

PDE formulation survey

Huyên PHAM

Laboratoire de Probabilités et

Modèles Aléatoires

CNRS, UMR 7599

Universités Paris 6-Paris 7

e-mail: pham@math.jussieu.fr,

CREST and

Institut Universitaire de France

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1 Introduction

Historically in their seminal paper [3] of 1973, Black and Scholes derived a partial differential equation (PDE) for the call option price by considering a portfolio containing the option and the underlying asset, and by using arbitrage-free arguments. Later in the eighties, Harrison, Kreps and Pliska [12], [13] pioneered the use of stochastic calculus in mathematical finance, and introduce in particular the fundamental notion of risk-neutral probability. The relation between the PDE approach of Black-Scholes and the risk-neutral martingale method is made through the Feynman-Kac formula, and this complementary aspect between PDE and stochastic calculus was largely extended in the mathematical finance literature, providing new developments both from a theoretical and practical viewpoint.

In this survey article, we focus on the various contexts in finance where PDEs arise, in particular for option pricing, portfolio optimization, and calibration. We do not emphasize numerical issues. Our plan is to first recall the basic derivation of the Black-Scholes PDE. We then present the PDEs for various exotic options, and in particular American options. A paragraph is devoted to Hamilton-Jacobi-Bellman equations, which are nonlinear PDEs

arising from stochastic control and portfolio optimization. We finally discuss the PDE associated with calibration problems and indicate further issues for PDE in finance.

2 The Black-Scholes PDE

We recall the basic historical arguments in the derivation of the Black-Scholes PDE for option pricing.

We start from the standard Black-Scholes model with a risk free bond of constant interest rate r , and a stock with a price process S evolving according to a geometric Brownian motion :

$$dS_t = bS_t dt + \sigma S_t dW_t, \quad (2.1)$$

where the drift rate b and the volatility $\sigma > 0$ are assumed to be constant, and W is a standard Brownian motion. We now consider the simplest financial option, the European call option, characterized by its payoff $(S_T - K)_+$ at the maturity T , and with strike K . We denote by V the value of the Call option : $V = V(t, S_t)$ is a function of the spot price of the underlying asset S_t at time t . At the expiry date of the option, we have $V(T, S_T) = (S_T - K)_+$, and for $t < T$, by assuming that the function V is smooth, we get by Itô's formula :

$$\begin{aligned} dV &= \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} dt \\ &= \left(\frac{\partial V}{\partial t} + bS \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \sigma S \frac{\partial V}{\partial S} dW \end{aligned}$$

Now, let us consider a portfolio consisting of the option and of a short position Δ in the asset : the portfolio value is then equal to $\Pi = V - \Delta S$, and its self-financed dynamics is given by :

$$\begin{aligned} d\Pi &= dV - \Delta dS \\ &= \left(\frac{\partial V}{\partial t} + bS \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - \Delta bS \right) dt + \sigma S \left(\frac{\partial V}{\partial S} - \Delta \right) dW \end{aligned}$$

The random component in the evolution of the portfolio Π may be eliminated by choosing

$$\Delta = \frac{\partial V}{\partial S}.$$

This results in a portfolio with deterministic increment :

$$d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt. \quad (2.2)$$

Now, by arbitrage-free arguments, the rate of return of the riskless portfolio Π must be equal to the interest rate r of the bond, i.e. :

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = r\Pi,$$

and recalling that $\Pi = V - \frac{\partial V}{\partial S}S$, this leads to the *Black-Scholes partial differential equation* :

$$rV - \frac{\partial V}{\partial t} - rS \frac{\partial V}{\partial S} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = 0.$$

This PDE together with the terminal condition $V(T, S) = (S - K)_+$ is a linear parabolic Cauchy problem, whose solution is analytically known : this is the celebrated Black-Scholes formula. Moreover, this formula can also be computed as an expectation :

$$V(t, S) = \mathbb{E}^{\mathbb{Q}} \left[e^{-r(T-t)} (S_T - K)_+ \middle| S_t = S \right],$$

where $\mathbb{E}^{\mathbb{Q}}$ denotes the expectation under which the drift rate b in (2.1) is replaced by the interest rate r . This expectation representation was first derived by Harrison, Kreps and Pliska, and \mathbb{Q} is called risk-neutral probability.

3 PDE for European options and linear Feynman-Kac solutions

The derivation presented in the previous paragraph is prototypical. Besides the arbitrage-free arguments, the key points for the derivation of the PDE satisfied by the option price is the Markov property of suitable stochastic processes related to the market model. It is also known that the option price may be represented by a risk-neutral expectation, and the equivalence with the PDE approach is achieved through the Feynman-Kac formula. Notice that in the PDE literature, the Feynman-Kac formula is more familiarly known as the Green's function solution, see e.g. [10]. In its basic (multidimensional) version, the Feynman-Kac representation is formulated as follows : let us consider the stochastic differential equation on \mathbb{R}^n

$$dX_s = b(X_s)ds + \sigma(X_s)dW_s, \tag{3.3}$$

where b and σ are measurable functions valued respectively on \mathbb{R}^n and $\mathbb{R}^{n \times d}$, and W is a n -dimensional Brownian motion. Consider the Cauchy problem

$$rv - \frac{\partial v}{\partial t} - b(x) \cdot D_x v - \frac{1}{2} \text{tr}(\sigma \sigma'(t, x) D_x^2 v) = 0 \quad \text{on } [0, T) \times \mathbb{R}^n, \tag{3.4}$$

$$v(T, x) = g(x) \quad \text{on } \mathbb{R}^n. \tag{3.5}$$

Here, $D_x v$ is the gradient, $D_x^2 v$ is the Hessian matrix of v with respect to the x -variable, σ' is the transpose of σ , and tr denotes the trace of a matrix. Then, the solution to this Cauchy problem may be represented as

$$v(t, x) = \mathbb{E} \left[e^{-\int_t^T r(u, X_u^{t,x}) ds} g(X_T^{t,x}) \right], \quad (3.6)$$

where $X_s^{t,x}$ is the solution to (3.3) starting from x at $s = t$. The Black-Scholes price is a particular case with r constant, $b(t, x) = rx$, $\sigma(t, x) = \sigma x$, and $g(x) = (x - K)_+$. More generally, the interest rate r and the volatility σ may depend on time and spot price. In the general case, we do not have analytical expression for v , and we have to resort on numerical methods for option pricing. The probabilistic representation (3.6) is the basis for Monte-Carlo methods in option pricing while deterministic numerical methods are based on the PDE (3.4).

In the sequel, we state the PDEs for the prices of various European options without providing all the details of the derivation.

Barrier options

The payoff of these options depends on the fact that the underlying asset crossed or not some given barriers during the time interval $[0, T]$. For example, a down-and-out call option has a payoff $(S_T - K)_+ 1_{\inf_{t \in [0, T]} S_t > L}$. Its price $v(t, x)$ at time t for a spot price $S_t = x$, satisfies the boundary value problem :

$$\begin{aligned} rv - \frac{\partial v}{\partial t} - rx \frac{\partial v}{\partial x} - \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 v}{\partial x^2} &= 0, \quad (t, x) \in [0, T) \times (L, \infty) \\ v(t, L) &= 0 \\ v(T, x) &= (x - K)_+. \end{aligned}$$

Lookback options

The payoff of these options involve the maximum or minimum of the underlying asset. For instance, the floating strike lookback put option pays at maturity $M_T - S_T$ where $M_t = \sup_{0 \leq t \leq T} S_t$. The pair (S_t, M_t) is a Markov process, and the price at time t of this lookback option is equal to $v(t, S_t, M_t)$ where the function v is defined for $t \in [0, T]$, $(S, M) \in \{(S, M) \in \mathbb{R}_+^2 : 0 \leq S \leq M\}$, and satisfies the Neumann problem :

$$\begin{aligned} rv - \frac{\partial v}{\partial t} - rS \frac{\partial v}{\partial S} - \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 v}{\partial S^2} &= 0 \\ \frac{\partial v}{\partial M}(t, M, M) &= 0 \\ v(T, S, M) &= M - S. \end{aligned}$$

Asian options

These options involve the average of the risky asset. For example, the payoff of an Asian call option is $(A_T - K)_+$ where $A_t = \frac{1}{t} \int_0^t S_u du$. The pair (S_t, A_t) is a Markov process, and the price at time t of this Asian option is equal to $v(t, S_t, A_t)$ where the function v is defined for $t \in [0, T]$, $(S, A) \in \mathbb{R}_+^2$, and satisfies the Cauchy problem, see e.g. [16] :

$$\begin{aligned} rv - \frac{\partial v}{\partial t} - rS \frac{\partial v}{\partial S} - \frac{1}{t}(S - A) \frac{\partial v}{\partial A} - \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 v}{\partial S^2} &= 0 \\ v(T, S, A) &= (A - K)_+. \end{aligned}$$

4 American options and free boundary problems

With respect to European options presented so far, American options give the holder the right to exercise his right at any time up to maturity. For an American put option of payoff $(K - S_t)_+$, $0 \leq t \leq T$, its price at time t and for a spot stock price $S_t = x$ is given by :

$$v(t, x) = \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}^{\mathbb{Q}} \left[e^{-r(\tau-t)} (K - S_\tau)_+ \middle| S_t = x \right],$$

where $\mathcal{T}_{t,T}$ denotes the set of stopping times valued in $[t, T]$. In terms of PDE, and within the Black-Scholes framework, this leads via the dynamic programming principle (see [7]) to a variational inequality :

$$\min \left[rv - \frac{\partial v}{\partial t} - rx \frac{\partial v}{\partial x} - \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 v}{\partial x^2}, v(t, x) - (K - x)_+ \right] = 0, \quad (4.7)$$

together with the terminal condition : $v(T, x) = (K - x)_+$. This variational inequality may be written equivalently as :

$$rv - \frac{\partial v}{\partial t} - rx \frac{\partial v}{\partial x} - \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 v}{\partial x^2} \geq 0, \quad (4.8)$$

which corresponds to the supermartingale property of the discounted price process $e^{-rt}v(t, S_t)$,

$$v(t, x) \geq (K - x)_+, \quad (4.9)$$

which results directly from the fact that by exercising his right immediately, one receives the option payoff, and :

$$rv - \frac{\partial v}{\partial t} - rx \frac{\partial v}{\partial x} - \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 v}{\partial x^2} = 0, \quad \text{for } (t, x) \in \mathcal{C} = \{v(t, x) > (K - x)_+\}, \quad (4.10)$$

which means that as long as we are in the continuation region \mathcal{C} , i.e. the value of the American option price is strictly greater than its payoff, the holder does not early exercise

his right. The formulation (4.8)-(4.9)-(4.10) is also called a free-boundary problem, and in the case of the American Put, there is an increasing function $x^*(t)$, the free-boundary or critical price, which is smaller than K , and such that $\mathcal{C} = \{(t, x) : x > x^*(t)\}$. This free boundary is an unknown part of the PDE, and separates the continuation region from the exercise region where the option is exercised, i.e. $v(t, x) = (K - x)_+$. The above conditions do not determine the unknown free boundary $x^*(t)$. An additional condition is required, which is the continuous differentiability of the option price across the boundary $x^*(t)$:

$$v(t, x^*(t)) = K - x^*(t), \quad \frac{\partial v}{\partial x}(t, x^*(t)) = -1.$$

This general property is known in optimal stopping theory as the *smooth fit principle*. However, the American option price v is not C^2 , and the nonlinear PDE (4.7) should be interpreted in a weak sense by means of distributions, see [2] or [11], or in the viscosity sense, see [5]. Notice that the main difference between PDE for American options and European options is the nonlinearity of the equation in the former case. This makes the theory and the numerical implementation more difficult than for European options.

5 Stochastic control in finance and Bellman equation

Stochastic control is a traditional domain of applied mathematics, and is an important field in mathematical finance arising typically in portfolio management. This may be formulated in a roughly general framework as follows : we consider a controlled diffusion process in the form

$$dX_s = b(X_s, \alpha_s)ds + \sigma(X_s, \alpha_s)dW_s, \quad \text{in } \mathbb{R}^n, \quad (5.11)$$

where W is a d -dimensional Brownian motion on some filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t), \mathbb{P})$, and $\alpha = (\alpha_t)$ is an adapted process valued in a Borel set $A \subset \mathbb{R}^m$, the so-called control process, which influences the dynamics of the state process X through the drift coefficient b and the diffusion coefficient σ . A stochastic control problem (in a finite horizon) consists of maximizing over the control processes a functional objective in the form

$$\mathbb{E} \left[\int_0^T f(X_t, \alpha_t)dt + g(X_T) \right], \quad (5.12)$$

where f and g are real-valued measurable functions. The method used to solve this problem, and initiated by Richard Bellman in the 50's, is to introduce the value function $v(t, x)$, that is the maximum of the objective when starting from state x at time t , and to apply the dynamic

programming principle (DPP). The DPP formally states that if a control is optimal from time t until T , then it is also optimal from time $t+h$ until T for any $t+h > t$. Mathematically, the DPP relates the value functions at two different dates t and $t+h$, and by studying the behavior of the value functions when h goes to zero, one obtains a PDE satisfied by v , the so-called Hamilton-Jacobi-Bellman equation :

$$-\frac{\partial v}{\partial t} - \sup_{a \in A} \left[b(x, a) \cdot D_x v + \frac{1}{2} \text{tr}(\sigma \sigma'(x, a) D_x^2 v) + f(x, a) \right] = 0, \text{ on } [0, T) \times \mathbb{R}^n, (5.13)$$

together with the terminal condition $v(T, x) = g(x)$. Here, and in the sequel, the symbol prime $'$ is for the transpose. The most famous application of Bellman equation in finance is the portfolio selection of Merton [14]. In this problem, an investor can choose to invest at any time between a riskless bond of interest rate r or a stock with Black-Scholes dynamics of rate of return β and volatility γ . By denoting α_t the proportion of wealth X_t invested in the stock, this corresponds to a controlled wealth process X in (5.11) with $b(x, a) = ax(\beta - r) + rx$ and $\sigma(x, a) = ax\gamma$. The objective of the investor is to maximize his expected utility from terminal wealth, which corresponds to a functional objective of the form (5.12) with $f = 0$, and g an increasing, concave function. A usual choice of utility function is $g(x) = x^p$, with $0 < p < 1$, in which case, there is an explicit solution to the corresponding Hamilton-Jacobi-Bellman equation (5.13). Moreover, the optimal control attaining the argument maximum in (5.13) is a constant equal to $\frac{b-r}{(1-p)\gamma^2}$. In the general case, there is no explicit solution to the HJB equation. Moreover, the solution is not smooth C^2 and one proves actually that the value function is characterized as the unique weak solution to the HJB equation in the viscosity sense. We refer to the books by [9] and [15] for an overview of stochastic control and viscosity solutions in finance.

6 Calibration

It is well known that the constant coefficient Black-Scholes model is not consistent with empirical observations in the markets. Indeed, given a call option with quoted price C_M on the market, one may associate the so-called *implied volatility*, i.e. the volatility σ_{imp} such that the price given by the Black-Scholes formula coincides with C_M . If the Black-Scholes model was sharp, then the implied volatility would not depend on the strike and maturity of the option. However, it is often observed that the implied volatility is far from constant, and is actually a convex function of the strike price, a phenomenon known as the *volatility smile*. Several extensions of the Black-Scholes model have been proposed in the literature. We focus here on local volatility models : the volatility is a function of time and of the spot

price, i.e. $\sigma(t, S_t)$. In this model, the price $C(t, S_t)$ of a call option of strike K and maturity T , satisfies the PDE :

$$\begin{aligned} rC - \frac{\partial C}{\partial t} - rS \frac{\partial C}{\partial S} - \frac{1}{2} \sigma^2(t, S) S^2 \frac{\partial^2 C}{\partial S^2} &= 0, \quad (t, S) \in [0, T) \times (0, \infty) \\ C(T, S) &= (S - K)_+. \end{aligned}$$

The calibration problem in this local volatility model consists of finding a function $\sigma(t, S)$, which reproduces the observed call option prices on the market for all strikes and maturities. In other words, we want to determine $\sigma(t, S)$ in such a way that the prices computed e.g. with the above PDE coincide with the observed prices. The solution to this problem has been provided by B. Dupire [6]. By fixing the date t and the spot price S , and by denoting $C(t, S, T, K)$ the call option price with strike K and maturity $T \geq t$, Dupire showed that it satisfies the forward (with initial condition) parabolic PDE :

$$\frac{\partial C}{\partial T} + rK \frac{\partial C}{\partial K} - \frac{1}{2} \sigma^2(T, K) K^2 \frac{\partial^2 C}{\partial K^2} = 0, \quad (T, K) \in [t, \infty) \times \mathbb{R}_+, \quad (6.14)$$

$$C(t, S, t, K) = (S - K)_+. \quad (6.15)$$

This PDE may be obtained at least by two methods. The first one is based on Itô-Tanaka formula on the risk-neutral expectation representation of the call option price, while the second one is derived by PDE arguments based on the Fokker-Planck equation for the diffusion process (S_t) . This PDE proof can be found e.g. in the book [1]. The PDE (6.14) is useful for local volatility calibration since the local volatility could be computed from the call options prices observed at various strikes K and maturities T :

$$\sigma^2(T, K) = 2 \frac{\frac{\partial C}{\partial T} + rK \frac{\partial C}{\partial K}}{K^2 \frac{\partial^2 C}{\partial K^2}}. \quad (6.16)$$

This is known as Dupire's formula. Notice that (6.16) cannot be used directly since only a finite number of options are quoted on the market. We refer to [4] or [1] for a more advanced discussion and study on calibration problems.

7 Further issues

We have presented so far essentially three types of PDE in finance : linear PDEs arising from European options and Feynman-Kac formulas, variational inequalities arising from American options and optimal stopping, and nonlinear Hamilton-Jacobi-Bellman arising

from stochastic control problems. We now consider semilinear PDEs in the form

$$\begin{aligned} -\frac{\partial v}{\partial t} - b(x) \cdot D_x v - \frac{1}{2} \text{tr}(\sigma \sigma'(x) D_x^2 v) - f(x, v, D_x v' \sigma) &= 0, \quad (t, x) \in [0, T) \times \mathbb{R}^n, \\ v(T, x) &= g(x), \quad x \in \mathbb{R}^n. \end{aligned}$$

Such PDEs arise for instance in finance in option pricing with large investor models or in indifference pricing. From a probabilistic viewpoint, they are connected with the solution (Y, Z) to the backward stochastic differential equation (BSDE) :

$$Y_t = g(X_T) + \int_t^T f(X_s, Y_s, Z_s) ds - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T, \quad (7.17)$$

with

$$X_t = X_0 + \int_0^t b(X_s) ds + \int_0^t \sigma(X_s) dW_s,$$

through

$$Y_t = v(t, X_t), \quad Z_t = D_x v'(t, X_t).$$

By solution to BSDE, we mean a pair of adapted processes (Y, Z) satisfying (7.17). We refer to the seminal paper [8] for an introduction to BSDE and applications in finance. BSDE is a very active field of research both from a theoretical and numerical viewpoint, since they provide alternative probabilistic approaches for the resolution of PDEs arising especially in finance.

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