1 Exam 3 - Stochastics Solutions

1. *Show that the process defined by*

$$Y(t) = \sinh(C + t + X(t))$$

where $X(t)$ *is a Brownian motion,* $\sinh$ *is the hyperbolic sine function and* $C = \sinh^{-1}(Y_0)$ *is a solution of the stochastic differential equation*

$$dY(t) = \left(\sqrt{1 + Y(t)^2} + \frac{1}{2}Y(t)\right) dt + \left(\sqrt{1 + Y(t)^2}\right) dX(t)$$

with initial condition $Y(0) = Y_0$

Hint: *Use the Itô formula with* $F(t, z) = \sinh(t + z)$.

Let $F(t, z) = \sinh(t + z)$, then

$$\begin{aligned} \frac{\partial F}{\partial t}(t, z) &= \cosh(t + z) \\ \frac{\partial F}{\partial z}(t, z) &= \cosh(t + z) \\ \frac{\partial^2 F}{\partial z^2}(t, z) &= \sinh(t + z) \end{aligned}$$

Applying Itô's formula to $F(t, C + X(t))$,

$$\begin{aligned} dY(t) &= dF(t, C + X(t)) \\ &= \left(\frac{\partial F}{\partial t}(t, C + X(t)) + \frac{1}{2}\frac{\partial^2 F}{\partial z^2}(t, C + X(t))\right) dt + \frac{\partial F}{\partial z}(t, C + X(t))dX(t) \\ &= \left(\cosh(t + C + X(t)) + \frac{1}{2}\sinh(t + C + X(t))\right) dt + \cosh(t + C + X(t))dX(t) \end{aligned}$$

Since $\cosh(z) = \sqrt{1 + \sinh^2(z)}$, then

$$dY(t) = \left(\sqrt{1 + \sinh^2(t + C + X(t))} + \frac{1}{2}\sinh(t + C + X(t))\right) dt + \sqrt{1 + \sinh^2(t + C + X(t))}dX(t)$$

Substituting $Y(t) = \sinh(C + t + X(t))$ in the equation, we can conclude that

$$dY(t) = \left(\sqrt{1 + Y(t)^2} + \frac{1}{2}Y(t)\right) dt + \left(\sqrt{1 + Y(t)^2}\right) dX(t)$$

2. i. *Let* ξ_n *be a sequence of* $\mathcal{F}_n$*-adapted random variables. Show that if* ξ_n *is a martingale with respect to* $\mathcal{F}_n$*, then*

$$\mathbf{E}[\xi_1] = \mathbf{E}[\xi_1] = \ldots = \mathbf{E}[\xi_n] = \ldots$$

Hint: *go back to the definition of martingales and consider the* $\mathbf{E}[\xi_{n+1}|\mathcal{F}_n]$

By definition of martingales,

$$\mathbf{E}[\xi_{n+1}|\mathcal{F}_n] = \xi_n$$

for all n.
Taking the (unconditional) expectation on both sides and applying the third property of conditional expectations,

$$\mathbf{E}[\xi_n] = \mathbf{E}\left[\mathbf{E}[\xi_{n+1}|\mathcal{F}_n]\right] = \mathbf{E}[\xi_{n+1}]$$

for all n. Hence

$$\mathbf{E}[\xi_1] = \mathbf{E}[\xi_2] = \ldots = \mathbf{E}[\xi_n] = \ldots$$

as required.

ii. *Let* ξ_n *be a sequence of* $\mathcal{F}_n$*-adapted random variables such that* $\mathbf{E}[|\xi_n^2|] < \infty$*. Show that if* ξ_n *is a martingale with respect to* $\mathcal{F}_n$*, then* ξ_n^2 *is a* ***sub****martingale with respect to* $\mathcal{F}_n$*.*
Hint: *use Jensen's inequality with* $f(x) = x^2$*.*

If ξ_n is adapted to $\mathcal{F}_n$, then so is ξ_n^2. By definition of martingales,

$$\mathbf{E}[\xi_{n+1}|\mathcal{F}_n] = \xi_n$$

Let f be a convex function, then by Jensen's inequality,

$$f(\xi_n) = f\left(\mathbf{E}[\xi_{n+1}|\mathcal{F}_n]\right) \leq \mathbf{E}[f\left(\xi_{n+1}\right)|\mathcal{F}_n]$$

for all n.

In particular, if $f(x) = x^2$, then

$$\xi_n^2 = \left(\mathbf{E}[\xi_{n+1}|\mathcal{F}_n]\right)^2 \leq \mathbf{E}[\xi_{n+1}^2|\mathcal{F}_n]$$

for all n. This is the definition of submartingales and therefore ξ_n^2 is indeed submartingale with respect to $\mathcal{F}_n$