4	17.8
RnF2d	90.6%,30.9%,4.0%,-2.79,-2.15	500	82.8	27.9	26.4	14.9	15.3
3-Aug-05							
BC	16.3%,29.1%,38.1%,45.4%,61.9%	500	79.8	27.5	14.8	9.9	-
Gauss	ρ=16.3%	500	187.5	44.5	11.8	1.4	136.1
Clayton	θ=0.076	500	187.9	43.9	11.4	1.26	136.5
NIG	α=0.269,λ=0.217,ρ=18.5%	500	79.5	22.7	10.9	2.3	16.9
Double-t	ρ=21.2%,ν=3,$\bar{\nu}$=3	500	97.9	34.8	21.4	11.9	35.6
RndRR	ρ=13.3%,μ=0.835,σ=0.159,γ=35.6%	500	86.6	35.3	15.5	1.36	24.6
RnF1d	ρ_h=61.4%,ρ_l=6.4%,θ=-2.41	500	79.7	13.9	11.9	11.5	18.2
RnF2d	80.2%,26.9%,5.6%,-2.81,-2.06	500	79.7	27.5	24.4	9.9	9.7
7-Sep-05							
BC	11.8%,22.5%,30.2%,36.3%,50.3%	500	81	25.8	14.3	8.2	-
Gauss	ρ=11.8%	500	159.4	25.8	4.5	0.3	96.1
Clayton	θ=0.054	500	159.3	25.5	4.4	0.3	96.3
NIG	α=0.280,λ=0.221,ρ=12.8%	500	78.2	15.3	4.6	0.3	31.5
Double-t	ρ=14.9%,ν=3,$\bar{\nu}$=6	500	100.3	26.9	14.3	7.0	21.7
RndRR	ρ=10.7%,μ=0.869,σ=0.178,γ=31.0%	500	86.0	28.7	8.3	0.6	22.0
RnF1d	ρ_h=52.7%,ρ_l=5.3%,θ=-2.554	500	81.0	13.4	11.8	11.4	18.0
RnF2d	92.7%,18.3%,4.6%,-2.935,-2.132	500	81.0	24.9	15.5	5.4	4.9

Results iTraxx 5 Year							
5-Oct-05	Best Fit Model Parameters	Model Spreads					Abs Err
BC	12.7%,24.9%,33.9%,41.6%,60.5%	500	91.0	28.0	11.8	6.0	-
Gauss	ρ=12.7%	500	197.6	38.5	8.0	0.7	126.1
Clayton	θ=0.059	500	197.5	38.0	7.8	0.6	125.9
NIG	α=0.234,λ=0.196,ρ=15.5%	500	91.0	20.8	9.2	1.61	14.2
Double-t	ρ=18.0%,ν=3,$\bar{\nu}$=4	500	121.8	34.6	19.0	9.6	48.2
RndRR	ρ=10.2%,μ=0.84,σ=0.163,γ=28.9%	493	93.3	24.3	6.6	0.6	22.8
RnF1d	ρ_h=54.3%,ρ_l=4.8%,θ=-2.4	500	91.0	13.3	11.3	10.9	20.15
RnF2d	71.4%,22.6%,4.8%,-2.85,-2.05	500	97.3	26.4	21.9	7.9	19.9
2-Nov-05							
BC	12.4%,24.7%,34.3%,42.1%,61.4%	500	93.0	24.0	12.0	5.8	-
Gauss	ρ=12.4%	500	201.3	38.3	7.8	0.6	132.0
Clayton	θ=0.058	500	201.7	37.9	7.6	0.6	131.7
NIG	α=0.22,λ=0.199,ρ=15.6%	500	92.9	20.4	8.9	1.5	11.1
Double-t	ρ=17.8%,ν=3,$\bar{\nu}$=4	500	125.9	34.9	19.0	9.5	55.1
RndRR	ρ=14.8%,μ=0.887,σ=0.138,γ=38.4%	491.9	103.7	9.5	6.7	2.5	41.9
RnF1d	ρ_h=76.5%,ρ_l=4.5%,θ=-2.5	500	93.0	13.3	11.5	11.3	16.7
RnF2d	79.8%,61.6%,4.5%,-2.64,-2.51	500	93.0	13.3	11.5	11.3	16.7
7-Dec-05							
BC	15.0%,28.9%,38.9%,47.4%,67.1%	500	77.0	23.8	10.0	5.3	-
Gauss	ρ=15.0%	500	197.5	44.5	11.0	1.2	146.4
Clayton	θ=0.0708	500	197.4	44.1	10.8	1.1	145.8
NIG	α=0.221,λ=0.165,ρ=18.5%	500	76.9	21.0	10.7	2.9	5.9
Double-t	ρ=20.6%,ν=3,$\bar{\nu}$=4	500	117.0	36.8	20.9	10.9	69.7
RndRR	ρ=10.2%,μ=0.776,σ=0.135,γ=21.7%	500	78.5	10.2	1.8	0.12	28.6
RnF1d	ρ_h=67.1%,ρ_l=4.8%,θ=-2.4	500	77.0	12.7	11.4	11.1	18.2
RnF2d	81.4%,23.5%,4.2%,-2.697,-2.05	500	77.43	25.6	21.6	7.7	16.5
4-Jan-06							
BC	13.1%,25.2%,33.9%,41.3%,59.7%	500	81.7	26.3	11.5	5.8	-
Gauss	ρ=13.1%	500	177.7	32.9	6.6	0.5	112.7
Clayton	θ=0.0608	500	177.4	32.7	6.6	0.5	112.3
NIG	α=0.249,λ=0.212,ρ=15.0%	500	81.2	18.0	6.9	0.8	18.7
Double-t	ρ=16.2%,ν=4,$\bar{\nu}$=6	500	126.5	33.3	15.9	6.8	57.4
RndRR	ρ=17.6%,μ=0.812,σ=0.011,γ=21.9%	500	110.0	9.3	0.6	0.01	62.8
RnF1d	ρ_h=48.4%,ρ_l=5.2%,θ=-2.4	500	81.7	13.4	11.7	11.2	18.6
RnF2d	76.5%,19.2%,4.4%,-2.765,-2.10	501	81.4	25.3	16.5	5.5	7.9
1-Feb-06							
BC	10.7%,21.3%,28.3%,34.7%,50.2%	500	75.5	26.0	10.0	5.5	-
Gauss	ρ=10.7%	500	144.83	19.7	2.9	0.2	88.1
Clayton	θ=0.049	500	144.4	19.7	2.8	0.1	87.8
NIG	α=0.295,λ=0.227,ρ=11.3%	500	75.5	11.9	2.5	0.1	27.2
Double-t	ρ=13.5%,ν=3,$\bar{\nu}$=6	500	93.1	23.0	12.0	5.8	23.0
RndRR	ρ=12.2%,μ=0.844,σ=0.142,γ=20.1%	500	75.2	6.4	0.8	0.04	34.6
RnF1d	ρ_h=53.3%,ρ_l=4.9%,θ=-2.62	500	75.5	13.0	11.8	11.3	20.6
RnF2d	96.9%,21.0%,4.1%,-3.3,-2.09	500	75.5	25.7	18.5	4.3	10.0

Results iTraxx 5 Year							
1-Mar-06	Best Fit Model Parameters	Model Spreads					Abs Err
BC	10.9%,22.3%,30.3%,36.8%,54.8%	500	70.0	22.0	11.3	4.3	-
Gauss	ρ=10.9%	500	143.7	19.8	2.9	0.2	88.3
Clayton	θ=0.05	500	143.2	19.8	2.9	0.2	87.8
NIG	α=0.262,λ=0.220,ρ=11.7%	500	70.0	11.2	2.5	0.1	23.7
Double-t	ρ=13.2%,ν=3,$\bar{\nu}$=7	500	93.5	23.1	11.9	5.6	26.9
RndRR	ρ=11.8%,μ=0.852,σ=0.138,γ=25.3%	500	69.5	7.0	1.2	0.1	29.8
RnF1d	ρ_h=51.1%,ρ_l=4.6%,θ=-2.56	500	70.0	12.7	11.7	11.2	16.7
RnF2d	92.6%,19.4%,4.3%,-3.15,-2.06	500	74.6	24.7	15.6	3.3	12.6
5-Apr-06							
BC	14.9%,28.5%,38.6%,46.8%,66.6%	500	62.3	16.5	8.3	3.5	-
Gauss	ρ=14.9%	500	164.8	35.3	8.4	0.9	124.2
Clayton	θ=0.069	500	165.11	34.8	8.1	0.8	124.2
NIG	α=0.255,λ=0.164,ρ=16.6%	500	62.1	16.2	7.1	1.1	4.1
Double-t	ρ=19.6%,ν=3,$\bar{\nu}$=3	500	86.6	29.5	17.9	10.1	53.5
RndRR	ρ=10.7%,μ=0.691,σ=0.139,γ=7.7%	500	62.3	3.9	0.3	0.01	24.1
RnF1d	ρ_h=71.6%,ρ_l=5.1%,θ=-2.5	500	64.2	12.4	11.4	11.1	16.8
RnF2d	98.5%,17.2%,3.9%,-2.73,-2.05	501	62.0	22.3	12.7	4.8	13.2
10-May-06							
BC	14.2%,27.5%,37.1%,44.9%,64.7%	500	51.0	15.5	7.0	3.3	-
Gauss	ρ=14.2%	500	137.5	26.4	5.7	0.5	101.5
Clayton	θ=0.0642	500	137.7	26.0	5.5	0.5	101.7
NIG	α=0.251,λ=0.1805,ρ=14.8%	500	51.0	12.1	4.3	0.4	9.0
Double-t	ρ=18.4%,ν=3,$\bar{\nu}$=4	500	75.0	24.4	14.4	7.7	44.8
RndRR	ρ=6.3%,μ=0.8,σ=0.157,γ=22.8%	500	51.6	8.5	0.9	0.01	17.2
RnF1d	ρ_h=46.2%,ρ_l=4.8%,θ=-2.38	500	51.0	11.7	11.0	10.4	14.9
RnF2d	95.4%,13.2%,3.7%,-2.71,-2.13	496	51.0	15.5	6.4	4.3	5.3

Results CDX 5 Year							
5-Jan-05	Best Fit Model Parameters	Model Spreads					Abs Err
BC	18.4%,26.8%,30.8%,37.8%,52.9%	500	188.5	63.3	23.0	8.3	-
Gauss	ρ=18.4%	500	251.0	58.9	14.7	1.0	82.4
Clayton	θ=0.0934	500	249.8	58.7	14.9	1.0	81.2
NIG	α=0.604,λ=0.328,ρ=21.0%	500	188.5	45.6	16.4	1.5	31.0
Double-t	ρ=22.6%,ν=5,$\bar{\nu}$=7	500	198.3	56.4	25.2	7.7	19.3
RndRR	ρ=17.8%,μ=0.936,σ=0.263,γ=38.9%	500	188.0	61.0	23.7	1.7	10.3
RnF1d	ρ_h=47.0%,ρ_l=12.8%,θ=-2.306	500	189.4	38.7	27.5	18.4	40.1
RnF2d	47.6%,22.2%,12.4%,-2.43,-1.98	500	188.6	53.4	24.2	10.3	13.2
1-Feb-05							
BC	15.8%,19.9%,17.8%,0.70%,0.30%	500	205.0	72.0	26.5	8.5	-
Gauss	ρ=15.8%	500	231.7	45.2	9.4	0.5	78.7
Clayton	θ=0.0787	500	230.5	45.3	9.5	0.5	77.3
NIG	α=0.910,λ=0.450,ρ=16.8%	500	205.1	37.0	8.0	0.2	61.8
Double-t	ρ=21.8%,ν=9,$\bar{\nu}$=9	500	199.5	49.3	18.4	4.1	40.8
RndRR	ρ=23.5%,μ=1.093,σ=0.250,γ=47.2%	500	204.8	41.0	21.4	3.4	41.4
RnF1d	ρ_h=28.3%,ρ_l=13.4%,θ=-2.346	500	204.8	40.5	24.2	5.6	36.9
RnF2d	66.6%,22.3%,11.1%,-3.97,-1.72	500	201.9	71.7	24.1	2.2	12.2
2-Mar-05							
BC	17.3%,25.0%,28.2%,33.8%,28.7%	500	165.5	53.0	19.0	7.5	-
Gauss	ρ=17.3%	500	212.7	43.6	9.7	0.5	72.9
Clayton	θ=0.0861	500	211.1	44.1	9.9	0.5	70.6
NIG	α=0.769,λ=0.300,ρ=18.7%	500	165.5	35.0	9.8	0.4	34.3
Double-t	ρ=21.9%,ν=5,$\bar{\nu}$=6	500	165.5	44.3	19.7	6.2	10.8
RndRR	ρ=19.1%,μ=0.996,σ=0.267,γ=39.3%	500	165.5	44.4	19.0	1.6	14.6
RnF1d	ρ_h=47.4%,ρ_l=12.9%,θ=-2.481	500	165.2	25.4	12.1	9.8	37.8
RnF2d	70.6%,21.5%,9.61%,-4.16,-1.44	500	165.1	58.1	17.2	1.3	13.9
6-Apr-05							
BC	19.0%,29.1%,34.7%,43.6%,63.1%	500	207.5	66.0	27.0	11.0	-
Gauss	ρ=19.0%	500	293.7	75.8	20.5	1.6	111.9
Clayton	θ=0.0998	500	291.8	75.5	20.9	1.6	109.4
NIG	α=0.465,λ=0.304,ρ=23.5%	500	207.5	54.0	23.3	4.0	22.8
Double-t	ρ=25.1%,ν=4,$\bar{\nu}$=5	500	217.9	65.7	32.0	11.9	16.7
RndRR	ρ=19.4%,μ=0.930,σ=0.248,γ=40.0%	500	207.5	66.0	27.7	2.4	9.3
RnF1d	ρ_h=50.3%,ρ_l=11.8%,θ=-2.174	500	207.5	39.7	26.7	20.3	35.9
RnF2d	53.4%,36.9%,11.76%,-2.39,-2.11	500	207.2	39.1	26.0	14.8	32.7

Table 5.2: Best Fit Parameters and Tranche Spreads for the five year CDX index

Results CDX 5 Year							
4-May-05	Best Fit Model Parameters	Model Spreads					Abs Err
BC	15.4%,28.6%,37.3%,49.0%,73.4%	500	213.8	49.0	22.8	8.8	-
Gauss	ρ=15.4%	500	336.7	73.6	16.3	0.9	161.9
Clayton	θ=0.0825	500	333.1	74.5	17.2	1.0	158.2
NIG	α=0.310,λ=0.212,ρ=22.3%	500	213.8	47.1	21.7	4.9	6.9
Double-t	ρ=23.7%,ν=3,$\bar{\nu}$=4	500	240.9	62.4	30.8	12.7	55.0
RndRR	ρ=17.1%,μ=0.907,σ=0.208,γ=37.5%	500	213.9	48.7	22.0	1.8	8.9
RnF1d	ρ_h=45.5%,ρ_l=7.9%,θ=-2.050	500	217.2	34.2	26.2	18.6	31.6
RnF2d	46.1%,45.3%,7.946%,-3.46,-2.05	500	215.8	34.0	26.2	18.5	32.9
1-Jun-05							
BC	8.9%,24.1%,31.6%,41.6%,61.1%	500	180.0	55.0	25.5	12.5	-
Gauss	ρ=8.9%	500	321.5	40.1	4.5	0.1	189.8
Clayton	θ=0.0456	500	320.0	40.8	4.8	0.1	187.3
NIG	α=0.139,λ=0.123,ρ=19.0%	500	180.0	29.2	15.1	5.3	43.5
Double-t	ρ=12.5%,ν=4,$\bar{\nu}$=6	500	274.4	46.3	16.5	4.6	120.4
RndRR	ρ=7.0%,μ=0.853,σ=0.199,γ=26.3%	498	182.2	54.8	9.2	0.1	32.7
RnF1d	ρ_h=37.6%,ρ_l=2.7%,θ=-2.046	500	180.0	27.6	25.7	15.2	30.3
RnF2d	80.3%,25.9%,1.97%,-4.03,-1.74	500	180.1	54.9	42.4	7.1	22.5
6-Jul-05							
BC	11.8%,33.0%,43.7%,56.5%,79.4%	500	135.0	39.5	24.0	14.8	-
Gauss	ρ=11.8%	500	347.2	59.2	9.7	0.3	260.6
Clayton	θ=0.0613	500	345.0	60.0	10.2	0.3	258.7
NIG	α=0.122,λ=0.111,ρ=26.0%	500	167.7	37.6	22.6	11.3	39.5
Double-t	ρ=15.8%,ν=4,$\bar{\nu}$=7	500	288.3	60.3	23.4	6.9	182.6
RndRR	ρ=7.9%,μ=0.791,σ=0.145,γ=27.9%	500	134.9	34.7	9.7	0.1	33.8
RnF1d	ρ_h=62.7%,ρ_l=1.6%,θ=-2.093	500	135.0	29.1	28.4	26.0	25.9
RnF2d	59.8%,11.8%,0.6%,-2.07,-1.86	499	134.6	40.5	28.9	25.2	17.9
3-Aug-05							
BC	15.4%,34.7%,44.5%,57.3%,80.4%	500	123.0	38.0	18.5	11.8	-
Gauss	ρ=15.4%	500	291.4	60.7	13.0	0.7	207.7
Clayton	θ=0.0791	500	289.7	60.9	13.3	0.7	205.8
NIG	α=0.127,λ=0.101,ρ=23.0%	497	115.2	27.8	16.0	6.6	28.2
Double-t	ρ=22.4%,ν=3,$\bar{\nu}$=3	500	192.0	53.5	28.3	12.7	95.2
RndRR	ρ=13.1%,μ=0.818,σ=0.151,γ=35.8%	500	123.0	38.0	18.6	0.6	11.3
RnF1d	ρ_h=56.5%,ρ_l=4.1%,θ=-2.090	500	123.0	29.2	27.9	23.3	29.7
RnF2d	77.7%,29.5%,4.2%,-2.37,-2.05	500	123.9	28.1	26.0	13.5	20.3
7-Sep-05							
BC	8.7%,22.2%,29.1%,36.7%,46.3%	500	132.0	36.0	21.5	10.3	-
Gauss	ρ=8.7%	500	219.8	20.0	1.8	0.01	133.8
Clayton	θ=0.0428	500	218.7	20.5	1.9	0.01	132.0
NIG	α=0.234,λ=0.165,ρ=12.2%	500	131.9	16.4	4.3	0.1	47.1
Double-t	ρ=15.8%,ν=4,$\bar{\nu}$=7	500	180	60.3	23.5	6.9	77.9
RndRR	ρ=12.3%,μ=0.934,σ=0.186,γ=35.1%	500	132.5	38.1	19.2	0.6	14.7
RnF1d	ρ_h=38.7%,ρ_l=3.5%,θ=-2.261	500	132.0	28.3	26.7	13.7	16.4
RnF2d	74.6%,23.3%,3.5%,-2.98,-2.11	500	132.9	27.1	21.5	5.9	14.3

Results CDX 5 Year							
5-Oct-05	Best Fit Model Parameters	Model Spreads					Abs Err
BC	9.9%,26.6%,35.7%,47.6%,73.9%	500	108.0	27.0	14.0	5.0	-
Gauss	ρ=9.9%	500	232.9	28.8	3.5	0.1	142.2
Clayton	θ=0.0482	500	232.5	28.7	3.5	0.1	141.7
NIG	α=0.146,λ=0.123,ρ=15.6%	500	108.0	19.2	9.4	1.8	15.9
Double-t	ρ=22.4%,ν=3,$\bar{\nu}$=3	500	135	53.5	28.3	12.7	75.5
RndRR	ρ=9.9%,μ=0.892,σ=0.197,γ=31.4%	501	108.0	34.4	12.7	0.2	15.3
RnF1d	ρ_h=33.6%,ρ_l=2.9%,θ=-2.046	500	114.8	26.4	24.9	10.5	24.0
RnF2d	96.8%,24.1%,2.3%,-2.71,-2.08	500	107.9	25.6	22.0	6.4	10.9
2-Nov-05							
BC	10.7%,29.7%,40.3%,54.1%,81.1%	500	112.5	25.5	11.5	5.5	-
Gauss	ρ=10.7%	500	273.7	39.9	5.6	0.1	186.9
Clayton	θ=0.0537	500	273.0	40.0	5.7	0.1	186.1
NIG	α=0.128,λ=0.045,ρ=19.4%	500	113.8	23.0	12.7	3.3	7.7
Double-t	ρ=13.3%,ν=3,$\bar{\nu}$=3	500	160.9	29.2	13.9	5.7	54.7
RndRR	ρ=9.5%,μ=0.824,σ=0.130,γ=30.7%	500	112.2	25.8	6.1	0.2	11.7
RnF1d	ρ_h=35.0%,ρ_l=4.4%,θ=-2.046	500	160.1	27.7	25.1	12.2	70.1
RnF2d	49.3%,45.4%,1.8%,-2.44,-2.08	500	112.3	27.0	26.0	18.8	29.6
7-Dec-05							
BC	10.3%,28.3%,37.6%,50.3%,75.8%	500	114.5	32.5	14.0	7.5	-
Gauss	ρ=10.3%	500	256.5	33.9	4.3	0.1	160.4
Clayton	θ=0.0513	500	255.5	34.1	4.5	0.1	159.6
NIG	α=0.130,λ=0.076,ρ=21.2%	500	110.5	25.0	14.7	5.0	14.8
Double-t	ρ=15.1%,ν=3,$\bar{\nu}$=3	500	161.4	35.2	17.8	7.5	53.4
RndRR	ρ=9.8%,μ=0.860,σ=0.150,γ=31.3%	500	113.0	34.6	13.0	0.2	11.9
RnF1d	ρ_h=41.7%,ρ_l=2.2%,θ=-2.065	500	114.6	26.7	25.5	15.7	25.6
RnF2d	87.6%,31.1%,2.2%,-2.71,-2.06	500	114.5	26.1	24.7	9.9	19.6
4-Jan-06							
BC	9.1%,21.7%,28.5%,37.7%,56.7%	500	113.5	28.5	13.0	6.0	-
Gauss	ρ=9.1%	500	184.4	16.5	1.5	0.01	100.5
Clayton	θ=0.0436	500	183.7	16.7	1.6	0.01	99.5
NIG	α=0.269,λ=0.213,ρ=11.5%	500	113.6	14.3	3.0	0.01	30.3
Double-t	ρ=16.2%,ν=3,$\bar{\nu}$=3	500	170	43.6	21.5	8.8	83.9
RndRR	ρ=10.7%,μ=0.931,σ=0.203,γ=30.1%	500	113.5	28.6	6.7	0.2	12.2
RnF1d	ρ_h=48.0%,ρ_l=4.6%,θ=-2.491	500	113.5	12.8	11.6	10.1	21.3
RnF2d	91.2%,25.3%,4.1%,-3.27,-2.15	500	113.5	26.8	21.9	4.4	12.4
1-Feb-06							
BC	9.8%,23.9%,31.8%,42.3%,66.4%	500	105.0	24.5	12.0	4.5	-
Gauss	ρ=9.8%	500	186.4	18.9	2.0	0.01	101.6
Clayton	θ=0.0476	500	185.7	19.1	2.0	0.01	100.6
NIG	α=0.247,λ=0.188,ρ=12.9%	500	105.0	15.7	4.2	0.1	21.0
Double-t	ρ=15.8%,ν=3,$\bar{\nu}$=3	500	130	39.2	19.4	8.1	50.6
RndRR	ρ=10.7%,μ=0.905,σ=0.195,γ=31.6%	500	105.0	31.9	8.2	0.2	15.6
RnF1d	ρ_h=47.2%,ρ_l=4.5%,θ=-2.411	500	105.0	12.6	11.5	9.9	17.8
RnF2d	90.9%,21.9%,3.9%,-2.99,-2.08	500	105.0	26.2	18.3	5.2	9.3

Results CDX 5 Year							
1-Mar-06	Best Fit Model Parameters	Model Spreads					Abs Err
BC	8.6%,20.5%,27.2%,35.6%,55.6%	500	92.0	20.5	10.3	3.8	-
Gauss	ρ=8.6%	500	143.3	9.9	0.7	0.01	75.1
Clayton	θ=0.0406	500	142.7	10.2	0.8	0.01	74.2
NIG	α=0.3027λ=0.250,ρ=10.0%	500	92.0	8.1	0.8	0.01	25.7
Double-t	ρ=12.7%,ν=3,$\bar{\nu}$=3	500	98.7	19.4	9.9	4.1	8.6
RndRR	ρ=13.2%,μ=0.941,σ=0.162,γ=34.7%	500	91.1	8.3	4.6	0.2	22.3
RnF1d	ρ_h=58.9%,ρ_l=4.6%,θ=-2.688	500	91.2	12.5	11.8	11.0	17.6
RnF2d	67.8%,17.5%,4.1%,-2.97,-2.20	500	92.0	23.0	10.6	4.4	3.6
5-Apr-06							
BC	12.6%,29.1%,39.1%,51.9%,77.4%	500	101.0	18.0	9.5	6.0	-
Gauss	ρ=12.6%	500	216.5	34.8	5.7	0.2	141.9
Clayton	θ=0.0620	500	216.3	34.4	5.6	0.2	141.5
NIG	α=0.2376λ=0.132,ρ=17.5%	500	101.0	22.1	9.5	1.0	9.2
Double-t	ρ=16.8%,ν=3,$\bar{\nu}$=8	500	154.8	37.3	18.0	6.9	82.6
RndRR	ρ=6.87%,μ=0.776,σ=0.157,γ=23.8%	500	101.0	18.0	1.9	0.01	13.7
RnF1d	ρ_h=47.9%,ρ_l=3.9%,θ=-2.186	500	100.9	27.1	25.8	18.1	37.6
RnF2d	64.5%,11.2%,3.9%,-2.37,-2.15	500	99.6	18.1	11.8	10.7	10.6
10-May-06							
BC	16.3%,35.6%,46.0%,59.5%,84.4%	500	72.5	16.8	7.0	5.0	-
Gauss	ρ=16.3%	500	200.0	40.7	8.8	0.5	157.7
Clayton	θ=0.0796	500	200.0	40.2	8.5	0.4	157.0
NIG	α=0.200,λ=0.115,ρ=21.9%	500	72.5	21.2	10.5	2.1	11.3
Double-t	ρ=20.0%,ν=5,$\bar{\nu}$=7	500	157.3	42.2	18.4	5.6	122.2
RndRR	ρ=8.6%,μ=0.723,σ=0.128,γ=27.6%	500	71.3	16.7	2.2	0.01	11.1
RnF1d	ρ_h=88.2%,ρ_l=4.6%,θ=-2.387	500	72.5	12.1	11.5	11.0	15.7
RnF2d	94.6%,6.82%,2.5%,-2.40,-1.58	500	72.3	13.3	11.6	11.1	14.5