Exercise for Session 5.6
Case Study: developing a simple CDS pricing model
Solution

CQF

In this case study, we will develop a simple model for pricing Credit Default Swaps (CDS)

1 Background

A Credit Default Swap (CDS) is an agreement between 2 parties: the protection buyer and the protection seller. The protection seller agrees to pay the protection buyer a given amount in the event of the default of a reference credit security. In exchange, the protection buyer will pay a periodic premium to the protection seller up until maturity of the CDS or default of the reference credit security, whichever comes first.

Pricing CDS is very similar to pricing interest rates swaps. Interest rate swaps are quoted in terms of their fixed rate. In an interest rate swap, we compute the value of the fixed rate leg and the floating rate leg of the swaps and deduce the fixed rate by equating the values of the two legs.

CDS are quoted in terms of the rate of the premium (or coupon)the protection buyer needs to pay the protection seller. Note that the premium is generally paid in arrears.

In this exercise, we will price a CDS **from the perspective of the protection buyer**, i.e. from the perspective of an investor who pays a premium in order to gain a protection from the default of a reference entity. To price a CDS, we will need to compute the value of the premium leg (the leg of premium payments) and the value of the protection leg (the leg paying a given amount in the event of default) and deduce the premium by equating the values of the two legs.

2 Setup

Recall that in the general case, the probability of default of the reference entity between the time s and the time t (with $0 \leq s \leq t$) is given by

$$P(s \leq \text{default time} \leq t) = 1 - \exp\left\{-\int_s^t \lambda(u)du\right\} \quad (1)$$

where $\lambda(u)$ is the **risk-neutral default intensity**.

In the following questions, to make the pricing exercise simpler, we will assume that:

- the **risk-neutral default intensity** λ is constant (hence the default probability of the reference entity is exponentially distributed);
- the CDS expires at time T;
- the CDS has n coupon payment dates $t_i, i = 1, \ldots, n$;
- the discount rate for maturity date t_i is denoted by r_i;
- the expected loss given default is denoted by L.

We will now price each leg of the transaction.

3 Pricing the CDS

1. Cash flow diagram *- Draw a cash flow diagram for the CDS. Make sure to consider the two cases: (1) the underlying asset does not default, and (2) the underlying asset does default*

Hint: *the cash flow diagram for a defaultable coupon bond is given in figure 1.*

Answer: the cash flow diagram for a CDS is given in figure 2.

Hint: *this is what the cash flow diagram would look like for a defaultable coupon bond:*

2. The premium leg (Part I) *- Imagine we are now at time 0 and that we consider only the coupon paid at some time t_i. What is the time 0 expected value, a_i, of GBP 1 paid at the coupon date t_i?*

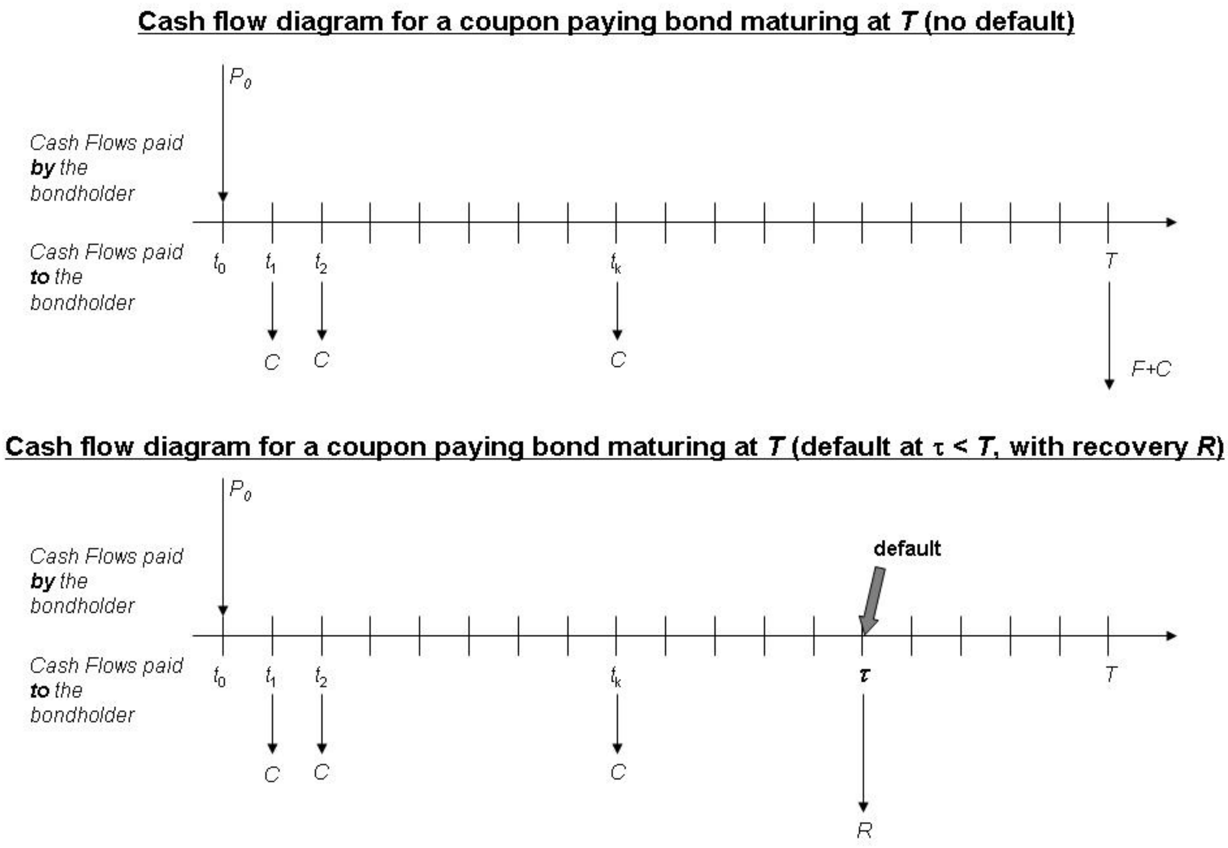

Figure 1: Cash Flow Diagram for a Defaultable Coupon Paying Bond

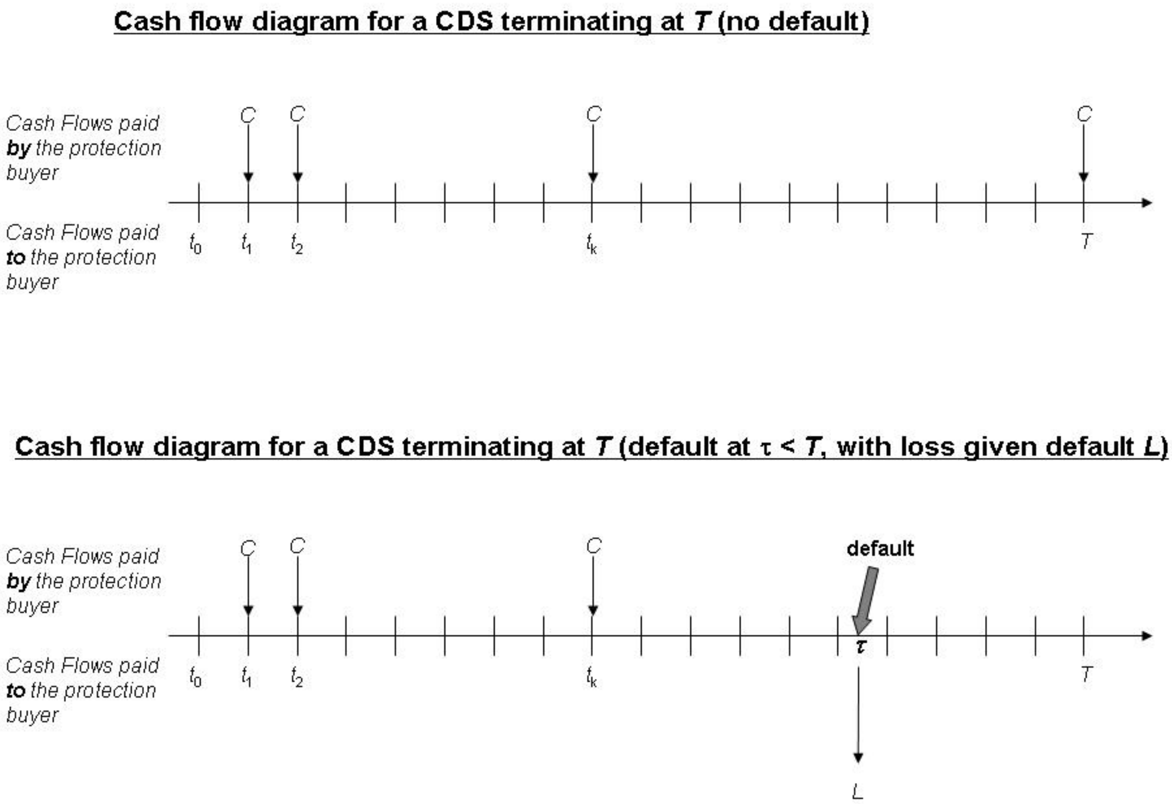

Figure 2: Cash Flow Diagram for a Defaultable Coupon Paying Bond

Hint: *You need to consider two possibilities: either the reference entity has defaulted by time t_i, in which case the coupon is not paid) or the reference entity has not defaulted by time t_i.*

Answer: The time 0 expected value, a_i, of GBP 1 paid at the coupon date t_i is equal to:

$$a_i := a(\lambda, r_i, t_i) = e^{-(\lambda + r_i)t_i}$$

To see this recall that the time 0 value of GBP 1 paid **with certainty** at time t_i is equal to the discounted value of the cash flow, i.e. $e^{-r_i t_i}$.

Now we have seen that the probability that the issuer defaults between time 0 and time t_i is given by

$$P(0 \leq \text{default time} \leq t_i, b) = 1 - \exp\{-\lambda t_i\}$$

Hence, the probability that the issuer has not defaulted before time t_i is equal to $e^{-\lambda t_i}$.

$$P(0 \leq \text{does not default} \leq t_i) = \exp\{-\lambda t_i\}$$

Therefore, a_i is computed as the discounted expected value of GBP 1 paid at time t_i:

$$a_i = e^{-r_i t_i}\left[e^{-\lambda t_i} \times 1 + (1 - e^{-\lambda t_i}) \times 0\right]$$

since naturally, if default has occurred at time $t_j < t_i$ all subsequent coupon are forfeited and in particular nothing is paid at time t_i.

*3. **The Premium Leg (Part II)** - If the premium / coupon payment was GBP 1, what would be the total present value of the premium leg?*

Answer: Summing across all coupon payment times, the time 0 value of an annuity paying 1 unit of currency at each coupon date if default has not occurred by the coupon payment date is equal to:

$$\begin{aligned} A &:= A(\lambda; r_i, i = 1, \ldots, n; t_i, i = 1, \ldots, n) \\ &= \sum_{i=1}^{n} a_i \end{aligned}$$

*4. **The Protection or Default Leg (Part I)** - Imagine we are now at time 0 and that we consider only the protection amount paid at some time t_i. What is the value, b_i, of GBP 1 paid at the coupon date t_i in case of default?*

Hint: *The protection leg only pays at time t_i if default occurs between t_{i-1} and t_i.*

Answer: The protection leg only pays a cash flow at the coupon date when default occurs. The probability of defaulting between time t_{i-1} and time t_i is the probability of surviving up to time t_{i-1} and of having defaulted by time t_i. It is equal to: the probability that the issuer defaults between time 0 and time t_i is given by

$$P(t_{i-1} \le \text{default time} \le t_i) = 1 - \exp\{-\lambda(t_i - t_{i-1})\}$$

Thus, the time 0 expected value, b_i, of a 1 currency unit payment at the coupon date t_i in the event of default is equal to:

$$b_i := b(\lambda, r_i, t_i) = e^{-r_i t_i}\left[1 - \exp\{-\lambda(t_i - t_{i-1})\}\right]$$

5. The Protection or Default Leg (Part II) - *If the protection amount paid in the event of default was GBP 1, what would be the total present value of the premium leg?*

Answer: Summing across all coupon payment times, the time 0 expected value of 1 paid in the event of a default up to an including time T is given by:

$$\begin{aligned} B &:= B(\lambda; r_i, i = 1, \ldots, n; t_i, i = 1, \ldots, n) \\ &= \sum_{i=1}^{n} b_i \end{aligned}$$

6. Pricing the CDS - *Let U be the rate at which the premium is paid. Express the value V of the CDS at inception (time 0). What is the value of the fair premium rate C^*?*

Answer: Given a premium U, the current value V of the CDS to the protection buyer is equal to

$$V = B \times L - A \times C$$

At inception, the value of the CDS must be zero. This implies that the premium is set at:

$$C^* = \frac{B \times L}{A} \tag{2}$$