Complete Working for Lecture 3.1

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These notes are to be read in conjunction with Paul Wilmott's lecture 3.1 on the Black-Scholes Model and Greeks. The first section discusses some important results on the fundamental solution when working through the heat equation as an initial value problem. Earlier we solved the diffusion equation for a general solution, but here we look at this function as a *Green's function* which is used later to solve the BSE.

After this initial discussion the detailed derivation of the BSE and its subsequent method of solution is presented with all steps in the working given. These notes conclude with details of early exercise, i.e. American options given, by considering the perpetual American put and call, which gives an insight to the idea of the *smooth pasting condition*.

1 Mathematical Methods

1.1 The Heat Equation

We have seen on numerous occasions the one dimensional heat/diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

for the unknown function of the form $u(x,t) = t^{\alpha}\phi\left(\frac{x}{t^{\beta}}\right)$. The corresponding solution derived using the similarity reduction technique is the *fundamental solution*

$$u(x,t) = \frac{1}{2\sqrt{\pi t}} \exp\left(-\frac{x^2}{4t}\right).$$

Some books refer to this as a *source solution.*

Let's consider the following integral

$$\lim_{t \longrightarrow 0} \int_{-\infty}^{\infty} u\left(y, t\right) f\left(y\right) dy$$

which can be simplified by the substitution

$$s = \frac{y}{2\sqrt{t}} \Longrightarrow 2\sqrt{t} ds = dy$$

to give

$$\lim_{t \longrightarrow 0} \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} \exp\left(-s^2\right) f\left(2\sqrt{t}s\right) 2\sqrt{t} ds.$$

In the limiting process we get

$$\begin{aligned} f\left(0\right) \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp\left(-s^2\right) ds &= f\left(0\right) \frac{1}{\sqrt{\pi}} \sqrt{\pi} \\ &= f\left(0\right). \end{aligned}$$

Hence

$$\lim_{t \longrightarrow 0} \int_{-\infty}^{\infty} u(y,t)\, f(y)\, dy = f(0)\,.$$

A slight extension of the above shows that

$$\lim_{t \longrightarrow 0} \int_{-\infty}^{\infty} u(x-y,t)\, f(y)\, dy = f(x)\,,$$

where

$$u(x-y,t) = \frac{1}{2\sqrt{\pi t}} \exp\left(-\frac{(x-y)^2}{4t}\right).$$

Let's derive the result above. As earlier we begin by writing $s = \dfrac{x-y}{2\sqrt{t}} \Longrightarrow y = x - 2\sqrt{t}s$ and hence $dy = -2\sqrt{t}ds$. Under this transformation the limits are

$$\begin{aligned} y &= \infty \longrightarrow s = -\infty \\ y &= -\infty \longrightarrow s = \infty \end{aligned}$$

$$\frac{1}{2\sqrt{\pi t}}\int_{\infty}^{-\infty} \exp\left(-s^2\right) f\left(x - 2\sqrt{t}s\right)\left(-2\sqrt{t}ds\right) ds$$

$$\begin{aligned} & \lim_{t \longrightarrow 0} \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} \exp\left(-s^2\right) f\left(x - 2\sqrt{t}s\right) ds \\ = \; & f(x) \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} \exp\left(-s^2\right) ds \\ = \; & f(x) \frac{1}{\sqrt{\pi}}\sqrt{\pi} \end{aligned}$$

and

$$\lim_{t \longrightarrow 0}\int_{-\infty}^{\infty} u(x-y,t)\, f(y)\, dy = f(x)\,.$$

Since the heat equation is a constant coefficient PDE, if $u(x,t)$ satisfies it, then $u(x-y,t)$ is also a solution for any y.

Recall what it means for an equation to be linear:

Since the heat equation is linear,

1. if $u(x-y,t)$ is a solution, so is a multiple $f(y)\,u(x-y,t)$

2. we can add up solutions. Since $f(y)\,u(x-y,t)$ is a solution for any y, so too is the integral

   $$\int_{-\infty}^{\infty} u(x-y,t)\, f(y)\, dy.$$

   Recall, adding can be done in terms of an integral. So we we can summarize by specifying the following initial value problem

   $$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} \\ u(x,0) &= f(x) \end{aligned}$$

   which has a solution

   $$u(x,t) = \frac{1}{2\sqrt{\pi t}} \int_{\infty}^{-\infty} \exp\left(-\frac{(x-y)^2}{4t}\right) f(y)\, dy.$$

This satisfies the initial condition at $t = 0$ because we have shown that at that point the value of this integral is $f(x)$. Putting $t < 0$ gives a non-existent solution, i.e. the integrand will blow up.

Example 1 *Consider the IVP*

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} \\ u(x,0) &= \begin{cases} 0 & \textit{if } x > 0 \\ 1 & \textit{if } x < 0 \end{cases} \end{aligned}$$

We can write down the solution as

$$\begin{aligned} u(x,t) &= \frac{1}{2\sqrt{\pi t}} \int_{\infty}^{-\infty} \exp\left(-\frac{(x-y)^2}{4t}\right) \underbrace{u(y,0)}_{=f(y)} dy \\ &= \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{0} \exp\left(-\frac{(x-y)^2}{4t}\right) .1 dy \end{aligned}$$

put

$$s = \frac{y - x}{\sqrt{2t}}$$

$\int_{-\infty}^{0}$ *becomes* $\int_{-\infty}^{\frac{-x}{2\sqrt{t}}}$

$$\begin{aligned} & \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\frac{-x}{2\sqrt{t}}} \exp\left(-s^2/2\right) \sqrt{2t} ds \\ = & \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{-x}{2\sqrt{t}}} \exp\left(-s^2/2\right) ds \\ = & N\left(\frac{-x}{2\sqrt{t}}\right) \end{aligned}$$

So we have expressed the solution in terms of the CDF.

This can also be solved by using the substitution

$$\widehat{s} = \frac{-(y - x)}{\sqrt{2t}} \longrightarrow -dy = \sqrt{2t} d\widehat{s}$$

$\int_{-\infty}^{0}$ becomes $\int_{\infty}^{\frac{x}{2\sqrt{t}}}$

$$
\begin{aligned}
& -\frac{1}{2\sqrt{\pi t}}\int_{\infty}^{\frac{x}{2\sqrt{t}}} \exp\left(-\widehat{s}^2/2\right)\sqrt{2t}d\widehat{s} \\
= \; & \frac{1}{2\sqrt{\pi t}}\int_{\frac{x}{2\sqrt{t}}}^{\infty} \exp\left(-\widehat{s}^2/2\right)d\widehat{s} \\
= \; & \mathrm{erf\,c}\left(\frac{x}{2\sqrt{t}}\right)
\end{aligned}
$$

so now we have a solution in terms of the complimentary error function.

2 Black-Scholes World

2.1 Introduction

The field of mathematical finance has become particularly prominent due to the much celebrated Black-Scholes equation written in 1973 by Fisher Black, Myron Scholes and Robert Merton, for which they were awarded the Nobel prize for economics, in 1997. The origins of quantitative finance can however be traced back to the start of the twentieth century. Louis Jean-Baptiste Alphonse Bachelier (March 11, 1870 - April 28, 1946) is credited with being the first person to derive the price of an option where the share price movement was modelled by Brownian motion, as part of his PhD, entitled The Theory of Speculation (published 1900). Thus, Bachelier is considered a pioneer in the study of financial mathematics and one of the earliest exponents of Brownian Motion.

Some of the ideas introduced in the Binomial Model, i.e. delta hedging and risk-neutral world are now used in studying the Black–Scholes equation (BSE).

In this section we will derive the Black-Scholes equation(s) and find formulae for vanilla call and put options.

This work is fundamental to pricing in the Black-Scholes environment.

2.1.1 Notation

Consider an options contract

$$V(S, t; \sigma, \mu; E, T; r).$$

Semi-colons separate different types of variables and parameters.

- S and t are variables;
- σ and μ are parameters associated with the asset price;
- E and T are parameters associated with the particular contract;
- r is a parameter associated with the currency.

For the moment just use $V(S, t)$ to denote the option value.

2.2 The Black–Scholes assumptions

- The underlying follows a lognormal random walk with known volatility

  $$\sigma = \sqrt{\frac{1}{T-t}\int_t^T \sigma^2(\tau)\,d\tau}$$

  So this assumption ignores the fact that stocks can jump giving rise to jump diffusion models

- The risk-free interest rate is a known function of time

  $$r = \frac{1}{T-t}\int_t^T r(\tau)\,d\tau$$

- There are no dividends on the underlying (we can drop this assumption)

- Delta hedging is done continuously (later we do discrete hedging)

- There are no transaction costs on the underlying, no taxes or limits to trading (when you delta hedge stock must be bought and sold - which costs) - later in the course we will study transaction costs.

- There are no arbitrage opportunities. (A portfolio consisting of an option and stock is constructed. Delta hedging eliminates risk hence it can only grow at the risk free rate)

The resulting PDE is essentially the Binomial Model in a continuous time setting.

2.3 Constructing the portfolio

A call option will $\left\{ \begin{array}{c} \text{rise} \\ \text{fall} \end{array} \right.$ in value if the underlying asset $\left\{ \begin{array}{c} \text{rises} \\ \text{falls} \end{array} \right.$ — positive correlation

A put option will $\left\{ \begin{array}{c} \text{rise} \\ \text{fall} \end{array} \right.$ in value if the underlying asset $\left\{ \begin{array}{c} \text{falls} \\ \text{rises} \end{array} \right.$ — negative correlation

Set up the following portfolio Π consisting of one long option position and a short position in some quantity Δ, **Delta**, of the underlying asset:

$$\Pi = V(S,t) - \Delta S.$$

The asset evolves according to the SDE

$$dS = \mu S dt + \sigma S dX$$

So the question we ask is how does the value of the portfolio change over one time-step dt?

$$d\Pi = dV - \Delta dS.$$

We hold Δ fixed during the time step and change when rehedging. Itô for $V(S,t)$ gives

$$dV = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial S} dS + \frac{1}{2}\frac{\partial^2 V}{\partial S^2} dS^2.$$

and using the form for dS yields

$$dV = \left(\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \sigma S \frac{\partial V}{\partial S} dX.$$

Substituting in $d\Pi$ gives the following portfolio change

$$\begin{aligned} d\Pi &= \left(\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \\ &\quad \sigma S \frac{\partial V}{\partial S} dX - \Delta \left(\mu S dt + \sigma S dX \right) \end{aligned}$$

So we note that the change contains risk which is present due to

$$\sigma S \frac{\partial V}{\partial S} dX - \Delta \left(\sigma S dX \right),$$

i.e. coefficients of dX. Ideally we want this expression to vanish,

$$\sigma S \frac{\partial V}{\partial S} dX - \Delta \left(\sigma S dX \right) = 0$$

which gives

$$\Delta = \frac{\partial V}{\partial S}.$$

This choice of Δ renders the randomness zero.

More importantly we term the reduction of risk as **hedging**. The perfect elimination of risk, by exploiting correlation between two instruments (in this case an option and its underlying) is generally called **Delta hedging**.

Delta hedging is an example of a **dynamic hedging** strategy.

From one time step to the next the quantity $\frac{\partial V}{\partial S}$ changes, since it is, like V a function of the ever-changing variables S and t.

This means that the perfect hedge must be continually rebalanced.

After choosing the quantity Δ as suggested above, we hold a portfolio whose value changes by the amount

$$d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt.$$

This change is completely *riskless*.

If we have a completely risk-free change $d\Pi$ in the value Π then it must be the same as the growth we would get if we put the equivalent amount of cash in a risk-free interest-bearing account:

$$d\Pi = r\Pi \, dt.$$

This is an example of the **no arbitrage** principle.

Hence we find that

$$\begin{aligned}\left(\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}\right)dt &= r\left(V-\Delta S\right)dt\\ &= r\left(V-S\frac{\partial V}{\partial S}\right)dt.\end{aligned}$$

On dividing by dt and rearranging we get the **Black–Scholes equation (BSE)** for the price of an option

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+rS\frac{\partial V}{\partial S}-rV=0$$

The Black-Scholes equation is a **linear parabolic partial differential equation.** This means that

- if V_1 and V_2 are solutions of the BSE then so is $V_1 + V_2$ and
- if V is a solution of the BSE and k is any constant then kV is also a solution

Two simple solutions of the BSE are

1. Asset $V(S, t) = S$
2. Cash $V(S, t) = S_0 e^{rt}$

2.4 Final and Boundary conditions

To solve the Black-Scholes PDE we need to impose suitable boundary and final conditions. Until we do so the BSE knows nothing about what kind of option we are pricing.

If we remind ourselves of the structure of this equation, i.e. first order in time and second order in asset price - this tells us that we need one time condition and two boundary conditions.

1. **Final Condition** provides information on t. This is called the *Payoff.*

2. **Boundary Condition** tells us something about the underlying for two values of S. In this case we choose $S = 0$ and $S \longrightarrow \infty$ (i.e. when the underlying becomes large).

Recall that in the absence of such conditions we obtain a general solution. PDEs (unlike ODEs) are generally solved for particular solutions, as most equations are obtained from physical situations hence we have some information about their behaviour. This is dealt with by the **final condition**. We must specify the option value V as a function of the underlying at the expiry date T. That is, we must prescribe $V(S,T)$, the payoff.

2.5 Calls and Puts

For a **call option** we use the following:

Payoff:

$$V(S,T) = \max(S - E, 0).$$

Boundary Conditions:

$$S = 0 \Longrightarrow V(S,t) = 0$$

If we put $S = 0$ in $dS = \mu S dt + \sigma S dX$ then the change will be zero.

$$S \longrightarrow \infty \Longrightarrow V(S,t) \sim S$$

As S becomes very large if we look at $\max(S-E,0)$ then we find that $S >> E$, hence V is approximately similar to S.

For a **put option** we use the following:

Payoff:

$$V(S,T) = \max(E-S,0).$$

Boundary Conditions:

$$S = 0 \Longrightarrow V(S,t) = Ee^{-r(T-t)}$$

This is obtained from the **put call parity:**

$$C(S,t) - P(S,t) = S - Ee^{-r(T-t)}.$$

where C and P represent a call and put in turn. We know when $S = 0$, $C = 0$.

$$S \longrightarrow \infty \Longrightarrow V(S,t) \sim 0$$

This is all the information we need to solve the BSE.

2.6 The PDE Approach

Why use a PDE?

- This method opens door to many fast accurate numerical schemes. FDM to come soon.
- Provides a simple framework for formulating complex contracts as BVPs
- A deterministic approach focusing on the relationship between V and S. The evolution of the asset is now unimportant - the model takes account of all possible price histories. The delta hedging has eliminated randomness to give us a deterministic scheme.

2.7 Solving the Equation

The Black–Scholes equation is now solved for plain vanilla calls and puts. Starting with

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} rS \frac{\partial V}{\partial S} - rV = 0$$

The three main steps are:

- Turn the BSE into a one dimensional heat equation by a series of transformations.

- Use a known solution of the heat equation called the *fundamental solution*.

- Reverse the transformations.

Step 1

Recalling that the payoff is received at time T but that we are valuing the option at time t this suggests that we write

$$V(S,t) = e^{-r(T-t)}U(S,t)$$

$$\begin{aligned}
\frac{\partial V}{\partial t} &= re^{-r(T-t)}U + e^{-r(T-t)}\frac{\partial U}{\partial t}. \\
\frac{\partial V}{\partial S} &= e^{-r(T-t)}\frac{\partial U}{\partial S} \longrightarrow \frac{\partial^2 V}{\partial S^2} = e^{-r(T-t)}\frac{\partial^2 U}{\partial S^2}
\end{aligned}$$

This takes our differential equation to

$$\frac{\partial U}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 U}{\partial S^2} + rS\frac{\partial U}{\partial S} = 0.$$

Step 2

As we are solving a backward equation we can write

$$\tau = T - t.$$

The time to expiry is more useful in an option's value than simply the time. We can use the chain rule to rewrite the equation in the new time variable τ

$$\begin{aligned}\frac{\partial}{\partial t} &\equiv \frac{\partial \tau}{\partial t}\frac{\partial}{\partial \tau} \\ &= -\frac{\partial}{\partial \tau}\end{aligned}$$

Under the new time variable

$$\tau = 0 \Longrightarrow t = T \ \text{(expiry)}$$

so that now τ will be increasing from zero. So as $t \uparrow \tau \downarrow$.

The BSE becomes

$$\frac{\partial U}{\partial \tau} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 U}{\partial S^2} + rS\frac{\partial U}{\partial S},$$

which is simply the Kolmogorov equation. So $V(S,t)$ is the discounted solution of the Kolmogorov equation.

Step 3

We now wish to cancel out the variable coefficients S and S^2. When we first started modelling equity prices we used intuition about the asset price *return* and the idea of a lognormal random walk. Let's write

$$\xi = \log S.$$

Again use the chain rule to write the stock in terms of ξ. With this as the new variable, we find that this is equivalent to $S = e^{-\xi}$

$$\frac{\partial}{\partial S} = \frac{\partial \xi}{\partial S}\frac{\partial}{\partial \xi} = \frac{1}{S}\frac{\partial}{\partial \xi}$$

$$\begin{aligned}\frac{\partial^2}{\partial S^2} &= \frac{\partial}{\partial S}\left(\frac{1}{S}\frac{\partial}{\partial \xi}\right) = \frac{1}{S}\frac{\partial}{\partial S}\left(\frac{\partial}{\partial \xi}\right) - \frac{1}{S^2}\frac{\partial}{\partial \xi} \\ &= \frac{1}{S}\frac{\partial \xi}{\partial S}\frac{\partial}{\partial \xi}\left(\frac{\partial}{\partial \xi}\right) - \frac{1}{S^2}\frac{\partial}{\partial \xi} \\ &= \frac{1}{S^2}\frac{\partial^2}{\partial \xi^2} - \frac{1}{S^2}\frac{\partial}{\partial \xi} = \frac{1}{S^2}\left(\frac{\partial^2}{\partial \xi^2} - \frac{\partial}{\partial \xi}\right)\end{aligned}$$

Now the Black–Scholes equation can be written under this transformation as

$$\frac{\partial U}{\partial \tau} = \frac{1}{2}\sigma^2 S^2 \frac{1}{S^2}\left(\frac{\partial^2}{\partial \xi^2} - \frac{\partial}{\partial \xi}\right) U + rS\frac{1}{S}\frac{\partial}{\partial \xi}U$$

which simplifies to

$$\frac{\partial U}{\partial \tau} = \frac{1}{2}\sigma^2 \frac{\partial^2 U}{\partial \xi^2} + \left(r - \frac{1}{2}\sigma\right)\frac{\partial U}{\partial \xi}$$

We need to eliminate the first order derivative term in ξ.

Final Step

Perform a translation of the co-ordinate system

$$x = \xi + \left(r - \frac{1}{2}\sigma^2\right)\tau$$

So we are transforming from (ξ, τ) to (x, τ). So apply chain rule I

$$\begin{aligned}
\frac{\partial}{\partial \tau} &= \frac{\partial \tau}{\partial \tau}\frac{\partial}{\partial \tau} + \frac{\partial x}{\partial \tau}\frac{\partial}{\partial x} = 1.\frac{\partial}{\partial \tau} + \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial}{\partial x} \\
\frac{\partial}{\partial \xi} &= \frac{\partial \tau}{\partial \xi}\frac{\partial}{\partial \tau} + \frac{\partial x}{\partial \xi}\frac{\partial}{\partial x} = 0.\frac{\partial}{\partial \tau} + 1.\frac{\partial}{\partial x} \Longrightarrow \frac{\partial^2}{\partial \xi^2} = \frac{\partial^2}{\partial x^2}
\end{aligned}$$

So $U = W(x, \tau)$. $\dfrac{\partial U}{\partial \tau} = \frac{1}{2}\sigma^2 \dfrac{\partial^2 U}{\partial \xi^2} + \left(r - \frac{1}{2}\sigma\right)\dfrac{\partial U}{\partial \xi}$ becomes

$$\left(\frac{\partial}{\partial \tau} + \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial}{\partial x}\right) W = \frac{1}{2}\sigma^2 \frac{\partial^2 W}{\partial \xi^2} + \left(r - \frac{1}{2}\sigma\right)\frac{\partial W}{\partial \xi}$$

After this change of variables the BSE becomes

$$\frac{\partial W}{\partial \tau} = \frac{1}{2}\sigma^2 \frac{\partial^2 W}{\partial x^2}. \tag{1}$$

To summarize the steps taken to get this 1D heat equation:,

$$\begin{aligned} V(S,t) &= \\ e^{-r(T-t)}U(S,t) &= e^{-r\tau}U(S, T-\tau) = e^{-r\tau}U(e^{\xi}, T-\tau) \end{aligned}$$

$$= e^{-r\tau}U\left(e^{x-\left(r-\frac{1}{2}\sigma^2\right)\tau}, T-\tau\right) = e^{-r\tau}W(x,\tau).$$

So we will start by solving for $W(x,\tau)$. The equation for this function is solved using the similarity reduction method, for the *fundamental* *solution* $W_f(x,\tau;x')$. This is all familiar methodology. We define

$$W_f(x,\tau;x') = \tau^{\alpha} f\left(\frac{(x-x')}{\tau^{\beta}}\right),$$

where x' is an arbitrary constant, and the parameters α and β are constant, to be chosen shortly. We choose $\frac{(x-x')}{\tau^{\beta}}$ because it is a constant coefficient problem.

Note that the unknown function depends on only *one* variable

$$\eta = (x-x')/\tau^{\beta}$$

Again we use a combination of product and chain rule to write the PDE in terms of an ODE:

$$W_f(x,\tau;x') = \tau^{\alpha} f(\eta); \quad \eta = (x-x')/\tau^{\beta}$$

The following working has already been seen in Riaz's maths primer and Paul's lecture 1.2 - all familiar stuff.

So

$$\frac{d\eta}{d\tau} = -\beta\tau^{-\beta-1}(x - x'); \quad \frac{d\eta}{dx} = \tau^{-\beta}$$

$$\begin{aligned}
\frac{\partial W}{\partial \tau} &= \tau^{\alpha}\frac{\partial}{d\tau}f\left(\eta\right) + \alpha\tau^{\alpha-1}f\left(\eta\right) \\
&= \tau^{\alpha}\frac{df}{d\eta}\frac{d\eta}{d\tau} + \alpha\tau^{\alpha-1}f\left(\eta\right) \\
&= -\tau^{\alpha-1-\beta}\beta\frac{df}{d\eta}.\left(x - x'\right) + \alpha\tau^{\alpha-1}f\left(\eta\right) \\
&= \tau^{\alpha-1}\left(-\beta\eta\frac{df}{d\eta} + \alpha f\left(\eta\right)\right)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial W}{\partial x} &= \tau^{\alpha}\frac{\partial}{dx}f(\eta) \\
&= \tau^{\alpha}\frac{df}{d\eta}\frac{d\eta}{dx} = \tau^{\alpha}\tau^{-\beta}\frac{df}{d\eta} \\
&= \tau^{\alpha-\beta}\frac{df}{d\eta} \\
\frac{\partial^2 W}{\partial x^2} &= \tau^{\alpha-\beta}\frac{\partial}{dx}\left(\frac{df}{d\eta}\right) \\
&= \tau^{\alpha-\beta}\frac{d}{d\eta}\frac{d\eta}{dx}\left(\frac{df}{d\eta}\right) = \tau^{\alpha-\beta}\tau^{-\beta}\frac{d^2 f}{d\eta^2} \\
&= \tau^{\alpha-2\beta}\frac{d^2 f}{d\eta^2}
\end{aligned}$$

So

$$\frac{\partial W}{\partial \tau} = \tau^{\alpha-1}\left(-\beta\eta\frac{df}{d\eta} + \alpha f(\eta)\right) \tag{2}$$

$$\frac{\partial^2 W}{\partial x^2} = \tau^{\alpha-2\beta}\frac{d^2 f}{d\eta^2} \tag{3}$$

Substituting (2) , (3) into (1) gives the 2nd order equation

$$\tau^{\alpha-1}\left(\alpha f - \beta\eta\frac{df}{d\eta}\right) = \frac{1}{2}\sigma^2\tau^{\alpha-2\beta}\frac{d^2 f}{d\eta^2} \tag{4}$$

We still have a τ term in (4) and for similarity reduction we need to reduce the dimension of the problem. This implies

$$\alpha - 1 = \alpha - 2\beta \quad \Longrightarrow \quad \beta = \frac{1}{2},$$

to give

$$\left(\alpha f - \frac{1}{2}\eta\frac{df}{d\eta}\right) = \frac{1}{2}\sigma^2\frac{d^2 f}{d\eta^2}$$

With the correct choice of $\alpha,\ \beta$ we want

$$\int_{-\infty}^{\infty} W_f(x,\tau;x')dx = 1\ \forall\tau$$

So

$$\int_{-\infty}^{\infty} W_f(x,\tau;x')dx = \tau^{\alpha}\int_{\mathbb{R}} f\left(\frac{x-x'}{\sqrt{\tau}}\right)dx$$

$$\begin{aligned} \eta &= \frac{x-x'}{\sqrt{\tau}} \\ \sqrt{\tau}d\eta &= dx \end{aligned}$$

So the integral becomes

$$\tau^{\alpha}\int_{\mathbb{R}} f(\eta)\sqrt{\tau}d\eta = \tau^{\alpha+1/2}\int_{\mathbb{R}} f(\eta)\,d\eta = 1$$

This implies that $\tau^{\alpha+1/2}$ should equal one, in order for the solution to be normalised regardless of time. Therefore $\alpha = -1/2$.

$f(\eta)$ becomes our PDF.

The function f now satisfies

$$-\frac{1}{2}\left(f+\eta\frac{df}{d\eta}\right)=\frac{1}{2}\sigma^2\frac{d^2f}{d\eta^2}.$$

where the left hand side can be expressed as an exact derivative

$$-\frac{d}{d\eta}\left(\eta f\right)=\sigma^2\frac{d^2f}{d\eta^2}.$$

This can be integrated

$$\eta f+\sigma^2\frac{df}{d\eta}=A$$

where the constant $A=0$ because as η becomes large, both $f\left(\eta\right)$ and $f\prime\left(\eta\right)$ tend to zero.

This is variable separable

$$
\begin{aligned}
\eta f &= -\sigma^2 \frac{df}{d\eta} \\
\int \frac{df}{f} &= -\frac{1}{\sigma^2} \int \eta d\eta \\
\ln f &= -\frac{1}{2\sigma^2}\eta^2 + K
\end{aligned}
$$

Taking exponentials of both sides gives

$$f(\xi) = C \exp\left(-\frac{\eta^2}{2\sigma^2}\right)$$

C is a normalising constant such that

$$C \int_{\mathbb{R}} \exp\left(-\frac{\eta^2}{2\sigma^2}\right) d\eta = 1.$$

Easy to solve by substituting $u = \dfrac{\eta}{\sqrt{2}\sigma} \longrightarrow \sqrt{2}\sigma du = d\eta$, and converts the integral to

$$\begin{aligned} C\sqrt{2}\sigma \int_{\mathbb{R}} e^{-u^2} du &= 1 \\ C\sqrt{2}\sigma\sqrt{\pi} &= 1 \\ C &= \frac{1}{\sqrt{2\pi}\sigma} \end{aligned}$$

$$f(\eta) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{\eta^2}{2\sigma^2}}.$$

Replacing η gives us the fundamental solution :

$$W_f(x, \tau; x') = \frac{1}{\sqrt{2\pi\tau}\ \sigma} e^{-\frac{(x-x')^2}{2\sigma^2\tau}}. \tag{5}$$

This is the probability density function for a Normal random variable x having mean of x' and standard deviation $\sigma\sqrt{\tau}$. For $\tau \neq 0$, W_f

represents a series of Gaussian curves. (5) allows us to find the solution of the BSE at different points (e.g. $x' = 2$; $x' = -17$, etc).

2.8 Properties of The Solution

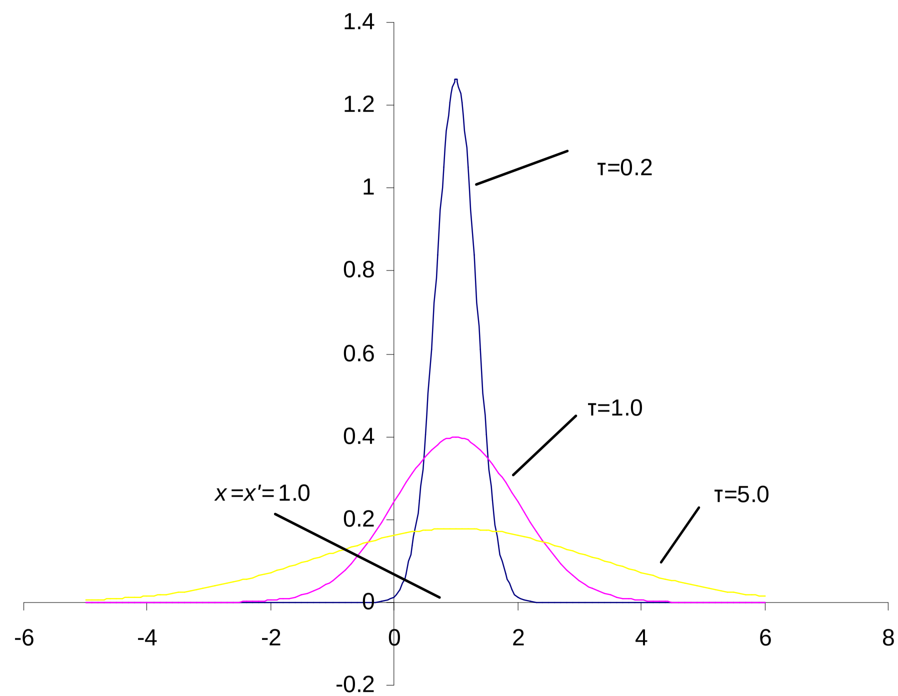
We have made sure from our solution method that

$$\int_{\mathbb{R}} W_f \; dx = 1$$

this has been fixed. At $x' = x$ ($\exp 0 = 1$)

$$W_f(x, \tau; x') = \frac{1}{\sqrt{2\pi\tau}\;\sigma}.$$

Then as $\tau \to 0$ (close to expiration), $W_f \longrightarrow \infty$, the Gaussian curve becomes taller but the area is confined to unity therefore it becomes slimmer to compensate. As x moves away from x', $\exp(-\infty) \longrightarrow 0$. $W_f(x, \tau; x')$ is plotted below for different values of τ. If τ is large then W_f is flat, as τ gets smaller W_f is increasingly peaked, and focused on x'.



This behaviour of decay away from one point x', unbounded growth at that point and constant area means that W_f has turned in to a **Dirac delta function** $\delta(x' - x)$ as $\tau \to 0$.

2.9 Dirac delta function

This is written $\delta(x - x') = \lim_{\tau \longrightarrow 0} \delta(x - x')$, such that

$$\delta\left(x - x'\right) = \begin{cases} \infty & x = x' \\ 0 & x \neq x' \end{cases}$$

$$\int_{-\infty}^{\infty} \delta\left(x - x'\right) dx = 1$$

or $\int_0^{\infty} \delta(x - x')\, dx = 1$

If $g\left(x\right)$ is a continuous function then

$$\int_{-\infty}^{\infty} g\left(x\right) \delta\left(x - x'\right) dx = g\left(x'\right)$$

So if we take a delta function and multiply it by any other function - and calculate the area under this product - this is simply the function $g\left(x\right)$ evaluated at the point $x = x'$. What is happening here?

The delta function picks out the value of the function at which it is singular (in this case x'). All other points are irrelevant because we are multiplying by zero.

In the limit as $\tau \longrightarrow 0$ the function W_f becomes a delta function at $x = x'$. This means that

$$\lim_{\tau \to 0} \frac{1}{\sigma\sqrt{2\pi\tau}} \int_{-\infty}^{\infty} e^{-\frac{(x'-x)^2}{2\sigma^2\tau}} g(x') dx' = g(x).$$

Here we have swapped x and x' - it makes no difference due to the $(x'-x)^2$ term hence either can be the spatial variable.

So

$$\frac{1}{\sigma\sqrt{2\pi\tau}}e^{-\frac{(x'-x)^2}{2\sigma^2\tau}}$$

is a delta function and $g(x')$ will be replaced by the payoff function. The term above is also an example of a *Green's function*, which allows us to write down the general solution of the BSE in integral form.

So as we get closer to expiration, i.e. $\tau \longrightarrow 0$, the delta function picks out the value of $g(x')$ at which $x' = x$

Now introduce the payoff at $t = T$ $(\tau = 0)$:

$$V(S,T) = \text{Payoff}(S).$$

Recall $x = \xi + \left(r - \frac{1}{2}\sigma^2\right)\tau$, so at expiry $\tau = 0 \Longrightarrow x = \xi = \log S$. Hence $S = e^x$ to give

$$W(x,0) = \text{Payoff}(e^x).$$

The solution of this for $\tau > 0$ is

$$W(x,\tau) = \int_{-\infty}^{\infty} W_f(x,\tau;x')\text{Payoff}(e^{x'})\,dx'.$$

We have converted the backward BSE to the Forward Equation. Look at

$$\text{Payoff}(e^{x'})\,dx'.$$

We know

$$\begin{aligned} x' &= \log S' \Longrightarrow dx' = \frac{dS'}{S'} \text{ and} \\ e^{x'} &= S' \end{aligned}$$

therefore Payoff($e^{x'}$).dx' becomes

$$\text{Payoff}(S').\frac{dS'}{S'}$$

This result is important. As $\log S$ does not exist in the negative plane the integral goes from 0 to infinity, with the lower limit acting as an asymptote.

Let's start unravelling some of the early steps and transformations. Returning to our Green's function

$$\frac{1}{\sigma\sqrt{2\pi(T-t)}}\,\mathrm{exp}\left(-\frac{(x'-x)^2}{2\sigma^2\,(T-t)}\right)=$$

$$\frac{1}{\sigma\sqrt{2\pi(T-t)}}\,\mathrm{exp}\left(-\frac{1}{2\sigma^2(T-t)}\cdot\left(\underbrace{\left(\log S+\left(r-\frac{1}{2}\sigma^2\right)\right)(T-t)}_{x}-\underbrace{\log S'}_{x'}\right)^2\right)=$$

$$\frac{1}{\sigma\sqrt{2\pi(T-t)}}\,\mathrm{exp}^{\left(-\frac{1}{2\sigma^2(T-t)}\cdot\left(\left(\log\frac{S}{S'}+\left(r-\frac{1}{2}\sigma^2\right)\right)(T-t)\right)^2\right)}$$

So putting this together with the Payoff function as an integrand we have

$$\frac{1}{\sigma\sqrt{2\pi(T-t)}}\int_0^\infty e^{\left(-\frac{1}{2\sigma^2(T-t)}\cdot\left(\left(\log\frac{S}{S'}+\left(r-\frac{1}{2}\sigma^2\right)\right)(T-t)\right)^2\right)}\mathrm{Payoff}(S').\frac{dS'}{S'}$$

$$V(S,t) = \frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \times$$

$$\int_0^\infty e^{-\left(\log(S/S') + \left(r - \frac{1}{2}\sigma^2\right)(T-t)\right)^2 / 2\sigma^2(T-t)} \text{Payoff}(S') \frac{dS'}{S'}. \qquad (6)$$

This expression works because the equation is linear - so we just need to specify the payoff condition. It can be applied to any European option on a single lognormal underlying asset.

2.10 Formula for a call

The call option has the payoff function

$$\text{Payoff}(S) = \max(S - E, 0).$$

When $S < E$, $\max(S - E, 0) = 0$ therefore

$$\int_0^\infty \equiv \int_0^E + \int_E^\infty = \int_E^\infty \because \int_0^E 0 = 0$$

Expression (6) can then be written as

$$\frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \int_E^\infty e^{-\left(\log(S/S') + \left(r - \frac{1}{2}\sigma^2\right)(T-t)\right)^2 / 2\sigma^2(T-t)} (S' - E)\frac{dS'}{S'}.$$

Return to the variable $x' = \log S' \implies -x' = \log 1/S'$ so we can write the above integral as

$$\frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}}\int_{\log E}^{\infty} e^{-\left(-x'+\log S+\left(r-\frac{1}{2}\sigma^2\right)(T-t)\right)^2/2\sigma^2(T-t)}\left(e^{x'}-E\right)dx'$$

$$= \frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}}\int_{\log E}^{\infty} e^{-\left(-x'+\log S+\left(r-\frac{1}{2}\sigma^2\right)(T-t)\right)^2/2\sigma^2(T-t)}e^{x'}\,dx'$$

$$-E\frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}}\int_{\log E}^{\infty} e^{-\left(-x'+\log S+\left(r-\frac{1}{2}\sigma^2\right)(T-t)\right)^2/2\sigma^2(T-t)}\,dx'.$$

Just a couple more steps are required to simplify these messy looking integrals. Let's look at the second integral

$$-E\frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}}\int_{\log E}^{\infty} e^{-\frac{1}{2}\left(-x'+\log S+\left(r-\frac{1}{2}\sigma^2\right)(T-t)\right)^2/\sigma^2(T-t)}\,dx'$$

use the substitution

$$
\begin{aligned}
u &= \frac{\left(-x'+\log S+\left(r-\frac{1}{2}\sigma^2\right)(T-t)\right)}{\sigma\sqrt{(T-t)}} \\
du &= \frac{-1}{\sigma\sqrt{(T-t)}}dx' \longrightarrow -\sigma\sqrt{(T-t)}du = dx'
\end{aligned}
$$

and the limits:

$$
\begin{aligned}
x' &= \infty \longrightarrow u = -\infty \\
u &= \log E \longrightarrow u = \frac{\left(-\log E+\log S+\left(r-\frac{1}{2}\sigma^2\right)(T-t)\right)}{\sigma\sqrt{(T-t)}}
\end{aligned}
$$

$$
\begin{aligned}
& -E\frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}}\int_{\frac{\left(\log S/E+\left(r-\frac{1}{2}\sigma^2\right)(T-t)\right)}{\sigma\sqrt{(T-t)}}}^{-\infty} e^{-\frac{1}{2}u^2}. -\sigma\sqrt{(T-t)}du \\
= & -E\frac{e^{-r(T-t)}}{\sqrt{2\pi}}\int_{\frac{\left(\log S/E+\left(r-\frac{1}{2}\sigma^2\right)(T-t)\right)}{\sigma\sqrt{(T-t)}}}^{-\infty} e^{-\frac{1}{2}u^2}. -du \\
= & -E\frac{e^{-r(T-t)}}{\sqrt{2\pi}}\int_{-\infty}^{\frac{\left(\log S/E+\left(r-\frac{1}{2}\sigma^2\right)(T-t)\right)}{\sigma\sqrt{(T-t)}}} e^{-\frac{1}{2}u^2}du \\
= & -Ee^{-r(T-t)}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_2} e^{-\frac{1}{2}u^2}du \\
= & -Ee^{-r(T-t)}N\left(d_2\right)
\end{aligned}
$$

The first integral requires similar treatment however before we do that we complete the square on the exponent. The integrand is

$$
= e^{-\left(-x'+\log S+\left(r-\frac{1}{2}\sigma^2\right)(T-t)\right)^2/2\sigma^2(T-t)}e^{x'-r(T-t)}
$$

Now just work on the exponent, and put $\tau = T - t$ temporarily to simplify working

$$
\begin{aligned}
& \frac{-\left(-x'+\log S+\left(r-\frac{1}{2}\sigma^2\right)\tau\right)^2}{2\sigma^2\tau} + x' - r\tau \\
= \quad & \frac{-\left(-x'+\log S+\left(r-\frac{1}{2}\sigma^2\right)\tau\right)^2 + 2\left(x'-r\tau\right)\sigma^2\tau}{2\sigma^2\tau} \\
= \quad & \frac{-\left(\left(-x'+\log S+\left(r-\frac{1}{2}\sigma^2\right)\tau\right)^2 - 2\left(x'-r\tau\right)\sigma^2\tau\right)}{2\sigma^2\tau} \\
= \quad & -\frac{1}{2}\frac{\left(\left(-x'+\log S+r\tau-\frac{1}{2}\sigma^2\tau\right)^2 - 2\left(x'-r\tau\right)\sigma^2\tau\right)}{\sigma^2\tau}
\end{aligned}
$$

Now expand the bracket in the numerator

$$
\begin{aligned}
& \left(x'^2 + \log^2 S + r^2\tau^2 + \frac{1}{4}\sigma^2\tau^2 - 2x' \log S - 2x' r\tau + x'\sigma^2\tau + 2r\tau \log S \right. \\
& \left. -\sigma^2\tau \log S - r\sigma^2\tau^2\right) - 2x'\sigma^2\tau + 2r\sigma^2\tau^2 \\
& \left(x'^2 + \log^2 S + r^2\tau^2 + \frac{1}{4}\sigma^2\tau^2 - 2x' \log S - 2x' r\tau - x'\sigma^2\tau + 2r\tau \log S \right. \\
& \left. -\sigma^2\tau \log S + r\sigma^2\tau^2\right) \quad \text{now complete the square} \\
= \; & \left(-x' + \log S + r\tau + \frac{1}{2}\sigma^2\tau\right)^2 - 2\sigma^2\tau \log S \\
= \; & \left(-x' + \log S + \left(r + \frac{1}{2}\sigma^2\right)\tau\right)^2 - 2\sigma^2\tau \log S
\end{aligned}
$$

Let's return to the integral

$$\begin{aligned}
& \frac{1}{\sigma\sqrt{2\pi\tau)}}\int_{\log E}^{\infty} e^{-\frac{1}{2\sigma^2\tau}\left(\left(-x'+\log S+\left(r+\frac{1}{2}\sigma^2\right)\tau\right)^2-2\sigma^2\tau\log S\right)}dx' \\
= \quad & \frac{S}{\sigma\sqrt{2\pi(T-t)}}\int_{\log E}^{\infty} e^{-\frac{1}{2\sigma^2(T-t)}\left(\left(-x'+\log S+\left(r+\frac{1}{2}\sigma^2\right)(T-t)\right)^2\right)}dx'
\end{aligned}$$

and as before use a similar substitution

$$\begin{aligned}
v &= \frac{\left(-x'+\log S+\left(r+\frac{1}{2}\sigma^2\right)(T-t)\right)}{\sigma\sqrt{(T-t)}} \\
dv &= \frac{-1}{\sigma\sqrt{(T-t)}}dx' \longrightarrow -\sigma\sqrt{(T-t)}dv = dx'
\end{aligned}$$

and the limits as before:

$$\begin{aligned}
x' &= \infty \longrightarrow u = -\infty \\
x' &= \log E \longrightarrow u = \frac{\left(-\log E+\log S+\left(r+\frac{1}{2}\sigma^2\right)(T-t)\right)}{\sigma\sqrt{(T-t)}}.
\end{aligned}$$

Following the earlier working reduces this to

$$
\begin{aligned}
& S\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\frac{\left(\log S/E+\left(r+\frac{1}{2}\sigma^2\right)(T-t)\right)}{\sigma\sqrt{(T-t)}}} e^{-\frac{1}{2}v^2}dv \\
= \; & S\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_1} e^{-\frac{1}{2}v^2}dv \\
= \; & SN\left(d_1\right)
\end{aligned}
$$

Thus the option price can be written as two separate terms involving the cumulative distribution function for a Normal distribution:

$$V\left(S,t\right) = SN(d_1) - Ee^{-r(T-t)}N(d_2)$$

where

$$d_1 = \frac{\log(S/E) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \text{ and}$$

$$d_2 = \frac{\log(S/E) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t}.$$

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}\phi^2} d\phi.$$

2.11 Formula for a put

The put option has payoff

$$\text{Payoff}(S) = \max(E - S, 0).$$

A similar working as in the case of a call yields

$$V(S,t) = -SN(-d_1) + Ee^{-r(T-t)}N(-d_2),$$

with the same d_1 and d_2. Naturally the more sensible approach is to exploit the put-call parity. If the price of a call and put are denoted in turn by $C(S,t)$ and $P(S,t)$

$$C - P = S - Ee^{-r(T-t)}$$

hence rearranging, using the formula for a call together with $N(x) + N(-x) = 1$, gives

$$\begin{aligned} P &= C - S + Ee^{-r(T-t)} \\ &= SN(d_1) - Ee^{-r(T-t)}N(d_2) - S + Ee^{-r(T-t)} \\ &= S\underbrace{(N(d_1) - 1)}_{=-N(-d_1)} + Ee^{-r(T-t)}\underbrace{(1 - N(d_2))}_{=N(-d_2)} \\ &= -SN(-d_1) + Ee^{-r(T-t)}N(-d_2) \end{aligned}$$

2.12 The greeks

A comprehensive list of basic greeks was presented in lecture 3.1. More advanced greeks are discussed in Espen's lecture and an exhaustive collection can be found in his book. Here we use basic differentiation techniques to demonstrate the simplicity in obtaining (for example) the delta of a **European Put**. The idea is to show how straightforward it actually is to produce complex looking formulae. We have just written the price of a put

$$P(S,t) = Ee^{-r(T-t)}N(-d_2) - Se^{-D(T-t)}N(-d_1)$$

So we want $\Delta = \frac{\partial P}{\partial S}$.

Useful results:

If $N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} \exp(-\phi^2/2)d\phi$ then $\frac{dN}{dx} = \exp(-x^2/2)$: Leibniz Rule

$$\left.\begin{array}{c} d_1 = \dfrac{\log\left(S/E\right) + \left(r - D + \frac{1}{2}\sigma^2\right)\left(T-t\right)}{\sigma\sqrt{T-t}}, \\ d_2 = d_1 - \sigma\sqrt{T-t} \end{array}\right\} \Longrightarrow$$

$$\frac{\partial\left(d_1\right)}{\partial S} = \frac{\partial\left(d_1\right)}{\partial S}$$

Another result of importance (messy to prove)

$$Se^{-D(T-t)}\frac{1}{\sqrt{2\pi}}\exp(-{d_1}^2/2) = Ee^{-r(T-t)}\frac{1}{\sqrt{2\pi}}\exp(-{d_2}^2/2)$$

Write

$$Ee^{-r(T-t)}N\left(-d_2\right) \tag{a}$$

$$Se^{-D(T-t)}N\left(-d_1\right) \tag{b}$$

and

$$\frac{\partial}{\partial S}\text{(a)} = Ee^{-r(T-t)}\frac{\partial}{\partial S}N\left(-d_2\right)$$

now use chain rule

$$\begin{aligned} & Ee^{-r(T-t)}\frac{\partial}{\partial d_2}N(-d_2)\frac{\partial(-d_2)}{\partial S} \\ = \; & -Ee^{-r(T-t)}\frac{1}{\sqrt{2\pi}}\exp(-d_2{}^2/2)\frac{\partial(d_2)}{\partial S} \end{aligned}$$

$$\frac{\partial}{\partial S}(\mathrm{b}) = e^{-D(T-t)}\frac{\partial}{\partial S}SN(-d_1)$$

use product rule then chain rule

$$\begin{aligned} & e^{-D(T-t)}\left(N(-d_1) + S\frac{\partial}{\partial S}N(-d_1)\right) \\ = \; & e^{-D(T-t)}\left(N(-d_1) + S\frac{\partial}{\partial d_1}N(-d_1)\frac{\partial(-d_1)}{\partial S}\right) \\ = \; & e^{-D(T-t)}\left(N(-d_1) - S\frac{1}{\sqrt{2\pi}}\exp(-d_1{}^2/2)\frac{\partial(d_1)}{\partial S}\right) \end{aligned}$$

So now

$$
\begin{aligned}
\Delta &= \frac{\partial}{\partial S}(\mathrm{a}) - \frac{\partial}{\partial S}(\mathrm{b}) \\
&= -Ee^{-r(T-t)}\frac{1}{\sqrt{2\pi}}\exp(-d_2{}^2/2)\frac{\partial(d_2)}{\partial S} - \\
&\quad e^{-D(T-t)}\left(N(-d_1) - S\frac{1}{\sqrt{2\pi}}\exp(-d_1{}^2/2)\frac{\partial(d_1)}{\partial S}\right) \\
&= -e^{-D(T-t)}N(-d_1) + \\
&\quad \frac{\partial(d_1)}{\partial S}\left(Se^{-D(T-t)}\frac{1}{\sqrt{2\pi}}\exp(-d_1{}^2/2) - Ee^{-r(T-t)}\frac{1}{\sqrt{2\pi}}\exp(-d_2{}^2/2)\right) \\
&= -e^{-D(T-t)}N(-d_1) + \frac{\partial(d_1)}{\partial S}(0)
\end{aligned}
$$

Using

$$N(x) + N(-x) = 1 \Longrightarrow N(-x) = 1 - N(x)$$

$$
\begin{aligned}
\Delta &= -e^{-D(T-t)}(1 - N(d_1)) \\
&= e^{-D(T-t)}(N(d_1) - 1)
\end{aligned}
$$

2.13 The perpetual American put

We continue working in a Black-Scholes environment to look at American options. Solving the BSE for an early exercise condition is essentially a numerical problem. This means that the pricing equation has to be solved computationally. To get an insight into options with American features there is a simpler problem we can solve which assists in understanding the dynamics of exercising early. This can be done by studying the **perpetual American put**.

This contract can be exercised for a put payoff at *any* time. There is no expiry. So we can, at any time of *our* choosing, sell the underlying and receive an amount E. That is, the payoff is $\max(E - S, 0)$.

We want to find the value of this option before exercise.

- The solution is independent of time, $V(S)$. It depends only on the level of the underlying.

- The option value can never go below the early-exercise payoff.

So this requires

$$V \geq \max(E - S, 0). \tag{1}$$

Consider what would happen if this ‘constraint’ were violated. Suppose that the option value were less than $\max(E - S, 0)$, we could buy the option for $\max(E - S, 0)$, immediately exercise it by handing over the asset (worth S) and receive an amount E. We thus make

$$-\text{cost of put} - \text{cost of asset} + \text{strike price} = -V - S + E > 0.$$

This is a riskless profit.

Recalling that the option is perpetual and therefore that the value is independent of t, it must satisfy

$$\tfrac{1}{2}\sigma^2 S^2 \frac{d^2V}{dS^2} + rS\frac{dV}{dS} - rV = 0.$$

This is the ordinary differential equation obtained when the option value is a function of S only. We note it has a nice Cauchy-Euler structure so can be solved analytically. By looking for solutions of the form $V = S^{\lambda}$, we obtain the auxiliary equation

$$\begin{aligned} \tfrac{1}{2}\sigma^2 S^2 \lambda^2 + \left(r - \tfrac{1}{2}\sigma^2\right)\lambda - r &= 0 \longrightarrow \\ \left(\frac{1}{2}\sigma^2\lambda + r\right)(\lambda - 1) &= 0 \end{aligned}$$

We want $r \neq -\frac{1}{2}\sigma^2$ for two distinct roots

$$\lambda_1 = 1 \text{ and } \lambda_2 = -\frac{2r}{\sigma^2}$$

and the general solution is

$$V(S) = AS + BS^{-2r/\sigma^2}.$$

where A and B are arbitrary constants.

We know for a put option that

$$\lim_{S \longrightarrow \infty} V(S) \longrightarrow 0$$

Hence, for the perpetual American put the coefficient A must be zero; as $S \rightarrow \infty$ the value of the option must tend to zero. Else the condition is not satisfied as $V(S)$ grows without bound. Hence $V(S) = BS^{-2r/\sigma^2}$.

How can we calculate B?

Postulate that while the asset value is 'high' we won't exercise the option, because Payoff tends to zero. But if it falls too low we immediately exercise the option, receiving $E - S$. Suppose that we decide that $S = S^*$ is the value at which we exercise, i.e. as soon as S reaches this value from above we exercise.

- How do we choose S^*?

When $S = S^*$ the option value must be the same as the exercise payoff:

$$V(S^*) = E - S^*.$$

It cannot be less, that would result in an arbitrage opportunity, and it cannot be more or we wouldn't exercise.

This expression involving S^* is telling us that at this point there is continuity of the option value with the payoff. This gives us one equation:

$$V(S^*) = B(S^*)^{-2r/\sigma^2} = E - S^*.$$

We simply put S^* in $V(S)$ and the payoff. But since both B and S^* are unknown, we need one more equation. We can rearrange this equation to get B in terms of S^* :

$$B = \frac{(E - S^*)}{(S^*)^{-2r/\sigma^2}}.$$

Now substitute this in to

$$V(S) = BS^{-2r/\sigma^2}$$

to get

$$V(S) = \frac{(E - S^*)}{(S^*)^{-2r/\sigma^2}} \left(S^{-2r/\sigma^2} \right)$$

So we find for $S > S^*$

$$V(S) = (E - S^*) \left(\frac{S}{S^*} \right)^{-2r/\sigma^2} . \qquad (2)$$

Now choose S^* to *maximize the option's value at any time before exercise*. In other words, what choice of S^* makes V given by (2) as large as possible? We do this by differentiating (2) with respect to S^* and setting the resulting expression equal to zero:

$$
\begin{aligned}
& \frac{\partial}{\partial S^*}(E - S^*)\left(\frac{S}{S^*}\right)^{-2r/\sigma^2} \\
= \;& -\left(\frac{S}{S^*}\right)^{-2r/\sigma^2} + (E - S^*)\, S^{\,-2r/\sigma^2} \frac{\partial}{\partial S^*} (S^*)^{2r/\sigma^2} \\
= \;& -\left(\frac{S}{S^*}\right)^{-2r/\sigma^2} + (E - S^*)\, S^{\,-2r/\sigma^2} \frac{2r}{\sigma^2} (S^*)^{(2r/\sigma^2)-1} \\
= \;& -S^{-2r/\sigma^2} (S^*)^{2r/\sigma^2} + \frac{2r}{\sigma^2} (E - S^*)\, S^{\,-2r/\sigma^2} (S^*)^{2r/\sigma^2} (S^*)^{-1} \\
= \;& \frac{1}{S^*}\left(\frac{S}{S^*}\right)^{-2r/\sigma^2} \left(-S^* + \frac{2r}{\sigma^2}(E - S^*)\right) \\
= \;& 0.
\end{aligned}
$$

This implies

$$\begin{aligned} S^* &= \frac{2r}{\sigma^2}(E - S^*) \longrightarrow \frac{\sigma^2}{2r}S^* = E - S^* \\ &\longrightarrow S^*\left(1 + \sigma^2/2r\right) = E \end{aligned}$$

hence

$$S^* = \frac{E}{1 + \sigma^2/2r}.$$

This choice maximizes $V(S)$ for *all* $S \geq S^*$. The solution with this choice for S^* is shown below.

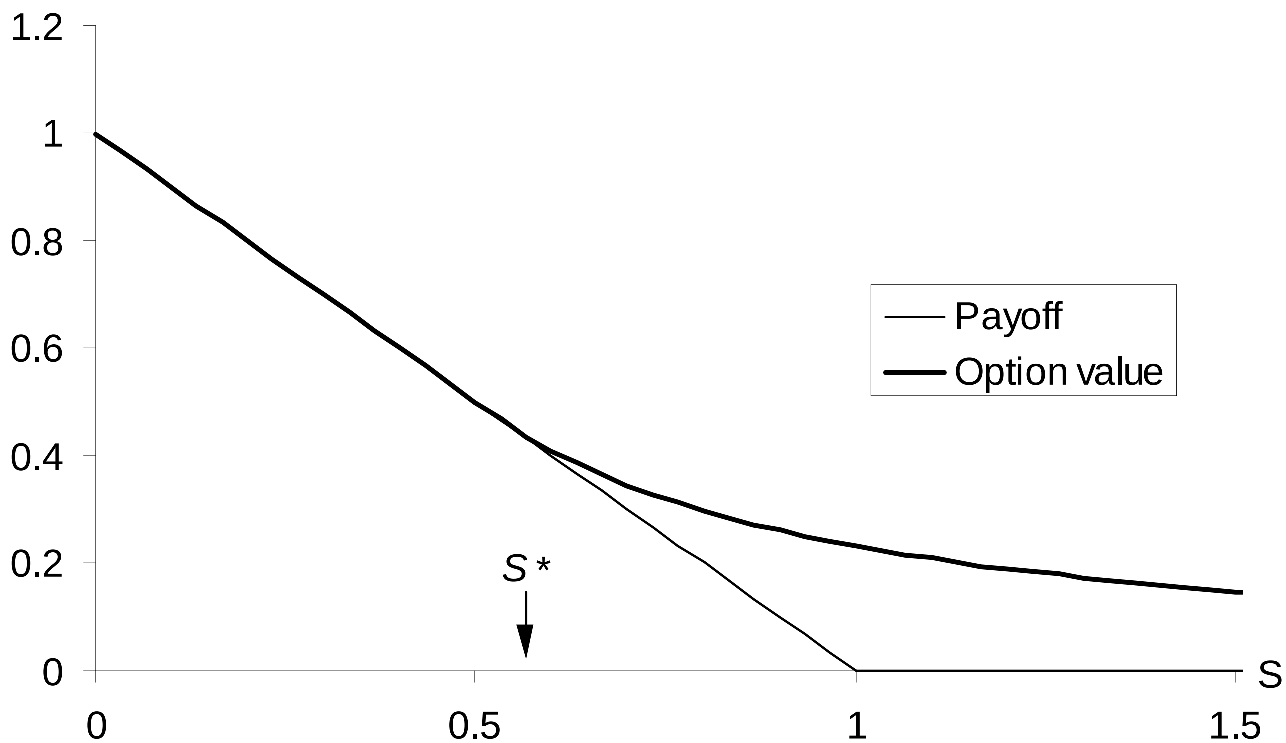


Figure 1: The solution for the perpetual American put.

At $S = S^*$, $V(S^*)$ has a $\Delta = -1$ the payoff also has a delta $= -1$. So this tells us that there is something special about this function.

To see that both have a slope of -1 at this point S^* examine the difference between the option value and the payoff function:

$$V(S) - \text{Payoff}$$

$$(E - S^*)\left(\frac{S}{S^*}\right)^{-2r/\sigma^2} - (E - S).$$

which at $S = S^*$ becomes

$$(E - S^*)\left(\frac{S^*}{S^*}\right)^{-2r/\sigma^2} - (E - S^*).$$

and collapses to zero.

So to maximize $V(S)$ we choose S^* which makes the option value $V(S)$ and its delta $\left(\Delta = \frac{\partial V}{\partial S}\right)$ continuous. This exercise strategy, is equivalent to solving the Black–Scholes equation with continuity of option *value* and option *delta*, the slope. This is called the **high-contact** or **smooth-pasting condition**.

We exercise the option as soon as the asset price reaches the level at which the option price and the payoff meet. This position, S^*, is called the **optimal exercise point**.

Consider what happens if the delta is not continuous at the exercise point. The two possibilities are shown on the next page.

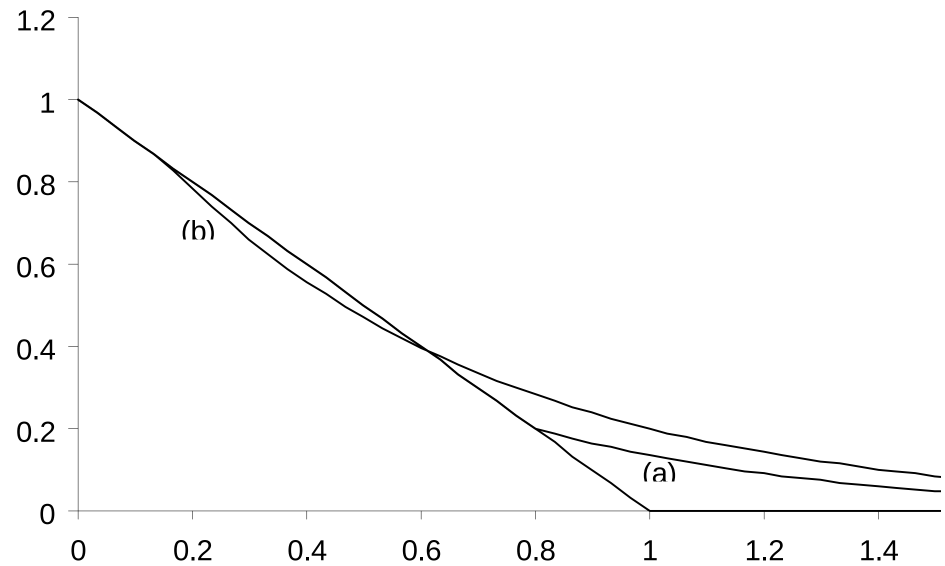


Figure 2: Option price when exercise is (a) too soon (b) too late.

Case (a) corresponds to exercise that is not optimal because it is premature, the option value is lower than it could be.

In case (b) there is clearly an arbitrage opportunity.

Move to the right of S^* option value goes down and we have discontinuity

Move to the left of S^* and there is discontinuity and the option is beneath the payoff.

To summarize:

For a put solve

$$\frac{1}{2}\sigma^2 S^2 \frac{d^2V}{dS^2} + rS\frac{dV}{dS} - rV = 0.$$

subject to

$$\left.\begin{array}{c} V(S^*) = E - S^* \\ \frac{dV}{dS}(S^*) = -1 \end{array}\right\} \text{Smooth pasting condition}$$

$$\lim_{S \longrightarrow \infty} V(S) \longrightarrow 0$$

For a **call** solve the same BSE together with

$$\left.\begin{array}{c} V(S^*) = S^* - E \\ \frac{dV}{dS}(S^*) = +1 \end{array}\right\} \text{Smooth pasting condition}$$

$$V(S = 0) = 0 \Longrightarrow B = 0$$

2.13.1 Dividends

When there is a continuously paid and constant dividend yield on the asset, or the asset is a foreign currency, the relevant ordinary differential equation for the perpetual option is

$$\tfrac{1}{2}\sigma^2 S^2 \frac{d^2V}{dS^2} + (r - D)S\frac{dV}{dS} - rV = 0.$$

The general solution is now $AS^{\alpha^+} + BS^{\alpha^-}$, where

$$\alpha^{\pm} = \tfrac{1}{2}\left(-\frac{2r}{\sigma^2}\left(r - D - \tfrac{1}{2}\sigma^2\right) \pm \sqrt{\frac{4r^2}{\sigma^4}\left(r - D - \tfrac{1}{2}\sigma^2\right)^2 + \frac{8r}{\sigma^2}}\right),$$

with $\alpha^- < 0 < \alpha^+$.

The perpetual American put now has value

$$BS^{\alpha^-},$$

where

$$B = -\frac{1}{\alpha^-}\left(\frac{E}{1 - 1/\alpha^-}\right)^{1+\alpha^-} .$$

It is optimal to exercise when S reaches the value

$$\frac{E}{1 - 1/\alpha^-}.$$

2.13.2 Perpetual American call with dividends

The solution for the American perpetual call is

$$AS^{\alpha^+},$$

where

$$A = \frac{1}{\alpha^+}\left(\frac{E}{1-1/\alpha^+}\right)^{1+\alpha^+}$$

and it is optimal to exercise as soon as S reaches

$$S^* = \frac{E}{1-1/\alpha^+}$$

from below.

An interesting special case is when $D = 0$. Then the solution is $V = S$ and S^* becomes infinite. Thus it is never optimal to exercise the American perpetual call when there are no dividends on the underlying; its value is the same as the underlying.