Certificate in Quantitative Finance

Module 6 Assessed Assignment

2009 Jan Cohort

Assessment for Module 6 of the CQF is through a programming project. The project is designed to test your understanding of the two main numerical methods used in finance and your proficiency in completing practical, useful products. You must submit working code and a report.

The two parts of this project are as follows.

1. **HJM model by Monte Carlo simulation**
2. **Uncertain volatility with static hedge by Finite Difference**

Notes:

- Your software must be written in VBA, Excel or C++ (or other language if prior permission is received)
- The code must be thoroughly tested and well documented
- The printed report must contain a full description of the models used, both mathematical and numerical, together with sample results
- The final report will be between 15 and 30 pages, not including the code
- The report must be soft bound in thesis style with the CD attached in an easily accessible manner to the inside back cover. Two copies must be submitted to 7city. (7city will not be able to return the submitted copies so you may want to make additional copies for yourself.)
- Submission date is **20th July 2009**.

You are being given this project at this stage so that you can begin reading and preparing in advance. The first part of the project uses material up to Module 5. The second part uses material up to the start of Module 6.

The next two pages describe the models and requirements in detail.

Implement HJM Model

Summary

This project requires you to implement HJM model to price zero coupon bond, interest rate options including caps and floors by Monte Carlo simulation. Before that you should use historical data, with appropriate parametrization, to estimate HJM model with at least 3 factors.

Introduction to HJM

Unlike many other short rate models, such as Vasicek and CIR, that focus on spot interest rate, the HJM model works on instantaneous forward rate. So its underlying stochastic process is forward rate process of $F(t, T)$ which can be written in terms of the following SDE

$$dF(t, T) = m(t, T)dt + \nu(t, T)dX$$

For the time being the SDE didn't show what probability measure the it is using. However one thing crystal clear is that under no arbitrage condition, there exists a restriction on risk-neutral drift of forward rate.

$$m(t, T) = \nu(t, T) \int_t^T \nu(t, s)ds$$

One can see that the risk-neutral drift $m(t, T)$ only depends on volatility of forward rate, so that it is crucial to determine volatility function $\nu(t, T)$ when implementing HJM.

In addition to modeling on $F(t, T)$ which is one point on the forward curve, in theory the HJM model actually evolves the whole forward curve, i.e., T varies from t to T^*(the longest time horizon you are interested in). However it is certainly impossible in practice, one can only evolve discrete points on the forward curve.

Multi-factor HJM

The HJM model we have introduced so far are one factor model, which implies that all rates are perfectly correlated. To incorporate richer dynamics multi-factor can be adopted. The SDE of n-factor HJM can be written as

$$dF(t,T) = m(t,T)dt + \sum_{i=1}^{n} \nu_i(t,T)dX_i$$

where dX_is are uncorrelated BM, and the risk-neutral drift of $F(t,T)$ is

$$m(t,T) = \sum_{i=1}^{n} \nu_i(t,T) \int_t^T \nu_i(t,s)ds$$

PCA approach to find volatility

To implement HJM model, whether one factor or multi-factor, one need to estimate volatility function $\nu(t,T)$. Principal Component Analysis is one of the approaches to achieve this task. It identifies the most prominent factors that determines multi-dimensional stochastic processes by orthogonalization of it historical time series data.

Suppose we are modeling a vector of stochastic process $(F_1(t), \cdots, F_n(t))$ with associated SDEs,

$$\begin{aligned} dF_1(t) &= m_1(t)dt + \sigma_1(t)dW_1 \qquad (0.1) \\ \vdots \quad &\quad \vdots \\ dF_n(t) &= m_n(t)dt + \sigma_n(t)dW_n \end{aligned}$$

where m_i and σ_i are drift and volatility(they can be function of F as well), and $(W_1, \cdots, W_n)$ are n-dimensional standard BM with correlation matrix Σ, i.e.,

$$\Sigma_{ij} = \rho_{ij}$$

Since Σ is a symmetric matrix, it can be factorized into the following form

$$\Sigma = V\Lambda V'$$

where V is an orthogonal matrix with Σ's eigenvector in its column, and Λ is a diagonal matrix whose diagonal corresponds to eigenvalues of Σ. So

$$\Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

and $\lambda_1 > \cdots > \lambda_n$.

What's more, since Σ is semi-positive definite, we can have

$$\Sigma = V\Lambda^{\frac{1}{2}} \left(V\Lambda^{\frac{1}{2}}\right)'$$

The diffusions in the block of equations (0.1) are originally in terms of correlated BM can then be reformed in terms of independent BM like this

$$V\Lambda^{\frac{1}{2}}\mathbf{dX} \tag{0.2}$$

Where $\mathbf{dX}$ is independent BM vector. If one only chooses the first d factors, expand (0.2) equation (0.1) can be rewritten as

$$\begin{aligned} dF_1(t) &= m_1(t)dt + \sum_{i=1}^{d} \sqrt{\lambda_i} v_{i1} dX_i \qquad (0.3) \\ \vdots \quad & \quad \vdots \\ dF_n(t) &= m_n(t)dt + \sum_{i=1}^{d} \sqrt{\lambda_i} v_{in} dX_i \end{aligned}$$

where v_{ij} is the jth element in the ith column of V.

Simulating forward rate

Now consider those $F_j(t)$ is the forward rate with the jth time to maturity. Once we can compute eigenvalue and eigenvector of F_is, then volatility function for is obtained by

$$\nu_i(t,T) = \overline{\nu}(\tau_j) = \sqrt{\lambda_i} v_{ij}$$

The implication is that when simulating, one will model volatility function at each maturity τ_j, hence Musiela parameterization can be used.

Pricing derivative

Bond pricing is very easy in HJM framework, it is

$$Z(t,T) = \exp\{-\int_t^T F(t,s)ds\}$$

Give zero coupon bond price one can easily calculate its corresponding yield $Y(t,T)$, one can consider it as 6 month LIBOR if time to maturity is half a year.

We also need to price simple interest rate options such as caps and floors. The caps can be regarded as a portfolio of caplets, and similarly floors are portfolio of floorlets.

The payoff a typical caplet of 6 month LIBOR with strike K_c, maturity T is

$$\max\left(L(T) - K_c, 0\right)$$

Similar payoff for floorlet with strike K_f is

$$\max\left(K_f - L(T), 0\right)$$

Finite Difference

Consider the problem of pricing a binary option using the uncertainty volatility model for the worst case value V^-, which satisfies

$$\frac{\partial V^-}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V^-}{\partial S^2} + rS\frac{\partial V^-}{\partial S} - rV^- = 0$$

where

$$\Gamma = \frac{\partial^2 V^-}{\partial S^2}$$

$$\sigma(\Gamma) = \begin{cases} \sigma^-, & \text{if} \quad \Gamma > 0 \\ \sigma^+, & \text{if} \quad \Gamma < 0 \end{cases}$$

A binary option has a Heaviside payoff $H(S-K)$, a step function, for a call and a put has payoff $H(K-S)$.

Consider the following three steps in the problem:

1. Use the explicit finite difference method to solve the above problem for each binary option in isolation.

2. Now price the binary option when statically hedged with vanilla options. This will require you to solve numerically the general problem of a portfolio consisting of a single binary option with an arbitrary number of vanilla calls and puts with arbitrary strikes and expiries. To price a portfolio of options using a single pass through of the finite-difference code, you must implement a jump condition across the date/time step at which each option expires. Remember that you will be working backwards in time, so you will start with expiration of the last contract. (This may be the binary option or one of the hedging vanillas, your code must be flexible enough to deal with both cases.) The jump condition is

$$[V]_{t-}^{t+} = \text{Payoff for ith option}$$

   where t_i is the expiry of the ith option

3. Subtract off from the solution of stage 2 above, the market price of the static hedge (you will have to specify the market prices of all of the

vanillas). Finally maximise this value by varying the quantities of the vanillas (you may use Excel's Solver for this).

Readings

Paul Wilmott On Quantitative Finance, second edition

- Chapter 52 (Uncertain Parameters)
- Chapter 60 (Static Hedging)