CQF Module 2, Session 4: Martingales I Solutions

CQF

1 Exercise 2.5 Solutions

1. *Let's say we want to price a given financial instrument deriving its value $V(t, S_1, S_2, S_3)$ from 3 stochastic processes S_1, S_2, S_3, where*

$$\begin{aligned} dS_i &= f_i(t, S_k, k = 1, \ldots, 3)dt \\ &\quad + g_i(t, S_k, k = 1, \ldots, 3)dX_i, \\ &\quad\quad i = 1, \ldots, 3 \end{aligned}$$

and where

$$dX_i dXj = \rho_{ij} dt, \quad i, j = 1, \ldots, 3, \ i < j$$

For simplicity, we will write

$$dS_i = f_i dt + g_i dX_i, \quad i = 1, \ldots, 3$$

Let $V(t, S_1(t), S_2(t), S_3(t))$ be a function on $[0, T]$ with $V(0, S_1(0), S_2(0), S_2(0)) = v$. Using Itô, compute the SDE for dV and deduce the stochastic integral for $V(T)$.

Since $dX_i \to t$ in the means square limit, we see that

$$dS_i^2 \to g_i^2 dt, \quad i = 1, \ldots, 3$$

Also, since $dX_i dXj = \rho_{ij} dt$, we see that

$$dS_i dS_j \to \rho_{ij} g_i g_j dt, \quad i = 1, \ldots, 3$$

By the multivariate version of Itô's Lemma,

$$\begin{aligned} dV &= \left(\frac{\partial V}{\partial t} + f_1 \frac{\partial V}{\partial S_1} + f_2 \frac{\partial V}{\partial S_2} + f_3 \frac{\partial V}{\partial S_3} + \frac{1}{2} g_1^2 \frac{\partial^2 V}{\partial S_1^2} + \frac{1}{2} g_2^2 \frac{\partial^2 V}{\partial S_2^2} + \frac{1}{2} g_3^2 \frac{\partial^2 V}{\partial S_3^2} \right. \\ &\quad \left. + \rho_{12} g_1 g_1 \frac{\partial^2 V}{\partial S_1 S_2} + \rho_{13} g_1 g_3 \frac{\partial^2 V}{\partial S_1 S_3} + \rho_{23} g_2 g_3 \frac{\partial^2 V}{\partial S_2 S_3} \right) dt \\ &\quad + g_1 \frac{\partial V}{\partial S_1} dX_1 + g_2 \frac{\partial V}{\partial S_2} dX_2 + g_3 \frac{\partial V}{\partial S_3} dX_3 \end{aligned}$$

Integrating over $[0,T]$, we get

$$\begin{aligned} V_T &= v + \int_0^T \left(\frac{\partial V}{\partial t} + f_1 \frac{\partial V}{\partial S_1} + f_2 \frac{\partial V}{\partial S_2} + f_3 \frac{\partial V}{\partial S_3} + \frac{1}{2} g_1^2 \frac{\partial^2 V}{\partial S_1^2} + \frac{1}{2} g_2^2 \frac{\partial^2 V}{\partial S_2^2} + \frac{1}{2} g_3^2 \frac{\partial^2 V}{\partial S_3^2} \right. \\ & \left. + \rho_{12} g_1 g_1 \frac{\partial^2 V}{\partial S_1 S_2} + \rho_{13} g_1 g_3 \frac{\partial^2 V}{\partial S_1 S_3} + \rho_{23} g_2 g_3 \frac{\partial^2 V}{\partial S_2 S_3} \right) dt \\ & + \int_0^T g_1 \frac{\partial V}{\partial S_1} dX_1 + \int_0^T g_2 \frac{\partial V}{\partial S_2} dX_2 + \int_0^T g_3 \frac{\partial V}{\partial S_3} dX_3 \end{aligned}$$

2. **The Heston Model.** *The Heston Model (1993) is a popular stochastic volatility model used for option valuation. In this model, the stock price dynamics follows a GBM in which the stock variance v is itself stochastic and follows a square root process* [1]*. The stock price dynamics is:*

$$dS_t = \mu S_t dt + \sqrt{v_t} S_t dX_1(t) \tag{1}$$

the dynamics of the stock variance is

$$dv_t = -\lambda(v_t - \bar{v})dt + \eta\sqrt{v_t} dX_2(t) \tag{2}$$

and the two processes have correlation ρ correlated, i.e.

$$dX_1(t) dX_2(t) = \rho dt \tag{3}$$

Let $F(t, S_t, v_t)$ be a function on $[0,T]$ with $F(0, S_0, v_0) = f$. Using Itô, compute the SDE for dV and deduce the stochastic integral for $F(T)$.

Since $dX_i \to t$ in the means square limit for $i = 1,2$, we see that

$$\begin{aligned} dS_t^2 &\to v_t S_t^2 dt, \\ dv_t^2 &\to \eta^2 v_t dt, \end{aligned}$$

Also, since $dX_1 dX2 = \rho dt$, we see that

$$dS_t dv_r \to \rho\eta v_t S_t dt$$

By the bivariate version of Itô's Lemma, the SDE for F is given by

$$\begin{aligned} dF &= \left(\frac{\partial F}{\partial t} + \mu S_t \frac{\partial F}{\partial S} - \lambda(v_t - \bar{v}) \frac{\partial F}{\partial v} \right. \\ & \left. + \frac{1}{2} v_t S_t^2 \frac{\partial^2 F}{\partial S^2} + \frac{1}{2} \eta^2 v_t \frac{\partial^2 F}{\partial v^2} + \rho\eta v_t S_t \frac{\partial^2 F}{\partial v \partial S} \right) dt \\ & + \sqrt{v_t} S_t \frac{\partial F}{\partial S} dX_1 + \eta\sqrt{v_t} \frac{\partial F}{\partial v} dX_2 \end{aligned}$$

[1] In the fixed income world, the square root process is called a Cox-Ingersoll-Ross process and is used to model short-term interest

Integrating over $[0,T]$, we get

$$\begin{aligned} F(T) &= f + \int_0^T \left(\frac{\partial F}{\partial t} + \mu S_t \frac{\partial F}{\partial S} - \lambda(v_t - \bar{v}) \frac{\partial F}{\partial v} \right. \\ & \left. + \frac{1}{2} v_t S_t^2 \frac{\partial^2 F}{\partial S^2} + \frac{1}{2}\eta^2 v_t \frac{\partial^2 F}{\partial v^2} + \rho \eta v_t S_t \frac{\partial^2 F}{\partial v \partial S} \right) dt \\ & + \int_0^T \sqrt{v_t} S_t \frac{\partial F}{\partial S} dX_1 + \int_0^T \eta \sqrt{v_t} \frac{\partial F}{\partial v} dX_2 \end{aligned}$$

3. *Let $Y_t = X_t^4$ where X_t is a Brownian motion. Using Itô's lemma, express the SDE for Y_t. Then, deduce the stochastic integral for Y_t over $[0,T]$. Finally, deduce from the stochastic integral an expression for $\mathbf{E}[Y_t]$.*

First, note that $Y_t = f(X_t)$ where $f(x) = x^4$. Hence,

$$\begin{aligned} \frac{\partial f}{\partial t} &= 0 \\ \frac{\partial f}{\partial x} &= 4x^3 \\ \frac{\partial f}{\partial x} &= 12x^2 \end{aligned}$$

By Itô's lemma,

$$dY_t = 6X_t^2 dt + 4X_t^3 dX_t$$

Since X_t is a Brownian motion then $X_0 = 0$ and therefore $Y_0 = 0$. Thus, integrating the SDE over $[0,T]$, we get

$$Y_T = 6\int_0^T X_t^2 dt + 4\int_0^T X_t^3 dX_t$$

taking the expectation and by linearity of the expectation operator,

$$\mathbf{E}[Y_T] = 6\mathbf{E}\left[\int_0^T X_t^2 dt\right] + 4\mathbf{E}\left[\int_0^T X_t^3 dX_t\right]$$

Now, the Itô integral $\int_0^T X_t^3 dX_t$ is a martingale and hence $\mathbf{E}\left[\int_0^T X_t^3 dX_t\right] = 0$. Also, by Fubini's Theorem, we can change the order of integration and therefore slide the expectation inside $\int_0^T X_t^2 dt$. Hence,

$$\mathbf{E}[Y_T] = 6\int_0^T \mathbf{E}\left[X_t^2\right] dt$$

Now, $\mathbf{E}[X_t^2] = \mathbf{E}[(X_t - X_0)^2] = t - 0 = t$. Therefore,

$$\mathbf{E}[Y_T] = 6 \int_0^T t dt = 3T^2$$

4. **Discrete Time Martingale**: *Let $Y_1, \ldots, Y_n$ be a sequence of independent random variables such that $\mathbf{E}[Y_i] = 0$ for $i = 1, \ldots, n$. Let $\mathcal{F}_n$ be the filtration generated by the sequence $Y_1, \ldots, Y_n$. Consider the random variable $S_n = \sum_{i=1}^n Y_i$. Prove that S_n is a martingale for all n.*

   Reminder - *proving that a process S_n is a martingale involves proving that $\mathbf{E}[|S_n|] < \infty$ and that $\mathbf{E}[S_{n+1}|\mathcal{F}_n] = S_n$*

   First,

$$\begin{aligned} \mathbf{E}[|S_n|] &= \mathbf{E}[|Y_1 + Y_2 + \ldots + Y_n|] \\ &\leq \mathbf{E}[|Y_1| + |Y_2| + \ldots + |Y_n|] \\ &= \mathbf{E}[|Y_1|] + \mathbf{E}[|Y_2|] + \ldots + \mathbf{E}[|Y_n|] \\ &< \infty \end{aligned}$$

   since we have a finite sum of finite finite numbers.

   Second,

$$\begin{aligned} \mathbf{E}[S_{n+1}|\mathcal{F}_n] &= \mathbf{E}[S_n + Y_{n+1}|\mathcal{F}_n] \\ &= \mathbf{E}[S_n|\mathcal{F}_n] + \mathbf{E}[Y_{n+1}|\mathcal{F}_n] \end{aligned}$$

   by linearity of the expectation operator.

   Now, since S_n is $\mathcal{F}_n$-measurable (i.e. if we have the filtration $\mathcal{F}_n$ we know what S_n is), then $\mathbf{E}[S_n|\mathcal{F}_n] = S_n$.

   Also, since $Y_1, \ldots, Y_n, Y_{n+1}$ are independent, then Y_{n+1} is independent from $\mathcal{F}_n$ and hence $\mathbf{E}[Y_{n+1}|\mathcal{F}_n] = \mathbf{E}[Y_{n+1}|\mathcal{F}_n]$.

   Therefore,

$$\mathbf{E}[S_{n+1}|\mathcal{F}_n] = \mathbf{E}[S_n]$$

   and we can conclude that S_n is a martingale for all n.