Certificate in Quantitative Finance (CQF)
**Session 5.4: Credit Default Swaps** *
Solutions

## 1 CDS: implied survival probabilities

An extended formulation of the CDS pricing can be found *Par Credit Default Swap Approximation from Default Probabilities* published by JP Morgan, which can be downloaded from:

`http://www.wilmott.com/attachments/CDS_JPM1.zip`

This methodology includes both computational advantages (only survival probabilities are used), payments can occur anytime (not only on an annual basis), and takes into account payment accruals in the premium leg.

**Formulation WITHOUT accruals**

In this formulation, the PV of the premium leg is

$$PL_N = S_N \sum_{n=1}^{N} D(0,T_n)P(T_n)\,(\Delta t_n)$$

where $\Delta_n$ is the year fraction corresponding to $T_{n-1} - T_n$ and $(P(T_{n-1}) - P(T_n))$ is the probability of the credit default event occuring during period $T_{n-1} - T_n$.

The PV of the default leg is

$$DL_N = (1-R)\sum_{n=1}^{N} D(0,T_n)\,(P(T_{n-1}) - P(T_n))$$

Therefore the spread $S_N$ for an $N-$period credit default swap is given by

*Tutor: Dr Alonso Peña (alonso.pena@sdabocconi.it )

$$S_N = \frac{(1-R)\sum_{n=1}^{N} D(0,T_n)\left(P(T_{n-1}) - P(T_n)\right)}{\sum_{n=1}^{N} D(0,T_n)P(T_n)\left(\Delta t_n\right)}$$

**Formulation WITH accruals**

The PV of the premium leg is now

$$\begin{aligned} PL_N = {} & S_N \sum_{n=1}^{N} D(0,T_n)P(T_n)\left(\Delta t_n\right) \\ & + S_N \sum_{n=1}^{N} D(0,T_n)\left(P(T_{n-1}) - P(T_n)\right)\frac{(\Delta t_n)}{2} \end{aligned}$$

which includes an extra payment accrual term.

The PV of the default leg is the same as before.

Therefore the par credit default swap spread $S_N$ is given by

$$S_N = \frac{(1-R)\sum_{n=1}^{N} D(0,T_n)\left(P(T_{n-1}) - P(T_n)\right)}{\sum_{n=1}^{N} D(0,T_n)P(T_n)\left(\Delta t_n\right) + D(0,T_n)\left(P(T_{n-1}) - P(T_n)\right)\frac{(\Delta t_n)}{2}}$$

You can find these formulas implemented in the accompanying Excel Spreadsheet *CDSExcelJPMORGAN.xls* , available from the course website.

**Bootstrapping WITHOUT accruals**

By equating the $PL_N$ and $DL_N$ for various $N$ we obtain the bootstrapping procedure to determine the survival probabilities from a given vector of market spreads $[S_1, S_2, ..., S_N]$ as follows.

**Step N=1**

For $T_1$ we have

$$P(T_1) = \frac{L}{L + \Delta t_1 S_1}$$

where $L = (1-R)$.

**Step N=2**

For $T_2$ we have

$$P(T_2) = \frac{D(0,T_1)\left[L(1) - (L + \Delta t_1 S_2)P(T_1)\right]}{D(0,T_2)(L + \Delta t_2 S_2)} + \frac{P(T_1)L}{L + \Delta t_2 S_2}$$

**Step N**

$$P(T_N) = \frac{\sum_{n=1}^{N-1} D(0,T_n)\left[LP(T_{n-1}) - (L + \Delta t_n S_N)P(T_n)\right]}{D(0,T_N)(L + \Delta t_n S_N)} + \frac{P(T_{N-1})L}{(L + \Delta t_N S_N)}.$$

You can find these formulas implemented in the accompanying Excel Spreadsheet *ImprovedBootstrappingExample.xls* , available from the course website.

### 1.1 Part a: implied survival probabilities with term-structure hazard rates

All bootstrapping calculations are done without accruals.

| Maturity | ABC | XYZ |
|---|---|---|
| 1Y | 99.42 | 49.72 |
| 2Y | 98.45 | 30.60 |
| 3Y | 97.26 | 18.87 |
| 4Y | 95.88 | 14.10 |
| 5Y | 94.37 | 11.52 |

Table 1: Survival probabilities for ABC and XYZ, in percent.

### 1.2 Part b: implied survival probabilities with flat hazard rates

All bootstrapping calculations are done without accruals.

| Maturity | ABC | XYZ |
|---|---|---|
| 1Y | 98.87 | 58.06 |
| 2Y | 97.76 | 33.71 |
| 3Y | 96.66 | 19.58 |
| 4Y | 95.57 | 11.37 |
| 5Y | 94.49 | 6.60 |

Table 2: Survival probabilities for ABC and XYZ, in percent.

### 1.3 Part c: implied survival probabilities for various recovery rates

All bootstrapping calculations are done without accruals.

| Maturity | $R = 20\%$ | $R = 50\%$ | $R = 65\%$ |
|---|---|---|---|
| 1Y | 99.64 | 99.42 | 99.18 |
| 2Y | 99.03 | 98.45 | 97.80 |
| 3Y | 98.28 | 97.26 | 96.12 |
| 4Y | 97.40 | 95.88 | 94.17 |
| 5Y | 96.44 | 94.37 | 92.06 |

Table 3: Survival probabilities for ABC for various recovery rate assumptions.

## 2 Expected Default Times

### 2.1 Part a

The expected default time can be calculated using integration by parts

$$E[\tau] = \lambda \int_0^\infty s \exp(-\lambda s)\, ds = \left[ se^{-\lambda s} \right]_0^\infty + \int_0^\infty e^{-\lambda s} ds = \frac{1}{\lambda}$$

### 2.2 Part b

The expected variance of the default time is

$$E[\tau^2] - E[\tau]^2 = \int_0^\infty s^2 \lambda \exp(-\lambda s)\, ds - \frac{1}{\lambda^2} = \left( \frac{2}{\lambda^2} - \frac{1}{\lambda^2} \right) = \frac{1}{\lambda^2}$$

### 2.3 Part c

With $\lambda = 1\%$ the expected default time is 100 years and its variance is 10,000 years.

## 3 The Credit Triangle

This problem is solved by assuming a continous approximation to the pricing of a CDS.

The premium leg ($PL$) is

$$PL(0,T) = S\int_0^T Z(0,T)P(0,t)dt$$

where $(0,t)$ is the survival probability aqs seen from time zero.
The default leg $(DL)$ is

$$DL(0,T) = (1-R)\int_0^T D(0,T)(-dP(0,t)dt)$$

with $D(0,T)$ the discount factor for time $T$.
Since $dP(0,t) = -\lambda(t)P(0,t)dt$ this can be written as

$$DL(0,T) = (1-R)\int_0^T D(0,t)\lambda(t)P(0,t)dt$$

and with a constant hazard rate this becomes

$$DL(0,T) = (1-R)\lambda\int_0^T D(0,t)P(0,t)dt$$

The value of the spread $S$ which makes the protection and the premium legs equal is given by

$$S = (1-R)\lambda$$

# 4 Upfront Credit Default Swap

The upfront CDS replaces the premium leg of a CDS with a single payment of $U(0)$ at the initiation of the contract. The two legs of the contract are therefore the payment of the upfront value and the protection (default) leg. We can therefore determine $U(0)$ by setting the net value of the CDS contract equal to zero at initiation. The value of the contract, from the point of view of a protection buyer who has paid the upfront at time $t = 0$ to buy protection to time $T$, is

$$(1-R)\int_0^T D(0,s)(-dP(0,s)) - U(0) = 0$$

such that

$$U(0) = (1-R)\int_0^T D(0,s)(-dP(0,s))$$

Once the upfront payment has been made, it goes into the cash account of the protection seller. The mark-to-market value of the contract for the protection buyer at a later time $t$ is given by

$$MTM(t) = (1-R)\int_0^T D(0,s)(-dP(0,s))$$

which is simply the value of the protection (default) leg of the standard CDS.