How to Value Interest Rate Derivatives in a No-Arbitrage Setting

Sirimon Treepongkaruna
Department of Commerce, University of Queensland
Stephen Gray
Department of Commerce, University of Queensland
and
Fuqua School of Business, Duke University

Abstract

This paper shows how to use the arbitrage-free trinomial tree approach of Hull and White (1994) to value a range of interest rate derivatives such as caps, floors, collars, and swaps. We also provide detailed numerical examples for expositional purposes.

1. Introduction

The recent literature on the term structure of interest rates has considered two different ways of modeling interest rates. One method involves the empirical estimation of the time series properties of short-term interest rates. These short rate models are relatively easy to estimate and are flexible in capturing the stylized facts of the time series properties of interest rates. A number of these short rate models provide a closed-form solution for discount bond prices. This enables the entire yield curve to be constructed from knowledge of the current short rate and the parameters of the model. There is no guarantee, however, that this model yield curve will match the actual yield curve. To the extent that these differ, prices computed using the model will admit arbitrage.

The second method of modeling interest rates is via no-arbitrage term structure models. These no-arbitrage models essentially take the current yield curve as an input and therefore exclude arbitrage automatically. These models often require complicated numerical computation and therefore allow only relatively simple dynamics for short-term interest rates. Nevertheless, these no-arbitrage term structure models have become popular among practitioners since it is crucial for them to quote prices in a way that does not admit arbitrage. Even though these no-arbitrage models rely on relatively simple dynamics for the short rate, they can only be implemented via relatively complicated numerical procedures. In this paper, we demonstrate how to construct the trinomial trees required to implement a range of no-arbitrage models. Our contribution is in providing a detailed exposition of this procedure within the context of a series of examples. We also demonstrate how to use this framework to value a range of interest rate derivative securities including options, caps, floors, collars, and swaps.

The balance of the paper is organized as follows. Section 2 describes a number of no-arbitrage models and their properties. Section 3 reviews the procedure of the Hull-White trinomial tree. Section 4 explains how to price different interest rate derivatives using the trinomial tree. Section 5 provides a summary.

2. No-arbitrage Models and Their Properties

No-arbitrage models have been modeled in three different ways. First, Heath, Jarrow, and Morton (HJM) (1992) model forward rates. However, their approach does not guarantee that interest rates will stay positive or finite. Moreover, derivative valuation is slow using

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their framework since most models under the HJM framework are non-Markov, resulting in non-recombining trees. Hence, we need to know not only the value of the interest rate at each node but also its path in reaching that node. This also severely limits the number of time steps that can be used and considerably slows the computational process. The second method involves modeling bond prices. This approach involves specifying the process followed by all discount bond prices at all times, such as in the Hull and White (1993) model. This approach also leads to non-recombining trees. The third method involves modeling the short-term interest rate directly. Focusing directly on the short rate allows us to model the interest rate process with a recombining tree. This eases the computational burden considerably. In this paper, we therefore focus on techniques that model the short rate directly.

2.1 The Ho-Lee model

The first no-arbitrage term structure model was proposed by Ho and Lee (1986). They start with the complete current term structure, and allow the entire term structure to shift up or down in a way that matches the empirically observed volatility of interest rates. Since they begin with the current term structure, their model does not admit arbitrage. Under the Ho-Lee model, interest rates have no mean-reversion and constant volatility. The short-rate standard deviation and the market price of risk of the short rate are parameters in the model. The model can be expressed as a stochastic differential equation:

$$dr=\theta(t)dt+\sigma dW \tag{1}$$

where σ is constant, and $\theta(t)$ is a function of time chosen to ensure that the resulting forward rate curve exactly matches the current term structure. As such, the average direction in which the short rate will move in the future is defined by:

$$\theta(t)=F_t(0,t)+\sigma^2 t \tag{1a}$$

where $F_t(0,t)$ denotes the first derivative with respect to time of the instantaneous forward rate at present for a contract maturing at time t. For instance, the average direction in which the short rate will move in the future is equal to the slope of the instantaneous forward curve when $\theta(t) = F_t(0,t)$.

In this model, the price at time t of a discount bond that matures at time T is:

$$P(t,T)=A(t,T)e^{-r(t)(T-t)} \tag{1b}$$

where

$$\ln A(t,T) = \ln\frac{P(0,T)}{P(0,t)} - (T-t)\frac{\partial \ln P(0,t)}{\partial t} - \frac{1}{2}\sigma^2 t(T-t)^2 \tag{1c}$$

The advantage of this model is that it is Markov and analytically tractable. However, the model gives the user very little flexibility to choose the volatility structure and does not incorporate any mean-reversion. The model also admits the possibility that the short rate may be negative for some t.

2.2 The Black-Derman-Toy model (BDT)

Black, Derman, and Toy (1990) develop a binomial model to fit both the initial term structure and its volatility structure. They assume a binomial evolution of the short rate in the same manner as the binomial model for stock prices developed by Cox, Ross, and Rubinstein (1979). Interest rates at the various nodes of the binomial tree are chosen by matching the currently observed yield and the yield volatilities. The model takes an array of long rates for various maturities, or the "yield curve," and an array of yield volatilities for the same bonds, or the "volatility curve" as inputs. The yield curve and the volatility curve together form the term structure. A binomial tree is constructed in such a way as to match these term structure inputs. The short rate is assumed to be log normally distributed. This model is virtually identical to the Ho and Lee model (1986), except that the underlying variable is ln(r) rather than r. The model can be expressed as a stochastic differential equation:

$$d\ln(r)=\theta(t)dt+\sigma dW \tag{2}$$

where σ is the instantaneous standard deviation of the short rate, and $\theta(t)$ is a function of time chosen to ensure that the resulting forward rates and volatility curve match those of the current term structure.

By using a lognormal distribution, the BDT model improves upon the Ho and Lee model by not allowing the interest rate to become negative. However, log-normality is strong assumption in that it does not allow negative forward rates, which can be found when the term structure is steeply downward-sloping. Moreover, the BDT model has no closed-form solution for bond prices, so it requires numerical methods to solve for future possible term structures. As the period grows, the system of solutions becomes more difficult to solve.

2.3 The Black-Karasinski model (BK)

Black and Karasinski(1991) extend the BDT model to incorporate mean reversion in short-term interest rates. For any given time, the distribution of possible short rates is lognormal, so the rate neither falls below nor reflects off a barrier at zero. This model allows one to match the yield curve, the volatility curve, and the cap curve. Yield and volatility curves are matched as in the BDT model. The cap curve is the price of an at-the-money differential cap for each maturity. The payoff of a differential cap is the positive difference between the short rate and the strike price, or otherwise zero. The model can be expressed as a stochastic differential equation:

$$d\ln(r)=\phi(t)[\ln(\mu(t))-\ln(r)]dt+\sigma(t)dW \quad (3)$$

where $\sigma(t)$ allows for some time-dependent volatility in the short rate process, $\phi(t)$ is the speed of adjustment towards the target rate, and $\mu(t)$ is the target interest rate and allows for some time-dependent long-run mean. That is, by incorporating mean reversion, $ln(r)$ tends to fall when it is above $ln(\mu(t))$ and rise when it is below $ln(\mu(t))$.

The advantage of this model is that the yield curve, the volatility curve, and the cap curve are observable and correspond to market prices. However, the BK model does not provide a closed-form solution for bond pricing. Moreover, as the period under examination grows, the system of equations becomes much more difficult to solve. The BK model also matches the volatilities of the yields on discount bonds only at time zero[1], and there is no guarantee that the pattern of discount bond yield volatilities in the future will be similar to the pattern at time zero. In fact, the volatility term structure at the future times is liable to be quite different from the volatility term structure today. It may not, therefore be appropriate to use the BK model to price a long-lived option, which can be sensitive to the way bond yield volatilities evolve.

1 If the volatilities of the yields on discount bonds are to have some particular pattern at all times, a Non-Markov model, which is essentially intractable and intensively computational, is required.

2.4 The Hull-White model (HW)

Hull and White (1994) propose a more general model that nests the above three models as special cases. Their model is based on the following stochastic differential equation:

$$dx=[\theta(t)-a(t)x]dt+\sigma(t)dW \quad (4)$$

where $x=f(r)$ is some function of the short rate, $\theta(t)$ is a function of time chosen to ensure that the theoretical (or model based) prices of all discount bonds match current market prices, $a(t)$ allows for some time-dependent mean-reversion, and $\sigma(t)$ allows for some time-dependent volatility in the short rate process. When $x=r$ and $a(t)=0$, this model reduces to the Ho and Lee (1986) model. When $x=ln(r)$ and $a(t)=-\sigma'(t)/\sigma'(t)=\partial\sigma/\partial t$, and, this model reduces to the Black, Derman, and Toy (1990) model. When $x=ln(r)$ and $a(t)$ and $\sigma(t)$ are allowed to be functions of time, this model becomes the Black and Karasinski (1991) model.

One of the key advantages of the HW (1994) model is that its more general form allows us to match the initial volatility of all zero-coupon rates and the volatility of the short rate at all future times. Moreover, the HW trinomial tree can be adjusted to price not only zero-coupon bonds, but also a set of interest rate derivatives.

3. Numerical Operationalization of the Hull-White Model

To represent the evolution of interest rates, Hull and White (1994) develop a two-stage trinomial tree. The first stage is to determine the overall shape of the tree as well as allocate the appropriate probabilities to each node of the tree. The second stage is to "shift" the first stage tree so that discount bond prices computed using the tree match observed market prices.

Hull and White (1994) begin with the following stochastic process for short-term interest rates:

$$dr=[\theta(t)-ar]dt+\sigma dW \quad (5)$$

Next, they define a new variable, called r*, obtained by setting both θ(t) and r(0) to be zero. The resulting process is:

$$dr^*=-ar^*+\sigma dW \quad (5a)$$

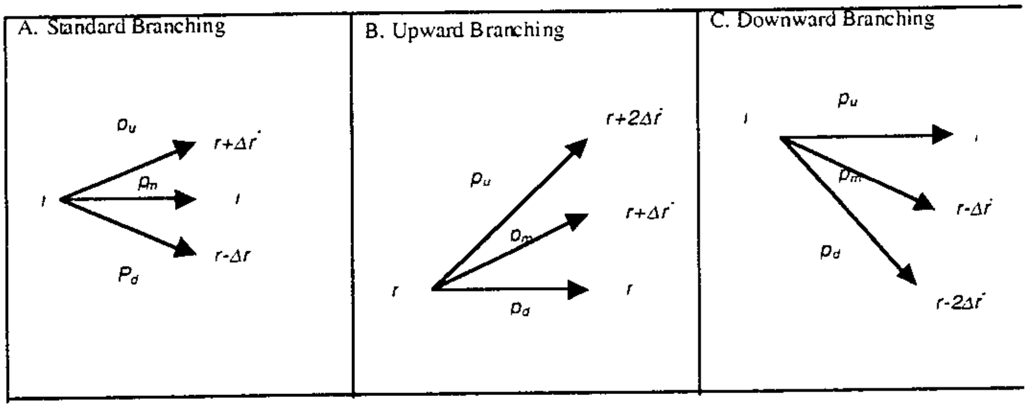
For this process, the expected change in r^* and the variance of the change in r^* over some time interval are given as follows:

$$E[dr^*]=Mr^*=(e^{-a\Delta t}-1)r^*, \quad (5b)$$

$Var[dr^*] = V = \sigma^2(1-e^{-2a\Delta t})/2a.$ (5c)

Next, they construct a trinomial tree so that at every node in the tree the expected change in r^* and the variance of the change in r^* match the quantities given by (5b) and (5c). Note that a trinomial tree is required because the expected change in r^* depends on the level of r^* and therefore varies across the different nodes of the tree. In a standard binomial tree the expected change in the underlying is the same at every node in the tree. The extra degree of freedom required to incorporate a time-varying conditional mean results in a trinomial tree being required.

Figure 1
Branching patterns in a trinomial tree



The HW trinomial tree procedure requires that one of three different branching patterns be used, as illustrated in Figure 1. Under the standard branching pattern, r^* either stays the same or increases or decreases by an amount of Δr^*. As the name suggests, this branching pattern is used for the majority of nodes in the tree. Under the upward branching pattern, r^* either stays the same, increases by an amount of r* or by an amount of $2\Delta r^*$. This branching pattern is used for nodes at the bottom of the tree. Under the downward branching pattern, r^* either stays the same, decreases by an amount of Δr^* or by an amount of $2\Delta r^*$. This branching pattern is used for nodes at the top of the tree.

Hull and White (1994) set the vertical distance between the nodes on the tree Δr^* as:

$$\Delta r^* = \sqrt{3V} \tag{5d}$$

This is done so that conditional mean and variance in (5b) and (5c) can be matched by proper specification of the probabilities of each of the three branches. Since we need to match the conditional mean and variance, and the three branch probabilities must sum to one, we have three equations and three unknowns. Solving simultaneously yields the three branching probabilities. Hull and White (1994) define (i,j) as the node for which $t=i\Delta t$ and $r^*=j\Delta r^*$. That is, i time steps have passed and we are j steps above the center of the tree (set by the position of the initial node). When r^* is at node (i,j) the expected change during the next time step of length Δt is equal to $j\Delta r^* M$, and the variance of the change is V. They also define p_u, p_m, and p_d as the probabilities of the top, middle, and bottom branches stemming from a node. For the majority of nodes, the standard branching pattern in Figure 1 is used. The probabilities associated with the standard branching pattern are:

$$p_u = \frac{1}{6} + \frac{j^2M^2 + jM}{2}, \tag{6a}$$

$$p_m = \frac{2}{3} - j^2M^2, \tag{6b}$$

$$p_d = \frac{1}{6} + \frac{j^2M^2 - jM}{2}. \tag{6c}$$

When $a>0$, the process exhibits mean reversion so that r^* is drawn back towards the center of the tree. At the top of the tree, r^* is large and mean reversion is strong so that further increases are essentially impossible. In this case, the downward branching pattern is used. In particular, Hull and White(1994) show that the downward branching pattern should be used at any node where $j>j_{max}$, where j_{max} is the smallest integer greater than -0.184/M. The probabilities associated with the downward branching pattern are:

$$p_u=\frac{7}{6}+\frac{j^2M^2+3jM}{2}, \quad (7a)$$

$$p_m=-\frac{1}{3}-j^2M^2-2jM, \quad (7b)$$

$$p_d=\frac{1}{6}+\frac{j^2M^2+jM}{2}. \quad (7c)$$

Conversely, at the bottom of the tree r^* is small and mean reversion is strong so that further decreases are essentially impossible. In this case the upward branching pattern is used. In particular, Hull and White (1994) show that the upward branching pattern should be used whenever $j<j_{min}$ where $j_{min}=-j_{max}$. The probabilities associated with the upward branching pattern are:

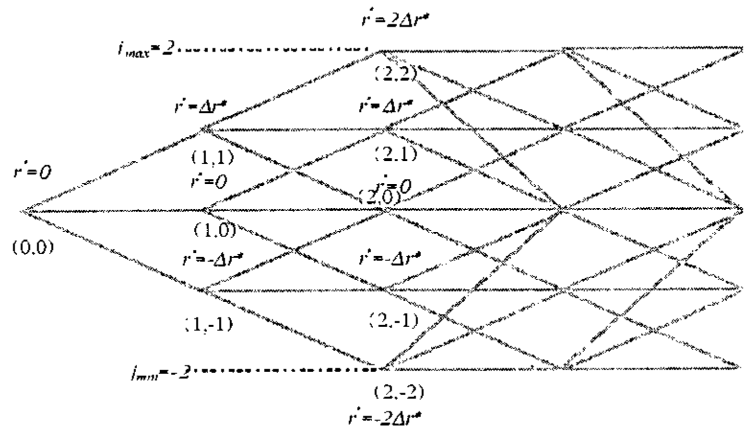
$$p_u=\frac{1}{6}+\frac{j^2M^2-jM}{2}, \quad (8a)$$

$$p_m=-\frac{1}{3}-j^2M^2+2jM, \quad (8b)$$

$$p_d=\frac{7}{6}+\frac{j^2M^2-3jM}{2}. \quad (8c)$$

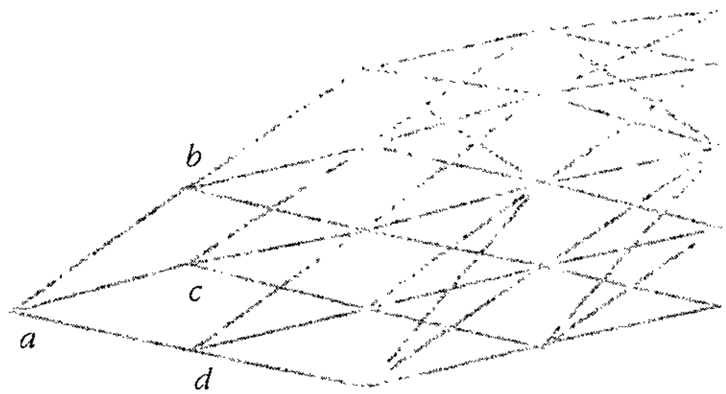
This concludes the first stage and results in a trinomial tree as illustrated in Figure 2. A tree drawn in this fashion approximates the stochastic process in Equation (5a). In Figure 2, the numbers in parentheses refer to the node index used by Hull and White (1994) (i,j) and $j_{max}=2$.

Figure 2
The first stage tree (r*)



The second stage involves converting the r^* tree into a tree for the short rate itself (the r tree) by using forward induction from time zero to the end of the tree, and adjusting the location of the nodes at each time step so as to match the current yield curve. This results in a tree similar that in Figure 3. Suppose the nodes in our tree are one-month apart (i.e., $\Delta t=1$ month). Also suppose that the currently observed 1-month and 2-month zero-coupon rates are 5.0% and 5.5%, respectively.

Figure 3
The second stage tree (r)



In this case, the short rate tree in Figure 3 begins with the current short (1-month) rate of 5.0% that is, Node *a* is 5.5%. The possible short rates at Nodes *b,c,* and *d* must be set in a way that is consistent with the current 2-month rate of 5.5%. If we knew that the short rate was going to branch up to Node *b*, the price of a 2-month zero-coupon bond would be $e^{-r_b(1/12)} e^{-0.05(1/12)}$. This is just the present value of \$1, discounted at r_b for one month and 5% for one month. If we knew the short rate was going to branch to Node *c*, the price of a zero-coupon bond would be $e^{-r_c(1/12)} e^{-0.05(1/12)}$, and so on. Since we already know the probabilities associated with each of the three branches (p_u, p_m, and p_d are specified during construction of the first-stage r^* tree), the short rates at Nodes *b,c,* and *d* must be chosen to satisfy:

$$e^{-0.055(2/12)} = p_u e^{-r_b(1/12)} e^{-0.05(1/12)} + p_m e^{-r_c(1/12)} e^{-0.05(1/12)} + p_d e^{-r_d(1/12)} e^{-0.05(1/12)}$$

That is, the one-month interest rates one month from now must be consistent with the current yield curve.

To achieve this, Hull and White (1994) define the value of r at node *(i,j)* on the second stage tree as the sum of the value of r^* at node *(i,j)* on the first stage tree and α_i. They also define $Q_{i,j}$ as the present value of security that pays off \$1 if node *(i,j)* is reached and zero otherwise. The mathematical definitions of α_i, and $Q_{i,j}$ are:

$$\alpha_i = \frac{\sum_{j=-n_i}^{n_i} Q_{i,j} e^{-j\Delta r\Delta t} - \ln P(0,i+1)}{\Delta t}, \qquad (9)$$

$$Q_{i+1,j} = \sum_k Q_{i,k}\, q(k,j) \exp[-(\alpha_i + k\Delta r)\Delta t]. \qquad (10)$$

where $Q_{i+1,j}$ is the present value of security that pays off \$1 if node *(i+1,j)* is reached and zero otherwise, *q(k,j)* is the probability of moving from node *(i,k)* to node *(i+1,j)*, and the summation is taken over all values of *k* for which this is non-zero.

Table 1
The Deutschemark zero-coupon yield curve , July 8, 1994

Maturity	Days	Rate	Discount Bond Price
3 days	3	5.01772	1
1 year	367	5.09276	0.950347521
2 year	731	5.79733	0.890557204
3 year	1096	6.30595	0.827673348
4 year	1461	6.73464	0.763884530
5 year	1826	6.94816	0.706517011
6 year	2194	7.08807	0.653584010
7 year	2558	7.27527	0.600934762
8 year	2922	7.30852	0.557283273
9 year	3287	7.39790	0.513856621
10 year	3653	7.49015	0.472832063

Note: This table is reconstructed from Hull and White (1996). The discount bond price is calculated from where $P(t)=e^{-r_t(t)}$ is the price of a discount bond that matures time t from now and r_t is the appropriate zero-rate.

4.Pricing Interest Rate Derivatives Using the HW Trinomial Tree

In section 4.1, we first illustrate how to build the HW trinomial tree to price interest rate derivatives using a numerical example from Hull and White (1996). However, the term structure model specified in Hull and White (1996) allows the interest rate to become negative. To overcome this, in Section 4.2 we build a tree for the natural logarithm of the short rate. Using the constructed $ln(r)$ tree, we then determine the expected payoff of various interest rate derivatives. Finally, we discount these expected payoffs back through the trinomial tree, to obtain the value of the interest rate derivatives in question. In particular, we examine a cap, a floor, a collar, and a swap.

4.1 Numerical implementation of the HW trinomial tree

Table 1 contains the Deutschemark zero-coupon yield curve given in Hull and White (1996), which forms the basis of our illustration. To illustrate how to construct the HW trinomial tree, we assume the interest rate follows a one-factor term structure model of the following form:

$$dr=[\theta(t)-0.1r]dt+0.01dW \qquad (11)$$

4.1.1 The first stage tree (r)*

The objective of this first stage tree is to set the overall shape of the tree and allocate the appropriate probabilities for each node of the tree. By setting both $\theta(t)$ and $r(0)$ in Equation 11 to be zero, we obtain a new variable, r^*, that follows the stochastic process:

$$dr^*=-0.1r^*dt+0.01dW \qquad (11a)$$

Applying this process and Equations 5b-d, yields $M=e^{-0.1(1)}-1=-0.095162582$, and $\Delta r^*=0.01\sqrt{3}=0.016489508$, where we have set $\Delta t=1$ year. Next, we determine points where nonstandard branching patterns must be used. In this case, -0.184/M =1.934 so $j_{max}=2$, and $j_{min}=-2$. To allocate appropriate probabilities, we use Equations 6a-c when $-2<j<2$, Equations 7a-c when $j=2$, and Equations 8a-c when $j=-2$. That is, the probabilities when $j=1$ are:

$$p_u=\frac{1}{6}+\frac{(1)^2(-0.095)^2+(1)(-0.095)}{2}=0.12361,$$

$$p_m=\frac{2}{3}-(1)^2(-0.095)^2=0.65761,$$

$$p_d=\frac{1}{6}+\frac{(1)^2(-0.095)^2-(1)(-0.095)}{2}=0.21878;$$

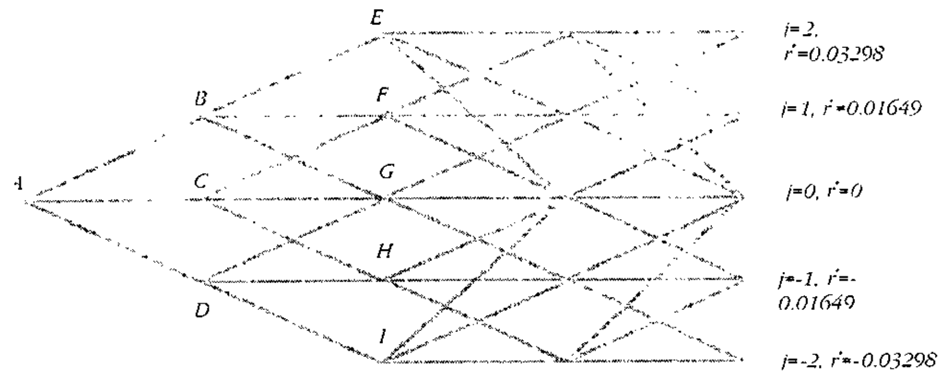
the probabilities when j=2 are:

$$p_u=\frac{7}{6}+\frac{(2)^2(-0.095)^2+3(2)(-0.095)}{2}=0.89929,$$

$$p_m=-\frac{1}{3}-(2)^2(-0.095)^2-2(2)(-0.095)=0.01109,$$

$$p_d=\frac{1}{6}+\frac{(2)^2(-0.095)^2+(2)(-0.095)}{2}=0.08962;$$

Figure 4
The first stage tree (r*)



j	*Node*	r^*	p_u	p_m	p_d
0	***A,C,G***	0.00000	0.16667	0.66667	0.16667
1	***B,F***	0.01649	0.12361	0.65761	0.21878
-1	***D,H***	-0.01649	0.21878	0.65761	0.12361
2	***E***	0.03298	0.89929	0.01109	0.08962
-2	***I***	-0.03298	0.08962	0.01109	0.89929

and the probabilities when j=-2 are:

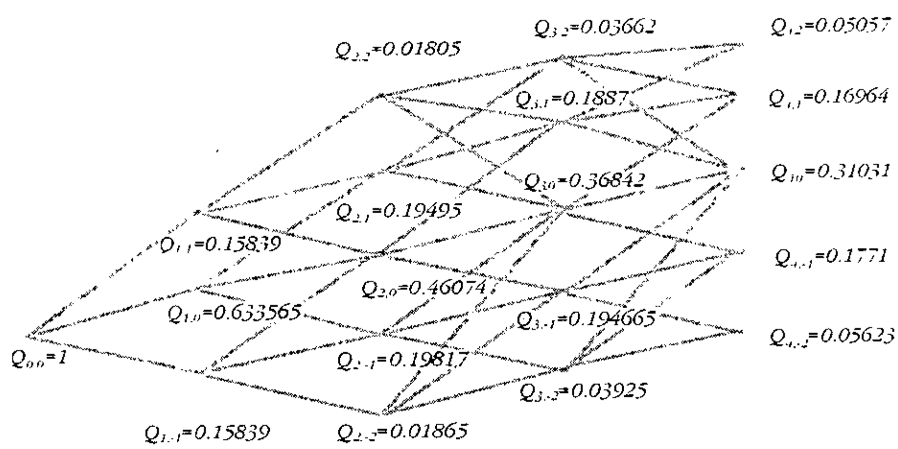
$$p_u = \frac{1}{6} + \frac{(-2)^2(-0.095)^2 - (-2)(-0.095)}{2} = 0.08962,$$

$$p_m = -\frac{1}{3} - (-2)^2(-0.095)^2 + 2(-2)(-0.095) = 0.01109,$$

$$p_d = \frac{7}{6} + \frac{(-2)^2(-0.095)^2 - 3(-2)(-0.095)}{2} = 0.89929.$$

This results in the tree in Figure 4.

Figure 5
The state price tree (Q)



4.1.2 The second stage tree (r)

This stage involves converting the r^* tree into the r tree by shifting the tree to match the initial yield curve. First, we set the initial node (node a in Figure 3) to be equal to the 1-year zero coupon rate, in other words setting α_0=5.092755%, and $Q_{0,0}$=1. Next, we use the first stage tree and its allocated probabilities in Figure 4, together with α_0, and $Q_{0,0}$, and apply Equation 10 to determine the state prices of securities that pay off $1 at nodes B, C, and D in Figure 4 as follows:

Node B: $Q_{1,1} = 0.166667e^{-(5.092755\%)(1)} = 0.15839$. (12a)

Node C: $Q_{1,0} = 0.666667e^{-(5.092755\%)(1)} = 0.633565$. (12b)

Node D: $Q_{1,-1} = 0.166667e^{-(5.092755\%)(1)} = 0.15839$. (12c)

Note that to match the initial yield curve the sum of $Q_{1,1}$, $Q_{1,0}$, and $Q_{1,-1}$ and are equal to the present value of a 1-year zero coupon bond, which is $e^{-(5.092755\%)(1)}$=0.950347521. Given $Q_{1,1}$, $Q_{1,0}$, and $Q_{1,-1}$, and the 2-year zero coupon bond price, we apply Equation 9 to determine the interest rate at nodes B, C, and D in Figure 4 as follows:

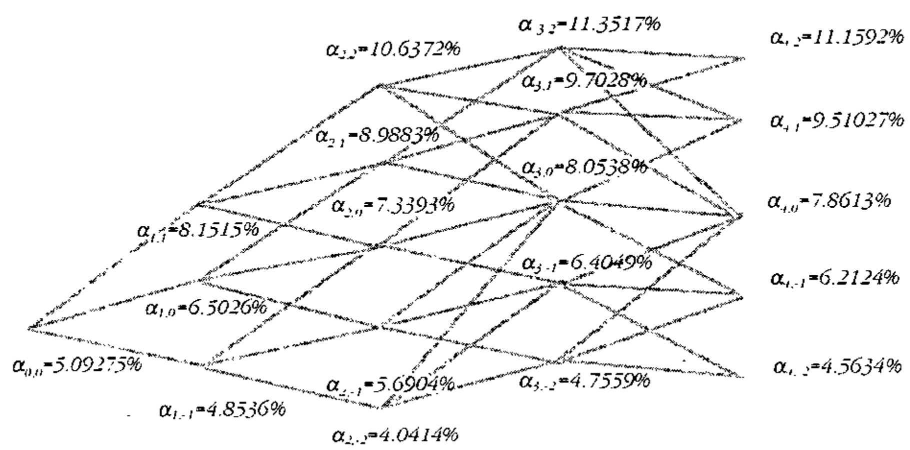
Node C: $\alpha_1 = \ln\left(\left(Q_{1,1}e^{-(1)(1.649\%)(1)} + Q_{0,0}e^{-(0)(1.649\%)(1)} + Q_{1,-1}e^{-(-1)(1.649\%)(1)}\right)/0.890557204\right) = 6.5026\%,$

Node B: $\alpha_1 + \Delta r^* = 6.5026\% + 1.649\% = 8.1515\%,$

Node D: $\alpha_1 - \Delta r^* = 6.5026\% - 1.649\% = 4.8536\%.$

To solve for nodes at the next time step, we repeat the whole process. This results in the tree of state prices in Figure 5, and the interest rate tree in Figure 6.

Figure 6
The second stage tree (r)



4.1.3 Pricing a put option

Hull and White (1996) price a three-year put option with strike price of $63 on a zero-coupon bond that will pay $100 in nine years. They assume that the interest rate follows the process specified in Equation 11, which allows them to use a closed-form solution for the bond price to determine the bond price at year 3 when the option matures. This means that the interest rate tree can terminate after three years. At this point, the short-term interest rate can be converted into the price of the underlying six-year zero-coupon bond via a closed-form solution. For a more complicated stochastic process of interest rates, a closed form solution is unavailable and the tree would have to be extended out to nine years - the maturity of the bond. Backward induction through the tree could then be used to compute the price of six-year bonds three years from now. Under the stochastic process used in this case, the price at time t of a zero-coupon bond maturing at time T, $P(t,T)$, is:

$$P(t,T) = A(t,T)e^{-B(t,T)R_t} \tag{13}$$

where R_t is the instantaneous (continuously compounded) interest rate at time t. Hull and White (1996) define $A(t,T)$, $B(t,T)$, and $F(0,t)$ as follows:

$$A(t,T) = \frac{P(0,T)}{P(0,t)}\exp\left[B(t,T)F(0,t) - \sigma^2 B(t,T)^2(1-e^{-2at})/(4a)\right]. \tag{13a}$$

$$B(t,T) = (1-e^{-a(T-t)})/a. \tag{13b}$$

$$F(0,t) = -\frac{\partial \ln\left[P(0,t)\right]}{\partial t}. \tag{13c}$$

To determine the expected payoff of the put option, we need to compute the possible value of six-year zero-coupon bond prices three years from now. Our trinomial tree, however, gives us possible one-year rates three years from now. We can convert these one-year rates into instantaneous rates using Equation 13:

$$P(t,t+\Delta t) = e^{-r_{t,t+\Delta t}\Delta t} = A(t,t+\Delta t)e^{-B(t,t+\Delta t)R_t}$$

in which case:

$$R_t = \frac{r_{t,t+\Delta t}\Delta t + \ln A(t,t+\Delta t)}{B(t,t+\Delta t)}. \tag{13d}$$

We can then use the expression in Equations 13-13c to compute possible values of six-year zero-coupon bond prices and therefore option payoffs.

To find the option payoff at year 3, we first need to determine $A(3,4)$ and $B(3,4)$ in order to transform $r_{t,t+\Delta t}$ at each node on the second stage tree at time t=3 to be the instantaneous rate R_t. Then we also have to determine $A(3,9)$ and $B(3,9)$ to obtain the price of a six-year bond at time t=3. This is the value of the underlying asset at the time the option matures, and thus determines the option payoffs. In particular, we subtract the resulting bond price from the strike price to obtain the put option payoff. If the bond price exceeds the strike price, the put option payoff is zero at that node.

Table 2
Put option payoff at year 3

j	One-year rate r	Instantaneous rate - R	Bond price	Option payoff
2	11.3517101%	11.3205517%	0.529172228	10.08277719
1	9.7027593%	9.58777794%	0.572203456	5.779654432
0	8.0538085%	7.8550072%	0.618733897	1.126610337
-1	6.4048576%	6.1222349%	0.669048100	0
-2	4.7559068%	4.3894627%	0.723453754	0

Note: r is the one-year zero-coupon rate at year 3, taken from node $i=3$ in Figure 6. R is calculated from Equation 13d. The bond price is computed from Equation 13. The expected payoff is 100* max(*strike–bondprice*,0).

Table 2 illustrates the put option payoff at year 3 at each node of the tree. We can numerically implement Equation 13c as:

$$F(0,3) \approx -\frac{\left[\ln\left(P(0,3+\varepsilon)\right) - \ln\left(P(0,3-\varepsilon)\right)\right]}{2\varepsilon}$$

For example, using =1 year and discount bond prices in Table 1 yields:

$$F(0,3) = -\frac{\ln(0.7639) - \ln(0.8906)}{2} = 7.6729\%$$

Hull and White (1996) use a finer interpolation procedure (not reported) that yields F(0,3)=7.8304%. In order to allow comparison with their work, we adopt this value throughout.

Next, we use Equations 13a-b to compute:

$B(3,4) = (1-e^{-0.1(1)})/0.1 = 0.95162582,$

$A(3,4) = \frac{0.7639}{0.8277}\exp\left[0.95162582(7.8304\%) - (0.0095)^2(0.95162582)^2(1-e^{-2(0.1)3})/4(0.1)\right] = 0.994229.$

$B(3,9) = (1-e^{-0.1(9-3)})/0.1 = 4.511883639,$

$A(3,9) = \frac{0.5139}{0.8277}\exp\left[4.51188(7.8304\%) - (0.0095)^2(4.51188)^2(1-e^{-2(0.1)3})/4(0.1)\right] = 0.881944.$

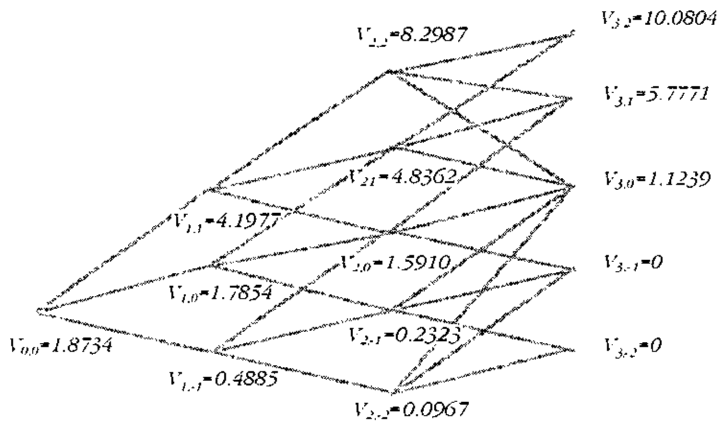
Finally, we discount the option payoff at year 3 back through the r tree in Figure 6:

$v_{2,2} = \left[p_u v_{3,2} + p_m v_{3,1} + p_d v_{3,0}\right]e^{-\alpha_{2,2}\Delta t},$

$v_{2,2} = \left[0.8993(10.0804) + 0.0111(5.771) + 0.0896(1.1239)\right]e^{-10.6372\%(1)} = 8.2987.$

The probabilities are obtained from Figure 4 and the discount rate is obtained from Figure 6. Similar calculations are applied to each node of the tree, resulting in the value of put option as shown in Figure 7.

Figure 7
The 3-year put option tree



4.2 Pricing Interest Rate Derivatives Using a Trinomial Tree for ln(r)

4.2.1 Constructing a trinomial tree for ln(r)

To assure the positiveness of interest rates, we define x in Equation 4 as $x=ln(r)$, resulting in the Black and Karasinski (1991) model:

$$d\ln r = \left[\theta(t) - a\ln r\right]dt + \sigma dW \qquad (14)$$

Following Hull and White (1994), we use a=0.22, σ=0.25,Δt=0.5, and the initial yield curve is computed as $0.08–0.05e^{-0.18t}$. Applying these parameters and the procedure described in Section 4.1.1, we build the first stage tree as shown in Figure 8. Using Equations 5b-d, yields

$V = (0.25)^2\left(1-e^{-2(0.22)0.5}\right)/2(0.22) = 0.028051307,$

$\Delta x^* = \sqrt{3(0.028051307)} = 0.290092953,$

$M = e^{-0.22(0.5)} - 1 = -0.10416586.$

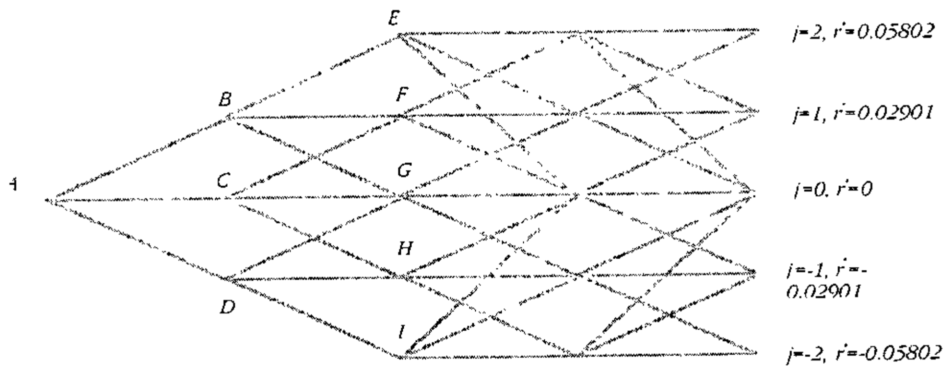
Next, we determine points where non-standard branching must be used. In this case, j_{max}=-0.184/-0.10416586=1.7664, and j_{max}=-j_{min}=2.

Since the first stage tree is written in terms of $ln(r)$, α_i is interpreted as the amount by which we need to "shift" our $ln(r)$ tree so that the tree is consistent with current zero-coupon bond prices. Consider, for example, a node one step above the centre of the first-stage tree. In this case j=1 and $x^*=1\Delta x^*$. This needs to be shifted by α_1 to obtain $ln(r)= \alpha_1+1\Delta x^*$. Next, we take exponents to recover the interest rate itself: $r=exp(\alpha_1+1\Delta x^*)$. The value of a one-period (Δt) bond at this interest rate is $exp(-exp(\alpha_1+1\Delta x^*)\Delta t)$. Summing over all possible nodes yields expressions analogous to Equation 9 and 10, but for a tree in terms of $ln(r)$ rather than r:

$$P(0,i+1)=\sum_{j=-n_i}^{n_i} Q_{i,j}e^{-g(\alpha_i+j\Delta x^*)\Delta t}, \quad (14a)$$

$$Q_{i+1,j}=\sum_k Q_{i,k}q(k,j)e^{-g(\alpha_i+k\Delta x^*)\Delta t}. \quad (14b)$$

Figure 8
The first stage tree (x*)



j	Node	r^*	p_u	p_m	p_d
0	***A,C,G***	0.00000	0.16667	0.66667	0.16667
1	***B,F***	0.02901	0.12001	0.65582	0.22417
-1	***D,H***	-0.02901	0.22417	0.65582	0.12001
2	***E***	0.05802	0.87587	0.03993	0.08420
-2	***I***	-0.05802	0.08420	0.03993	0.87587

where $g(x)=e^x$. Since Equation 14(a) cannot be easily rearranged to solve for α_1, we resort to numerical estimation. Figure 9 illustrates the state prices of securities that pay off \$1 at a particular node. We set the initial guess for $ln(r)$ to be zero (setting node a in Figure 3 to be zero), and numerically match the initial term structure. In this case, we use the first stage tree and the probabilities in Figure 8, together with $\alpha_0=0$ and $Q_{0,0}=1$, and apply Equation 14b to determine the state prices of securities that pay off \$1 at nodes *B*, *C*, and *D* in Figure 8 as follows:

Node B: $Q_{1,1}=0.166667e^{-e^{(\alpha_0+0)0.5}}$,

Node C: $Q_{1,0}=0.666667e^{-e^{(\alpha_0-0)0.5}}$,

Node D: $Q_{1,-1}=0.166667e^{-e^{(\alpha_0+0)0.5}}$.

Next, to determine the interest rate at nodes *B*, *C*, and *D* in Figure 8, we use Equation 14a:

$$e^{-0.5\alpha_0}=Q_{1,1}e^{-e^{(\alpha_0+0)0.5}}+Q_{1,0}e^{-e^{(\alpha_0+0)0.5}}+Q_{1,-1}e^{-e^{(\alpha_0+0)0.5}}$$

Figure 9
The state price tree (Q)

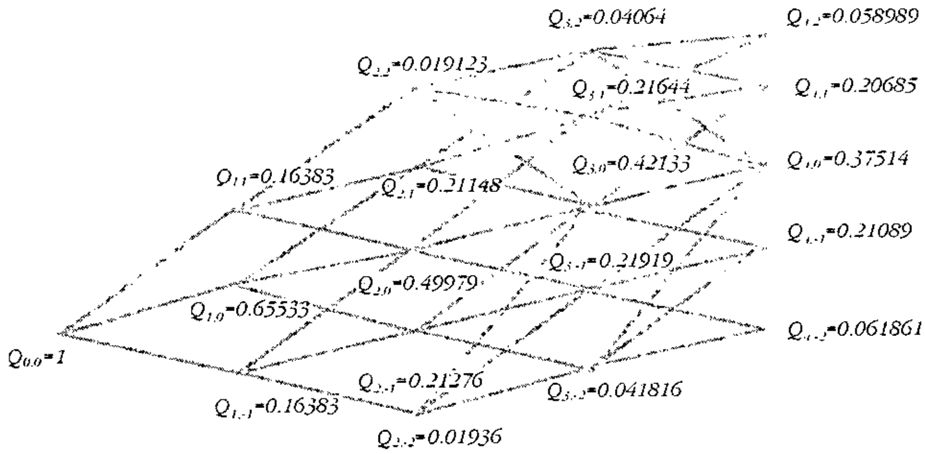


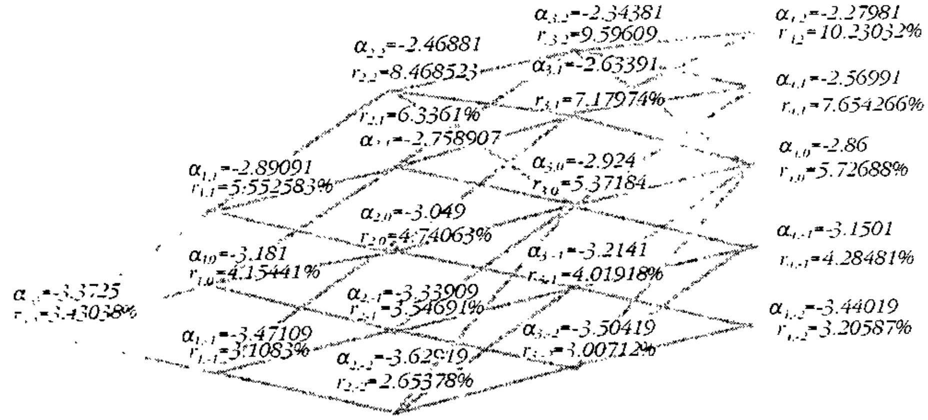
Table 3
Zero-coupon rate and discount bond prices.

Maturity	Days	Zero Rate	Discount Bond Price
1 year	367	3.823648943	0.962485296
2 year	731	4.511618370	0.913718842
3 year	1096	5.086258738	0.858483548
4 year	1461	5.566238720	0.800395300
5 year	1826	5.967151701	0.742035951
6 year	2194	6.302022372	0.685147358
7 year	2558	6.581729868	0.630828597
8 year	2922	6.815361207	0.579709075
9 year	3287	7.010506505	0.532088428
10 year	3653	7.173505559	0.488043589

Note: The zero rate is calculated as $0.08-0.05e^{-0.18t}$, as in Hull and White (1994). The discount bond price is calculated as $P(t)=e^{-r_t(t)}$ where $P(t)$ is the price of a discount bond that matures time t from now and r_t is the appropriate zero-rate.

We numerically change α_0 until this equation is satisfied. Finally, we repeat the whole process to solve for nodes at the next time step. This results in the state price tree *(Q)* in Figure 9, and the *ln(r)* tree (α) in Figure 10. The last step is to convert the *ln(r)* tree back to the *r* tree by taking the inverse function of the natural log at every node of the *r* tree. This results in the r tree in Figure 10. Our results are consistent with those in Hull and White (1994). We then extend this result to price a cap, a floor, a collar, and a swap.

Figure 10
The second stage tree (r)



4.2.2 Pricing an interest rate cap

In this section, we price a 2.5-year European interest rate cap with strike 7.5%, reset every 6 months. The cap holder will receive a positive payoff whenever the interest rate is higher than the strike price on a reset date. Any payment due will be made on the following reset date. Thus, if the interest rate is above the strike price today, this will trigger a payoff to be received on the next reset date. There is no payment made on the first reset date because that would be based on the interest rate now, which is already known for certain, and the strike is almost always set above the current spot rate so that this first potential payment is out of the money.

By paying an option premium, the cap holder effectively locks in a maximum borrowing rate of 7.5%. Thus, this cap is really a string of four separate options (known as "caplets") that may generate payoffs depending on the level of the interest rate 6, 12, 18, and 24 months from now. The respective payoffs will actually be received 12, 18, 24, and 30 months from now. The appropriate premium is the sum of the values of each caplet. For every reset date (every 6 months), we determine the expected payoff by comparing the interest rate from Figure 10 with the strike price or cap rate, and discounting it back one-period to the particular reset date. The caplet payoff, discounted to the reset date on which it is determined, is:

$$v = notional \times \max(r - strike, 0) \times \Delta t \times e^{-r\Delta t} \qquad (15)$$

Figure 11
Valuing of a 2-year caplet

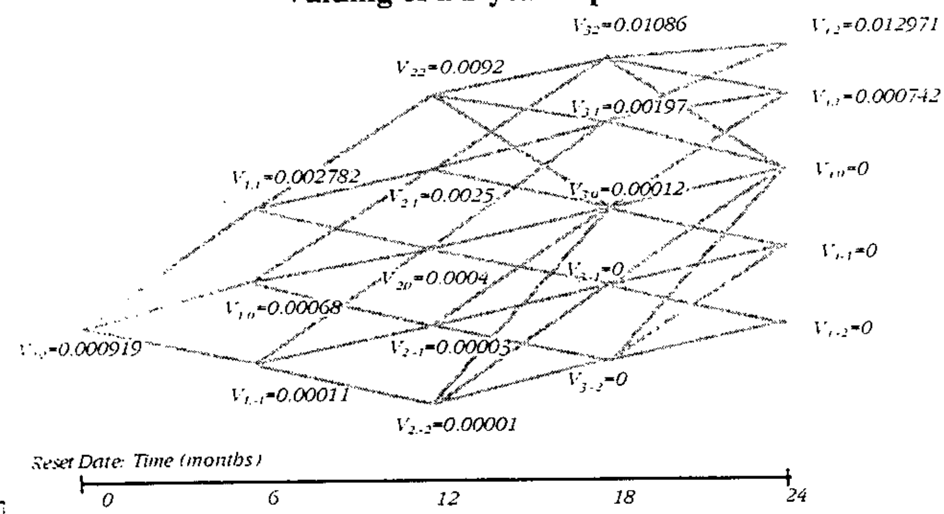


Figure 11 shows how to value a single caplet maturing two years from now assuming a $1 notional amount. At time t=24 months, the caplet is in-the-money only when the interest rate reaches either nodes *(4,2)* or *(4,1)*. That is, the caplet holder will exercise the option if the interest rate reaches one of these two nodes and obtain the following payoffs:

$v_{4,2}$=1(10.23032%–7.5%)0.5$e^{-10.23032\%(0.5)}$=0.012971,

$v_{4,1}$=1(7.654266%–7.5%)0.5$e^{-7.654266\%(0.5)}$=0.000742.

In both cases, the final term discounts the payoff back six months, reflecting the fact that the payment will be received six months after the reset date on which it is determined.

We then discount these expected payoffs back through the interest rate tree in Figure 10 using the appropriate probabilities from Figure 8. For example, the value of the 2-year caplet at time t=18 months (at the top node *(3,2)*) is:

$$v_{3,2} = \left[p_u v_{4,2} + p_m v_{4,1} + p_d v_{4,0}\right]e^{-r_{3,2}\Delta t},$$

$$v_{3,2} = \left[0.87587(0.012971) + 0.039928(0.000742) + 0.084202(0)\right]e^{-9.596099\%(0.5)} = 0.01086.$$

Similar calculations are applied to each node of the tree, resulting in the value of a 2-year caplet of 0.000919 as shown in Figure 11. To find the total value of a 2.5-year cap, we value each caplet according to the above procedure, and sum the values of each caplet. In this case, a 2.5-year cap is worth 0.001413468 as indicated in Table 4. That is, for $14,135 a borrower could lock in a maximum rate of 7.5% on a 2.5-year $10 million loan.

Table 4
The values of each caplet and a cap

Maturity (Reset Date)	Value of a caplet
6 months	0.000000000
12 months	0.000088766
18 months	0.000406000
24 months	0.000918702
Cap value	0.001413468

Note: The payment on each caplet occurs one-period after the reset date. For example, the payment on a 6-month caplet occurs at 12 months; the payment on a 12-month caplet at 18 months, and so on.

4.2.3 Pricing an interest rate floor

In this section, we price a 2.5-year European interest rate floor with strike 4.5%, reset every 6 months. This instrument is similar to an interest rate cap, except that a floor locks in a lower bound for the interest rate. The floor holder will receive a positive payoff whenever the strike rate is higher than the interest rate on a reset date. By paying an option premium, the floor holder effectively guarantees a minimum lending rate of 4.5%. A floor is really a string of four separate options (known as "floorlets") exerciseable 6, 12, 18, and 24 months from now. As for a cap, the payoffs will be received on the subsequent reset date. The appropriate premium is the sum of the values of each floorlet. For every reset date (every 6 months), we determine the expected payoff by comparing the interest rate from Figure 10 with the strike price or the floor rate and discounting it back one-period to the reset date. The floorlet payoff, discounted to the reset date on which it is determined, is:

$$v = notional \times \max(strike - r, 0) \times \Delta t \times e^{-r\Delta t} \qquad (16)$$

Figure 12
Valuing of a 2-year floorlet

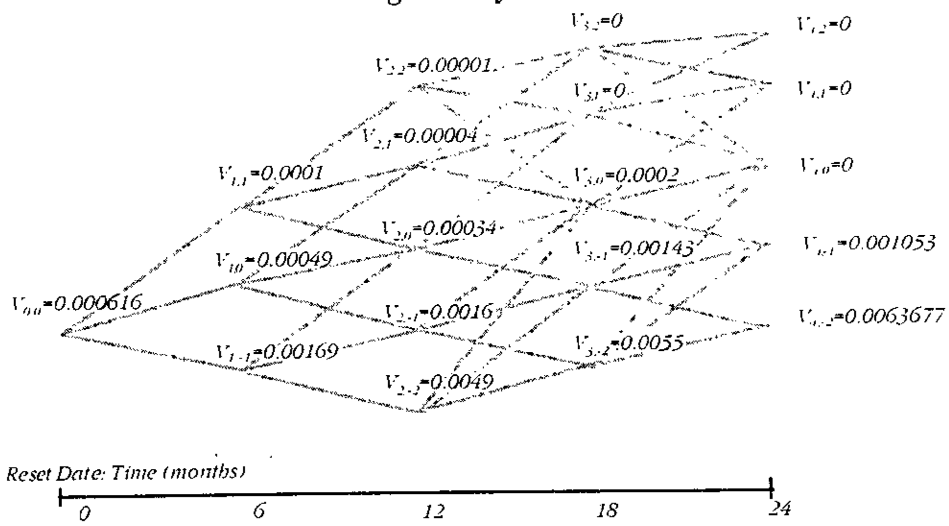


Figure 12 shows how to value a single floorlet maturing two years from now, assuming a $1 notional amount. At time t=24 months, the floorlet is in-the-money only when the interest rate reaches either nodes *(4,-1)* or *(4,-2)*. That is, the floorlet holder will exercise the option if the interest rate reaches one of these two nodes and obtain the following payoffs:

$$v_{4,-2} = 1(4.5\% - 3.20587\%)0.5e^{-3.20587\%(0.5)} = 0.0063677,$$

$$v_{4,-1} = 1(4.5\% - 4.28481\%)0.5e^{-4.28481\%(0.5)} = 0.001053.$$

We then discount these expected payoffs back through the interest rate tree in Figure 10 using the appropriate probabilities from Figure 8. For example, the value of the 2-year floorlet at time t=18 months (at the bottom node *(3,-2)*) is:

$$v_{3,-2} = \left[p_u v_{4,0} + p_m v_{4,1} + p_d v_{4,-2}\right]e^{-r_{3,-2}\Delta t},$$

$$v_{3,-2} = \left[0.084202(0) + 0.039928(0.001053) + 0.87587(0.0063677)\right]e^{-3.00712\%(0.5)} = 0.0055.$$

Similar calculations are applied to each node of the tree, resulting in the value of a 2-year floorlet of 0.000616 as shown in Figure 12. To find the total value of a 2.5-year floor, we value each floorlet according to the above procedure, and sum the values of each floorlet. In this case, a 2.5-year floor is worth 0.004843933 as indicated in Table 5. That is, for $48,439 a lender could lock in a minimum rate of 4.5% on a 2.5-year $10 million investment.

Table 5
The values of each floorlet and a floor

Maturity (Reset Date)	Value of a floorlet
6 months	0.002231544
12 months	0.001172430
18 months	0.000823950
24 months	0.000616009
Floor value	0.004843933

Note: The payment on each floorlet occurs one-period after the reset date. For example, the payment on a 6-month floorlet occurs at 12 months; the payment on a 12-month floorlet at 18 months, and so on.

4.2.4 Pricing an interest rate collar

In this section, we price a 2.5-year semiannual European collar with lower and upper bounds of 4.5% and 7.5% respectively. A collar can be viewed as a combination of a long cap and a short floor. A collar holder will receive a positive payoff to protect against high interest rates and sacrifice the benefits of low interest rates. When a borrower utilizes a collar, he or she is guaranteed of paying an interest rate between the lower (4.5%) and upper (7.5%) strikes. A collar is really a string of four caplets that the holder may exercise if interest rates are high, plus a string of four floorlets that might be exercised against the holder if interest rates are low. The value of the collar is the net value of a long cap and a short floor. Typically, the payment on a collar will be made one-period after the reset date. For every reset date (every 6 months), we determine the expected payoff by comparing the interest rate from Figure 10 with the two strike prices, and discounting them back one-period to the reset date. The payoff is defined as:

$$v = notional * \begin{cases} [r - upper\,strike]\Delta t e^{-r\Delta t} & if \quad r > upper\,strike \\ [r - lower\,strike]\Delta t e^{-r\Delta t} & if \quad r < lower\,strike \end{cases} \quad (17)$$

Figure 13
The expected payoffs of a collar at reset date t= 24 months and its present value

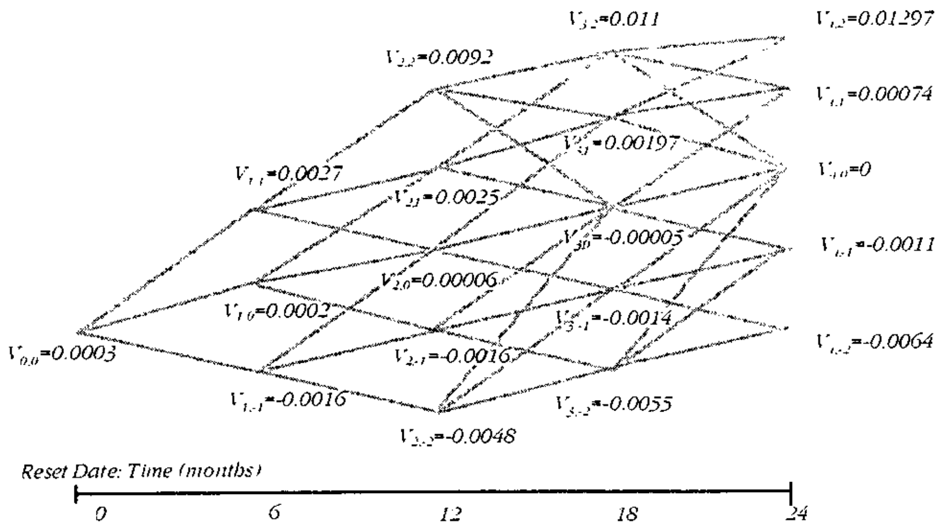


Figure 13 shows how to value the collar payoff that may arise from interest rates two years from now. At time t=24 months, the cap is in-the-money at either nodes *(4,2)*, or *(4,1)* whereas the floor is in-the-money at either node *(4,-2)*; or *(4,-1)*. That is, the collar holder will exercise the cap if the interest rate reaches either node *(4,2)* or *(4,1)*. When the interest rate reaches either node *(4,-2)* or *(4,-1)*, the floor will be exercised against the collar holder. The possible payoffs at the reset date t=24 months are:

$$v_{4,2} = 1(10.23032\% - 7.5\%)0.5e^{-10.23032\%(0.5)} = 0.01297,$$

$$v_{4,1} = 1(7.654266\% - 7.5\%)0.5e^{-7.654266\%(0.5)} = 0.00074,$$

$$v_{4,-1} = 1(4.28481\% - 4.5\%)0.5e^{-4.28481\%(0.5)} = -0.0011,$$

$$v_{4,-2} = 1(3.20587\% - 4.5\%)0.5e^{-3.20587\%(0.5)} = -0.0064.$$

Note that we have reflected the fact that the payoff will not be received for a further six months. We then discount these possible payoffs back through the interest rate tree in Figure 10 using the appropriate probabilities from Figure 8. For example, at time t=18 months, the value of the expected payoffs of a collar at reset date t=24 months (at node, (3,0)) is:

$$v_{3,0} = \left[p_u v_{4,1} + p_m v_{4,0} + p_d v_{4,-1}\right]e^{-r_{3,0}\Delta t},$$

$$v_{3,0} = \left[0.16667(0.00074) + 0.6667(0) + 0.16667(-0.0011)\right]e^{-5.371847\%(0.5)} = 0.00005.$$

Similar calculations are applied to each node of the tree, resulting in the present value of the expected payoffs of a collar at reset date t=24 months of 0.0003 as shown in Figure 13. The total value of a 2-year collar is the sum of the present values of the expected payoffs at each reset date, which are calculated using the above procedure. In this case, a 2.5-year collar is worth -0.003430465 as indicated in Table 6.

Table 6
The values of a collar at each reset date

Reset Date	Value of a collar
6 months	-0.002231544
12 months	-0.001083664
18 months	-0.000417950
24 months	0.000302693
Collar value	**-0.003430465**

Note that this collar value can also be constructed from the value of a long cap and a short floor: 0.001413468-0.004843933=-0.003430465. The reason that this collar has a negative value is that it is more likely that the floor will be exercised against the holder than the holder will exercise the cap. This is because the current interest rate is 3.43%. Interest rates below 4.5% are therefore more likely than interest rates above 7.5% over the next two years.

4.2.5 Valuing an interest rate swap

In this section, we value an existing 2.5-year swap paying fixed at 4.5% semiannually. A swap is really a string of five payments made on 6, 12, 18, 24, and 30 months. As for caps, floors, and collars, swap payments are made in arrears - on the following reset date. Unlike these other instruments, however, a payment is made on the first reset date and this is based on the current interest rate, which is already known. The first payment is therefore made after 6 months, rather than 12 months as in the case of the instruments discussed above. For every reset date (every 6 months), we determine the swap payoff by comparing the interest rate from Figure 10 with the strike price and then discounting it back one-period to the reset date on which it is determined. The payoff towards the fixed-rate payer is:

$$v = notional * (r - strike) * \Delta t * e^{-r\Delta t} \quad (18)$$

Figure 14
Valuing a swap payment

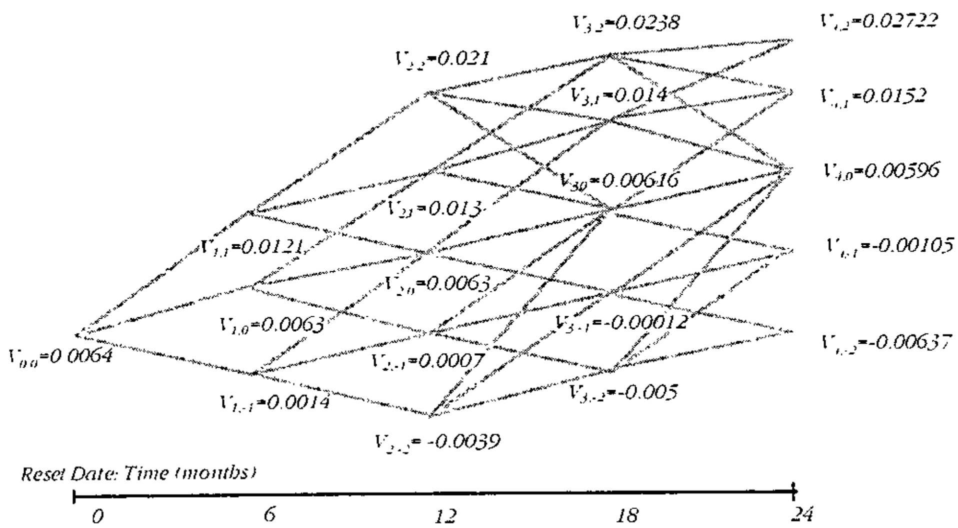


Figure 14 shows how to find the expected payoff towards a fixed-rate payer at reset date t = 24 months and its present value. That is, the fixed-rate payer incurs a positive payoff when the interest rate reaches either node *(4,2)*, *(4,1)*, or *(4,0)*, and a negative payoff when the interest rate reaches either node *(4,-2)*, or *(4,-1)*. . The expected payoff at reset date *t=24* months (discounted back to reflect the fact that it will not be received for a further six months) are:

$$v_{4,2} = 1(10.23032\% - 4.5\%)0.5e^{-10.23032\%(0.5)} = 0.02722,$$
$$v_{4,1} = 1(7.654266\% - 4.5\%)0.5e^{-7.654266\%(0.5)} = 0.0152,$$
$$v_{4,0} = 1(5.72688\% - 4.5\%)0.5e^{-5.72688\%(0.5)} = 0.00596,$$
$$v_{4,-1} = 1(4.28481\% - 4.5\%)0.5e^{-4.28481\%(0.5)} = -0.00105,$$
$$v_{4,-2} = 1(3.20587\% - 4.5\%)0.5e^{-3.20587\%(0.5)} = -0.00637.$$

We then discount these expected payoffs back through the interest rate tree in Figure 10 using the appropriate probabilities from Figure 8. For example, at time t=18 months, the value of the expected payoffs toward the fixed rate payer at reset date t=24 months (at node, (3,0) *3.0*) is:

$$v_{3,0} = \left[p_u v_{4,1} + p_m v_{4,0} + p_d v_{4,-1}\right]e^{-r_{3,0}\Delta t},$$

$$v_{3,0} = [0.16667(0.0152) + 0.6667(0.00596) + 0.16667(-0.00105)]e^{-5.371849\%(0.5)} = 0.00616.$$

Similar calculations are applied to each node of the tree, resulting in the present value of the expected payoffs towards the fixed-rate payer at reset date t=24 months of 0.0064 as shown in Figure 14.

The total value of a 2.5-year swap is the sum of the present value of the expected payoffs at each reset date, which are calculated using the above procedure. In this case, a 2.5-year swap is worth 0.006124221 towards the fixed-rate payer as indicated in Table 7. That is, if the swap were based on $10 million notional, the fixed-rate payer would show this swap as an asset worth $61,242. Note that this swap becomes an asset towards the fixed-rate payer because the current interest rate is higher than the rate when this swap was initiated. Typically when two counter-parties enter a swap contract the present value is zero - the swap rate is set so that the present value of the fixed payments equals the expected present value of the floating payments. As interest rates move over time, the value of the swap will move in favor of one party or the other.

Table 7
The values of the expected payoffs of a swap at each reset date

Reset Date	Value of expected payoffs
0 months	-0.005257167
6 months	-0.001392918
12 months	0.001659455
18 months	0.004748910
24 months	0.006365940
Swap value	**0.006124221**

3.5 Summary

In this paper, we review a range of no-arbitrage term structure models and methodologies employed to determine the future path of interest rates. We demonstrate how to model the evolution of interest rates using the trinomial tree methodology proposed by Hull and White (1994, 1996). Applying the Hull and White trinomial tree technique, we value a number of interest rate derivatives: options, caps, floors, collars, and plain-vanilla interest rate swaps. We also provide a number of detailed numerical examples for expositional purposes.

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