# CDO$^2$ Pricing Using Gaussian Mixture Model with Transformation of Loss Distribution *

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September 5, 2005

**Abstract**

We present a new approach to price CDOˆ2-type transactions consistently with the pricing of the underlying CDOs. We first present an extension to the current market standard model using a Gaussian mixture (GM) copula model instead of one parameter single Gaussian Copula model. It shows that GM broadly captures the correlation skew shown in the index tranche market, but not exactly and across time or across term structure. Then using an analogy to the option pricing for CDO tranche pricing we extract an *implied loss distribution* from the observed index tranche market or a set of bespoke pricing of the underlying baby portfolios. To strike a balance of matching the underlying baby CDO pricing and having a plausible economic correlation model to price CDOˆ2-type trades we create a *loss distribution transformation* for each baby CDO portfolio between the implied loss distribution from the index tranche market or bespoke pricing and the loss distribution from the GM model. This way, we can match the

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pricing for all baby portfolios or tranches, and at the same time, we price bespoke CDO and CDOˆ2 with a broadly skew aware correlation model. In the future we could further calibrate the parameters in the GM model to the CDOˆ2 market if its pricing becomes more observable.

# 1 Introduction

Our aim here is two folds. First we extend the framework of one copula function approach to a mixture of a few copula functions. Second, we present a new approach to price a CDOˆ2-type transactions consistently with the pricing of the underlying CDO tranches or portfolio. The examples of the CDOˆ2-type transactions include the standard CDOˆ2 transactions, CDOˆ2 of long and short tranches or credits, and CDOˆ2 consisting of tranches from different standard indices. There are various attempts to come up with a ”grand unified” portfolio model that can calibrate to price all index tranches to match the market observed break-even spreads, and then we can use it price all bespoke portfolio as well. The market practice for valuing market index tranches is to use one-parameter Gaussian copula function, a special case of the general copula function framework to credit portfolio modeling introduced by Li[3], and a method called "the base correlation". The base correlation method is to treat a CDO tranche as a derived security since a CDO tranche can be treated as a call spread on the total loss distribution of the underlying portfolio. For example, long a 3-7% tranche protection on the credit index is equivalent to long a call option on the total loss with a strike price of 3% of the total notional amount of the underlying portfolio and short a call with a strike price of 7% of the total notional amount of the underlying portfolio. Then we just need to price all equity tranches at different detachment levels or equity tranche sizes. Correspondingly the implied correlation could be given just for equity tranche with different detachment levels or tranche sizes. This way of quoting correlation and pricing CDOs are called the *base correlation method* introduced by J. P. Morgan [2].

However, the base correlation method does not guarantee that it is an arbitrage free model. We have observed that for two tranches with the same tranche size, the one with more support or higher attachment level could have a break-even spread higher than the one with a lower attachment level in the base correlation framework. Also the base correlation method along with the Gaussian copula model does not tell us too much about how to price non-index portfolio tranches, neither more complicated structures such as CDOˆ2 or CDO trades with both long and short credits as collateral. Currently, the market practice for bespoke pricing is to use a base correlation curve mapping so that the base correlation curve scaled by the portfolio loss or the tranche loss are equivalent for all portfolios. This mapping approach is difficult to justify from a theoretical perspective. Ideally we'd still like to have a model to have the "look through" capability in a similar fashion as a Gaussian copula model with a set of correlation parameters, but no base correlation method, so that we can manage all portfolio transactions in a consistent way.

Some attempts have been made along this direction. One example is the approach by Andersen and Sidenius [7], which we will call Gaussian extension (GE) method hereafter. This is done in a "non-copula" way since the correlation structure in the standardized asset returns is not consistent with the copula function used. The joint distribution of asset returns are not normally distributed, however the normal distribution is still used to translate a return scenario to a default time scenario. Due to this inconsistency this approach lacks the essential advantage of copula functions used in credit portfolio modeling: to preserve individual credit information when we model the correlation for the credit portfolio. For example, we cannot preserve the total loss if we simply do the implementation by taking a set of parameters in that approach. To overcome the problem it employs a calibration to maintain the single name default property from the credit curve input. But on the other hand it has introduced a flexible way with respect to the model formulation: we completely separate the modeling of correlation from the transformation function used to link the asset return to the survival time. The empirical studies of this model show that we capture the skew to a certain extend, but we still cannot match the index tranche market spreads using 5 parameters under three correlation regimes classified by two threshold values of an economic factor variable. We have also obtained different sets of parameters for different indices (CDX or ITraxx) or the same index with different tenors (5, 7 or 10 years). We could have two interpretations for this result: either the model stills misses certain aspect of the market or the current market is still a new market under development, and does not trade fully rationally.

We present an alternative approach to extend the current market standard Gaussian copula framework. We still maintain the advantage of copula function approach to preserve individual credit default property when we model the credit portfolio. Instead of using one copula function we use a mixture of a few copula functions. We can still work in the Gaussian copula function framework, that is, we can use a mixture of two Gaussian copula functions with two different correlation parameters. That is, in a good state, all credits are less correlated, but in a bad state, all credits are highly correlated. Then we have a mixing parameter which gives the percentage of time that the economy is in the good state or the bad state. In the case of mixing a few Gaussian copula functions with different correlation parameters we call the model "Gaussian Mixture" model or simply GM model. We are not confined to the normal copula function at all. Any mixture of two copula functions is still a copula function. For example, we could mix a Gaussian copula function with a Marshall-Olkin copula function. The advantage of this approach is that we still preserve the individual default information while gaining more flexible parametrization to help us to do calibration against what are observed in the market. The calibration will focus only on the correlation parameters in the copula function and the mixing parameters.

To mechanically calibrate a model to a developing market might not necessarily result in a rational model and a set of meaningful and stable parameters. However it is hard to use any extension model if we cannot calibrate the model to match the market consistently. What we accomplish here is to create a balance between the economically plausible model and a stable set of parameters, but at the same time, price the existing standard tranches to the market. This is achieved in the second part of the paper. We derive the implied loss distribution from the index tranche market or bespoke pricing result, and then create a mapping between the market implied or bespoke pricing implied loss distribution and the loss distribution from the GM model. This mapping would only change the marginal loss distributions of the underlying sub-portfolio, the correlation among all credits is still governed by our GM correlation model or other extension models. If the GM model captures the market perfectly, the transformation would have no impact on the marginal loss distribution observed from the market. Otherwise, the mapping would correct the difference between the market and the model implied loss distributions. This way we would price all baby CDOs consistently with our current pricing of CDOs, and also use a calibrated, theoretically sound correlation model to price the $\text{CDO}^2$ transactions.

This is a general idea in the sense that other default correlation extension models to the basic Gaussian copula model can be used in this framework, we could also mix different types of copula functions, such as Gaussian copula function with the Marshall-Olkin copula function. This idea could be used in the pricing of multi-asset options in which the risk neutral distribution of each asset is preserved through a probability transformation introduced in this paper while a correlation structure is introduced through a correlation model such as GM in this paper.

The rest of the paper is organized as follows: in Section 2 we first present our basic model setup and the basic one factor Gaussian Copula model. We then present a general setup in the mixture copula model, and a specific mixture model based on survival distributions. In section 3, we show that to price a CDO and $\text{CDO}^2$ we only need to have the loss distribution of the underlying portfolios at different times in the future. Section 4 presents our major result on implied loss distribution, basic property of portfolio loss distribution, and the transformation of loss distribution from the market implied loss distribution to a model implied loss distribution. In section 5 we present our numerical calibration result of our GM model and make some comparison of a $\text{CDO}^2$ pricing exercise between the market consensus and our proposed approach using GM plus loss distribution mapping. Section 6 concludes the paper.

# 2 Gaussian Copula and Its Extension of Gaussian Mixture

## 2.1 Standard Gaussian Copula Model

We consider a credit portfolio consisting of $n$ underlying credits whose notional amounts are $N_i$ and fixed recovery rates are $R_i$ $(i = 1, \cdots n)$. We consider the aggregate loss from today to time $t$ as a fixed sum of random variables:

$$\begin{aligned} L_n(t) &= \sum_{i=1}^{n} X_i \\ &= \sum_{i=1}^{n}(1 - R_i) \cdot N_i \cdot I_{[\tau_i < t]} \\ &= \sum_{i=1}^{n} B_j \cdot I_{[\tau_i < t]} \end{aligned} \tag{1a}$$

where $\tau_i$ is the survival time for the $i$th credit in the credit portfolio. The distribution function of survival time $\tau_i$ is denoted as $F_i(t) = \Pr[\tau_i \leq t]$. The specification of the survival time distribution $F_i(t)$ is usually called a credit curve, which can be derived from market credit default swap spreads, and $B_j = (1 - R_i) \cdot N_i$ can be called *recovery adjusted notional amount*.

We introduce the correlating structure through a copula function $C(u_1, \cdots, u_n; \rho)$ where $\rho$ is the correlation parameter set which could be one value or a matrix. Then the joint distribution of survival times $t_1, t_2, ..., t_n$ can be written as

$$F(t_1, t_2, ..., t_n) = C(F_1(t), F_2(t), \cdots, F_n(t); \rho)$$

This general framework of using copula function for credit portfolio modeling is introduced in Li [3]. The current market standard copula model is the Gaussian copula model where

$$C(u_1, \cdots, u_n; \rho) = \Phi_n\left(\Phi^{-1}(u_1), \Phi^{-1}(u_1), \cdots, \Phi^{-1}(u_1); \Sigma\right)$$

where $\Phi_n()$ is the $n$-dimension normal distribution and $\Phi^{-1}()$ is the inverse of one dimension normal distribution function, and $\Sigma$ is the correlation matrix. We can introduce another set of normal random variables $Y_1, Y_2, \cdots, Y_n$ whose correlation matrix is $\Sigma$. The relationship between the survival times $\tau_i$ and the normal random variables $Y_i$ is as follows:

$$\tau_i = F^{-1}\left(\Phi(Y_i)\right). \tag{2}$$

It can be seen from this setup that the whole correlation structure is based on a set of normal random variable $Y_1, Y_2, \cdots, Y_n$ whose correlation matrix is $\Sigma$.

Various correlation structure or specification of the correlation matrix $\Sigma$ has been used in credit portfolio modeling. The simplest one is called a one factor model where the correlation matrix is a constant $\rho$. It is called a one factor model since it corresponds to a factor model representation of the correlating matrix as follows

$$Y_i = \sqrt{\rho} \cdot Y_M + \sqrt{1-\rho} \cdot \epsilon_i,$$

where $Y_M$ is the market common factor and $\epsilon_i$ is the residual risk and both of them are standard normal distribution. This one factor approach is studied by Finger [1]. We could also represent two correlation parameter matrix, intra-industry correlation $\rho_I$ and inter-industry correlation $\rho_0$ in terms of an $m$ industry factors $Y_k$ and a common market factor $Y_m$ as follows:

$$Y_i = \sqrt{\rho_I - \rho_O} \cdot Y_k + \sqrt{\rho_O} \cdot Y_M + \sqrt{1-\rho_I} \cdot \epsilon_i, \rho_I \geq \rho_O,$$

if $i$th credit is from the $k$th industry. This is given by Li and Skarabot [4].

In the general case we can use principal axis analysis to reduce a whole correlation matrix to a few factors, a method introduced by Anderson and Sidenius [8].

## 2.2 Mixture of Copula Functions

Suppose we have chosen an $n-$dimension copula function $C(u_1, u_2, \cdots, u_n; \rho)$ where $\rho$ is a set of parameters in the copula function. Instead of having a constant set of parameters $\rho$ we assume that $\rho$ is stochastic and distributed over its range of possible values according to a distribution function $V(\rho)$ or a probability density function $v(\rho)$. Then we define another copula function using the mixing distribution $V(\rho)$ as follows

$$C(u) = \int_\rho C(u|\rho) dV(\rho) = E_\rho \left[ C(u|\rho) \right]$$

where $C(u|\rho)$ can be any copula function with random correlation parameters $\rho$. A discrete form of a mixture of two copula functions or uniform distributions can be constructed as follows. Let $C_1(u_1, u_2, \cdots, u_n; \rho_1)$ and $C_2(u_1, u_2, \cdots, u_n; \rho_2)$ denote as two copula functions and a mixing parameter $0 \leq \alpha \leq 1$.Then we can create another copula function as follows:

$$\alpha \cdot C_1(u_1, u_2, \cdots, u_n; \rho_1) + (1 - \alpha) \cdot C_2(u_1, u_2, \cdots, u_n; \rho_2).$$

This idea can then be extended to a mixture distribution of $m$ copula functions. That is

$$C(u_1, u_2, \cdots, u_n; \rho, \alpha) = \sum_{j=1}^{m} \alpha_j \cdot C_j(u_1, u_2, \cdots, u_n; \rho_j)$$
$$\sum_{j=1}^{m} \alpha_j = 1$$
$$\alpha_j \geq 0$$

is still a copula function. For example we could mix two Gaussian copula functions with two different pair-wise constant correlations. If we use three Gaussian copula functions with three pair-wise constant correlation parameters, then we would have 5 parameters since we have two independent mixing parameters. Using these 5 parameters we can calibrate to the 5 tranches frequently traded in the index tranche market. Going from a copula function to the joint distribution of survival times we just need to have the distribution function of survival times for all single names or credit curves expressed in terms of distribution functions $F_i(t)$, or density function $f_i(t)$, $i = 1, 2, \cdots n$. The joint distribution of survival times using copula function $C_i$ can be written as

$$F_i(t_1, t_2, \cdots, t_n) = C_i(F_1(t_1), F_2(t_2), \cdots, F_n(t_n); \rho)$$

## 2.3 Implementation of Mixture Copula Model

Here we still model survival times since we link all survival times by the mixture copula function. The payoff function of any portfolio transaction, depending on the underlying portfolio loss distribution, which in turn, depends on the survival times. That is, we can view any tranche payoff as a function in terms of survival times. Suppose the payoff function of a tranche is $Payoff(T_1, T_2, \cdots, T_n) = \max(L - K_L, 0) - \max(L - K_U, 0)$ where $L$ is the loss of the underlying portfolio, and $K_L$ and $K_U$ are the detachment and attachments points. The distribution of loss $L$ depends on the survival times of the underlying names. Then the expected loss of the tranche can be written as

$$\begin{aligned}
&\int \cdots \int Payoff(t_1, t_2, \cdots, t_n) f(t_1, t_2, \cdots, t_n) dt_1 \cdots dt_n \\
&= \sum_{i=1}^{m} \alpha_i \int \cdots \int Payoff(t_1, t_2, \cdots, t_n) \cdot F_i(t_1, t_2, \cdots, t_n) \, dt_1 \cdots dt_n.
\end{aligned}$$

In the analytical approach to the valuation of CDOs this is the essential step. Once this is assessed, the outstanding notional amount is obtained, we can then proceed to calculate the premium side. This expression shows that using this approach we can still use the current framework of valuation to the large extent. We simply apply the valuation of loss sides a few times with different correlation parameters, and then proceed to value the premium side.

## 2.4 Mixture of Expected Losses

In this approach we simply mixes the expected losses with respect to each Gaussian copulas. For a given time $t$, the expected tranche loss is,

$$E[L^T(t)] = \sum_{i=1}^{m} w_i E_{\rho_i}[L^T(t)]$$

Here $E_{\rho_i}[L^T(t)]$ is the tranche expected loss corresponding to the copula correlation $\rho_i$. Once the tranche expected loss is obtained across different time horizon both the default leg and premium leg of a CDO tranche can be easily calculated.

## 2.5 Mixture of Default Time Distributions

Suppose that we still want to maintain a one-parameter Gaussian copula function for each given one correlation parameter $\rho_j$ where we assume that the asset return is as follows:

$$X_i^j = \sqrt{\rho_j} \cdot Y_m + \sqrt{1-\rho_j} \cdot \varepsilon_{ij}$$

where $\rho_j$ is the one factor correlation parameter and $Y_m$ is the market common parameter. We still maintain that

$$T_i = F_i^{-1}(\Phi(X_i^j)),$$

then we can calculate the conditional default probability for any name as follow

$$\begin{aligned}\Pr\left[T_i \leq t | Y_m\right] &= \int \Phi\left(\frac{\Phi^{-1}\left(F_i\left(t\right)\right) - \sqrt{\rho} \cdot Y}{\sqrt{1-\rho}}\right) \cdot v\left(\rho\right) d\rho \\ &= \sum_{i=1}^{m} w_i \cdot \Phi\left(\frac{\Phi^{-1}\left(F_i\left(t\right)\right) - \sqrt{\rho_i} \cdot Y}{\sqrt{1-\rho_i}}\right).\end{aligned}$$

This is equivalent to having a specification of a copula function as follows

$$\begin{aligned}&\Pr\left[T_1 \leq t_1, T_2 \leq t_2, \cdots, T_n \leq t_n\right] \\ &= \sum_{i=1}^{m} w_i \cdot \Pr\left[X_1^i \leq \Phi^{-1}\left(F_i\left(t_1\right)\right), X_2^i \leq \Phi^{-1}\left(F_i\left(t_2\right)\right), \cdots, X_n^i \leq \Phi^{-1}\left(F_i\left(t_n\right)\right)\right]\end{aligned}$$

The marginal distribution of survival time $T_i$ is as follows

$$\Pr\left[T_i \leq t_i\right] = \sum_{j=1}^{m} w_j \cdot \Pr\left[X_i^j \leq \Phi^{-1}\left(F_i\left(t_i\right)\right)\right]$$

## 2.6 Calibration Algorithm

The calibration is around the parameters of $\alpha_i$, $i = 1, 2, \cdots, m-1$ and the correlation parameters $\rho_i$, $i = 1, 2, \cdots, m$ in the $m$ copula functions. If we use constant correlation for each of the three Gaussian copula functions we have $5$ parameters, $\alpha_1, \alpha_2$ and $\rho_1$, $\rho_2$ , $\rho_3$. If we try to match the break-even spreads of 5 index tranches we can have an objective function as follows:

$$f(\alpha_1, \alpha_2, \rho_1, \rho_2\,, \rho_3) = \sum_{i=1}^{5} \left(S^M - S^O\right)^2 \tag{3}$$

where $S^M$ is the model implied spreads while $S^O$ is the market observable spreads. Sometimes we'd like to give more weights to certain tranches due to our belief that certain tranches are traded more frequently and/or more fairly than other tranches. We can modify the above function as follows:

$$f(\alpha_1, \alpha_2, \rho_1, \rho_2\,, \rho_3) = \sum_{i=1}^{5} w_i \left(S^M - S^O\right)^2 \tag{4}$$

where $w_i' s$ are the weights that we assign to tranches.

This function can not be minimized using traditional optimization algorithm due to the potential problem of many local optimal solution. For example, even in the simple case of one correlation parameter Gaussian copula function we have observed that two different correlation parameters could result in the same spread for a mezzanine tranche. To find the global minimum in expressions (3) or (4) we use a sequential algorithm of optimization based on good lattice point (glp) or other Quasi Monte Carlo sequences. More details can be found in [9].

# 3 CDO and CDO$^2$ Pricing and Loss Distribution

## 3.1 CDO Pricing

A synthetic CDO tranche is a portfolio transaction between two parties, the protection buyer and the protection seller. The protection seller gets paid a fixed percentage of the outstanding notional amount of the chosen tranche, and incurs a loss only after the cumulative loss of the total portfolio becomes more than the attachment point, but the loss of the tranche to the protection sellers is capped by the tranche size.

We assume that the cumulative tranche loss up to time $t$ is $L^T(t)$, with the attachment level $K_L$ and detachment level $K_U$, and the tranche size is $\Delta^T = K_U - K_L$. If we ignore the interest rate impact, the payoff and its expected value can be expressed as follows:

$$L^T(t) = \max(L(t) - K_L, 0) - \max(L(t) - K_U, 0)$$
$$E[L^T(t)] = \int_{K_L}^{K_U} S_L(x)dx$$

where $S_L(t,x) = \Pr[L(t) > x]$ is the excess loss probability. This shows that the expected loss of a tranche is simply the size of the area of the excess loss distribution bounded by the attachment and detachment levels. Once we consider the impact of the interest rate we have to resort to some numerical method for the calculation. Here we present the details for the loss leg calculation.

We first obtain the unconditional tranche expected loss $E[L^T(t)]$ across different time horizon $t$, then we can calculate both default leg and premium leg of a CDO tranche. Leg $T_1$ and $T_2$ be the beginning and the end of a premium payment period, then the present value of the tranche expected loss for this period is

$$\begin{aligned} DefaultLeg[T_1, T_2] &= E\left[\int_{T_1}^{T_2} \exp(-\int_0^t r(u)du) \cdot dL_t\right] \\ &= \int_{T_1}^{T_2} \exp(-\int_0^t r(u)du) \cdot dE[L_t] \\ &= -\int_{T_1}^{T_2} \exp(-\int_0^t r(u)du) \cdot d\overline{O}_t \end{aligned}$$

Here $\overline{O}_t = \Delta^T - E[\overline{L}_t]$ is the tranche outstanding notional at time $t$. Assuming log-linear dependency between $\overline{O}_{T_1}$ and $\overline{O}_{T_2}$ over the time interval $[T_1, T_2]$ we can write

$$\overline{O}_t = \overline{O}_{T_1} \cdot \exp(-h(t - T_1))$$
$$h = \frac{\log(\overline{O}_{T_1}/\overline{O}_{T_2})}{T_2 - T_1}$$

Here $h$ can be viewed as the tranche "hazard rate" for the loss over the time interval $[T_1, T_2]$. Then the default leg present value for this period is,

$$DefaultLeg[T_1, T_2] = \frac{h}{h+f}\left[df(T_1)\overline{O}_{T_1} - df(T_2)\overline{O}_{T_2}\right]$$

where $df(t)$ is the risk-free discount factor at time $t$, $f = \frac{\log(df(T_1)/df(T_2))}{T_2 - T_1}$ is the forward risk free rate .

The premium leg for this period with market standard continuous settlement (accrual coupon is paid when default) convention can be written as

$$PremiumLeg[T_1, T_2] = \pi^T \cdot \Delta T \cdot df(T_2)\overline{O}_{T_2} + \pi^T \cdot \frac{h}{h+f} \cdot \left\{ \frac{1}{h+f} \left[ df(T_1)\overline{O}_{T_1} - df(T_2)\overline{O}_{T_2} \right] - \Delta T \cdot df(T_2)\overline{O}_{T_2} \right\}$$

where$\pi^T$ is the premium coupon and $\Delta T = T_2 - T_1$.

Both tranche default leg value and premium leg value are completed determined by the tranche expected loss (or equivalently tranche outstanding notional) along the time horizon. We can define tranche survival probability as the relative outstanding notional (outstanding notional divided by the original tranche size) then the tranche pricing is the same as a single name default swap pricing assuming zero recovery. All techniques in the single name CDS pricing can be used. As all the derivations in this section do not use any correlation assumption, one may also directly describe the portfolio loss process and use formulas here to compute CDO prices.

## 3.2 CDO$^2$ Pricing

Considering a CDO$^2$ tranche with an $m$ baby or underlying CDOs, and with a total of $N$ underlying credits. For each baby CDO we have the loss up to time $t$ as

$$L_B^N(t) = \sum_{i=1}^{N} N_i^k (1 - R_i) \cdot 1_{\tau_i < t}$$

where $N_i^k$ is the notional amount of the $i$th credit in the $k$th baby CDO, $\tau_i$ is the default time of the $i$th credit. The tranche loss, $\overline{L}_B^k$, for the $k$th baby CDO with the attachment level $K_L^B$ and detachment point $K_U^B$ is,

$$\overline{L}_B^k = \max(L_B^k(t) - K_L^B, 0) - \max(L_B^k(t) - K_U^B, 0)$$

And the tranche loss for the CDO$^2$ or "mother CDO" with the attachment point $K_L^M$ and detachment point $K_U^M$ is,

$$\overline{L} = \max(\sum_{k=1}^{n} \overline{L}_B^k - K_L^M, 0) - \max(\sum_{k} \overline{L}_b^k - K_U^M, 0)$$

It can also be seen from the above payoff function that the pricing of $CDO^2$ depends on the joint loss distribution from all underlying baby CDO portfolios.

## 3.3 Fast Loss Calculation for CDO and $CDO^2$ by Conditional Normal Approximation

Conditioning on the common factor $Y_M$ the loss from each credit is independent from each other. The total loss distribution as a sum of independent variables can then be calculated using a few methods such as Fourier transformation, recursive method or conditional normal approximation. All these calculation methods can be found in actuarial science literatures and books by Klugman, Panjer and Willmot [5]. Here we present the conditional normal approximation approach to facilitate our presentation.

we first compute the conditional mean and variance of the total loss variable, $L|Y_M$

$$M_v = \sum_{i=1}^{n} N_i(1 - R_i) \cdot q_i(t \mid Y_M)$$
$$\sigma_v^2 = \sum_{i=1}^{n} N_i^2(1 - R_i)^2 \cdot q_i(t \mid Y_M)(1 - q_i(t \mid Y_M))$$

where $q_i(t \mid Y_M) = Pr[\tau_i \leq t \mid Y_M]$ is the conditional marginal default probability for credit $i$ before time $t$. The conditional Normal approximation approach uses a Normal distributions to approximate the above conditional total loss distribution by assuming that the Normal distribution has the same mean and variance as computed above. The normal distribution is chosen due to the central limit theorem, which states that the sum of independent distributions (but not identical distribution) approaches to a normal distribution as the number of the independent distributions increases.

Given the conditional normal approach the conditional expected loss for the tranche can be easily computed in closed form as follow:

$$E[L^T(t)|v] = (M_v - K_L^T)\Phi\left(\frac{M_v - K_L^T}{\sigma_v}\right) + \sigma_v \cdot \phi\left(\frac{M_v - K_L^T}{\sigma_v}\right) \\ - \left(M_v - K_U^T\right)\Phi\left(\frac{M_v - K_U^T}{\sigma_v}\right) - \sigma_v \cdot \phi\left(\frac{M_v - K_U^T}{\sigma_v}\right)$$

where $\phi$ is one-dimensional normal density function. With the calculated conditional expected loss the unconditional expected loss is obtained simply by integrating over the common factor $Y_m$:

$$E[L^T(t)] = \int_{-\infty}^{\infty} E[L^T(t)|y] \cdot \phi(y)dy$$

For an equity tranche one can use the following method to preserve the expected loss of the whole portfolio, and to achieve a better approximation. We allow negative loss introduced in the conditional normal approximation in the equity tranche payoff function. An equity tranche with detachment point $K_U^T$ in this case can be expressed as follow:

$$L_{equity}^T(t) = L(t) - \max(L(t) - K_U^T, 0)$$

The conditional expected loss for the equity tranche is,

$$E[L_{equity}^T(t)|v] = M_v - \left(M_v - K_U^T\right)\Phi\left(\frac{M_v - K_U^T}{\sigma_v}\right) - \sigma_v \cdot \phi\left(\frac{M_v - K_U^T}{\sigma_v}\right)$$

This has been proven to work well for index equity tranche of size more than 3%. Alternatively we can also use inverse Gaussian distribution to approximate for the equity tranche since inverse Gaussian distribution takes only positive value.

Similarly, synthetic $\text{CDO}^2$ pricing depends on the joint loss distribution from all underlying baby CDO portfolio. Conditioning on the normal variable $Y_M$, we can compute the mean and covariance of loss variables, $L^k|Y_M$, $k = 1$,...,$n$

$$Mean_k^v = \sum_{i=1}^{N} N_i^k (1 - R_i) \cdot q_i(t \mid Y_M)$$

$$Covariance_{k,k'}^v = \sum_{i=1}^{N} N_i^k N_i^{k'} (1 - R_i)^2 \cdot q_i(t \mid Y_M)(1 - q_i(t \mid Y_M))$$

It can be seen from this expression that the conditional covariance of two loss distributions of sub-portfolios would depend on the overlapping of the two portfolios. Even though each credit default is independent to each other conditional on common factor, the overlapping of names in the two portfolios would result in a high correlation between the losses of the two portfolios. The conditional normal approximation approach uses multi-normal distributions to approximate the conditional loss distributions. The multi-normal distributions have the same means and covariance matrix as computed above. As the CDO$^2$ loss is a function of underlying CDO loss variables and with the conditional normal specification the conditional expected loss can be calculated with a multi-dimensional integration.

$$E[\overline{L}_t|v] = \int \cdots \int \overline{L}(L_b^1, ..., L_b^n) dL_b^1 dL_b^2 ... dL_b^n$$

The dimension of this integration equals to the number of underlying CDOs. A Monte Carlo simulation is used to compute this multi-dimensional integration. With the calculated conditional expected loss the unconditional expected loss is obtained simply by integrating crossing the common factor $Y_M$,

$$E[\overline{L}_t] = \int_{-\infty}^{\infty} E[\overline{L}_t|y] \cdot n(y) dy.$$

where $n(y) = \frac{1}{\sqrt{2\pi}} \exp(-y^2/2)$ is the standard normal density function.

# 4 Portfolio Loss Distribution

As we see in the above sections the valuation of CDO or CDO$^2$ only depends on the portfolio cumulative loss. For the pricing of a CDO or CDO$^2$ it is sufficient to know the portfolio loss distributions over different time horizons. Start with an equity tranche with a strike $K$, the tranche loss up to time $t$, $L^T(t)$, is,

$$L^T(t) = \max(L(t), 0) - \max(L(t) - K, 0) = L(t) - \max(L(t) - K, 0)$$

The expected tranche loss is

$$E[L^T(t)] = \int_0^K S_L(t, x) dx$$

where $S_L(t, x) = \Pr[L(t) > x]$ is the excess loss distribution. If we differentiate the above equation with respect to $K$, we get

$$\frac{\partial E[L^T(t)]}{\partial K} = S_L(t, K),$$

and therefore

$$\frac{\partial^2 E[L^T(t)]}{\partial K^2} = -f(K).$$

Here $f(x)$ is the density function of the portfolio loss variable $L(t)$. An important observation here is that the first derivative of the expected tranche loss with respect to strike $K$ is the portfolio excess loss probability, and the second derivative is the portfolio loss density function. Therefore the portfolio loss distribution is completely determined by the prices of equity tranches and this is in line with the base correlation concept [2]. There are several non-arbitrage constraints about the portfolio loss distribution, $L(t)$:

$$\begin{aligned}
\Pr(L(t) < 0) = 0,\ \Pr(L(t) < L_{\max}) = 1 \\
\frac{\partial S(t,x)}{\partial x} < 0 \\
\frac{\partial S(t,x)}{\partial t} \geq 0 \\
\int_0^{L_{\max}} S(t,x)dx = E[L(t)]
\end{aligned} \tag{5}$$

The first constrain states that the portfolio loss is non-negative and bounded by the maximum possible loss, which is $L_{\max} = \sum_{i=1}^n B_i$. The second constrain shows that the density function is always positive, i.e. the expected loss of equity tranche, $E[L^T(t)]$, is a concave function of the strike $K$. Or state in another way, for two tranches of equal size, the junior tranche always worths more than the senior tranche. The third constrain reflects that the cumulative loss is increasing in time. The last constrain comes from straightforward computation and states that the integration of loss probability across all the strikes equals to the whole portfolio expected loss, which is independent of default correlation assumptions.

## 4.1 A Fast and Accurate Loss Distribution Computation

The computation of a portfolio loss distribution involves the first-order derivative of tranche expected loss with respect to strikes, an accurate implementation is important for any practical use. Here we provide a semi-analytic approach to compute the loss distribution for a general portfolio using conditional normal approximation.

Conditioning on $v$ the expected loss for the equity tranche with strike $K$ is calculated as,

$$
\begin{aligned}
E[L^T(t)|v] &= E[L(t)|v] - (M_v - K)\,\Phi\left(\frac{M_v - K}{\sigma_v}\right) - \sigma_v \cdot \phi\left(\frac{M_v - K}{\sigma_v}\right) \\
&= E[L(t)|v] - A(K, v) \\
A(K, v) &= (M_v - K)\,\Phi\left(\frac{M_v - K}{\sigma_v}\right) + \sigma_v \cdot \phi\left(\frac{M_v - K}{\sigma_v}\right)
\end{aligned}
$$

and the unconditional expected loss is

$$
E[L^T(t)] = E[E[L^T(t)|v]] = E[L(t)] - \int_{-\infty}^{\infty} A(K, v) \cdot \phi(v) dv
$$

Then

$$
\Pr(L(t) > K) = \frac{\partial E[L^T(t)]}{\partial K} = \int_{-\infty}^{\infty} \frac{\partial A(K, v)}{\partial K} \cdot \phi(v) dv
$$

where $M_v$ and $\sigma_v$ are calculated as in section 3.3 and their derivatives to strike $K$ can be obtained using finite differences. The loss distribution computation only involves equity tranche prices for various strikes and can be applied to generate market implied portfolio loss distribution for a given credit portfolio. Equity tranches prices at standard strikes (such as 3%, 7% and etc. for CDX) can be obtained through bootstrapping, which is in line with the base correlation approach. For an non-standard strike the corresponding base correlation can be interpolated via cubic spline and the equity tranche price for this strike can be therefore computed.

## 4.2 Transformation of Loss Variables

Let $L_{imp}(t)$ be the market implied loss distribution from a CDO portfolio, and $L(t)$ the loss distribution from a model such as the GM model or any other extension of the basic Gaussian copula model. We introduce a transformation or mapping function from the model implied loss variable $L(t)$ to the market implied loss variable $L_{imp}(t)$ on a cumulative probability to a cumulative probability basis. Specifically, let $P_{imp}(K) = \Pr(L_{imp}(t) > K)$ and $P(K) = \Pr(L(t) > K)$, be the respective probability distribution functions for loss variables $L(t)$ and $L_{imp}(t)$. We define a transformation function from $L$ to $L_{imp}$ by $F(L) = P_{imp}^{-1} \circ P(L)$. This transformation satisfies the *Fundamental Transformation Law of Probabilities*, which is simply

$$f_{imp}(L_{imp})dL_{imp} = f(L)dL$$

here $f_{imp}$ and $f$ are the density functions for $L_{imp}$ and $L$. The transform function $F$ is invariant under two boundary points, 0 and $L_{\max}$, that means $F(0) = 0$ and $F(L_{\max}) = L_{\max}$.

It is easy to see that this transformation preserves CDO prices. As mentioned before the valuation of a CDO tranche is completely determined once the tranche expected loss $E[L^T(t)]$ is obtained across different time horizon $t$. Therefore, it is enough to show, for a given time $t$, the expected loss of an equity tranche with strike $K$ is invariant under the transformation. Let $E_{imp}[L^T(t)]$ be the expected loss of the equity tranche under the market implied distribution. From equation (5) for an equity tranche we have

$$\frac{\partial E[L^T(t)]}{\partial K} = \Pr(L(t) > K) \text{ and } \frac{\partial E_{imp}[L^T(t)]}{\partial K} = \Pr{}_{imp}(L(t) > K)$$

then

$$E[L^T(t)] = \int_0^K \Pr(L(t) > u)du = \int_0^K \Pr{}_{imp}(L(t) > w)dw = E_{imp}[L^T(t)]$$

The second equality of the above equation comes from the transformation definition. The transformation preserves tranche expected loss for every equity tranche therefore preserves the value of any CDO tranche.

This transformation can be naturally extended to multi-portfolio cases. Given $m$ CDO portfolios, let $L^k(t)$ be the loss variable and $P^k$ a loss probability function for the $k$th underlying CDO portfolio, $k = 1, 2, ...m$. Let $P^k_{imp}$ be the market implied loss probability function and $F^k = (P^k_{imp})^{-1} \circ P^k$ the respective transform function for the $k$th underlying CDO credit portfolio. We also assume that $L^k(t)$, $k = 1, 2, ...m$ follows some joint distribution generated by some correlation model such as GM with a joint probability function

$$\overline{P}(K^1, K^2, ..., K^m) = \Pr(L^1(t) < K^1, L^2(t) < K^2, ..., L^m(t) < K^m)$$

Then, for every realization of $L^k(t)$, $k = 1, 2, ...m$, according to the above joint distribution, we transform each $L^k(t)$ to its market skew aware counter party via $L^k_{imp}(t) = F^k(L^k(t))$, $k = 1, 2, ...m$. Also, for each $k$, $L^k_{imp}(t)$ has the same loss distribution as the market implied and jointly $L^k_{imp}$, $k = 1, 2, ...m$ follow the same joint distribution as that for $L^k(t)$, $k = 1, 2, ...m$, which means,

$$\begin{aligned}\Pr(L^1_{imp}(t) < K^1_{imp}, L^2_{imp}(t) < K^2_{imp}, ..., L^m_{imp}(t) < K^m_{imp}) \\ = \Pr(L^1(t) < K^1, L^2(t) < K^2, ..., L^m(t) < K^m)\end{aligned}$$

where $K^k_{imp} = F(K^k)$, $k = 1, 2, ...m$.

As the joint distribution of the loss variables $L^k(t)$'s, is naturally obtained through modeling default of each credit name in the portfolios through our GM model, and this is done at each path or scenario level, the dispersion or overlapping characteristics among the $m$ CDO portfolios and the order of defaults are naturally incorporated in this approach. This is fundamentally different from a so called "copula of copula" approach. In the copula of copula approach we simply use another copula function or a second layer of copula to combine the implied loss distributions from the underlying baby portfolios. This approach would maintain the marginal distributions or baby skews due to the nature of copula function. However, the second layer copula function is exogenously imposed to the joint loss distribution from the $m$ underlying CDO portfolios. The copula of copula approach can not consistently model the overlapping and the order of default in all underlying portfolio - two important factors in the valuation of a $\text{CDO}^2$ type transactions, especially when there exist an overlapping of names in different baby portfolios. Also in the copula of copula approach the parameter in the second layer of copula lacks any economic interpretation while in our approach we still maintain the basic economic interpretation of all parameters.

This transformation idea has been used to price bespoke portfolio from a base correlation from an index. Two popular approaches used in practice are normalized strike and normalized loss ratio approaches. We can use these approaches to price bespoke transactions first, and then to get implied loss distributions from these pricing results. One alternative way to price bespoke CDOs is to use the loss variable transformation approach mentioned above. We first find the GM parameters through the calibration and try to fit the relevant index tranche market as close as possible. Then we map the portfolio loss derived from the GM model for the bespoke portfolio to the loss distribution extracted from the index tranche market to price each bespoke portfolio. The idea is summarized in Figure 1

Or mathematically,

$$L_{imp} = P^{-1} \circ (P^{index}) \circ (P^{index}_{imp})^{-1} \circ P(L)$$

Here $L$ and $L_{imp}$ are bespoke losses, $P^{index}$ and $P^{index}_{imp}$ are the index portfolio loss probability, in the Gaussian Mixture measure and the index tranche market

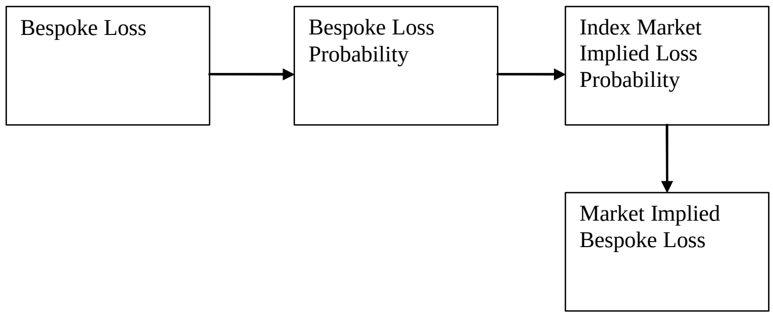


Figure 1: Graphic Illustration For Pricing Bespoke Portfolio

measure respectively. $P$ is the bespoke portfolio loss probability function in the Gaussian Mixture measure.

# 5 Calibration Result and Numerical Comparison

Using above optimization algorithm we can calibrate the Gaussian mixture model against the North American 5-year credit spread index (CDX) tranche market. The index tranche is quoted as running off annual spread in basis points (bps) with the exception that equity tranche is quoted as a front premium in terms of a fixed percentage plus a 500 bps running spread. The bid/ask and mid spreads are given in the following table for 5-year CDX standard tranches. We also list the spreads from the calibrated Gaussian mixture model. It shows that the GM model matches the index well with the model spread within bid/ask spread, the only exception is for senior tranche 15-30% the model spread is outside the bid/ask spread from the market.

Similarly we can calibrate the GM model to the European index market such as 5-year Itraxx market. The market quotes and the calibrated model spreads are as follows:

The calibrated GM parameters for 5-year CDX and ITRAXX are in Table 3

It shows that there is 76.07% chance with a correlation of zero, 21.56% with

Table 1: CDX 5-Year Index Tranche Quotes: Index Spread 51 bp on August 31, 2005

| Tranche | Market Bid | Market Offer | Market Mid | Model Spread |
|---|---|---|---|---|
| 0-3% | 40.1% | 40.6% | 40.4% | 40.4% |
| 3-7% | 131.0 | 135.0 | 133.0 | 133.8 |
| 7-10% | 34.0 | 37.0 | 35.5 | 35.7 |
| 10-15% | 19.5 | 21.0 | 20.3 | 21.6 |
| 15-30% | 9.5 | 11.0 | 10.3 | 15.4 |

Table 2: ITRAXX 5-Year Index Tranche Quotes: Index Spread 36 bp on August 31, 2005

| Tranche | Market Bid | Market Offer | Market Mid | Model Spread |
|---|---|---|---|---|
| 0-3% | 24.0% | 24.5% | 24.3% | 24.3% |
| 3-6% | 81.0 | 83.0 | 82.0 | 82.5 |
| 6-9% | 25.0 | 27.0 | 26.0 | 24.4 |
| 9-12% | 14.0 | 16.0 | 15.0 | 18.2 |
| 12-22% | 8.3 | 9.0 | 8.6 | 12.9 |

a high correlation of 88.5% and a very small chance 2.37% with a medium correlation of 26.4%. In the current market environment with no clear direction in the future we find that we very often fall into these two extreme states: either zero correlation or high correlation states. We have similar result for Itraxx with details

To compare the market against model we plot the implied loss distribution and the calibrated GM model distribution for both CDX and Itraxx and plot them in the following paragraph. It shows that the GM model can produce a reasonable skew observed in the index tranche market. But we still underestimate the small loss and large loss and overestimate medium size loss comparing to the implied loss distribution observed from the index market.

We could explain the result in two ways: the GM model still does not capture the index market dynamics or the index market does not trade "reasonably". But if we cannot arbitrage the market it is hard to say that market is wrong. Another way of studying the difference between the basic Gaussian copula model, plus the base correlation method against the GM model is to compare the tranche "leverage ratio" in the two models. The leverage ratio is defined as the break-even spread

Table 3: Calibration Result of 5-Year CDX and ITRAXX

| CDX Correlation | CDX Weight | ITRAXX Correlation | ITRAXX Weight |
|---|---|---|---|
| 0% | 76.07% | 1.8% | 71.9% |
| 26.4% | 2.37% | 21.2% | 9.81% |
| 88.5% | 21.56% | 94.8% | 18.29% |

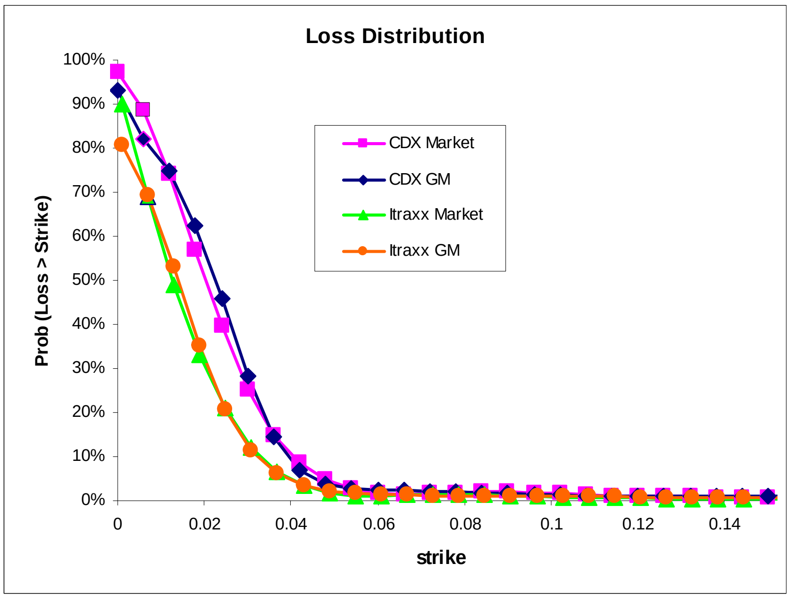


Figure 2: Implied Loss Distribution and GM Loss Distribution for CDX and Itraxx

changes when the index spread changes by one basis point. The empirical studies shows that GM model captures the actual index tranche market better than the basic Gaussian copula model plus base correlation.

In order to price a CDO$^2$ correctly there are several issues needed to be addressed. The correlation skew of each underlying baby CDO must be reproduced. The outer correlation, the correlation of joint distribution, among the losses of the baby CDOs must be considered. The outer correlation, which can be interpreted as a global skew, depends on the overlapping characteristics, the spread level for each credit in the portfolio, and the correlations structure among the credits.

By the transformation of loss variables the model reproduces the individual skew of each baby CDO automatically. The GM parameters can then be calibrated to fit the model to the outer correlation structure. The outer correlation can either be implied from market CDO$^2$ prices or indirectly from the ”global skew” of the master CDO, which, for each credit, has the same notional as that by aggregating the notional across all the baby CDOs. This mapping would only change the marginal loss distributions of the underlying sub-portfolio, the correlation among all credits is still governed by our GM correlation model or other extension models. The overlapping characteristics among baby CDOs and the default order, which have large impact on CDO$^2$ pricing, are not altered.

We use this model to price 5 years and 10 years CDO$^2$ deals and compare the results with market consensus provided by Mark-It Partners [1]. The underlying individual credits in the sample deal have an average of 50 basis points (bps) at the tenor of 5-year and 74 bps at the tenor of 10-year. The portfolio has 225 names in total with six underlying baby CDOs whose attachment and attachment points are 5% and 7% respectively, and each baby CDO has 75 names as the underlying. We price a whole range of CDO$^2$ tranches from an equity tranche (0-10%) to a super senior tranche (50%-100%). The market consensus is the average tranche spreads provided by participants of the Mark-It Partner CDO$^2$ exercise at the end of July 2005. Mark-It Partner also provides one standard deviation from the market consensus. We can see that there exist some variation among the quotes provided by different participants.

We use the following Gaussian mixture parameters to value various CDO$^2$ tranches using our GM plus loss distribution mapping approach. The model

Table 4: Calibration Result for GM Model

| Model Parameter | 5 year | 10 year |
|---|---|---|
| Correlation 1 | 0% | 8% |
| Correlation 2 | 10% | 15% |
| Correlation 3 | 85% | 90% |
| Weight 1 | 70% | 70% |
| Weight 2 | 20% | 10% |
| Weight 3 | 10% | 20% |

prices were all within one standard deviation to market consensus. One could get more close result if the five model parameters are fitted to market consensus in a

[1]Our special thanks go to Mark-It Partners for its allowance to use its poll data in an aggregate form and on a relative basis for the tranches spread quotes.

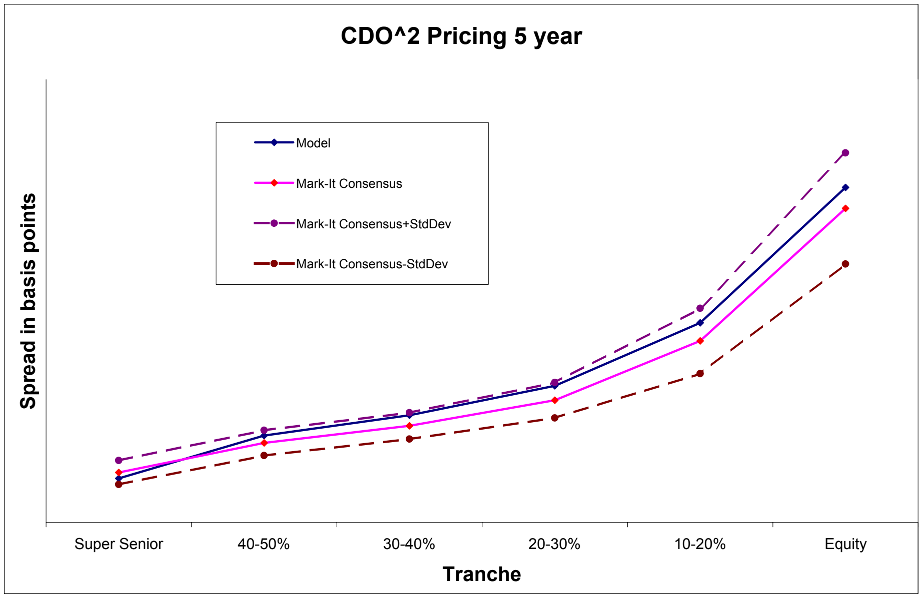


Figure 3: Comparison of a 5-Year $CDO^2$ Pricing

least square sense. The model shows the superior property of not only reproducing baby CDO correlation skew but also having the flexibility to fit outer correlation (or global skew) among baby CDOs. Figure 3 and Figure 4 show the 5-year and 10-year CDO$^2$ pricing comparison.

It has been observed that the GM model sometimes fits all five tranches, and sometimes it misses one or two tranches. Since it could not fit the standard tranche market consistently it is hard to trust the model to price bespoke CDO directly. We use GM plus loss mapping to overcome this problem. We have studied the "robustness" of the model with respect to CDO$^2$ pricing. In one case we use the same set of GM parameters to price CDO$^2$ transactions over 5-month period and compared with Mark-It results. It shows that we price the trades within the market consensus. Another approach we take is to study the robustness of the CDO$^2$ pricing with respect to the GM parameters. Within certain ranges of the GM

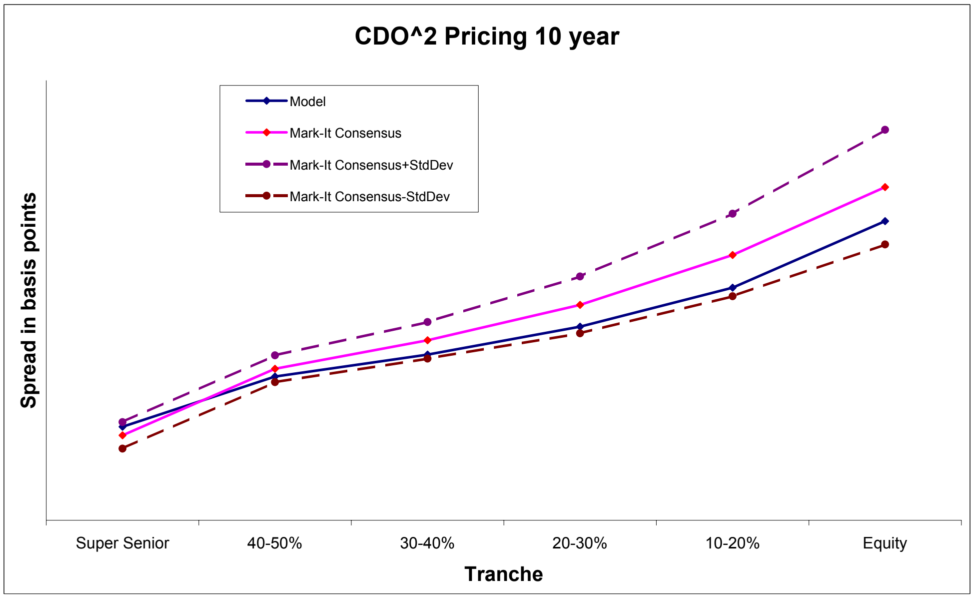


Figure 4: Comparison of a 10-Year $CDO^2$ Pricing

parameters the final pricing result for CDO$^2$ is not sensitive to GM parameters.

# 6 Summary and Conclusion

In summary we have presented an extension to the one-factor Gaussian Copula model in which we allow the correlation parameter to be random instead of fixed. The model has been shown to broadly capture the correlating skew observed in the index tranche market. But the model still has difficulty to calibrate to the market exactly, across tenors and all the times. This could be due to the model limitation in its flexibility to capture all sections of loss distributions implied from the index tranche market or due to relative young index market in which the supply and demand for different tranches are not all balanced.

It has been shown that we simply need the loss distribution of portfolio across times to value any CDO transactions, and joint loss distributions of all baby portfolios to price a $CDO^2$ trade. To this regard using market observed index tranche market or bespoke pricing we could obtain implied loss distributions for all baby CDO portfolios. This implied loss distribution is very close to the model implied loss distribution from the Gaussian mixture model if the GM model calibrates to the market well. Using a loss distribution mapping between the market implied loss distribution and the model implied loss distribution we have achieved a balance between matching the current market spread and also having a plausible economic model. This approach is totally different from the so-called "copula of copula" approach where a copula function is mechanically applied to the loss distributions of the baby portfolios directly. The "copula of copula" approach lacks a correlation structure, cannot deal with the overlapping of names in different portfolios, and the parameter has to vary from portfolio to portfolio or tranche to tranche to match market observed price. It has been shown that our combination of GM model plus loss distribution transformation has provided a robust framework to price CDOˆ2-type trades consistently with the pricing of the underlying CDOs. Other portfolio transactions such as CDO of long and short credits or tranches, combo index of a few subindexes could also be priced in this framework.

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