Further Mathematical Methods: II

In this lecture ...

- Double Integration
  - Introduction and examples
  - Applications to joint probability distributions
  - The gamma function
- Fourier Transforms
  - Definition and standard results

  - Application to the heat equation

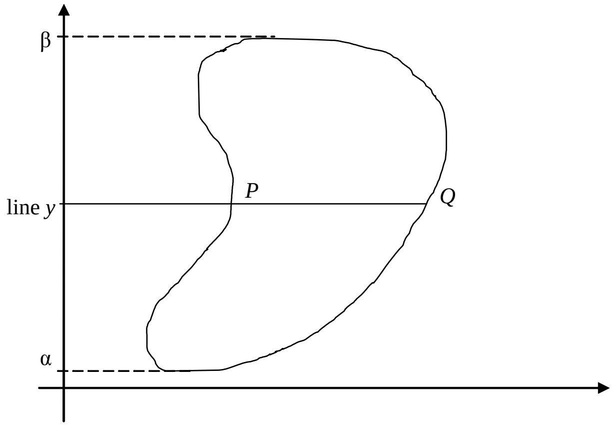
- Power series solutions of Ordinary Differential Equations

1 Double Integration

Evaluation of

$$\iint_A f(x,y)\,dx\,dy$$

where A is the region drawn below.

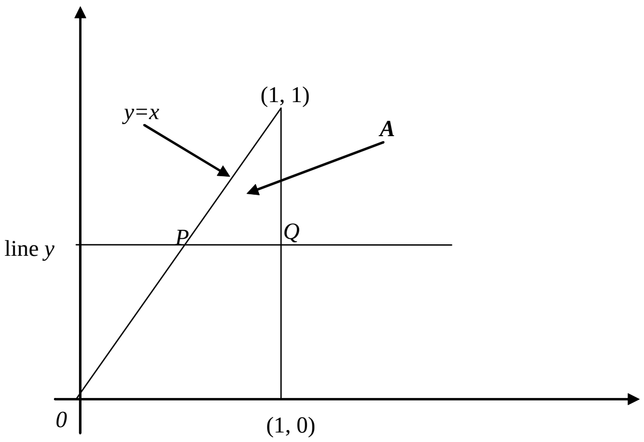


<u>So</u> limits are given by: $\int_{\alpha}^{\beta} \left\{ f(x,y)\Big|_{x_P(y)}^{x_Q(y)} dx \right\} dy$

Example: Evaluate

$$\iint_A (x+y)\,dxdy$$

where A is the $\boldsymbol{\Delta}$ in the following diagram:



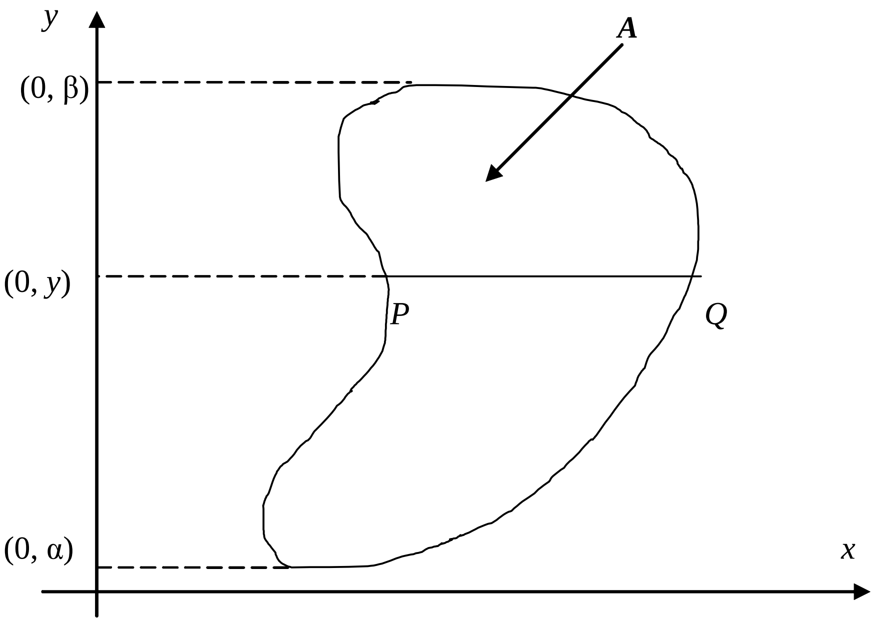
$$\begin{array}{rclc} x_P & = & y & P\,(y,\ y) \\ x_Q & = & 1 & Q\,(1,\ y) \end{array}$$

$$
\begin{aligned}
I &= \int_{y=0}^{y=1} (x+y)\big|_{x_P=y}^{x_Q=1} \; dxdy \\
\int_y^1 (x+y)\,dx &= \left[\frac{x^2}{2}+xy\right]_y^1 = \left(\frac{1}{2}+y\right) - \left(\frac{y^2}{2}+y^2\right) \\
I &= \int_0^1 \left(\frac{1}{2}+y-\frac{3y^2}{2}\right) dy = \left(\frac{y}{2}+\frac{y^2}{2}-\frac{y^3}{2}\right)_0^1 \\
&= \frac{1}{2}
\end{aligned}
$$

So generally

$$\iint_A f(x,y)dxdy$$

where A is defined as



$x_P,\ x_Q$ are functions of y

$$= \underbrace{\int_{\alpha}^{\beta} \left\{ \int_{x_P}^{x_Q} f\left(x,\ y\right) \right\}}_{\text{repeated integral}} dy$$

We note in passing that

$$\iint_A f \ dx \ dy = \iint_{A_1} f \ dx \ dy + \iint_{A_2} f \ dx \ dy$$

A: $A_1 + A_2$

The main problem lies in obtaining the limits. We consider the following examples –

Examples:

1. A Rectangle

$a \leq x \leq b$

$\alpha \leq y \leq \beta$

Here $x_P = a, \; x_Q = b$

$\alpha \leq y \leq \beta$

$\therefore$

$$\iint_A f \; dx \; dy = \int_\alpha^\beta \left\{ \int_a^b f \; dx \right\} dy$$

2. A Triangle

with sides

$$
\begin{aligned}
x + y &= 0 \\
x - y &= 0 \\
y &= 2
\end{aligned}
$$

In this case

$$
\begin{aligned}
x_P &= -y;\ x_Q = y \\
\alpha &= 0;\ \beta = 2
\end{aligned}
$$

$$\iint_A f\ dx\ dy = \int_0^2 \left\{ \int_{-y}^{y} f\, dx \right\} dy$$

3 A is the region defined by

$x^2 + y^2 \leq 1, \ x, \ y \geq 0$

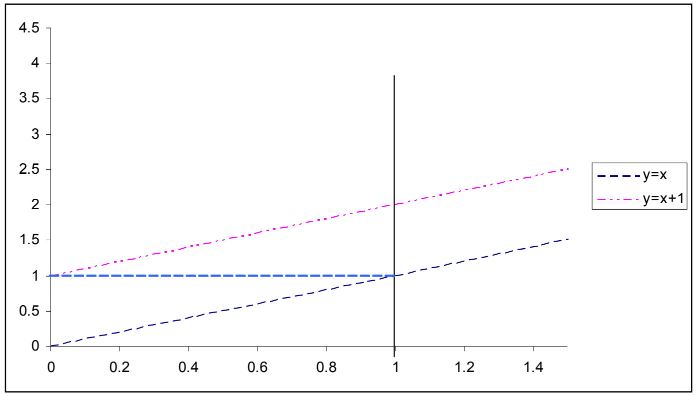
$$\iint_A f \ dx \ dy = \int_0^1 \left\{ \int_0^{\sqrt{1-y^2}} f \ dx \right\} dy$$

Difficulty: A parallelogram

For this A we do not have a simple value for x_P (or x_Q)

For A_1 $\qquad x_P = 0, \ x_P = y$

For A_2 $\qquad x_P = y - 1, \ x_Q = 1$



So

$$\iint_A f\ dx\ dy = \iint_{A_1} f\ dx\ dy + \iint_{A_2} f\ dx\ dy$$

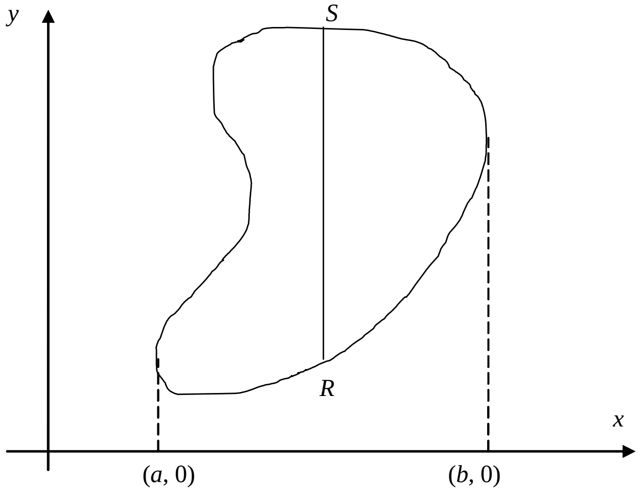
$$\iint_{A_1} = \int_0^1 \left\{ \int_0^y f\ dx \right\} dy \quad (0 \le y \le 1 \text{ in } A_1)$$

$$\iint_{A_2} = \int_1^2 \left\{ \int_{y-1}^1 f \ dx \right\} dy \quad (1 \le y \le 2 \text{ in } A_2)$$

Sometimes, then, we want to do the $y-$integration first:

$$\iint_A f \ dx \ dy =$$

$$\iint_A f \ dy \ dx = \int_a^b \left\{ \int_{y_R}^{y_S} f \ dy \right\} dx$$



Here $y_R,\ y_S$ depend on x

Example:

A is the parallelogram discussed earlier

$y_R = x \quad a = 0$

$y_S = x + 1 \quad b = 1$

$$\iint_A f \ dx \ dy = \int_0^1 \left\{ \int_x^{x+1} f(x, y) \, dy \right\} dx$$

1.1 Uses of Double Integration

<u>AREAS</u>

Theorem

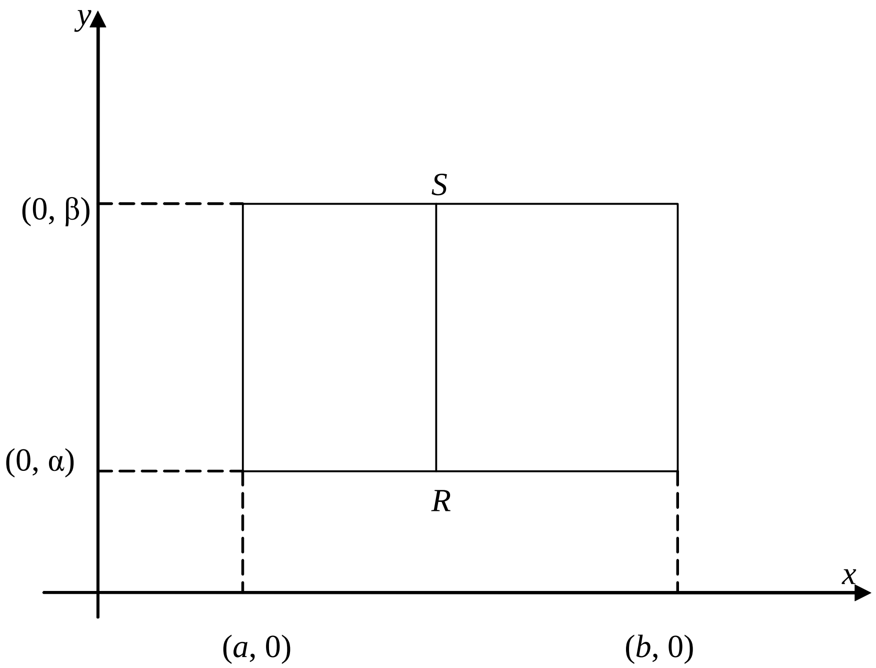
$$\iint_A 1 \; dx \; dy = \text{ area of } A$$

Here we have $f(x, \; y) = 1 \; \forall \, (x, \; y)$ in A

Example

A rectangle $a \leq x \leq b$, $\alpha \leq y \leq \beta$

$$\text{area} = \iint_A 1 \cdot dx dy = \int \left\{ \int 1 \; dy \right\} dx$$



$$
\begin{aligned}
&= \int_a^b [y]_\alpha^\beta \, dx = \int_a^b (\beta - \alpha) \, dx = (\beta - \alpha) \, [x]_a^b \\
&= (\beta - \alpha) \, (b - a)
\end{aligned}
$$

1.2 Changing to Plane Polars

If

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}$$

then

$$\iint_A f(x,y)\,dxdy = \iint_{A'} F(r,\theta)\,rdrd\theta$$

where

1. $F(r,\theta) = f(r\cos\theta, r\sin\theta)$

2. A' is the region A described in (r,θ) coordinates.

To prove

$$\int_0^{\infty} e^{-x^2} dx = \frac{1}{2}\sqrt{\pi}$$

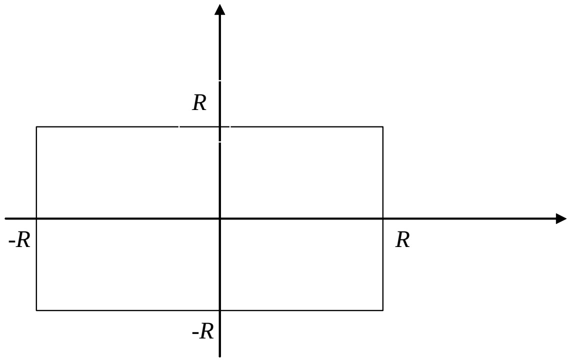
Let

$$I = \int_{-\infty}^{\infty} e^{-x^2} dx$$

then

$$\begin{aligned} I^2 &= \int_{-\infty}^{\infty} e^{-x^2} dx \times \int_{-\infty}^{\infty} e^{-y^2} dy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\left(x^2+y^2\right)} dx dy \end{aligned}$$

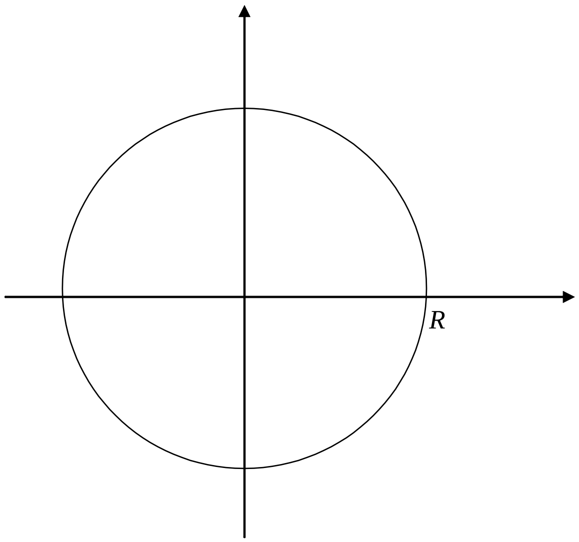
This region can be thought of as the square area as $R \longrightarrow \infty$ in



Now put

$$
\begin{aligned}
x &= r\cos\theta \\
y &= r\sin\theta
\end{aligned}
$$

for $r = R$ and $0 \leq \theta \leq 2\pi$, where $R \longrightarrow \infty$



So the integral becomes

$$I^2 = \int_{\theta=0}^{2\pi} \int_{r=0}^{\infty} e^{-r^2} r dr d\theta$$

$$\int_{r=0}^{\infty} e^{-r^2} r dr = \frac{1}{2}$$

so

$$\begin{aligned} I^2 &= \frac{1}{2} \int_{\theta=0}^{2\pi} d\theta = \pi \\ I &= \sqrt{\pi} \end{aligned}$$

hence

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-x^2} dx &= \sqrt{\pi} \\ \int_{0}^{\infty} e^{-x^2} dx &= \frac{1}{2}\sqrt{\pi} \end{aligned}$$

1.3 General Change of Variable in Double Integrals

Plane polars provide us with a useful change of variable technique when A has a circular boundary. For more general shapes we need a robust and generalised method. We know the double integral of a function $z = f(x, y)$ is

$$\iint_A f \, dx dy$$

If $f(x, y)$ is difficult to integrate, simplify the problem by making a change of variables u and v, given by

$$\iint_A f(x, y) \, dx dy = \iint_{A'} F(u, v) \, |J| \, du dv$$

where

1. $F(u, v) = f(x(u, v), y(u, v))$

2. A' is region A in terms of new coordinates

3. The *Jacobean* J is defined by the determinant

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

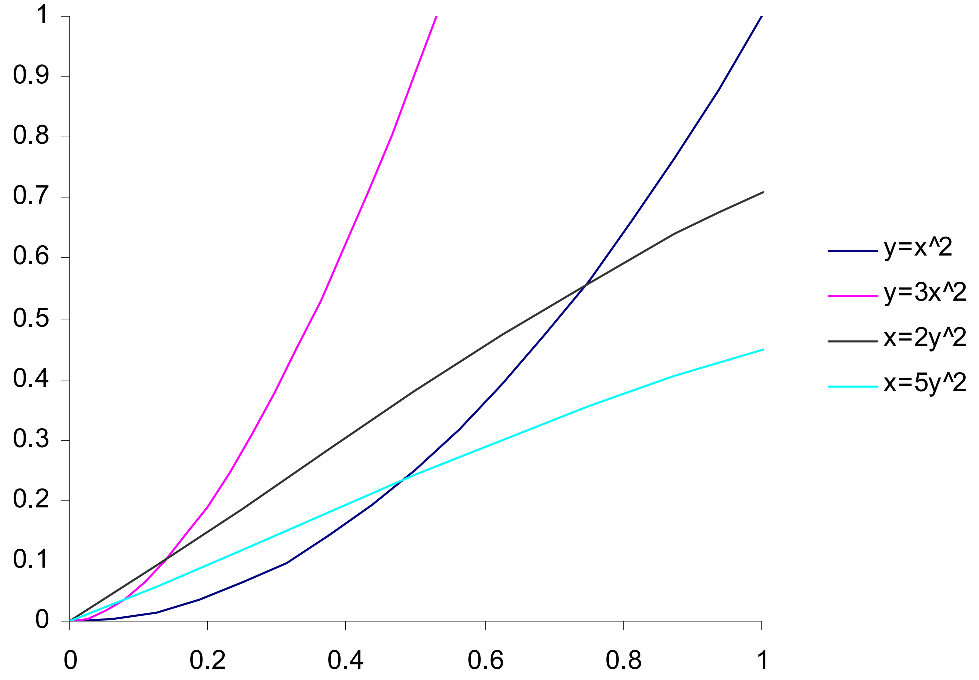
   we assume $J \neq 0$.

Example: Evaluate the integral

$$\iint_A xy\,dx\,dy$$

where A is the finite region in the first quadrant bounded by the four curves

$$y = x^2,\ y = 3x^2;\ x = 2y^2,\ x = 5y^2$$



Introduce new variables $u, \ v$ by

$$\left.\begin{array}{l} \frac{y}{x^2} = u \\ \frac{x}{y^2} = v \end{array}\right\} \longrightarrow \quad \left.\begin{array}{l} y = ux^2 \\ x = vy^2 \end{array}\right\}$$

combining these gives $x\,(u, v)$ and $y\,(u, v)$

$$\begin{aligned} x &= u^{-2/3}v^{-1/3} \\ y &= u^{-1/3}v^{-2/3} \end{aligned}$$

Now calculate the Jacobean $J =$

$$\begin{aligned} \left| \begin{array}{cc} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{array} \right| &= \left| \begin{array}{cc} -\frac{2}{3}u^{-5/3}v^{-1/3} & -\frac{1}{3}u^{-2/3}v^{-4/3} \\ -\frac{1}{3}u^{-4/3}v^{-2/3} & -\frac{2}{3}u^{-1/3}v^{-5/3} \end{array} \right| \\ &= \frac{1}{3u^2v^2} \end{aligned}$$

The integrand is $f\,(x, y) = xy$

$$F\,(u, v) = \underbrace{\left(u^{-2/3}v^{-1/3}\right)}_{x}\underbrace{\left(u^{-1/3}v^{-2/3}\right)}_{y} = 1/uv$$

hence

$$\begin{aligned}\iint_A xy\,dx\,dy &= \iint_{A'} (1/uv)\,\frac{1}{3u^2v^2}du\,dv \\ &= \frac{1}{3}\iint_{A'}\frac{1}{u^3v^3}du\,dv\end{aligned}$$

Now turn to A'. The parabolas

$$\left.\begin{array}{c} y = x^2 \\ y = 3x^2 \end{array}\right\} \text{ or } \left.\begin{array}{c} \frac{y}{x^2} = 1 \\ \frac{y}{x^2} = 3 \end{array}\right\}$$

become straight lines $u = 1$, $u = 3$ in the $u - v$ plane.

In a similar way

$$\left.\begin{array}{c} x = 2y^2 \\ x = 5y^2 \end{array}\right\} \text{ or } \left.\begin{array}{c} \frac{x}{y^2} = 2 \\ \frac{x}{y^2} = 5 \end{array}\right\}$$

become straight lines $v = 2$, $v = 5$ in the $u - v$ plane.

This very nicely takes a somewhat complex region in $x-y$ to a rectangular area in $u-v$, and our problem simplifies to

$$\frac{1}{3}\int_{2}^{5}\int_{1}^{3}\frac{1}{u^{3}v^{3}}dudv.$$

Example: Calculate the area of the finite region bounded by $y = x$ and $y = x^2$, between $x = 0$ and $x = 1$. Doing this as a simple A-level maths problem yields a value of $1/6$.

Here we will construct a double integral of the form $\iint_A 1 dx dy$.

$$\begin{aligned}
\int_0^1 \int_{x=y}^{x=\sqrt{y}} 1 dx dy &= \int_0^1 (\sqrt{y} - y)\, dy = \tfrac{2}{3} y^{3/2} - \tfrac{1}{2} y^2 \Big|_0^1 = 1/6 \\
\int_0^1 \int_{y=x^2}^{y=x} 1 dy dx &= \int_0^1 \left(x - x^2\right) dx = \tfrac{1}{2} x^2 - \tfrac{1}{3} x^3 \Big|_0^1 = 1/6
\end{aligned}$$

So 3 different methods of solution all arriving at the same answer!

1.4 Joint PDF for Continuous Random Variables

Recall that the cumulative distribution function $F(x)$ of a RV X is

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} p(s)\, ds$$

$F(x)$ is related to the PDF $p(x)$ by

$$p(x) = \frac{dF}{dx}.$$

Consider the pair (X, Y) with joint pdf $p_{XY}(x, y)$ and cdf $F_{XY}(x, y)$. They are related through a similar fashion

$$p_{XY}(x, y) = \frac{\partial^2}{\partial x \partial y} F_{XY}(x, y).$$

Integrating this (as before) gives the cdf as

$$F_{XY}(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} p_{XY}(s, t)\, dt ds$$

which allows us to calculate the probability

$$\mathbb{P}\left(X \leq x,\ Y \leq y\right).$$

We can extend the simple properties of $p_{XY}(x, y)$ to two dimensions:

- $p_{XY}(x, y) \geq 0$

- $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_{XY}(x, y)\, dx dy = 1$

- $\iint_R p_{X,\ Y}(x, y)\, dx dy = \mathbb{P}\left((X, Y) \in R\right)$ for all regions R

- $\mathbb{P}\left(a < X < b,\ c < Y < d\right) = \int_c^d \int_a^b p_{XY}(x, y)\, dx dy$

If X and Y are independent random variables the cdf can be expressed in separable form

$$F_{XY}(x,y) = F_X(x)\, F_Y(y)\,.$$

Then differentiating gives

$$\begin{aligned}
\frac{\partial^2 F_{XY}(x,y)}{\partial x \partial y} &= \frac{\partial F_X}{\partial x}\frac{\partial F_Y}{\partial y} \\
p_{XY}(x,y) &= p_X(x)\, p_Y(y)\,.
\end{aligned}$$

Example: Consider the joint pdf defined by $p_{XY}(x, y) = e^{-(x+y)}$. To calculate $\mathbb{P}(1 < X < 2,\ 0 < Y < 2)$ we solve

$$\begin{aligned}\int_1^2 \int_0^2 e^{-(x+y)} dx dy &= \int_1^2 e^{-x} dx \int_0^2 e^{-y} dy \\ &= \left(e^{-1} - e^{-2}\right)\left(1 - e^{-2}\right) = 0.2\end{aligned}$$

1.5 The Gamma Function Revisited

The Gamma Function $\Gamma(x)$ is defined as

$$\Gamma(x) = \int_0^\infty e^{-t}\, t^{x-1} dt \quad (x > 0)$$

The condition on x is a convergent criterion.

Theorem

$$\int_0^{\pi/2} \cos^{2m-1}\theta \sin^{2n-1}\theta \; d\theta = \frac{\Gamma(m)\,\Gamma(n)}{2\Gamma(m+n)}$$

Proof Start with the definition of the gamma function

$$\Gamma(m) = \int_0^\infty t^{m-1} e^{-t} dt$$

and make the substitution $t = x^2 \longrightarrow dt = 2x dx$ which gives

$$\begin{aligned}
\Gamma(m) &= \int_0^\infty \left(x^2\right)^{m-1} \exp\left(-x^2\right).2x dx \\
&= 2\int_0^\infty x^{2m-1} \exp\left(-x^2\right) dx
\end{aligned}$$

Similarly

$$\Gamma(n) = 2\int_0^\infty y^{2n-1} \exp\left(-y^2\right) dy$$

therefore

$$\begin{aligned}
\Gamma(m)\Gamma(n) &= 4\left(\int_0^\infty x^{2m-1} \exp\left(-x^2\right) dx\right)\left(\int_0^\infty y^{2n-1} \exp\left(-y^2\right) dy\right) \\
&= 4\int\int_A x^{2m-1} y^{2n-1} e^{-\left(x^2+y^2\right)} dx dy
\end{aligned}$$

where A is the region of integration defined by the first (positive) quadrant. Introduce polar coordinates

$$\begin{aligned}
x &= r\cos\theta \\
y &= r\sin\theta
\end{aligned}$$

to transform the integrand to

$$r^{2m+2n-2}\cos^{2m-1}\theta\sin^{2n-1}\theta\exp\left(-r^2\right)$$

and $dxdy \longrightarrow rdrd\theta$

$$\Gamma(m)\Gamma(n) = 4\underbrace{\int_0^{\pi/2}\cos^{2m-1}\theta\sin^{2n-1}\theta d\theta}_{\text{integral we want}}\underbrace{\int_0^{\infty} r^{2(m+n)-1}e^{-\left(r^2\right)}dr}_{\frac{1}{2}\Gamma(m+n)}$$

so rearranging gives the result

$$\int_0^{\pi/2}\cos^{2m-1}\theta\sin^{2n-1}\theta d\theta = \frac{\Gamma(m)\Gamma(n)}{2\Gamma(m+n)}$$

Example Calculate $\displaystyle\int_0^{\pi/2}\cos^4\theta\sin^3\theta\ d\theta$

Hence

$$\begin{aligned} 2m-1 &= 4 \longrightarrow m = 5/2 \\ 2n-1 &= 3 \longrightarrow n = 2 \end{aligned}$$

so integral equals

$$\frac{\Gamma\left(\frac{5}{2}\right)\Gamma(2)}{2\Gamma\left(\frac{9}{2}\right)} = \frac{\frac{3}{2}.\frac{\sqrt{\pi}}{2}.1}{2\left(\frac{7}{2}.\frac{5}{2}.\frac{3}{2}.\frac{\sqrt{\pi}}{2}\right)} = \frac{2}{35}$$

Example $I = \int_0^{\pi/2} \cos^6\theta \; d\theta$

$$\begin{aligned} 2m-1 &= 6 \longrightarrow m = 7/2 \\ 2n-1 &= 0 \longrightarrow n = 1/2 \end{aligned}$$

Hence $I =$

$$\frac{\Gamma\left(\frac{7}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma(4)} = \frac{\frac{5}{2}.\frac{3}{2}.\frac{1}{2}.\frac{\sqrt{\pi}}{2}.\sqrt{\pi}}{2(3.2)} = \frac{5\pi}{32}$$

2 The Fourier Transform

If $f = f(x)$ then consider

$$\widehat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{ix\omega} dx.$$

If this integral converges, it is called the *Fourier Transform* of $f(x)$. Similar to the case of Laplace Transforms, it is denoted as $\mathcal{F}(f)$, i.e.

$$\mathcal{F}(f) = \int_{-\infty}^{\infty} f(x)\, e^{ix\omega} dx = \widehat{f}(\omega).$$

The *Inverse Fourier Transform* is then

$$\mathcal{F}^{-1}\left(\widehat{f}(\omega)\right) = \int_{-\infty}^{\infty} \widehat{f}(\omega)\, e^{-ix\omega} d\omega = f(x).$$

The convergent property means that $\widehat{f}(\omega)$ is bounded and we have

$$\int_{-\infty}^{\infty} |f(x)|\, dx < \infty.$$

Functions of this type $f(x) \in L_1(-\infty, \infty)$ and are called *square integrable.*

We know from integration (basic property of Riemann integral) that

$$\left| \int_a^b f(x)\, dx \right| \leq \int_a^b |f(x)|\, dx.$$

Hence

$$\begin{aligned} \left|\widehat{f}(\omega)\right| &= \left| \int_{\mathbb{R}} f(x)\, e^{ix\omega} dx \right| \\ &\leq \int_{\mathbb{R}} \left| f(x)\, e^{ix\omega} \right| dx \end{aligned}$$

and Euler's identity $e^{i\theta} = \cos\theta + i\sin\theta$ implies that $\left|e^{i\theta}\right| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$, therefore

$$\left|\widehat{f}(\omega)\right| \leq \int_{\mathbb{R}} |f(x)|\, dx < \infty.$$

In addition to the boundedness of $\widehat{f}(\omega)$, it is also continuous (requires a $\delta - \epsilon$ proof).

Example: Obtain the Fourier transform of $f\ (x) = e^{-|x|}$

$$
\begin{aligned}
\widehat{f}(\omega) &= \mathcal{F}(f) = \int_{-\infty}^{\infty} f\ (x)\, e^{ix\omega} dx \\
&= \int_{-\infty}^{\infty} e^{-|x|} e^{ix\omega} dx \\
&= \int_{-\infty}^{0} e^{-|x|} e^{ix\omega} dx + \int_{0}^{\infty} e^{-|x|} e^{ix\omega} dx \\
&= \int_{-\infty}^{0} e^{x} e^{ix\omega} dx + \int_{0}^{\infty} e^{-x} e^{ix\omega} dx = \\
& \int_{-\infty}^{0} \exp\left[(1+i\omega)\, x\right] dx + \int_{0}^{\infty} \exp\left[-\,(1-i\omega)\, x\right] dx \\
&= \frac{1}{(1+i\omega)} \exp\left[(1+i\omega)\, x\right]\Big|_{-\infty}^{0} - \frac{1}{(1-i\omega)} \exp\left[-\,(1-i\omega)\, x\right]\Big|_{0}^{\infty} \\
&= \frac{1}{(1+i\omega)} + \frac{1}{(1-i\omega)} = \frac{2}{\left(1+\omega^{2}\right)}
\end{aligned}
$$

Our interest in differential equations continues, hence the reason for introducing this transform. We now look at obtaining Fourier transforms of derivative terms. We assume that $f\ \ (x)$ is continuous and $f(x) \to 0$ as $x \to \pm\infty$. Consider

$$\mathcal{F}\left\{f'(x)\right\} = \int_{\mathbb{R}} f'(x)\, e^{ix\omega} dx$$

which is simplified using integration by parts

$$f(x)\, e^{ix\omega}\Big|_{-\infty}^{\infty} - i\omega \int_{\mathbb{R}} f\ \ (x)\, e^{ix\omega} dx$$

so

$$\mathcal{F}\left\{f'(x)\right\} = -i\omega \int_{\mathbb{R}} f\ \ (x)\, e^{ix\omega} dx = -i\omega \widehat{f}(\omega).$$

We can obtain the Fourier transform for the second derivative by performing integration by parts (twice) to give

$$\mathcal{F}\left\{f''(x)\right\} = (-i\omega)^2\, \mathcal{F}\left\{f(x)\right\} = -\omega^2 \widehat{f}(\omega).$$

$$\boxed{\mathcal{F}\left\{f'(x)\right\} = -i\omega \widehat{f}(\omega)}$$
$$\boxed{\mathcal{F}\left\{f''(x)\right\} = -\omega^2 \widehat{f}(\omega)}$$

Example: Solve the diffusion equation problem

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad t > 0\\ u(x,0) &= e^{-|x|}, \quad -\infty < x < \infty\end{aligned}$$

Here $u = u(x,t)$, so we begin by defining

$$\mathcal{F}\left\{u(x,t)\right\} = \int_{-\infty}^{\infty} u(x,t)\, e^{ix\omega} dx = \widehat{u}(\omega, t).$$

Now take Fourier transforms of our PDE, i.e.

$$\mathcal{F}\left\{\frac{\partial u}{\partial t}\right\} = \mathcal{F}\left\{\frac{\partial^2 u}{\partial x^2}\right\}$$

to obtain

$$\frac{d\widehat{u}}{dt} = -\omega^2 \widehat{u}\,(\omega, t)\,.$$

We note that the second order PDE has been reduced to a first order equation of type variable separable. This has general solution

$$\widehat{u}\,(\omega, t) = Ce^{-\omega^2 t}.$$

We can find the constant C by transforming the initial condition

$$\begin{aligned}
\mathcal{F}\left\{u\,(x, 0)\right\} &= \mathcal{F}\left\{e^{-|x|}\right\} \\
\widehat{u}\,(\omega, 0) &= \int_{-\infty}^{\infty} e^{-|x|} e^{ix\omega} dx = \frac{2}{\left(1 + \omega^2\right)}.
\end{aligned}$$

Applying this to the solution $\widehat{u}\,(\omega, t)$ gives

$$\widehat{u}\,(\omega, 0) = C = \frac{2}{\left(1 + \omega^2\right)},$$

hence

$$\widehat{u}(\omega,t)=\frac{2}{\left(1+\omega^2\right)}e^{-\omega^2 t}.$$

We now use the inverse transform to get $u(x,t)=\mathcal{F}^{-1}(\widehat{u}(\omega,t))$

$$\begin{aligned}
&= \int_{-\infty}^{\infty}\widehat{u}(\omega,t)\,e^{-ix\omega}d\omega \\
&= 2\int_{-\infty}^{\infty}\tfrac{1}{(1+\omega^2)}e^{-\omega^2 t}e^{-ix\omega}d\omega \\
&= 2\int_{-\infty}^{\infty}\frac{1}{\left(1+\omega^2\right)}e^{-\omega^2 t}\left(\cos x\omega - i\sin x\right)d\omega \\
&= 2\int_{-\infty}^{\infty}\tfrac{1}{(1+\omega^2)}e^{-\omega^2 t}\cos x\omega\ d\omega - 2i\int_{-\infty}^{\infty}\tfrac{1}{(1+\omega^2)}e^{-\omega^2 t}\sin x\omega\ d\omega.
\end{aligned}$$

This now simplifies nicely because $\dfrac{1}{\left(1+\omega^2\right)}e^{-\omega^2 t}\sin x\omega$ is an odd function,

hence

$$\int_{-\infty}^{\infty} \frac{1}{\left(1+\omega^2\right)} e^{-\omega^2 t} \sin x\omega \ d\omega = 0.$$

Therefore

$$u\left(x,t\right) = 2\int_{-\infty}^{\infty} \frac{1}{\left(1+\omega^2\right)} e^{-\omega^2 t} \cos x\omega \ d\omega.$$

In order to solve this we now need to use *Residues* (Complex Analysis).

3 Power Series Solutions

3.1 Introduction

The Euler equation has a nice structure, i.e.

$$ax^2\frac{d^2y}{dx^2} + bx\frac{dy}{dx} + cy = 0$$

where the order of each derivative term and power of its coefficient in x is the same. The next step is to move away from this "nice pattern" and consider a more general equation of the form

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0 \tag{1}$$

We look for solutions in the neighbourhood of $x = 0$.

We say that $x = 0$ is an *ordinary point* of the differential equation (1) if both $p(x)$ and $q(x)$ have Taylor expansions about $x = 0$.

i.e.

$$
\begin{aligned}
p(x) &= p_0 + p_1 x + p_2 x^2 + O\left(x^3\right) \\
q(x) &= q_0 + q_1 x + q_2 x^2 + O\left(x^3\right)
\end{aligned}
$$

with both $p_i, \quad q_i \sim O(1)$ where $i = 0, 1, ..., n$.

If either or both $p(x), \ q(x)$ do not have Taylor expansions about $x = 0$, then $x = 0$ is a *singular point* for the D.E.

Regular Singular Point: $xp(x)$ and $x^2 p(x)$ have Taylor expansions about $x = 0$.

Irregular Singular Point: all other points.

Examples:

1. $x\frac{d^2y}{dx^2} + x^3\frac{dy}{dx} + xy = 0$

This can written in standard form as $\frac{d^2y}{dx^2} + x^2\frac{dy}{dx} + y = 0 \Rightarrow p(x) = x^2$ & $q(x) = 1$ which both have Taylor expansions about $x = 0$.

Therefore $x = 0$ is an ordinary point of the differential equation.

2. $x^3 \frac{d^2y}{dx^2} + 2x^2 \frac{dy}{dx} + 5x^2 y = 0$

which becomes $\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} + \frac{5}{x} y = 0$ and $p(x) = \frac{2}{x}$ & $q(x) = \frac{5}{x}$ do not have a Taylor expansion about $x = 0$ - however $xp(x) = 2$ & $x^2 q(x) = 5x$ do.

Therefore $x = 0$ is a regular singular point of the differential equation.

3. $\frac{d^2y}{dx^2} - \frac{1}{x^2}\frac{dy}{dx} + \frac{4}{x^3} y = 0$

$p(x) = O\left(\frac{1}{x^2}\right)$ & $xp(x) = O\left(\frac{1}{x}\right)$; $\qquad q(x) = O\left(\frac{1}{x^3}\right)$ & $x^2 q(x) = O\left(\frac{1}{x}\right)$

None of these expressions have a Taylor expansion about $x = 0$.

Therefore $x = 0$ is an irregular singular point of the given differential equation.

3.2 Ordinary Point

Assume a solution of (1) of the form

$$y(x) = \sum_{n=0}^{\infty} A_n \, x^n \quad (A_0 \neq 0) \tag{2}$$

with A_n constant and $A_{-n} = 0$.

Since no boundary conditions are imposed, the general solution involves two arbitrary constants - else the constants can be determined.

Substitute (2) into the equation given by (1) and equate to zero the coefficients of various powers of x.

$$
\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} A_n\, x^n \rightarrow q(x)\, y \sim \left(q_0 + q_1 x + q_2 x^2\right)\left(A_0 + A_1 x + A_2 x^2\right) \\
y'(x) &= \sum_{n=0}^{\infty} n\, A_n\, x^{n-1} \rightarrow p(x)\, y' \sim \left(p_0 + p_1 x + p_2 x^2\right)\left(A_1 + 2A_2 x + 3A_3 x^2\right) \\
y''(x) &= \sum_{n=0}^{\infty} n(n-1)\, A_n\, x^{n-2} \rightarrow y'' \sim 2A_2 + 6A_3 x + 12A_4 x^2
\end{aligned}
$$

$$
2A_2 + 6A_3 x + (p_0 + p_1 x)(A_1 + 2A_2 x) + (q_0 + q_1 x)(A_0 + A_1 x) = 0
$$

$O(1)$: $\quad A_0 q_0 + A_1 p_0 + 2A_2 = 0$

$O(x)$: $\quad q_0 A_1 + 2p_0\, A_2 + p_1 A_1 + q_1 A_0 + 6A_3 = 0$

All coefficients can be expressed in terms of A_0 and A_1 which can be arbitrary.

$\boxed{\text{Example}}$

Obtain the general solution of

$$y'' - 2xy' + y = 0$$

about the ordinary point $x = 0$.

We assume a solution of the form $y(x) = \sum_{n=0}^{\infty} A_n\ x^n$ and substitute the expression and its derivatives into the ODE to yield

$$\sum_{n=0}^{\infty} n(n-1) A_n x^{n-2} - 2x \sum_{n=0}^{\infty} n A_n x^{n-1} + \sum_{n=0}^{\infty} A_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1) A_n x^{n-2} + \sum_{n=0}^{\infty} (1-2n) A_n x^n = 0$$

We require a recurrence relation for which a "trick" is used in the summation. In the second summation above, the n term is changed to $(n-2)$ to give $\sum\limits_{n-2=0}^{\infty}(1-2(n-2))A_{n-2}x^{n-2}$ which is equivalent to having $\sum\limits_{n=2}^{\infty}$ ....

$$\sum_{n=0}^{\infty} n(n-1)A_n x^{n-2} + \sum_{n=2}^{\infty}(1-2(n-2))A_{n-2}x^{n-2} = 0$$

We are still unable to write the lhs of the expression above as one term of $O\left(x^{n-2}\right)$, because the lower limit of the first summation starts at $n=0$, whilst the other begins at $n=2$. This minor problem can be easily overcome by writing

$$\sum_{n=0}^{\infty} n(n-1)A_n x^{n-2} + \sum_{n=0}^{\infty}(5-2n)A_{n-2}x^{n-2} = 0 \qquad (\dagger)$$

because $A_{-2} = A_{-1} = 0$ and $A_0 \neq 0$, and (†) can now be expressed as

$$\sum_{n=0}^{\infty} \{n\,(n-1)\,A_n + (5-2n)\,A_{n-2}\}\ x^{n-2} = 0.$$

Collecting coefficients of x^{n-2} :

$$A_n = \frac{(2n-5)}{n\,(n-1)} A_{n-2} \quad (n \geq 2)$$

or

$$A_{n+2} = \frac{(2n-1)}{(n+2)\,(n+1)} A_n$$

which gives us the recurrence relationship which we sought.

$$n=0: \quad A_2 = -\frac{1}{2} A_0; \quad n=1: \quad A_3 = \frac{1}{6} A_1 = \frac{1}{3!} A_1$$

So we see that all terms A_{2k} will be in terms of A_0 and odd ones A_{2k} in terms of A_1.

$$
\begin{aligned}
n &= 2: \quad A_4 = \frac{3}{4.3} A_2 = -\frac{3}{4.3}\frac{1}{2} A_0 = -\frac{3}{4!} A_0 \\
n &= 3: \quad A_5 = \frac{5}{5.4} A_3 = \frac{5}{5.4}\frac{1}{3!} A_1 = \frac{5}{5!} A_1 \\
n &= 4: \quad A_6 = \frac{7}{6.5} A_4 = -\frac{7}{6.5}\frac{3}{4!} A_0 = -\frac{21}{6!} A_0 \\
n &= 5: \quad A_7 = \frac{9}{7.6} A_5 = \frac{9}{7.6}\frac{5}{5!} A_1 = \frac{45}{7!} A_1
\end{aligned}
$$

The solution is

$$
\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} A_n\, x^n = \sum_{k=0}^{\infty} \left(A_{2k}\, x^{2k} + A_{2k+1}\, x^{2k+1} \right) \\
&= A_0 \underbrace{\left[1 - \frac{1}{2!} x^2 - \frac{3}{4!} x^4 - \frac{21}{6!} x^6 + O\left(x^8\right) \right]}_{=y_1} + \\
& \quad A_1 \underbrace{\left[x + \frac{1}{3!} x^3 - \frac{5}{5!} x^5 + \frac{451}{7!} x^7 + O\left(x^9\right) \right]}_{=y_2} \\
&= A_0 y_1(x) + A_1 y_2.
\end{aligned}
$$

The linear combination $A_0 y_1(x) + A_1 y_2$ becomes the general solution of the equation. The terms A_0 , A_1 are arbitrary.

Appendix

$$\int_0^\infty r^{2(m+n)-1} e^{-\left(r^2\right)} dr = \frac{1}{2}\Gamma\left(m+n\right)$$

Put

$$\begin{aligned} r^2 &= t \longrightarrow 2r dr = dt \\ dr &= \frac{1}{2r} dt = \frac{1}{2} t^{-1/2} dt \end{aligned}$$

the integral

$$\int_0^\infty r^{2(m+n)-1} e^{-\left(r^2\right)} dr$$

becomes

$$
\begin{aligned}
& \int_0^\infty t^{(m+n)} t^{-1/2} e^{-t} \frac{1}{2} t^{-1/2} dt \\
= & \frac{1}{2} \int_0^\infty t^{(m+n)-1} e^{-t} dt \\
= & \frac{1}{2} \Gamma(m+n)
\end{aligned}
$$