Certificate in Quantitative Finance (CQF)

Session 5.1: Structural Models *

Problem Sheet

1 Merton (1974): Model Calibration

As we have seen in the context of the Merton (1974) model, at any time t the firm assets V_t are assumed to be sum of its debt D and its equity E_t,

$$V_t = E_t + D. \tag{1}$$

Consider the stituation at maturity. If $V_T < D$ the company will default on its debt at time $t = T$. The value of the equity is then zero. If $V_T > D$ the company can and should make its debt payment at time $t = T$. In this case the value of equity at maturity is $V_T - D$. In Merton's model therefore the value of the firm equity at time T is

$$E_T = max(V_T - D, 0) \tag{2}$$

The equity therefore can be interpreted as a call option on the value of the assets with a strike price equal to the repayment required by the debt. Using the Black-Scholes formula we can obtain the value of the equity as seen today ($t = 0$),

$$E_0 = V_0 N(d_1) - De^{-rT} N(d_2) \tag{3}$$

where

$$d_1 = \frac{1}{\sigma_V \sqrt{T}} \left[\ln\left(\frac{V_0}{D}\right) + \left(r + \frac{1}{2}\sigma_V^2\right) T \right]$$
$$d_2 = d_1 - \sigma_V \sqrt{T}$$

The value of the debt today is then $V_0 - E_0$. As have been seen in the class, the risk-neutral probability that the company will default on its debt

*Tutor: Dr Alonso Peña (alonso.pena@sdabocconi.it) is $N(-d_2)$. To calculate this we require both V_0 and σ_V, but neither of these is directly observable in the market. However, if the company is publicly traded we can observe E_0. Equation (3) thus offers us one condition that must be satisfied by V_0 and σ_V. The other condition can be obtained from applying Ito's lemma to obtain

$$\sigma_E E_0 = \frac{\partial E}{\partial V} \sigma_V V_0 \tag{4}$$

which is equivalent to

$$\sigma_E E_0 = N(d_1) \sigma_V V_0 \tag{5}$$

This represent a second equation that must be satisfied by V_0 and σ_V. Equation (3) and Equation (5) represent a system of simultaneous non-linear equations which can be solved to imply the value of V_0 and σ_V from the market and, with these, the implied probability of default of a company.

Tasks:

Assume that you are analysing a company described by the following market data: the value of the company's equity is 3 millon USD, the volatility of its equity is 70%. The debt that has to be paid is in 1 year is 10 million USD. The risk free rate is 5% per annum. Thus, $E_0 = 3$, $\sigma_E = 0.70$, $r = 0.05$, $T = 1$, and $D = 10$.

(a) Determine the values of V_0 and σ_V and the firm's probability of default at 1 year.

(b) To investigate the effect of the uncertainty in the input parameters, solve the same problem using a range of equity volatilities $\sigma_E = 10\%, 20\%, 30\%, 40\%, 50\%, 60\%$.

Hint: Setup the problem in an Excel spreadsheet and use the solver feature. Reference: Merton, R., 1974, "On the Pricing of Corporate Debt: the Risk Structure of Interest Rates," Journal of Finance 29, 449-470.

2 Black and Cox (1976): Default Probabilities

In the context of a simplified First Passage Model of the Black and Cox (1976) type, assume that the dynamics of the firm assets value under the risk neutral probability measure **P** are given by the diffusion process

$$dV_t = rV_t dt + \sigma_V V_t dW_t \tag{6}$$

and that there is a lower barrier level for the asset value such that the firm defaults once it reaches this level. Let us assume a constant default threshold $K > 0$. If we are at time $t \geq 0$ and default has not been triggered yet and $V_t > K$, then the time of default τ is given by

$$\tau = \inf \{ s \geq t | V_S \leq K \} \tag{7}$$

Using the properties of Brownian motion, it can be shown that the default probability from time t to time T is given by $P(t,T) = P[\tau \leq T | \tau > t]$ or

$$P(t,T) = N(h_1) + \exp\left\{ 2\left(r - \frac{\sigma_V^2}{2} \right) \ln\left(\frac{K}{V_t} \right) \frac{1}{\sigma_V^2} \right\} N(h_2) \tag{8}$$

where

$$h_1 = \frac{\ln\left(\frac{K}{e^{r(T-t)} V_t} \right) + \frac{\sigma_V^2}{2}(T-t)}{\sigma_V \sqrt{T-t}}$$

$$h_2 = h_1 - \sigma_V \sqrt{T-t}$$

Task: Prove equation (8).

Hint: Consider the stochastic properties of Brownian motion, in particular the reflection principle. See: (1) Black, F., and J. C. Cox, 1976 "Valuing Corporate Securities: Some Effects of Bond Indenture Provisions", Journal of Finance 31, 351-367, and (2) PWOQF 2nd Ed, Vol 1, Chapter 10: Probability Density Functions and First Exit Times, pp. 169-182.

3 Default Re-defined

Assume again the standard dynamics (6) for the firm value. Consider the case in which we re-define default as firm value falling below a lower barrier $D < K$ at any time before maturity *or* firm value falling below face value K of the debt at maturity. Formally, the default time is now given by

$$\tau = \min\left\{ \tau^1, \tau^2 \right\} \tag{9}$$

where τ^1 is the first passage time of assets to the barrier D and τ^2 is the maturity time T if assets $V_T < K$ at T and ∞ otherwise. In other words, the default time is defined as the minimum of the first-passage default time and Merton's default time.

In this model, even if the firm value does not fall below the barrier, if assets are below the bond's face value at maturity the firm defaults. See Figure 1.

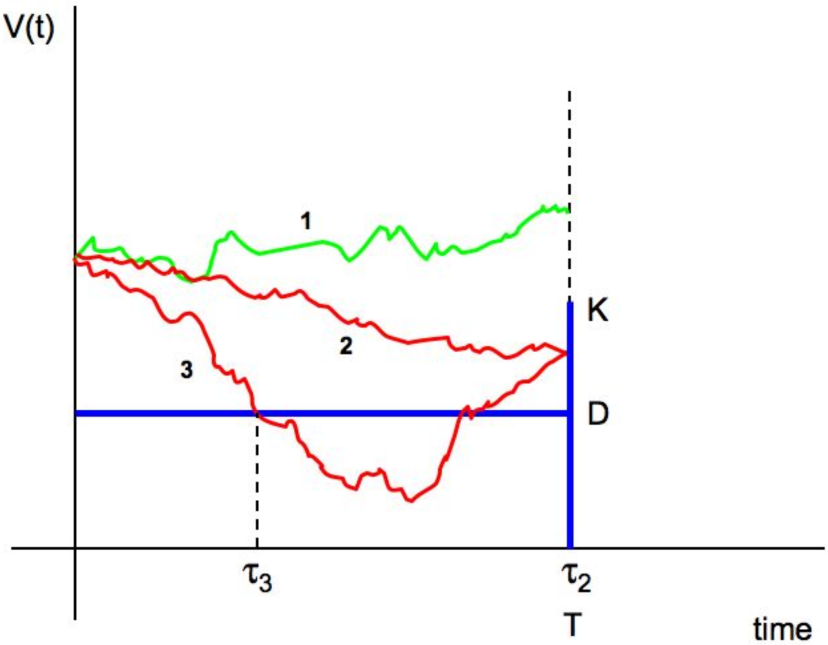

Figure 1: Default Re-defined. Three possible paths for the evolution of the firm. Path 1 does not default as its always above the lower barrier D during the life of the option and above K (the face value of debt) at maturity. Path 2 defaults as its below K at maturity. The default time is $\tau_2 = T$. Path 3 defaults the moment it touches the lower barrier τ_3.

Task: Construct a Monte Carlo simulation in Excel for the firm value process, Equation (6), and estimate the probability of default using condition (9). Include in your calculations a histogram of V_T and convergence graph for $P(t, T)$. Use $V_0 = 100$, $\sigma_V = 0.40$, $r = 0.05$, $T = 1$, $K = 90$ and $D = 80$.

Hint: PWOQF 2nd Ed, Vol 3, Chapter 80: Monte Carlo Simulation, pp. 1263-1284.