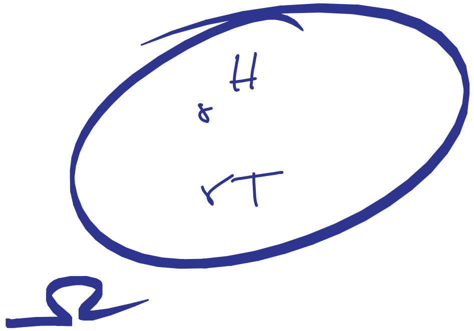
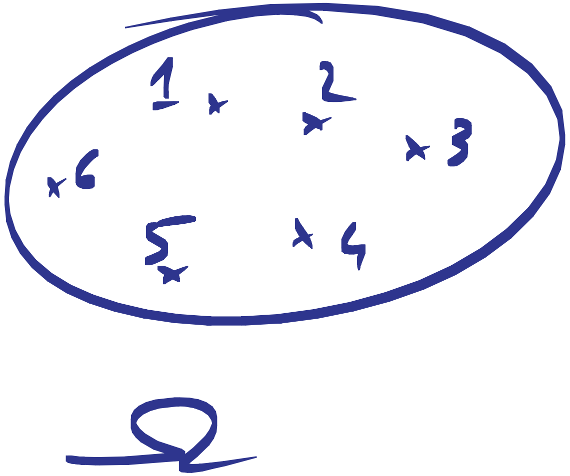
Coin Toss



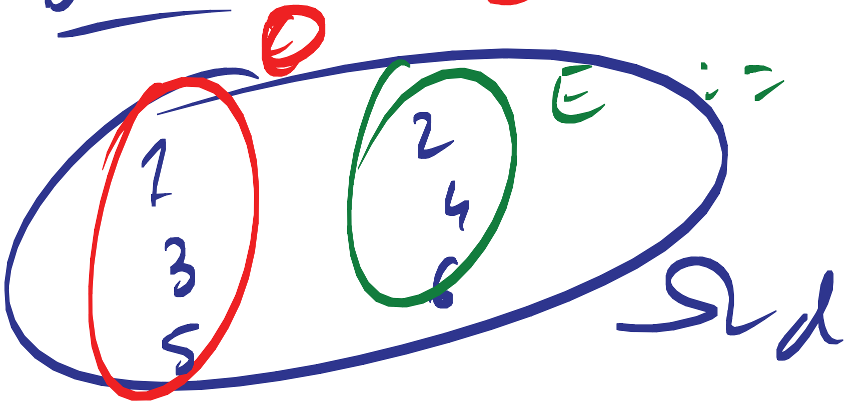
Dice



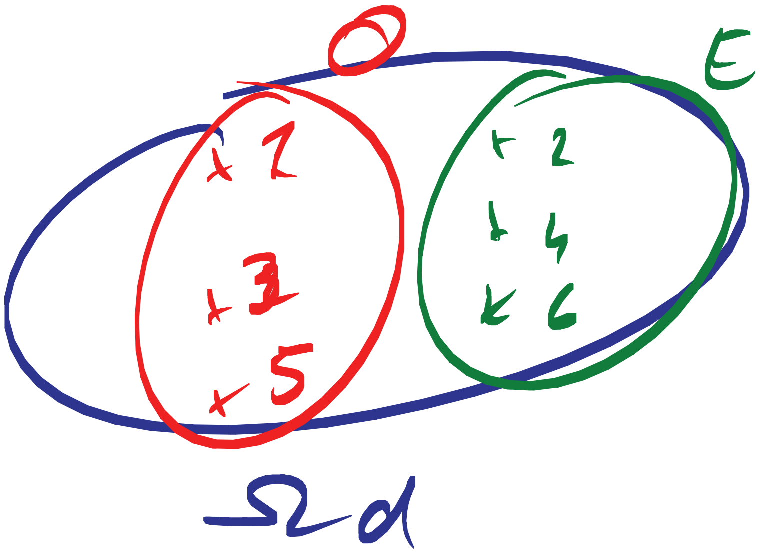
Dice :

O := subset of odd numbers

E := subset of all the even outcomes.



Complement

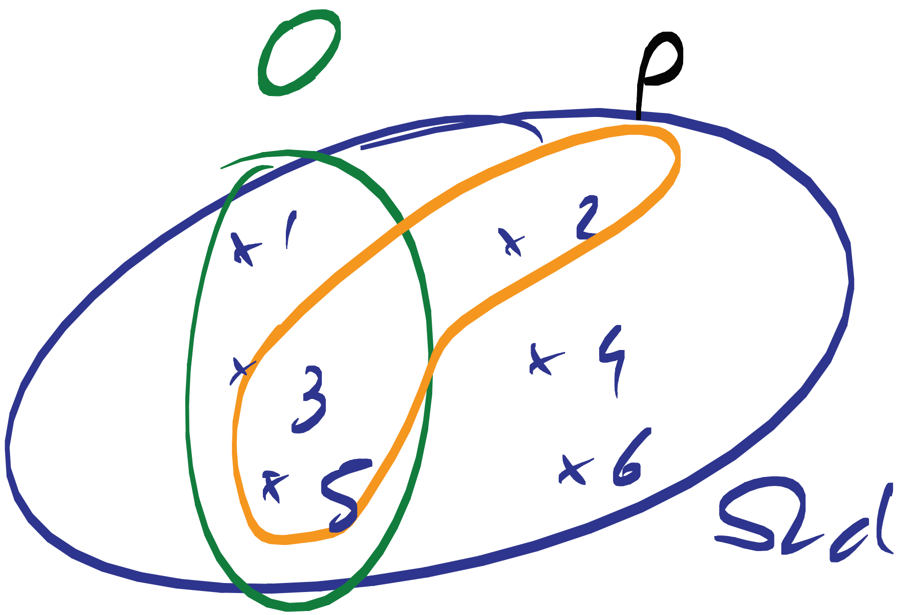


$E^c = \omega \in \Omega_d$ but **not** in E

$=: O$

Union $A \cup B$ "or"

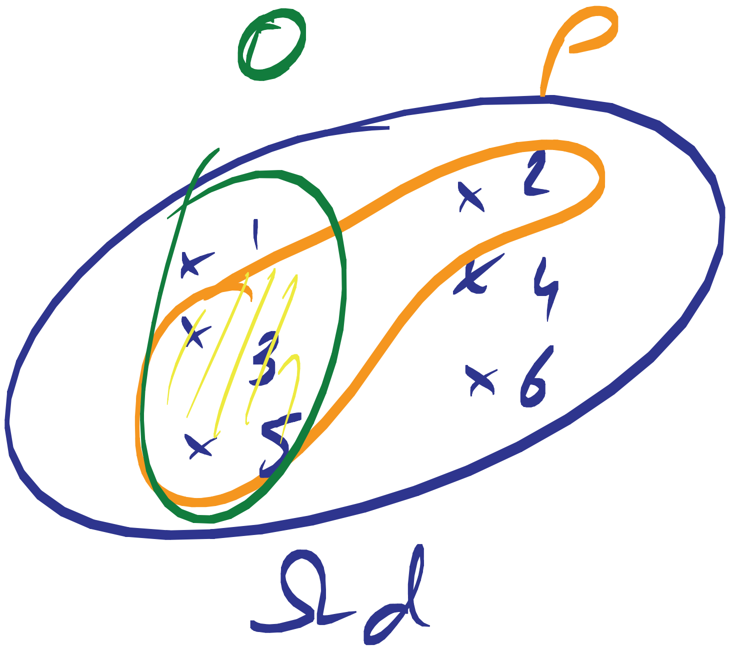
$\omega \in (A \cup B)$ if ω is in A
<u>or</u> ω is in B



$B = O \cup P$
$= \{1, 2, 3, 5\}$

Intersection : → "and"

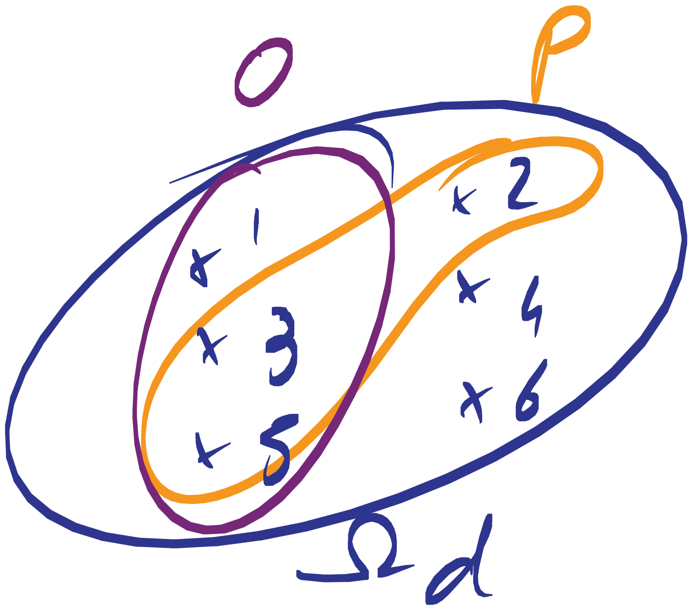
$\omega \in A \cap B$ if $\omega \in A$ and $\omega \in B$



$C = O \cap P$
$= \{3, 5\}$.

Set Difference $A \setminus B$

$\omega \in A$ but $\omega \notin B$.



$O \setminus P = \{1\}$

$P \setminus O = \{2\}$

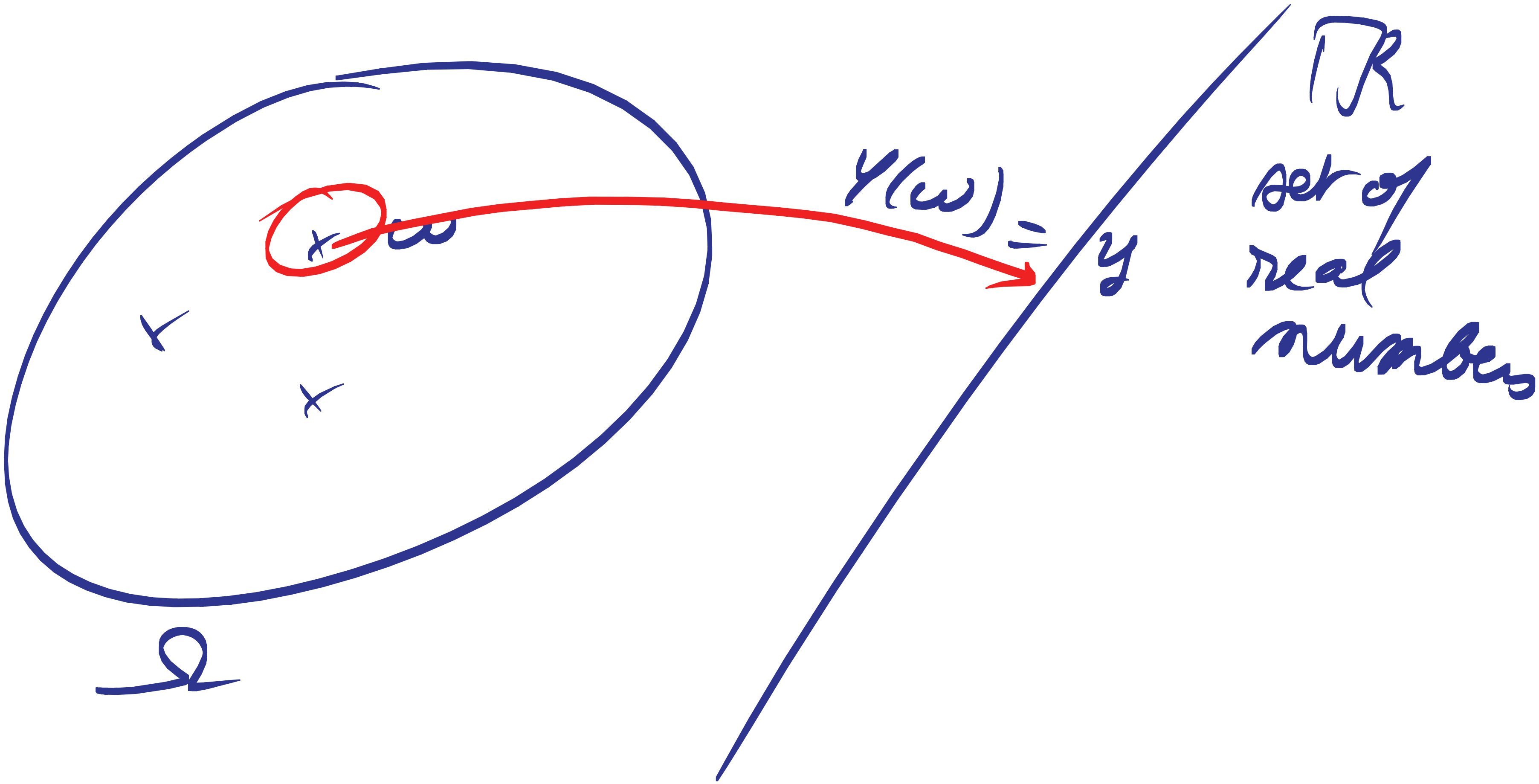
"zero"

→ "null set" $\emptyset$

→ $A \cup \{\emptyset\} = A$

→ $A \cap \{\emptyset\} = \{\emptyset\}$

→ $A \setminus \{\emptyset\} = A$



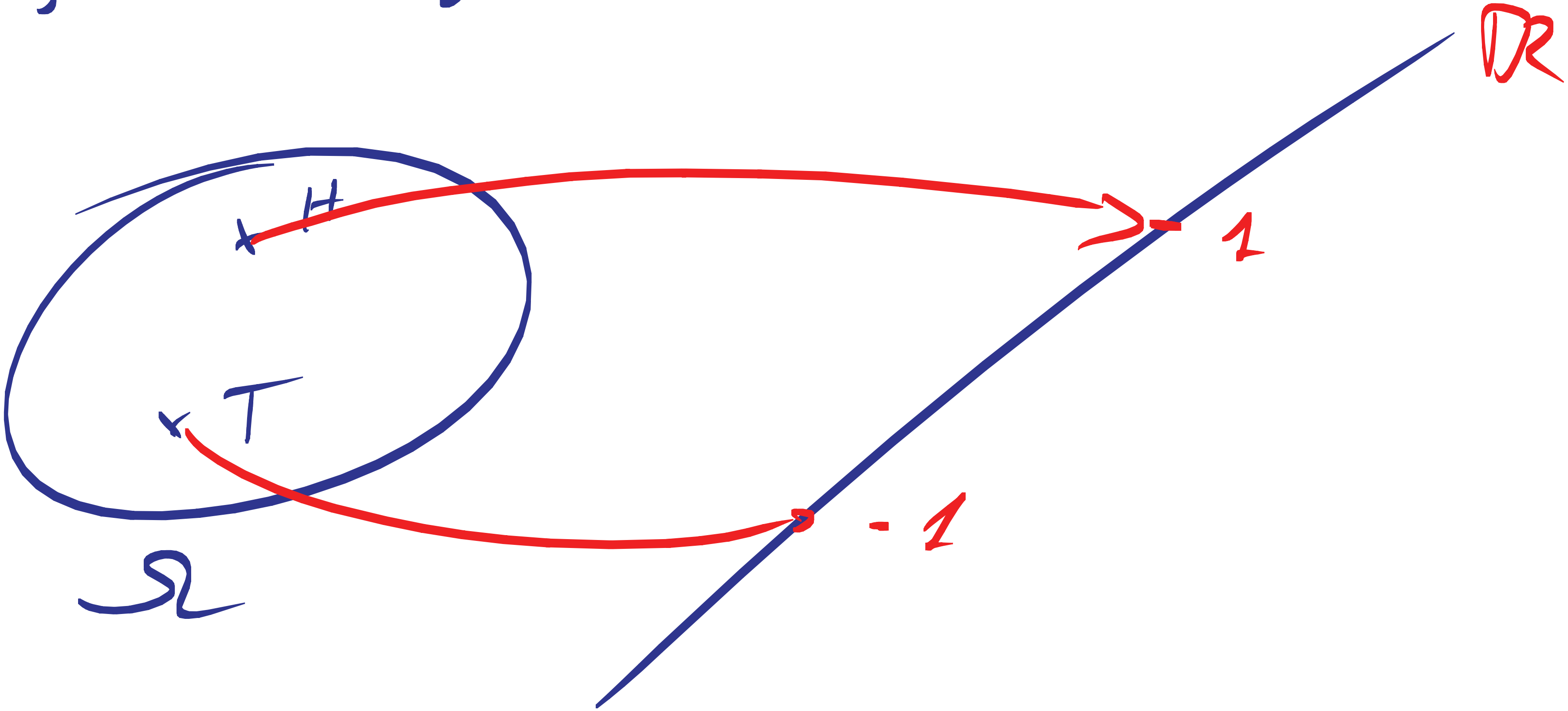
ω
$Y(\omega) = y$
$\mathbb{R}$
set of real numbers
Ω

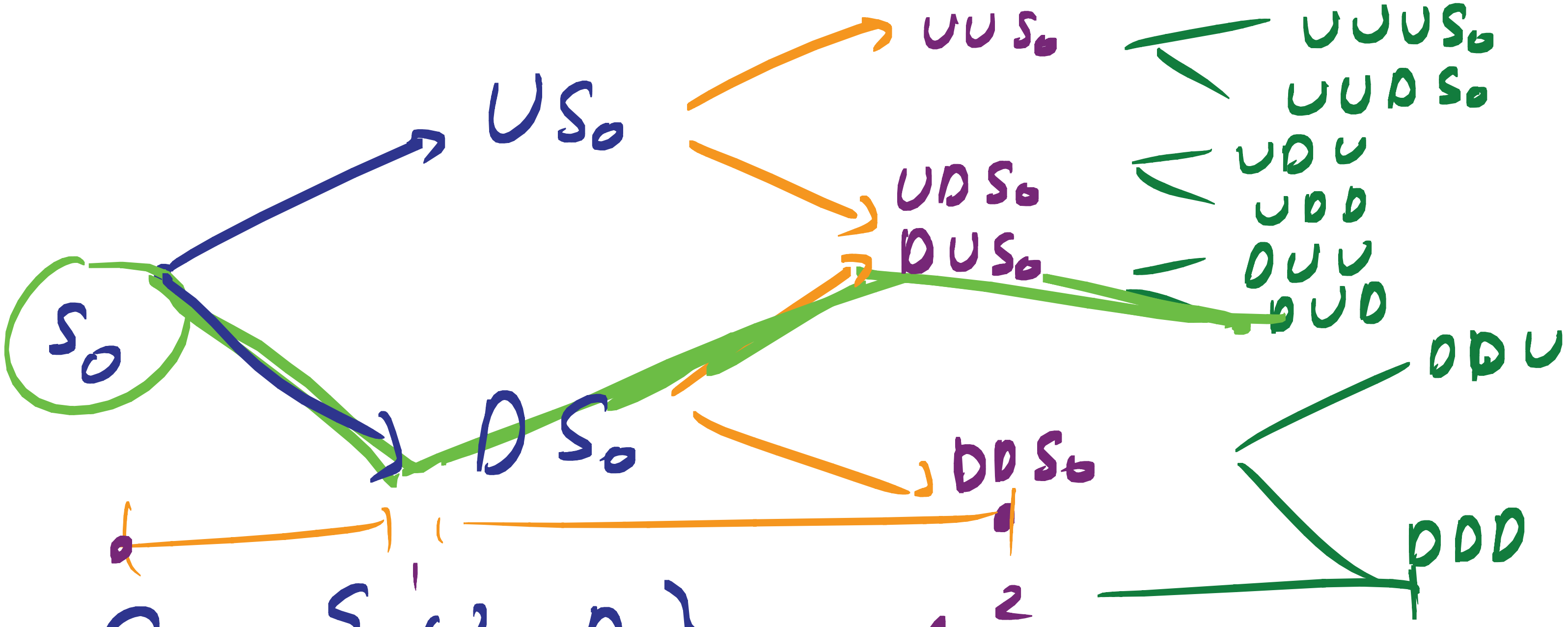
Simple Coin Toss Experiment

H $\rightarrow$ $+£1$

T $\rightarrow$ $-£1$

P&L is the RV Y



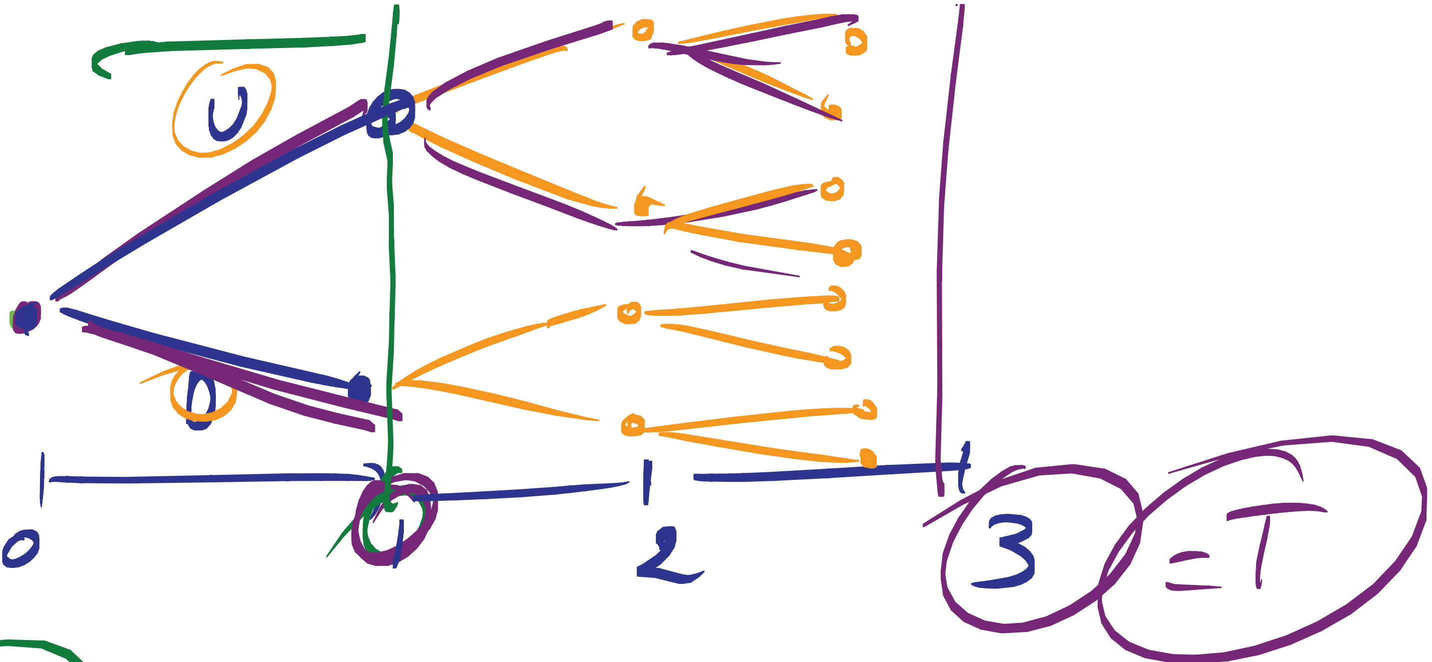


$\Omega_1 = \{U, D\}$ $\quad 2^1 = 2$

$\Omega_2 = \{UU, UD, DU, DD\}$ $\quad 2^2 = 4$

$\Omega_3 = \{UUU, UUD, UDU, UDD, DUU, DUD, DDU, DDD\}$ $\quad 2^3 = 8$

Ω_{10} $\quad 2^{10} = 1024$

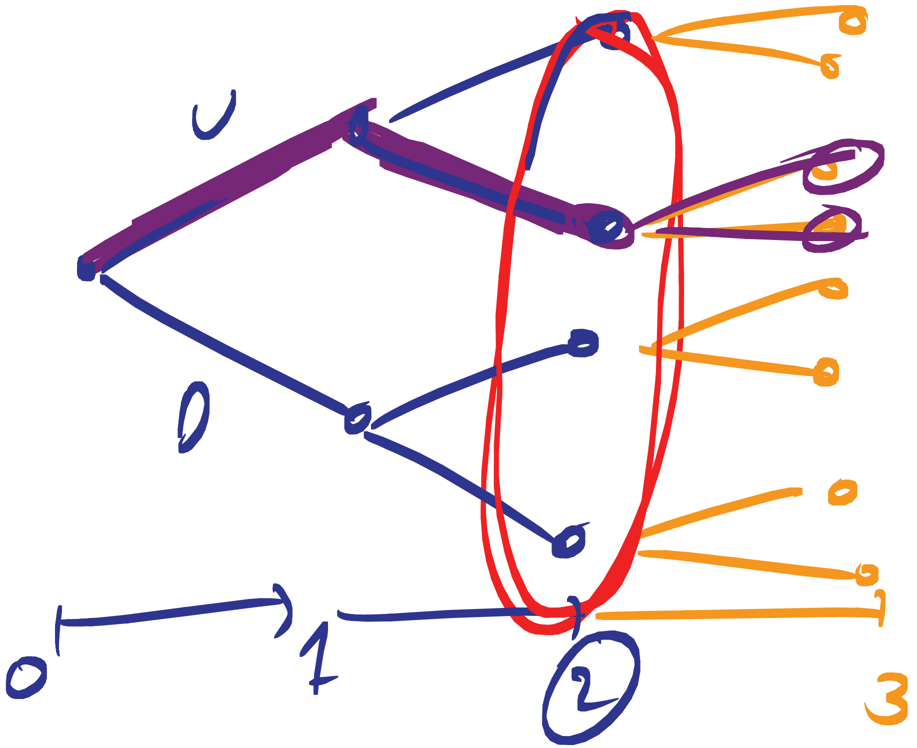


$\Omega_1 = \{U, D\}$

$\mathcal{F}_1 = \{\emptyset, \Omega, u, d\}$

$u = \{UUD, UDU, UDD, UUU\}$

$d = \{DUU, DUD, DDU, DDD\}$



$$\Omega_2 = \{UU, UD, DU, DD\}$$

$$\mathcal{F}_2 = \{\Omega, \emptyset, ud, du, dd, uu, \ldots\}$$

$$uu = \{UUU, UUD\} \qquad du = \{DUU, DUD\}$$

$$ud = \{UDU, UDD\} \qquad dd = \{DDU, DDD\}$$

$\mathbb{E}[h(X)] = \int_{\mathbb{R}} h(x) \underbrace{p(x)dx}$ ← PDF of X

CDF of X $\quad P \rightarrow \text{PDF}_P$ is ~~defined~~

$$p(\cdot) = \frac{dP}{dx}(\cdot)$$

$$\Leftrightarrow p(\cdot)\,dx = dP(\cdot)$$

$$\mathbb{E}[h(X)] = \int_{\mathbb{R}} h(x)\, d(P(x))$$

↖ interval

Measure :

$$\mathbb{E}[R(x)] = \int_{\Omega} R(x)\, d\mathbb{P}$$

set.

measure

"CDF"

Link between expectation & Proba.

$$\mathbb{E}[h(X)] = \int_\Omega h(x)\, d\mathbb{P}.$$

$$h(x) \longrightarrow \mathbb{1}_{\{x \in A\}} = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

$$\mathbb{E}[\mathbb{1}_{\{x\in A\}}] = \int_\Omega \mathbb{1}_{\{x \in A\}}\, d\mathbb{P}$$

$$= \int_A 1 \times d\mathbb{P} + \int_A 0 \times d\mathbb{P}$$

$$= \int_A d\mathbb{P} + 0$$

$$\mathbb{P}(X \in A)$$

Tonight

in $\mathbb{R}$

$$P[a \leq x \leq b] = \int_a^b p(x)\,dx$$

PDF

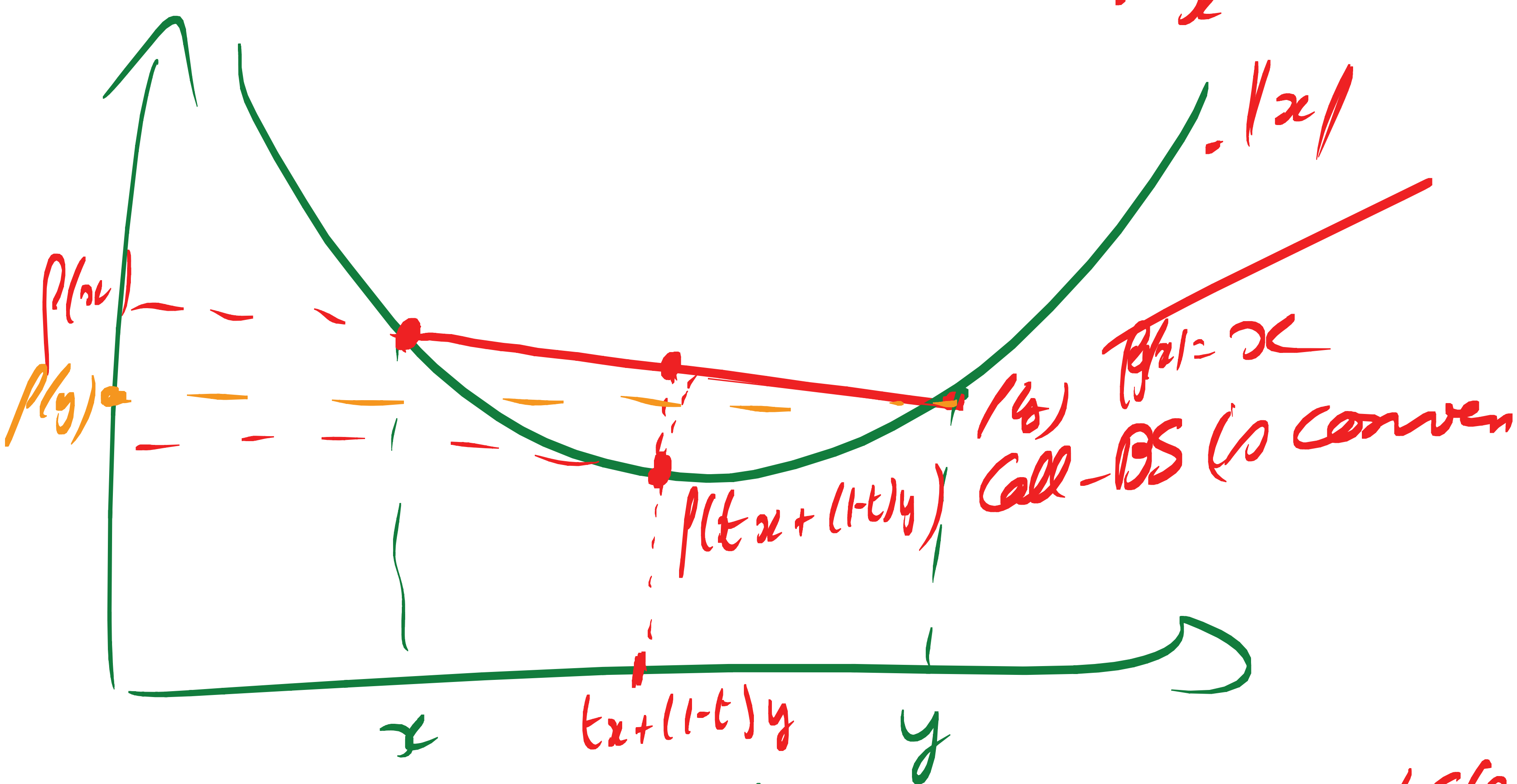
$$= \int_a^b d(P(x))$$

Normal variable

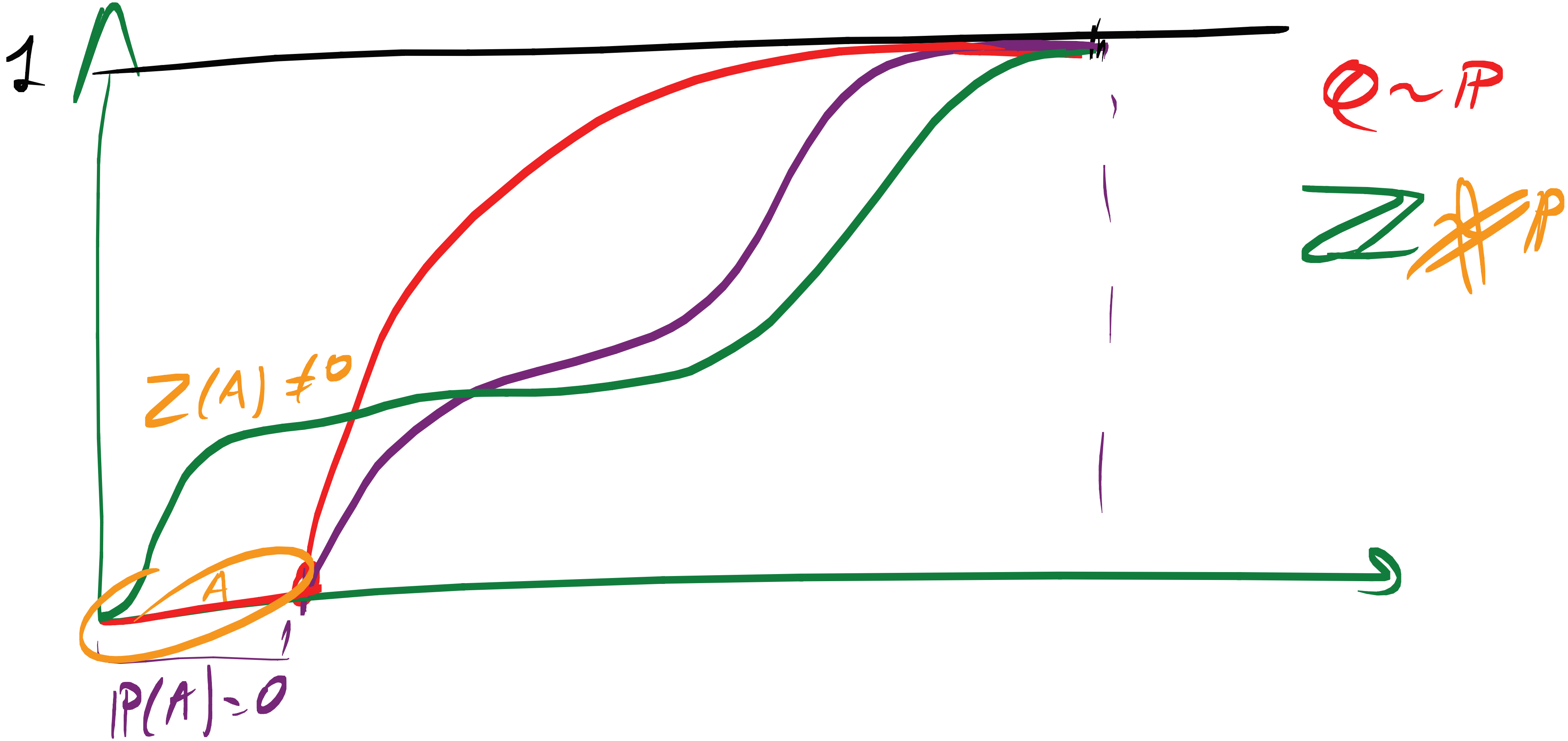
CDF

You know

Convex function f



$$f(tx+(1-t)y) \le t f(x) + (1-t) f(y) \quad t \in (0,1)$$


1
Q ~ P
Z ≁ P
Z(A) ≠ 0
A
P(A) = 0

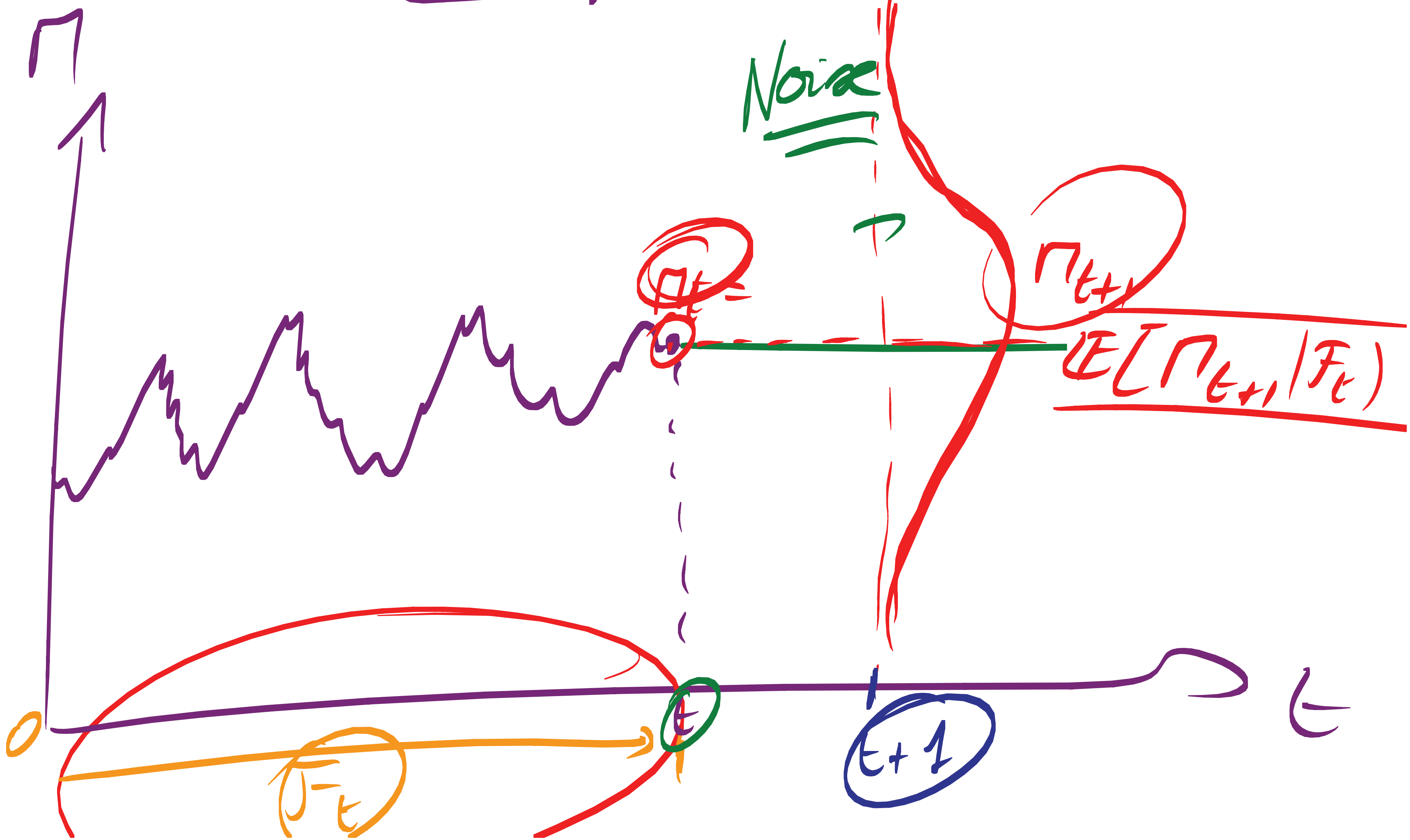
$$\mathbb{Q}(A) = \int_A \Lambda \, d\mathbb{P}$$

$$\Lambda = \frac{d\mathbb{Q}}{d\mathbb{P}}$$

$$\mathbb{Q}(A) = \int_A \frac{d\mathbb{Q}}{\cancel{d\mathbb{P}}} \cancel{d\mathbb{P}} = \int_A d\mathbb{Q}$$

=: def of proba of A under $\mathbb{Q}$

Martingale M_t



73:

Showing that a process X_n is a martingale involves 2 steps:

1. Showing that X_n is integrable

i.e. $$\mathbb{E}|X_n| < \infty$$

2. Showing that X_n satisfies the martingale property

$$\mathbb{E}[X_{n+1} | \mathcal{F}_n] = X_n$$

① - Integrability: To prove $\mathbb{E}|X_n| < \infty$

- Def of X_n:

$$\mathbb{E}[|X_n|] = \mathbb{E}[|\underbrace{\mathbb{E}[Y|\mathcal{F}_n]}_{=X_n}|]$$

absolute value → convex function

$\leq$

By Jensen's inequality

$$\mathbb{E}[|X_n|] \leq \| \mathbb{E}[\cancel{\mathbb{E}}[Y|\cancel{\mathcal{F}_n}]] \|$$

no information !!!

information

By Tower Property $\mathbb{E}[|X_n|] \leq | \mathbb{E}[Y] |$

$\mathbb{E}\left[|Y|\right] < \infty$ (By exercise)

$\Rightarrow \mathbb{E}\left[|X_n|\right] \leq \mathbb{E}\left[|Y|\right] < \infty$

$\boxed{X_n \text{ is integrable}}$

② - Martingale Property $\mathbb{E}[X_{n+1} | \mathcal{F}_n] = X_n$

$m < n$

$$\mathbb{E}[X_n | \mathcal{F}_m] = \mathbb{E}\Big[\underbrace{\cancel{\mathbb{E}}[Y | \cancel{\mathcal{F}_n}]}_{\text{def of } X_n} \Big| \mathcal{F}_m\Big]$$

By Tower property

$$\mathbb{E}[X_n | \mathcal{F}_m] = \underbrace{\mathbb{E}[Y | \mathcal{F}_m]}_{= X_m}$$

$$\mathbb{E}[X_n | \mathcal{F}_m] = X_m$$

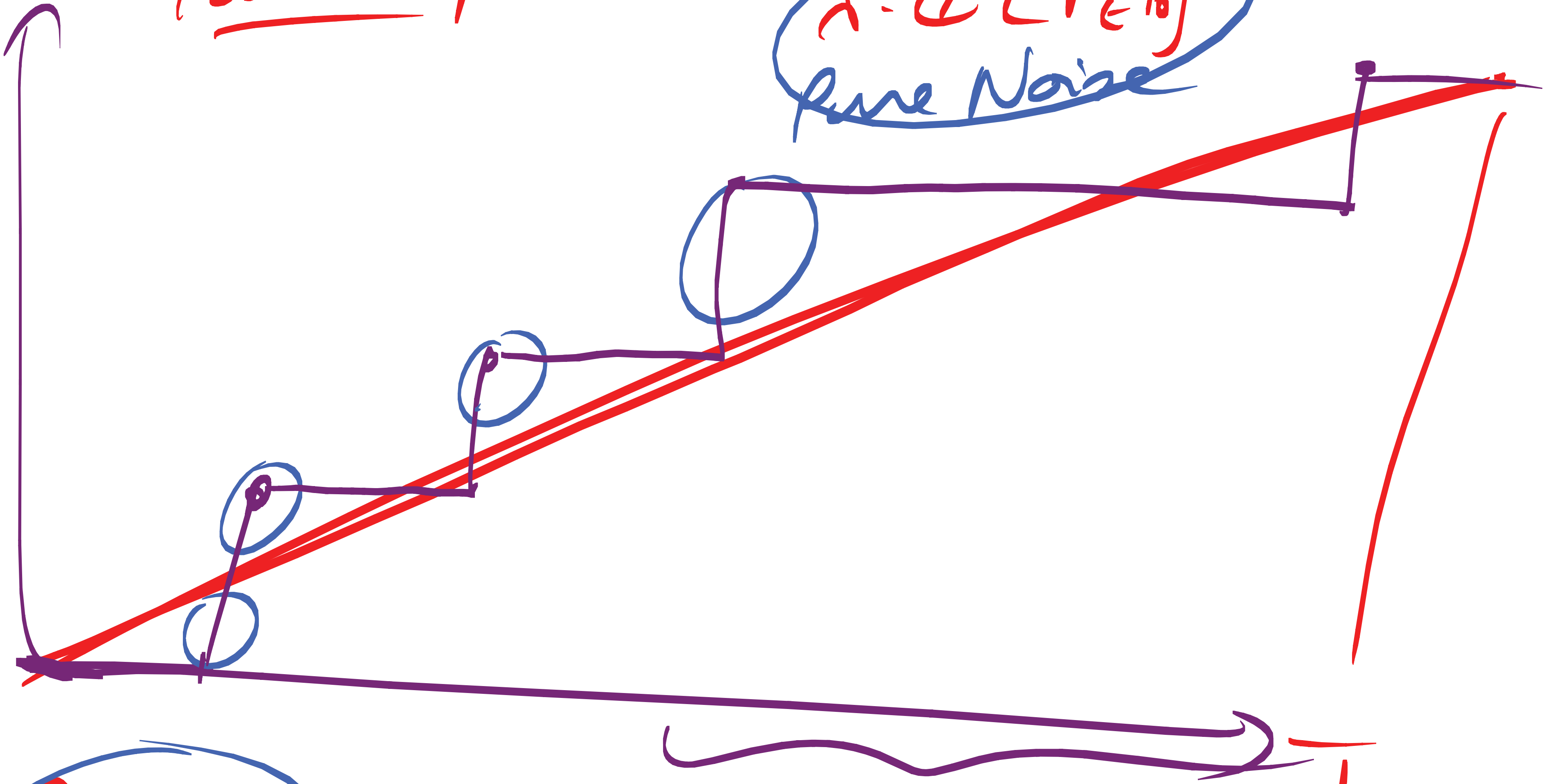
Choose $n = m + 1$ . . Done!

Supermartingale

Poisson process

$\lambda = \mathbb{E}[P_t - P_0]$

Pure Noise



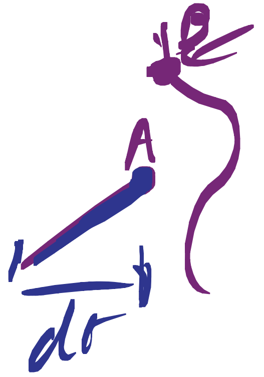
t

$P_t - \lambda t$ = Martingale Random

GBM

$$dS(t) = \underbrace{\mu S(t)\, dt}_{\text{drift}} + \underbrace{\sigma S(t)\, dX(t)}_{\text{volatility}}$$

A+B



$$\mathbb{E}\Big[\cancel{\mathbb{E}\big[}X\,\cancel{|\,\mathcal{F}_s\big]}\;\Big|\;\mathcal{F}_t\Big] \qquad t < s$$

$$= \mathbb{E}\big[X \,|\, \mathcal{F}_t\big]$$