Chapter 2

Option Pricing in Continuous-Time: The Black–Scholes–Merton Theory and Its Extensions

This chapter is organized as follows:

1. Section 2 provides an overview of the option pricing theory in the pre-Black–Scholes period.
2. Section 3 develops the foundations of the Black–Scholes–Merton Theory.
3. Section 4 reviews the main results in Black's (1976) model for the pricing of derivative assets when the underlying asset is traded on a forward or a futures market.
4. Section 5 develops the main results in Garman and Kohlhagen's (1983) model for the pricing of currency options.
5. Section 6 presents the main results in the models of Merton (1973) and Barone-Adesi and Whaley (1987) model for the pricing of European commodity and commodity futures options.
6. Section 7 develops option price sensitivities.
7. Section 8 presents Ito's lemma and some of its applications.
8. Section 9 develops Taylor series, Ito's theorem and the replication argument.
9. Section 10 derives the differential equation for a derivative security on a spot asset in the presence of a continuous dividend yield and information costs.
10. Section 11 develops a general context for the valuation of securities dependent on several variables in the presence of incomplete information.
11. Section 12 presents the general differential equation for the pricing of derivatives.

12. Section 13 extends the risk-neutral argument in the presence of information costs.
13. Section 14 extends the analysis to commodity futures prices within incomplete information.
14. Appendix 1 provides the risk measures in analytical models.
15. Appendix 2 gives the relationship between hedging parameters.
16. Appendix 3 introduces the valuation of options within information uncertainty.
17. Appendix 4 develops a general equation for the pricing of derivative securities.
18. Appendix 5 extends the risk-neutral valuation argument in the valuation of derivatives.
19. Appendix 6 provides an approximation of the cumulative normal distribution function.
20. Appendix 7 gives an approximation of the bivariate normal density function.

1. Introduction

The previous chapter presents the main concepts regarding option strategies, asset pricing and derivatives in a discrete-time framework. This chapter extends the valuation of derivatives to a continuous-time setting. Numerous researchers have worked on building a theory of rational option pricing and a general theory of contingent claims valuation. The story began in 1900, when the French Mathematician, Louis Bachelier, obtained an option pricing formula. His model is based on the assumption that stock prices follow a Brownian motion. Since then, numerous studies on option valuation have blossomed. The proposed formulas involve one or more arbitrary parameters. They were developed by Sprenkle (1961), Ayres (1963), Boness (1964), Samuelson (1965), Thorp and Kassouf (1967), Samuelson and Merton (1969) and Chen (1970) among others. The Black and Scholes (1973) formulation, hereafter B–S, solved a problem which has occupied economists for at least three-quarters of a century. This formulation represented a significant breakthrough in attacking the option pricing problem. In fact, the Black–Scholes theory is attractive since it delivers a closed-form solution to the pricing of European options.

Assuming that the option is a function of a single source of uncertainty, namely the underlying asset price, and using a portfolio which combines options and the underlying asset, Black–Scholes constructed a riskless hedge which allowed them to derive an analytical formula. This model provides a no arbitrage value for European options on shares. It is a function of the share price S, the strike price K, the time to maturity T, the risk free interest rate r and the volatility of the stock price, σ. This model involves only observable variables to the exception of volatility and it has become the benchmark for traders and market makers. It also contributed to the rapid growth of the options markets by making a brand new pricing technology available to market players.

About the same time, the necessary conditions to be satisfied by any rational option pricing theory were summarized in Merton's (1973) theorems. The post-Black–Scholes period has seen many theoretical developments. The contributions of many financial economists to the extensions and generalizations of Black–Scholes type models has enriched our understanding of derivative assets and their seemingly endless applications.

The first specific option pricing model for the valuation of futures options is introduced by Black (1976). Black (1976) derived the formula for futures and forward contracts and options under the assumption that investors create riskless hedges between options and the futures or forward contracts. The formula relies implicitly on the CAPM. Futures markets are not different in principal from the market for any other asset. The returns on any risky asset are governed by the asset contribution to the risk of a well diversified portfolio. The classic CAPM is applied by Dusak (1973) in the analysis of the risk premium and the valuation of futures contracts. Black (1976) model is used in Barone-Adesi and Whaley (1987) for the valuation of American commodity options. This model is referred to as the BAW (1987) model. It is helpful, as in Smithson (1991), to consider the Black–Scholes model within a family tree of option pricing models. This allows the identification of three major tribes within the family of option pricing models: analytical models, analytic approximations and numerical models.

Each analytical tribe can be divided into three distinct lineages, precursors to the Black–Scholes model, extensions of the Black–Scholes model and generalisations of the Black–Scholes model.This chapter presents in detail the basic theory of rational option pricing of European options in different contexts.

Starting with the analysis of the option pricing theory in the pre-Black–Scholes period, we develop the basic theory and its extensions in a continuous time framework. We develop option price sensitivities, Ito's lemma, Taylor series and Ito's theorem and the replication argument. We also derive the differential equation for a derivative security on a spot asset in the presence of a continuous dividend yield and information costs, a general context for the valuation of securities dependent on several variables in the presence of incomplete information. Finally, we present the general differential equation for the pricing of derivatives, extend the risk-neutral argument and the analysis to commodity futures prices within incomplete information. For the analysis of information costs and valuation, we can refer to Bellalah (2001), Bellalah, Prigent and Villa (2001), Bellalah M, Bellalah Ma and Portait (2001), Bellalah and Prigent (2001), Bellalah and Selmi (2001), etc.

2. Precursors to the Black–Scholes Model

The story begins in 1900 with a doctoral dissertation at the Sorbonne in Paris, France, in which Louis Bachelier gave an analytical valuation formula for options.

2.1. *Bachelier Formula*

Using an arithmetic Brownian motion for the dynamics of share prices and a normal distribution for share returns, he obtained the following formula for the valuation of a European call option on a non-dividend paying stock:

$$c(S,T) = SN\left(\frac{S-K}{\sigma\sqrt{T}}\right) - KN\left(\frac{S-K}{\sigma\sqrt{T}}\right) + \sigma\sqrt{T}n\left(\frac{K-S}{\sigma\sqrt{T}}\right)$$

where:

S: underlying common stock price,
K: option's strike price,
T: option's time to maturity,
σ: instantaneous standard deviation of return,
$N(.)$: cumulative normal density function, and
$n(.)$: density function of the normal distribution.

As pointed out by Merton (1973) and Smith (1976), this formulation allows for both negative security and option prices and does not account for the time value of money.

Sprenkle (1961) reformulated the option pricing problem by assuming that the dynamics of stock prices are log-normally distributed. By introducing a drift in the random walk, he ruled out negative security prices and allowed risk aversion. By letting asset prices have multiplicative, rather than additive fluctuations, the distribution of the option's underlying asset at maturity is log-normal rather than normal.

2.2. Sprenkle Formula

Sprenkle (1961) derived the following formula:

$$c(S, T) = Se^{\rho T} N(d_1) - (1 - Z) K N(d_2)$$

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + \left(\rho + \frac{\sigma^2}{2}\right) T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln\left(\frac{S}{K}\right) + \left(\rho - \frac{\sigma^2}{2}\right) T}{\sigma\sqrt{T}}$$

where ρ is the average rate of growth of the share price and Z corresponds to the degree of risk aversion. As it appears in this formula, the parameters corresponding to the average rate of growth of the share price and the degree of risk aversion must be estimated. This reduces considerably the use of this formula. Sprenkle (1961) tries to estimate the values of these parameters, but he was unable to do that.

2.3. Boness Formula

Boness (1964) presented an option pricing formula accounting for the time value of money through the discounting of the terminal stock price using the expected rate of return to the stock. The option pricing formula proposed is

$$c(S, T) = SN(d_1) - e^{-\rho T} K N(d_2)$$

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + \left(\rho + \frac{\sigma^2}{2}\right) T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln\left(\frac{S}{K}\right) + \left(\rho - \frac{\sigma^2}{2}\right) T}{\sigma\sqrt{T}}$$

where ρ is the expected rate of return to the stock.

2.4. Samuelson Formula

Samuelson (1965) allowed the option to have a different level of risk from the stock. Defining ρ as the average rate of growth of the share price and w

as the average rate of growth of the call's value, he proposed the following formula:

$$c(S, T) = Se^{(\rho-w)T}N(d_1) - e^{-wT}KN(d_2)$$

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + \left(\rho + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln\left(\frac{S}{K}\right) + \left(\rho - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}.$$

Note that all the proposed formulas show one or more arbitrary parameters, depending on the investors preferences toward risk or the rate of return on the stock.

Samuelson and Merton (1969) proposed a theory of option valuation by treating the option price as a function of the stock price. They advanced the theory by realizing that the discount rate must be determined in part by the requirement that investors hold all the amounts of stocks and the option. Their final formula depends on the utility function assumed for a "typical" investor.

2.5. *The Black–Scholes–Merton Theory*

In this theory, the main intuition behind the risk-free hedge is simple. Consider an *at-the-money* European call giving the right to its holder to buy one unit of the underlying asset in one month at a strike price of \$100. Assume that the final asset price is either 105 or 95. An investor selling a call on the unit of the asset will receive either 5 or 0. In this context, selling two calls against each unit of the asset will create a terminal portfolio value of 95. The certain terminal value of this portfolio must be equal today to the discounted value of 95 at the riskless interest rate. If this rate is 1%, the present value is (95/1.01) .

The current option value is (100 – (95/1.01))/2. If the observed market price is above (or below) the theoretical price, it is possible to implement anarbitrage strategy by selling the call and buying (selling) a portfolio comprising a long position in a half unit of the asset and a short position in risk-free bonds. The Black–Scholes–Merton model is the continuous-time version of this example. The theory assumes that the underlying asset follows a geometric Brownian motion and is based on the construction of a risk-free hedge between the option and its underlying asset. This implies that the call pay-out can be duplicated by a portfolio consisting of the asset and risk-free bonds. In this theory, the option value is the same for a risk-neutral investor and a risk-averse investor. Hence, options can be valued in a risk-neutral world, i.e. expected returns on all assets are equal to the risk-free rate of interest.

3. The Black–Scholes Model

Under the following assumptions, the value of the option will depend only on the price of the underlying asset S, time t and on other variables assumed constants. These assumptions or "ideal conditions" as expressed by Black–Scholes are the following:

— The option is European.
— The short term interest rate is known.
— The underlying asset follows a random walk with a variance rate proportional to the stock price. It pays no dividends or other distributions.
— There is no transaction costs and short selling is allowed, i.e. an investor can sell a security that he does not own.
— Trading takes place continuously and the standard form of the capital market model holds at each instant.

The main attractions of the Black–Scholes model are that their formula is a function of "observable" variables and that the model can be extended to the pricing of any type of option.

3.1. *The Black–Scholes Model and the Capital Asset Pricing Model, CAPM*

The capital asset pricing model of Sharpe (1964) can be stated as follows:

$$\bar{R}_S - r = \beta_S[\bar{R}_m - r]$$

where:

— $\bar{R}_S$: equilibrium expected return on security S,
— $\bar{R}_m$: equilibrium expected return on the market portfolio,
— r: 1 + the riskless rate of interest,

— $\beta_S = \frac{\text{cov}(\tilde{R}_S/\tilde{R}_m)}{\text{var}(\tilde{R}_m)}$: the beta of security S, that is the covariance of the return on that security with the return on the market portfolio, divided by the variance of market return.

The model gives a general method for discounting future cash flows under uncertainty. The model is referred to the previous chapter for more details about this model.

Denote by $C(S, t)$ the value of the option as a function of the underlying asset and time. To derive their valuation formula, B–S assumed that the hedged position was continuously rebalanced in order to remain riskless. They found that the price of a European call or put must verify a certain differential equation, which is based on the assumption that the price of the underlying asset follows a geometric Wiener process

$$\frac{\Delta S}{S} = \alpha\, \mathrm{d}t + \sigma \Delta z$$

where α and σ refer to the instantaneous rate of return and the standard deviation of the underlying asset, respectively, and z is a Brownian motion. The relationship between an option's beta and its underlying security's beta is

$$\beta_C = S\left(\frac{C_S}{C}\right)\beta_S$$

where:

— β_c: the option's beta,
— β_S: the stock's beta,
— C: the option value,
— C_S: the first derivative of the option with respect to its underlying asset. It is also the hedge ratio or the option's delta in a covered position.

According to the CAPM, the expected return on a security should be:

$$\bar{R}_S - r = \beta_S[\bar{R}_m - r]$$

where $\bar{R}_S$ is the expected return on the asset S and $\bar{R}_m$ is the expected return on the market portfolio. This equation may also be written as:

$$E\left(\frac{\Delta S}{S}\right) = [r + \beta_S(\bar{R}_m - r)]\Delta t.$$

Using the CAPM, the expected return on a call option should be:

$$E\left(\frac{\Delta C}{C}\right) = [r + \beta_C(\bar{R}_m - r)]\Delta t.$$

Multiplying the previous two equations by S and C gives

$$E(\Delta S) = [rS + S\beta_S(\bar{R}_m - r)]\Delta t$$
$$E(\Delta C) = [rC + C\beta_C(\bar{R}_m - r)]\Delta t.$$

When substituting for the option's elasticity β_c, the above equation becomes after transformation:

$$E(\Delta C) = [rC + SC_S\beta_S(\bar{R}_m - r)]\Delta t.$$

Assuming a hedged position is constructed and "continuously" rebalanced, and since ΔC is a continuous and differentiable function of two variables, it is possible to use Taylor series expansion to expand ΔC:

$$\Delta C = \frac{1}{2}C_{SS}(\Delta S)^2 + C_S\Delta S + C_t\Delta t.$$

This is just an extension of simple results to obtain Ito's lemma. Taking expectations of both sides of this equation and replacing ΔS, we obtain

$$E(\Delta C) = \frac{1}{2}\sigma^2 S^2 C_{SS}\Delta t + C_S E(\Delta S) + C_t\Delta t.$$

Replacing the expected value of ΔS from $E(\Delta S/S)$ gives:

$$E(\Delta C) = \frac{1}{2}\sigma^2 S^2 C_{SS}\Delta t + C_S[rS + S\beta_S(\bar{R}_m - r)]\Delta t + C_t\Delta t.$$

Combining $E(\Delta C)$ and this last equation and rearranging gives:

$$\frac{1}{2}\sigma^2 S^2 C_{SS} + rSC_S - rC + C_t = 0.$$

This partial differential equation corresponds to the Black–Scholes valuation equation.

Let T be the maturity date of the call and E be its strike price. The last equation subject to the following boundary condition at maturity

$$C(S, T) = S - K, \quad \text{if } S \geq K$$
$$C(S, T) = 0, \quad \text{if } S < K$$

is solved using standard methods for the price of a European call, which is found to be equal to

$$C(S, T) = SN(d_1) - Ke^{-rT}N(d_2)$$

with $d_1 = [\ln(\frac{S}{K}) + (r + \frac{1}{2}\sigma^2)T]/\sigma\sqrt{T}$, $d_2 = d_1 - \sigma\sqrt{T}$ and where $N(.)$ is the cumulative normal density function.

3.2. An Alternative Derivation of the Black–Scholes Model

Assuming that the option price is a function of the stock price and time to maturity, $c(S, t)$ and that over "short" time intervals, Δt, a hedged portfolio consisting of the option, the underlying asset and a riskless security can be formed, where portfolio weights are chosen to eliminate "market risk", Black–Scholes expressed the expected return on the option in terms of the option price function and its partial derivatives. In fact, following Black–Scholes, it is possible to create a hedged position consisting of a sale of $\frac{1}{[\partial c(S,t)/\partial S]}$ options against one share of stock long. If the stock price changes by a small amount ΔS, the option changes by an amount $[\partial c(S, t)/\partial S]\Delta S$. Hence, the change in value in the long position (the stock) is approximately offset by the change in $\frac{1}{[\partial c(S,t)/\partial S]}$ options.

This hedge can be maintained continuously so that the return on the hedged position becomes completely independent of the change in the underlying asset value, i.e. the return on the hedged position becomes certain.

The value of equity in a hedged position, containing a stock purchase and a sale of $\frac{1}{[\partial c(S,t)/\partial S]}$ options is $S - C(S, t)/\left[\frac{\partial c(S,t)}{\partial S}\right]$. Over a short interval Δt, the change in this position is

$$\Delta S - \frac{\Delta c(S, t)}{\left[\dfrac{\partial c(S, t)}{\partial S}\right]} \tag{1}$$

where $\Delta c(S, t)$ is given by $c(S + \Delta S, t + \Delta t) - c(S, t)$. Using stochastic calculus for $\Delta c(S, t)$ gives

$$\Delta c(S, t) = \frac{\partial c(S, t)}{\partial S}\Delta S + \frac{1}{2}\frac{\partial^2 c(S, t)}{\partial S^2}\sigma^2 S^2 \Delta t + \frac{\partial c(S, t)}{\partial t}\Delta t. \tag{2}$$

The change in the value of equity in the hedged position is found by substituting $\Delta c(S, t)$ from Eq. (2) into Eq. (1):

$$-\frac{\left(\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 c(S,t)}{\partial S^2} + \frac{\partial c(S,t)}{\partial t}\right)\Delta t}{\frac{\partial c(S,t)}{\partial S}}.$$

Since the return to the equity in the hedged position is certain, it must be equal to $r\Delta t$, where r stands for the short term interest rate. Hence, the change in the equity must be equal to the value of the equity times $r\Delta t$, or

$$-\frac{\left(\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 c(S,t)}{\partial S^2} + \frac{\partial c(S,t)}{\partial t}\right)\Delta t}{\frac{\partial c(S,t)}{\partial S}} = \left[S - \frac{c(S,t)}{\frac{\partial c(S,t)}{\partial S}}\right] r\Delta t.$$

Dropping the time and rearranging gives the Black–Scholes partial differential equation

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 c(S,t)}{\partial S^2} - rc(S,t) + \frac{\partial c(S,t)}{\partial t} + rS\frac{\partial c(S,t)}{\partial S} = 0.$$

This partial differential equation must be solved under the boundary condition expressing the call's value at maturity date: $c(S, t^*) = \max[0, S_{t^*} - K]$, where K is the option's strike price.

For the European put, the equation must be solved under the following maturity date condition:

$$P(S, t^*) = \max[0, K - S_{t^*}].$$

To solve this differential equation, under the call boundary condition, Black–Scholes make the following substitution:

$$c(S,t) = \mathrm{e}^{r(t-t^*)} y \left[\frac{2}{\sigma^2}\left(r - \frac{\sigma^2}{2}\right)\left(\ln\frac{S}{K} - \left(r - \frac{1}{2}\sigma^2\right)(t^* - t)\right),\right.$$
$$\left. - \frac{2(t^*-t)}{\sigma^2}\left(r - \frac{1}{2}\sigma^2\right)^2\right]. \tag{3}$$

Using this substitution, the differential equation becomes

$$\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial S^2}.$$

This differential equation is the heat transfer equation of physics. The boundary condition is rewritten as $y(u, 0) = 0$, if $u < 0$ otherwise,

$$y(u, 0) = K\left[\mathrm{e}^{\left(\frac{\frac{1}{2}u\sigma^2}{r-\frac{1}{2}\sigma^2}\right)} - 1\right].$$

The solution to this problem is the solution to the heat transfer equation given in Churchill (1963):

$$y(u, s) = \frac{1}{\sqrt{2\Pi}}\int_{\frac{-u}{\sqrt{2s}}}^{\infty} K\left[\mathrm{e}^{\left(\frac{\frac{1}{2}(u+q\sqrt{2s})\sigma^2}{r-\frac{1}{2}\sigma^2}\right)} - 1\right]\mathrm{e}^{\left(-\frac{q^2}{2}\right)} dq.$$

Substituting (4) gives the following solution for the European call price with $T = t^* - t$:

$$c(S, T) = SN(d_1) - K\mathrm{e}^{-rT}N(d_2)$$

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T}$$

where $N(.)$ is the cumulative normal density function given by $N(d) = \frac{1}{\sqrt{2\Pi}}\int_{-\infty}^{d} \mathrm{e}^{(-\frac{x^2}{2})}\mathrm{d}x$.

3.3. *The Put–Call Parity Relationship*

The put–call parity relationship can be derived as follows. Consider a portfolio A which comprises a call option with a maturity date t^* and a discount bond that pays K dollars at the option's maturity date. Consider also a portfolio B, with a put option and one share. The value of portfolio A at maturity is

$$\max[0, S_{t^*} - K] + K = \max[K, S_{t^*}].$$

The value of portfolio B at maturity is

$$\max[0, K - S_{t^*}] + S_{t^*} = \max[K, S_{t^*}].$$

Since both portfolios have the same value at maturity, they must have the same initial value at time t, otherwise arbitrage will be profitable. Therefore, the following put–call relationship must hold $c_t - p_t = S_t - K\mathrm{e}^{-r(t^*-t)}$, with $t^* - t = T$.

If this relationship does not hold, then arbitrage would be profitable. In fact, suppose for example, that

$$c_t - p_t > S_t - K\mathrm{e}^{-r(t^*-t)}.$$

At time t, the investor can construct a portfolio by buying the put and the underlying asset and selling the call. This strategy yields a result equal to $c_t - p_t - S_t$. If this amount is positive, it can be invested at the riskless rate until the maturity date t^*, otherwise it can be borrowed at the same rate for the same period.

At the option maturity date, the options will be *in-the-money* or *out-of-the-money* according to the position of the underlying asset S_{t^*} with respect to the strike price K.

If $S_{t^*} > K$, the call is worth its intrinsic value. Since the investor sold the call, he is assigned on that call. He will receive the strike price, delivers the stock and closes his position in the cash account. The put is worthless. Hence, the position is worth

$$K + \mathrm{e}^{r(t^*-t)}[c_t - p_t - S_t] > 0.$$

If $S_T < K$, the put is worth its intrinsic value. Since the investor is long the put, he exercises his option. He will receive upon exercise the strike price, delivers the stock and closes his position in the cash account. The call is worthless. Hence, the position is worth

$$K + \mathrm{e}^{r(t^*-t)}[c_t - p_t - S_t] > 0.$$

In both cases, the investor makes a profit without initial cashoutlay. This is a riskless arbitrage which must not exist in efficient markets. Therefore, the above put–call parity relationship must hold. Using this relationship, the European put option value is given by

$$p(S,T) = -SN(-d_1) + K\mathrm{e}^{-rT}N(-d_2)$$

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T}$$

where $N(.)$ is the cumulative normal density function given by $N(d) = \frac{1}{\sqrt{2\Pi}}\int_{-\infty}^{d}\mathrm{e}^{\left(-\frac{x^2}{2}\right)}\mathrm{d}x$.

We illustrate by the following examples the application of the Black–Scholes (1973) model for the determination of call and put prices.

Table 1: Simulations of Black and Scholes put prices.

S	Price	Delta	Gamma	Vega	Theta
80.00	19.44832	−0.81955	0.01676	0.21155	−0.00575
85.00	15.56924	−0.72938	0.01967	0.28240	−0.00769
90.00	12.17306	−0.62762	0.02105	0.34124	−0.00931
95.00	9.29821	−0.52216	0.02086	0.37895	−0.01035
100.00	6.94392	−0.42055	0.01933	0.39156	−0.01069
105.00	5.07582	−0.32847	0.01692	0.38031	−0.01038
110.00	3.63657	−0.24939	0.01412	0.35019	−0.00955
115.00	2.55742	−0.18449	0.01130	0.30789	−0.00838
120.00	1.76806	−0.13331	0.00871	0.26002	−0.00707

$S = 100$, $K = 100$, $t = 22/12/2002$, $T = 22/12/2003$, $r = 2\%$, $\sigma = 20\%$.

3.4. *Examples*

Tables 1–4 provide simulation results for European call and put prices using the Black–Scholes model. The tables provide also Greek-letters. The delta is given by the option's first partial derivative with respect to the underlying asset price. It represents the hedge ratio in the context of the Black–Scholes model. The option's gamma corresponds to the option second partial derivative with respect to the underlying asset or to the delta partial derivative with respect to the asset price. The option's theta is given by the option's first partial derivative with respect to the time remaining to maturity. The option's vega is given by the option price derivative with respect to the volatility parameter. The derivation of these parameters appears in Appendix 1.

4. The Black Model for Commodity Contracts

Using some assumptions similar to those used in deriving the original B–S option formula, Black (1976) presented a model for the pricing of commodity options and forward contracts.

In this model, the spot price $S(t)$ of an asset or a commodity is the price at which an investor can buy or sell it for an immediate delivery at current time, time t. This price may rise steadily, fall and fluctuate randomly.

The futures price $F(t, t^*)$ of a commodity can be defined as the price at which an investor agrees to buy or sell at a given time in the future, t^*, without putting up any money immediately.

Table 2: Simulations of Black Scholes call prices.

S	Price	Delta	Gamma	Vega	Theta
80.00	1.43382	0.18045	0.01676	0.21155	−0.00575
85.00	2.55474	0.27062	0.01967	0.28240	−0.00769
90.00	4.15856	0.37238	0.02105	0.34124	−0.00931
95.00	6.28371	0.47784	0.02086	0.37895	−0.01035
100.00	8.92943	0.57945	0.01933	0.39156	−0.01069
105.00	12.06132	0.67153	0.01692	0.38031	−0.01038
110.00	15.62208	0.75061	0.01412	0.35019	−0.00955
115.00	19.54292	0.81551	0.01130	0.30789	−0.00838
120.00	23.75356	0.86669	0.00871	0.26002	−0.00707

$S = 100$, $K = 100$, $t = 22/12/2002$, $T = 22/12/2003$, $r = 2\%$, $\sigma = 20\%$.

Table 3: Simulations of Black and Scholes call prices.

S	Price	Delta	Gamma	Vega	Theta
80.00	0.36459	0.07582	0.01332	0.08172	−0.00442
85.00	0.93156	0.15729	0.02070	0.14559	−0.00791
90.00	1.99540	0.27358	0.02656	0.21252	−0.01158
95.00	3.70489	0.41284	0.02900	0.26201	−0.01429
100.00	6.12966	0.55640	0.02763	0.27966	−0.01526
105.00	9.24535	0.68669	0.02332	0.26377	−0.01439
110.00	12.95409	0.79246	0.01781	0.22359	−0.01218
115.00	17.12243	0.87053	0.01245	0.17279	−0.00939
120.00	21.61673	0.92356	0.00807	0.12322	−0.00668

$S = 100$, $K = 100$, $t = 22/12/2002$, $T = 22/06/2003$, $r = 2\%$, $\sigma = 20\%$.

When $t = t^*$, the futures price is equal to the spot price. A forward contract is a contract to buy or sell at a price that stays fixed until the maturity date, whereas the futures contract is settled every day and rewritten at the new futures price.

Following Black (1976), let v be the value of the forward contract, u the value of the futures contract and c the value of an option contract. Each of these contracts, is a function of the futures price $F(t, t^*)$ as well as other variables. So, we can write at instant t, the values of these contracts, respectively, as $V(F, t)$, $u(F, t)$ and $c(F, t)$. The value of the forward contract depends also on the price of the underlying asset, K at time t^* and can be written $V(F, t, K, t^*)$. The futures price is the price at which a forward contract presents a zero current value. It is written as $V(F, t, F, t^*) = 0$.

Table 4: Simulations of Black and Scholes put prices.

S	Price	Delta	Gamma	Vega	Theta
80.00	19.36686	−0.92418	0.01332	0.08172	−0.00442
85.00	14.93383	−0.84271	0.02070	0.14559	−0.00791
90.00	10.99767	−0.72642	0.02656	0.21252	−0.01158
95.00	7.70716	−0.58716	0.02900	0.26201	−0.01429
100.00	5.13193	−0.44360	0.02763	0.27966	−0.01526
105.00	3.24762	−0.31331	0.02332	0.26377	−0.01439
110.00	1.95636	−0.20754	0.01781	0.22359	−0.01218
115.00	1.12471	−0.12947	0.01245	0.17279	−0.00939
120.00	0.61900	−0.07644	0.00807	0.12322	−0.00668

$S = 100$, $K = 100$, $t = 22/12/2002$, $T = 22/06/2003$, $r = 2\%$, $\sigma = 20\%$.

This equation says that the forward contract's value is zero when the contract is initiated and the contract price, K, is always equal to the current futures price $F(t, t^*)$. The main difference between a futures contract and a forward contract is that a futures contract may be assimilated to a series of forward contracts. This is because the futures contract is rewritten every day with a new contract price equal to the corresponding futures price. Hence when F rises, i.e. $F > K$, the forward contract has a positive value and when F falls, $F < K$, the forward contract has a negative value. When the transaction takes place, the futures price equals the spot price and the value of the forward contract equals the spot price minus the contract price or the spot price $V(F, t^*, K, t^*) = F - K$.

At maturity, the value of a commodity option is given by the maximum of zero and the difference between the spot price and the contract price. Since at that date, the futures price equals the spot price, it follows that if $F \geq K$, then

$$c(F, t^*) = F - K,$$

otherwise $c(F, t^*) = 0$.

In order to value commodity contracts and commodity options, Black (1976) assumes that:

— The futures price changes are distributed log-normally with a constant variance rate σ^2.
— All the parameters of the capital asset pricing model are constant through time.
— There are no transaction costs and no taxes.

Under these assumptions, it is possible to create a riskless hedge by taking a long position in the option and a short position in the futures contract. Let $[\partial c(F, t)/\partial F]$ be the weight affected to the short position in the futures contract, which is the derivative of $c(F, t)$ with respect to F. The change in the hedged position may be written as

$$\Delta c(F, t) - \left[\frac{\partial c(F, t)}{\partial F}\right]\Delta F.$$

Using the fact that the return to a hedged portfolio must be equal to the risk-free interest rate and expanding $\Delta c(F, t)$ gives the following partial differential equation

$$\frac{\partial c(F, t)}{\partial t} = rc(F, t) - \frac{1}{2}\sigma^2 F^2 \left[\frac{\partial^2 c(F, t)}{\partial F^2}\right]$$

or

$$\frac{1}{2}\sigma^2 F^2 \left[\frac{\partial^2 c(F, t)}{\partial F^2}\right] - rc(F, t) + \frac{\partial c(F, t)}{\partial t} = 0. \tag{4}$$

Denoting $T = t^* - t$, using the call's payoff and Eq. (4), the value of a commodity option is

$$c(F, T) = \mathrm{e}^{-rT}[FN(d_1) - KN(d_2)]$$

$$d_1 = \frac{\ln\left(\frac{F}{K}\right) + \frac{\sigma^2}{2}T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T}$$

where $N(.)$ is the cumulative normal density function.

It is convenient to note that the commodity option's value is the same as the value of an option on a security paying a continuous dividend. The rate of distribution is equal to the stock price times the interest rate. If $F\mathrm{e}^{-rT}$ is substituted in the original formula derived by Black–Scholes, the result is exactly the above formula. In the same context, the formula for the European put is

$$p(F, T) = \mathrm{e}^{-rT}[-FN(d_1) + KN(-d_2)]$$

$$d_1 = \frac{\ln\left(\frac{F}{K}\right) + \frac{\sigma^2}{2}T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T}$$

where $N(.)$ is the cumulative normal density function.

The value of the put option can be obtained directly from the put–call parity. The put–call parity relationship for futures options is

$$p - c = \mathrm{e}^{-rT}(K - F).$$

5. The Extension to Foreign Currencies: The Garman and Kohlhagen Model

Foreign currency options are priced along the lines of Black–Scholes (1973), Merton (1973) and Garman and Kohlhagen (1983). Using the same assumptions as in the Black–Scholes (1973) model, Garman and Kohlhagen (1983) presented the following formula for a European currency call:

$$c(S, T) = Se^{-r^*T}N(d_1) + Ke^{-rT}N(d_2)$$

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + \left(r - r^* + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

where S, the spot rate; K, the strike price; r, the domestic interest rate; r^*, the foreign interest rate; σ, the volatility of spot rates; and T, the option's time to maturity. The formula for a European currency put is

$$p(S, T) = -Se^{-r^*T}N(-d_1) - Ke^{-rT}N(-d_2)$$

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + \left(r - r^* + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T}.$$

Note that the main difference between these formulae and those of B–S for the pricing of equity options is that the foreign risk-free rate is used in the adjustment of the spot rate. The spot rate is adjusted by the known "dividend", i.e. the foreign interest earnings, whereas the domestic risk-free rate enters the calculation of the present value of the strike price since the domestic currency is paid over on exercise.

Examples

Tables 5–8 provide simulation results for option prices using the Garman–Kohlhagen model. The tables give also the Greek-letters. The reader can make comments about the values of the Greek-letters.

6. The Extension to Other Commodities: The Merton, Barone-Adesi and Whaley Model and Its Applications

The model presented in Barone-Adesi and Whaley (1987) is a direct extension of models presented by Black–Scholes (1973), Merton (1973) and

Table 5: Simulations of Garman–Kohlhagen call prices.

S	Price	Delta	Gamma	Vega	Theta
0.96	0.05384	0.43109	0.52931	0.00364	0.00008
0.97	0.05815	0.45081	0.50961	0.00370	0.00008
0.98	0.06266	0.47042	0.49003	0.00376	0.00009
0.99	0.06737	0.48984	0.47063	0.00380	0.00009
1.00	0.07227	0.50904	0.45145	0.00383	0.00009
1.01	0.07736	0.52798	0.43253	0.00386	0.00008
1.02	0.08264	0.54661	0.41391	0.00387	0.00008
1.03	0.08810	0.56492	0.39562	0.00388	0.00008
1.04	0.09375	0.58286	0.37769	0.00387	0.00008

$S = 1, K = 1, t = 07/02/2002, T = 07/02/2003, r = 3\%, r^* = 4\%,$
$\sigma = 20\%.$

Table 6: Simulations of Garman–Kohlhagen call prices.

S	Price	Delta	Gamma	Vega	Theta
1.06	0.10315	0.61071	0.34986	0.00385	0.00008
1.07	0.10988	0.62921	0.33137	0.00382	0.00008
1.08	0.11680	0.64714	0.31345	0.00378	0.00007
1.09	0.12392	0.66449	0.29611	0.00373	0.00007
1.10	0.13123	0.68123	0.27937	0.00368	0.00007
1.11	0.13872	0.69736	0.26325	0.00362	0.00007
1.12	0.14639	0.71287	0.24775	0.00355	0.00006
1.13	0.15423	0.72774	0.23289	0.00347	0.00006
1.14	0.16224	0.74198	0.21865	0.00339	0.00006

$S = 1.1, K = 1, t = 07/02/2002, T = 07/02/2003, r = 3\%,$
$r^* = 4\%, \sigma = 20\%.$

Black (1976). The absence of riskless arbitrage opportunities imply that the following relationship exists between the futures contract, F, and the price of its underlying spot commodity, S: $F = Se^{bT}$; where T is the time to expiration and b is the cost of carrying the commodity. When the underlying commodity dynamics are given by:

$$\frac{\mathrm{d}S}{S} = \alpha\,\mathrm{d}t + \sigma\,\mathrm{d}W$$

Table 7: Simulations of Garman–Kohlhagen put prices.

S	Price	Delta	Gamma	Vega	Theta
0.96	0.10195	−0.52960	0.52931	0.00364	0.00011
0.97	0.09666	−0.50987	0.50961	0.00370	0.00011
0.98	0.09156	−0.49027	0.49003	0.00376	0.00011
0.99	0.08666	−0.47084	0.47063	0.00380	0.00011
1.00	0.08195	−0.45164	0.45145	0.00383	0.00011
1.01	0.07744	−0.43271	0.43253	0.00386	0.00011
1.02	0.07311	−0.41407	0.41391	0.00387	0.00011
1.03	0.06896	−0.39577	0.39562	0.00388	0.00011
1.04	0.06500	−0.37783	0.37769	0.00387	0.00011

$S = 1$, $K = 1$, $t = 07/02/2002$, $T = 07/02/2003$, $r = 3\%$, $r^* = 4\%$, $\sigma = 20\%$.

Table 8: Simulations of Garman–Kohlhagen put prices and the Greek-letters.

S	Price	Delta	Gamma	Vega	Theta
1.06	0.05904	−0.34997	0.34986	0.00385	0.00011
1.07	0.05519	−0.33148	0.33137	0.00382	0.00011
1.08	0.05155	−0.31354	0.31345	0.00378	0.00011
1.09	0.04810	−0.29620	0.29611	0.00373	0.00011
1.10	0.04484	−0.27945	0.27937	0.00368	0.00011
1.11	0.04177	−0.26332	0.26325	0.00362	0.00010
1.12	0.03887	−0.24781	0.24775	0.00355	0.00010
1.13	0.03615	−0.23294	0.23289	0.00347	0.00010
1.14	0.03358	−0.21870	0.21865	0.00339	0.00010

$S = 1.1$, $K = 1$, $t = 07/02/2002$, $T = 07/02/2003$, $r = 3\%$, $r^* = 4\%$, $\sigma = 20\%$.

where α is the expected instantaneous relative price change of the commodity and σ is its standard deviation, then the dynamics of the futures price are given by the following differential equation:

$$\frac{\mathrm{d}F}{F} = (\alpha - b)\mathrm{d}t + \sigma\,\mathrm{d}W.$$

Assuming that a hedged portfolio containing the option and the underlying commodity can be constructed and adjusted continuously, the partial differential equation that must be satisfied by the option price, c, is

$$\frac{1}{2}\sigma^2 S^2 \left[\frac{\partial^2 c(S,t)}{\partial S^2}\right] - rc(S,t) + bS\left[\frac{\partial c(S,t)}{\partial S}\right] + \left[\frac{\partial c(S,t)}{\partial t}\right] = 0.$$

This equation appeared first indirectly in Merton (1973). When the cost of carry b is equal to the riskless interest rate, this equation reduces to that of B–S (1973).

When the cost of carry is zero, this equation reduces to that of Black (1976).

When the cost of carry is equal to the difference between the domestic and the foreign interest rate, this equation reduces to that in Garman and Kohlhagen (1983).

The short term interest rate r, and the cost of carrying the commodity, b, are assumed to be constant and proportional rates. Using the terminal boundary condition $c(S,T) = \max[0, S_T - K]$; Merton (1973) shows indirectly that the European call price is:

$$c(S,T) = Se^{(b-r)T}N(d_1) - Ke^{-rT}N(d_2)$$

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + (b + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T}.$$

Using the boundary condition for the put

$$p(S,T) = \max[0, K - S_T]$$

the European put price is given by

$$p(S,T) = -Se^{(b-r)T}N(-d_1) + Ke^{-rT}N(-d_2)$$

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + (b + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T}.$$

The call formula provides the composition of the asset-bond portfolio that mimics exactly the call's payoff. A long position in a call can be replicated by buying $e^{(b-r)T}N(d_1)$ units of the underlying asset and selling $N(d_2)$ units of risk-free bonds, each unit with strike price Ke^{-rT}. When the asset price varies, the units invested in the underlying asset and risk-free bonds will change. Using a continuous rebalancing of the portfolio, the pay-outs will be identical to those of the call. The same strategy can be used to duplicate the put's payoff.

7. Option Price Sensitivities: Some Specific Examples

7.1. *The Delta*

7.1.1. *The Call's Delta*

The call's delta is given by $\Delta_c = N(d_1)$. The use of this formula requires the computation of d_1 given by

$$d_1 = \frac{\ln \frac{S}{K} + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}.$$

Appendix 2 provides the detailed derivations of these parameters.

Example
Let the underlying asset price $S = 18$, the strike price $K = 15$, the short term interest rate $r = 10\%$, the maturity date $T = 0.25$ and the volatility $\sigma = 15\%$, the option's delta is given by $\Delta_c = N(d_1)$. Applying this formula needs the calculation of d_1:

$$d_1 = \frac{1}{0.15\sqrt{0.25}}\left[\ln\left(\frac{18}{15}\right) + \left(0.1 + \frac{1}{2} \times 0.5^2\right) 0.25\right] = 2.8017.$$

Hence, the delta is $\Delta_c = N(2.8017) = 0.997$. This delta value means that the hedge of the purchase of a call needs the sale of 0.997 units of the underlying asset.

When the underlying asset price rises by 1 unit, from 18 to 19, the option price rises from 3.3659 to approximately (3.3659+0.997), or 4.3629. When the asset price falls by one unit, the option price changes from 3.3659 to approximately (3.3659 – 0.997), or 2.3689.

7.1.2. *The Put's Delta*

The put's delta has the same meaning as the call's delta. It is also given by the option's first derivative with respect to the underlying asset price. When selling (buying) a put option, the hedge needs selling (buying) delta units of the underlying asset. The put's delta is given by

$$\Delta_p = \Delta_c - 1 = 0.0997 - 1 = -0.003.$$

The hedge ratio is –0.003. When the underlying asset price rises from 18 to 19, the put price falls from 0.0045 to approximately (0.0045 – 0.003), or 0.0015. When it falls from 18 to 17, the put price rises from 0.0045 to approximately (0.0045 + 0.003), or 0.0075.

Appendix 1 provides the derivation of the Greek letters in the context of analytical models.

7.2. The Gamma

7.2.1. The Call's Gamma

In the Black–Scholes model, the call's gamma is given by $\Gamma_c = \frac{\partial \Delta_c}{\partial S} = \frac{1}{S\sigma\sqrt{T}} n(d_1)$ with $n(d_1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_1^2}$. Using the same data as in the example $n(d_1) = \frac{1}{\sqrt{6.2831}} e^{-\frac{1}{2}(2.8017)^2} = 0.09826$ and $\Gamma_c = \frac{1}{18(0.15)\sqrt{0.25}} 0.09826 = 0.0727$. When the underlying asset price is 18 and its delta is 0.997, a fall in the asset price by one unit yields a change in the delta from 0.997 to approximately (0.997 − 0.0727), or 0.9243. Also, a rise in the asset price from 18 to 19, yields a change in the delta from 0.997 to (0.997+0.0727), or 1. This means that the option is *deeply-in-the-money*, and its value is given by its intrisic value $(S - K)$. The same arguments apply to put options. The call and the put have the same gamma.

7.2.2. The Put's Gamma

The put's gamma is given by $\Gamma_p = \frac{\partial \Delta_p}{\partial S} = \frac{1}{S\sigma\sqrt{T}} n(d_1)$ with $n(d_1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_1^2}$ or $\Gamma_p = \frac{1}{18(0.15)\sqrt{0.25}} 0.09826 = 0.0727$. When the asset price changes by one unit, the put price changes by the delta amount and the delta changes by an amount equals to the gamma.

7.3. The Theta

7.3.1. The Call's Theta

In the B-S model, the theta is given by

$$\Theta_c = \frac{\partial c}{\partial T} = \frac{-S\sigma n(d_1)}{2\sqrt{T}} - rK e^{-rT} N(d_2).$$

Using the same data as in the example above, we obtain:

$$\Theta_c = -0.2653 - 1.4571 = -1.1918.$$

When the time to maturity is shortened by 1% year, the call's price decreases by 0.01 (1.1918), or 0.011918 and its price changes from 3.3659 to approximately (3.3659 − 0.01918), or 3.3467.

7.3.2. *The Put's Theta*

In the B–S model, the put's theta is given by

$$\Theta_p = \frac{\partial p}{\partial T} = -\frac{S\sigma n(d_1)}{2\sqrt{T}} + rK\mathrm{e}^{-rT}N(d_2)$$

or $\Theta_p = -0.2653 + 0.0058 = -0.2594$. Using the same reasoning, the put price changes from 0.0045 to approximately (0.0045 − 0.0025), or 0.002.

7.4. *The Vega*

7.4.1. *The Call's Vega*

In the B–S model, the call's vega is given by $v_c = \frac{\partial c}{\partial \sigma} = S\sqrt{T}n(d_1)$ or using the above data $v_c = 18\sqrt{0.25}(0.09826) = 0.88434$.

Hence, when the volatility rises by 1 point, the call price increases by 0.88434.

The increase in volatility by 1% changes the option price from 3.3659 to (3.3659 + 1% (0.88434)), or 3.37474. In the same context, the put's vega is equal to the call's vega. The put price changes from 0.0045 to (0.0045 + 1% (0.88434)), or 0.0133434.

When the volatility falls by 1% the call's price changes from 3.3659 to (3.3659 − 1% (0.88434)), or 3.36156. In the same way, the put price is modified from 0.0045 to approximately (0.0045 − 1% (0.88434)), or zero since option prices cannot be negative.

7.4.2. *The Put's Vega*

In the Black–Scholes model, the put's vega is given by $v_p = \frac{\partial p}{\partial \sigma} = S\sqrt{T}n(d_1)$ or $v_p = 18\sqrt{0.25}(0.09826) = 0.88434$ and it has the same meaning as the call's vega.

Appendix 2 provides the relationships between hedging parameters.

8. Ito's Lemma and Its Applications

Financial models are rarely described by a function that depends on a single variable. In general, a function which is itself a function of more than one variable is used. Ito's lemma, which is the fundamental instrument in stochastic calculus, allows such functions to be differentiated. We first derive Ito's lemma with reference to simple results using Taylor series approximations. We then give a more rigorous definition of Ito's theorem. Let f be a continuous and differentiable function of a variable x.

If Δx is a small change in x, then using Taylor series, the resulting change in f is given by:

$$\Delta f \sim \left(\frac{\mathrm{d}f}{\mathrm{d}x}\right)\Delta x + \frac{1}{2}\left(\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}\right)\Delta x^2 + \frac{1}{6}\left(\frac{\mathrm{d}^3 f}{\mathrm{d}x^3}\right)\Delta x^3 + \cdots .$$

If f depends on two variables x and y, then Taylor series expansion of Δf is

$$\Delta f \sim \left(\frac{\partial f}{\partial x}\right)\Delta x + \left(\frac{\partial f}{\partial y}\right)\Delta y + \frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2}\right)\Delta x^2 + \frac{1}{2}\left(\frac{\partial^2 f}{\partial y^2}\right)\Delta y^2 + \left(\frac{\partial^2 f}{\partial y \partial y}\right)\Delta x \Delta y + \cdots .$$

In the limit case, when Δx and Δy are close to zero, Eq. (9) becomes

$$\Delta f \sim \left(\frac{\partial f}{\partial x}\right)\mathrm{d}x + \left(\frac{\partial f}{\partial y}\right)\mathrm{d}y.$$

Now, if f depends on two variables x and t in lieu of x and y, the analogous to Eq. (9) is

$$\Delta f \sim \left(\frac{\partial f}{\partial x}\right)\Delta x + \left(\frac{\partial f}{\partial t}\right)\Delta t + \frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2}\right)\Delta x^2 + \frac{1}{2}\left(\frac{\partial^2 f}{\partial t^2}\right)\Delta t^2 + \left(\frac{\partial^2 f}{\partial x \partial t}\right)\Delta x \Delta t + \cdots . \quad (5)$$

Consider a derivative security, $f(x, t)$, which value depends on time and on the asset price x. Assuming that x follows the general Ito process,

$$\mathrm{d}x = a(x, t)\mathrm{d}t + b(x, t)\mathrm{d}W$$

or

$$\Delta x = a(x, t)\Delta t + b\xi\sqrt{\Delta t}.$$

In the limit, when Δx and Δt are close to zero, we cannot ignore as before the term in Δx^2 since it is equal to $\Delta x^2 = b^2\xi^2\Delta t +$ terms in higher order in Δt. In this case, the term in Δt cannot be neglected. Since the term ξ is normally distributed with a zero mean, $E(\xi) = 0$ and a unit variance, $E(\xi^2) - E(\xi)^2 = 1$, then $E(\xi^2) = 1$ and $E(\xi)^2\Delta t$ is Δt . The variance of $\xi^2\Delta t$ is of order Δt^2 and consequently, as Δt approaches zero, $\xi^2\Delta t$ becomes certain and equals its expected value, Δt. In the limit, Eq. (5) becomes

$$\mathrm{d}f = \left(\frac{\partial f}{\partial x}\right)\Delta x + \left(\frac{\partial f}{\partial t}\right)\Delta t + \frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2}\right) b^2\,\mathrm{d}t.$$

This is exactly Ito's lemma.

Substituting $a(x,t)\mathrm{d}t + b(x,t)\mathrm{d}W$ for $\mathrm{d}x$ gives

$$\mathrm{d}f = \left[\left(\frac{\partial f}{\partial x}\right)a + \left(\frac{\partial f}{\partial t}\right) + \frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2}\right) b^2\right]\mathrm{d}t + \left(\frac{\partial f}{\partial x}\right) b\,\mathrm{d}W.$$

Example

Apply Ito's lemma to derive the process of $f = \ln(S)$.

First calculate the derivatives

$$\left(\frac{\partial f}{\partial S}\right) = \frac{1}{S};\quad \left(\frac{\partial^2 f}{\partial S^2}\right) = -\frac{1}{S^2};\quad \left(\frac{\partial f}{\partial t}\right) = 0.$$

Then from Ito's lemma, one obtains

$$\mathrm{d}f = \left[\left(\frac{\partial f}{\partial x}\right)\mu S + \left(\frac{\partial f}{\partial t}\right) + \frac{1}{2}\left(\frac{\partial^2 f}{\partial S^2}\right)\sigma^2 S^2\right]\mathrm{d}t + \left(\frac{\partial f}{\partial S}\right)\sigma S\,\mathrm{d}W$$

or

$$\mathrm{d}f = \left[\mu - \frac{1}{2}\sigma^2\right]\mathrm{d}t + \sigma\,\mathrm{d}W.$$

This last equation shows that, f follows a generalized Wiener process with a constant drift of $(\mu - \frac{1}{2}\sigma^2)$ and a variance rate of σ^2. The generalization of Ito's lemma is useful for a function that depends on n stochastic variables x_i, where i varies from 1 to n. Consider the following dynamics for the variables x_i:

$$\mathrm{d}x_i = a_i\,\mathrm{d}t + b_i\,\mathrm{d}z_i. \tag{6}$$

Using a Taylor series expansion of f gives

$$\Delta f = \sum_i \frac{\partial f}{\partial x_i}\Delta x_i + \frac{\partial f}{\partial t}\Delta t + \frac{1}{2}\sum_i\sum_j \frac{\partial^2 f}{\partial x_i \partial x_j}\Delta x_i \Delta x_j + \frac{\partial^2 f}{\partial x_i \partial t}\Delta x_i \Delta t + \cdots . \tag{7}$$

Equation (6) can be discretized as follows:

$$\Delta x_i = a_i \Delta t + b_i \epsilon_i \Delta\sqrt{z_i}$$

where the term ϵ_i corresponds to a random sample from a standardized normal distribution. The terms ϵ_i and ϵ_j reflecting the Wiener processes present a correlation coefficient $\rho_{i,j}$. It is possible to show that when the time interval tends to zero, in the limit, the term $\Delta x_i^2 = b_i^2\,\mathrm{d}t$ and the product $\Delta x_i \Delta x_j = b_i b_j \rho_{i,j}\,\mathrm{d}t$. Hence, in the limit, when the time interval becomes close to zero, Eq. (7) can be written as

$$\mathrm{d}f = \sum_i \frac{\partial f}{\partial x_i}\mathrm{d}x_i + \frac{\partial f}{\partial t}\mathrm{d}t + \frac{1}{2}\sum_i\sum_j \frac{\partial^2 f}{\partial x_i \partial x_j} b_i b_j \rho_{ij}\,\mathrm{d}t.$$

This gives the generalized version of Ito's lemma. Substituting Eq. (6) in the above equation gives:

$$\mathrm{d}f = \left(\sum_i \frac{\partial f}{\partial x_i}a_i + \frac{\partial f}{\partial t} + \frac{1}{2}\sum_i\sum_j \frac{\partial^2 f}{\partial x_i \partial x_j} b_i b_j \rho_{ij}\right) + \sum \frac{\partial f}{\partial x_i} b_i\,\mathrm{d}z_i.$$

Example
Use Ito's Lemma to show that:

$$\int_0^t \tau^m X^{n-1}(\tau)\mathrm{d}X(\tau) = \frac{1}{n}t^m X^n(t) - \frac{n-1}{2}\int_0^t t^m X^{n-2}(\tau)\mathrm{d}\tau - \frac{m}{n}\int_0^t t^{m-1}X^n(\tau)\mathrm{d}\tau.$$

Solution
When

$$F = X^n(t), \quad nm \in N^*$$

then

$$\int_0^t \mathrm{d}F = t^m X^n(t) - t^m X^n(0) = tmX^n(t)$$

since $X(0) = 0$.

Using Ito's Lemma gives:

$$\mathrm{d}F = nt^m X^{n-1}\,\mathrm{d}X + \frac{1}{2}n(n-1)X^{n-2}\,\mathrm{d}t + mt^{m-1}X^n(t)\mathrm{d}t.$$

Hence, we have:

$$\begin{aligned}\int_0^t \mathrm{d}F &= n\int_0^t \tau^m X^{n-1}(\tau)\mathrm{d}X(\tau) + \frac{1}{2}n(n-1)\int_0^t \tau^m X^{n-2}(\tau)\mathrm{d}\tau\\ &\quad + m\int_0^t \tau^{m-1}X^n(\tau)\mathrm{d}\tau = t^m X^n(t)\\ &\Leftrightarrow \int_0^t \tau^m X^{n-1}(\tau)\mathrm{d}X(\tau) + \frac{1}{2}(n-1)\tau^m X^{n-2}(\tau)\,\mathrm{d}\tau\\ &\quad + \frac{m}{n}\int_0^t \tau^{m-1}X^n(\tau)\mathrm{d}\tau = \frac{1}{n}t^m X^n(t)\\ &\Leftrightarrow \int_0^t \tau^m X^{n-1}(\tau)\mathrm{d}X(\tau) = \frac{1}{n}t^m X^n(t) - \int_0^t \frac{n-1}{2}\tau^m X^{n-2}(\tau)\,\mathrm{d}\tau\\ &\quad - \frac{m}{n}\int_0^t \tau^{m-1}X^n(\tau)\,\mathrm{d}\tau.\end{aligned}$$

9. Taylor Series, Ito's Theorem and the Replication Argument

We denote by $c(S, t)$ the option value at time t as a function of the underlying asset price S and time t. Assume that the underlying asset price follows a geometric Brownian motion:

$$\frac{\mathrm{d}S}{S} = \mu\,\mathrm{d}t + \sigma\,\mathrm{d}W(t)$$

where μ and σ^2 correspond, respectively, to the instantaneous mean and the variance of the rate of return of the stock.

9.1. *The Relationship Between Taylor Series and Ito's Differential*

Using Taylor series differential, it is possible to express the price change of the option over a small interval of time $[t, t + \mathrm{d}t]$ as:

$$\mathrm{d}c = \left(\frac{\partial c}{\partial S}\right)\mathrm{d}S + \left(\frac{\partial c}{\partial t}\right)\mathrm{d}t + \frac{1}{2}\left(\frac{\partial^2 c}{\partial^2 S}\right)(\mathrm{d}S)^2, \tag{8}$$

where the last term appears because $(\mathrm{d}S)^2$ is of order $\mathrm{d}t$. The last term in Eq. (8) appears because the term $(\mathrm{d}S)^2$ is of order $\mathrm{d}t$. Omberg (1991) makes a decomposition of the last term in Eq. (8) into its expected value and an error term.This allows one to establish a link between Taylor series ($\mathrm{d}c$) and Ito's differential $\mathrm{d}c_I$ as

$$\mathrm{d}c = \left(\frac{\partial c}{\partial S}\right)\mathrm{d}S + \left(\frac{\partial c}{\partial t}\right)\mathrm{d}t + \frac{1}{2}\left(\frac{\partial^2 c}{\partial^2 S}\right)\sigma^2 S^2\,\mathrm{d}W^2 + \mathrm{d}e(t),$$

which can be written as the sum of two components corresponding to the Ito's differential $\mathrm{d}c_I$ and an error term $\mathrm{d}e(t)$

$$\mathrm{d}c = \mathrm{d}c_I + \mathrm{d}e(t),$$

where

$$\mathrm{d}c_I = \left(\frac{\partial c}{\partial S}\right)\mathrm{d}S + \left(\frac{\partial c}{\partial t}\right)\mathrm{d}t + \frac{1}{2}\left(\frac{\partial^2 c}{\partial^2 S}\right)\sigma^2 S^2\,\mathrm{d}t \quad \text{and}$$
$$\mathrm{d}e(t) = \frac{1}{2}\left(\frac{\partial^2 c}{\partial^2 S}\right)\sigma^2 S^2\left[\mathrm{d}W^2 - \mathrm{d}t\right].$$

9.2. *Ito's Differential and the Replication Portfolio*

9.2.1. *The Standard Case in Frictionless Markets*

The pay-off of a derivative asset can be created using the discount bond, some options and the underlying asset. The portfolio which duplicates the pay-off of the asset is called the replicating portfolio. When using Ito's lemma, the error term $\mathrm{d}e(t)$ is often neglected and, the equation for the option is approximated only by the term $\mathrm{d}c_I$. The quantity $\mathrm{d}c_I$ is replicated by Q_S units of the underlying asset and an amount of cash Q_c with $Q_S = \left(\frac{\partial c}{\partial S}\right)$

and $Q_c = \frac{1}{r}\left[\left(\frac{\partial c}{\partial S}\right) + \frac{1}{2}\left(\frac{\partial^2 c}{\partial^2 S}\right)\sigma^2 S^2\right]$ where r stands for the risk-free rate of return. Hence, the dynamics of the replicating portfolio are given by

$$\mathrm{d}\Pi_R = \left(\frac{\partial c}{\partial S}\right)\mathrm{d}S + rQ_c\,\mathrm{d}t \tag{9}$$

where Π_R refers to the replicating portfolio.

9.2.2. *An Extension to Account for Information Costs in the Valuation of Derivatives*

Information costs are defined in the spirit of Merton (1987) as in the previous chapter. These costs appear in option pricing models in the analysis conducted by Bellalah and Jacquillat (1995) and Bellalah (1999, 2000a,b, 2001).

The trading of financial derivatives on organized exchanges has exploded since the beginning of 1970s. The trading on "over-the-counter" or OTC market has exploded since the mid-1980s. Since the publication of the pionnering papers by Black–Scholes (1973) and Merton (1973), three industries have blossomed: an exchange industry in derivatives, an OTC industry in structured products and an academic industry in derivative research. Each industry needs a specific knowledge regarding the pricing and the production costs of the products offered to the clients. As it appears in Scholes (1998), derivative instruments provide (and will provide) lower-cost solutions to investor problems than will competing alternatives. These solutions will involve the repackaging of coarse financial products into their constituent parts to serve the investor demands. The "commoditisation" of instruments and the increased competition in the *over-the-counter* (OTC) market reduce profit margins for different players. The inevitable result is that products become more and more complex requiring more and more expenses in information acquisition. The problems of information, liquidity, transparency, commissions and charges are specific features of these markets.

Differences in information are important in financial and real markets. They are used in several contexts to explain some puzzling phenomena like the "smile effect",[1] etc. Since Merton's CAPMI can explain several anomalies in financial markets, its application in the valuation of derivative securities can be useful in explaining some anomalies in the option markets as the smile effect.

[1] See the models in Bellalah and Jacquillat (1995) and Bellalah (1999).

As it appears in the work of Black (1989), Scholes (1998) and as Merton (1998) asserts:

> "Fisher Black always maintained with me that the CAPM-version of the option model derivation was more robust because continuous trading is not feasible and there are transaction costs."

This approach will be used here by applying the CAPMI of Merton (1987). As it is well-known, in all standard asset pricing models, assets that show only diversifiable risk or nonsystematic risk are valued to yield an expected return equal to the riskless rate. In Merton's context, the expected return is equal to the riskless rate plus the shadow costs of incomplete information. The derivation of an option pricing model is based on an arbitrage strategy which consists in hedging the underlying asset and rebalancing continuously until expiration. This strategy is only possible in a frictionless market. Investors spend time and money to gather information about the financial instruments and financial markets.

Consider for example a financial institution using a given market. If the costs of portfolio selection, models conception, etc. are computed, then it can require at least a return of say, for example $\lambda = 3\%$, before acting in this market. This cost is in some sense the minimal return required before implementing a given strategy. If you consider the above replicating strategy, then the returns from the replicating portfolio must be at least:

$$d\Pi_R = \left(\frac{\partial c}{\partial S}\right) dS + (r + \lambda) Q_c \, dt$$

with $Q_c = \frac{1}{r+\lambda}\left[\left(\frac{\partial c}{\partial S}\right) + \frac{1}{2}\left(\frac{\partial^2 c}{\partial^2 S}\right)\sigma^2 S^2\right]$, where Π_R refers to the replicating portfolio.

This shows that the required return must cover at least the costs necessary for constructing the replicating portfolio plus the risk-free rate. In fact, when constructing a portfolio, some money is spent and a return for that must be required. Hence, there must be a minimal cost and a minimal return required for investing in information at the aggregate market level. For this reason, the required return must be at least λ plus the riskless rate. For an introduction to information costs and their use in asset pricing, the reader can refer to Appendix 3.

9.3. Ito's Differential and the Arbitrage Portfolio

If one uses arbitrage arguments, then the option value must be equal to the value of its replicating portfolio.

9.3.1. The Standard Analysis

Using arbitrage arguments, we must have

$$c = Q_s S + Q_c \quad \text{or} \quad c = \left(\frac{\partial c}{\partial S}\right) S + \frac{1}{r}\left[\left(\frac{\partial c}{\partial t}\right) + \frac{1}{2}\left(\frac{\partial^2 c}{\partial^2 S}\right)\sigma^2 S^2\right]$$

and

$$\frac{1}{2}\frac{\partial^2 c}{\partial^2 S}\sigma^2 S^2 + rc - r\frac{\partial c}{\partial S}S + \left(\frac{\partial c}{\partial t}\right) = 0.$$

This equation is often referred to in financial economics as the Black–Scholes–Merton partial differential equation. Note that the value of the replicating portfolio is $\Pi_R = \left(\frac{\partial c}{\partial S}\right) S + Q_c$. It is possible to implement a hedged position by buying the derivative asset and selling delta units of the underlying asset:

$$\Pi_{\mathrm{H}} = c - \left(\frac{\partial c}{\partial S}\right) S = Q_c$$

where the subscript H refers to the hedged portfolio.

A hedged position or portfolio is a portfolio whose return at equilibrium must be equal in theory to the short-term risk-free rate of interest. This is the main contribution of Black–Scholes (1973) to the pricing of derivative assets. Merton (1973) uses the same argument as Black–Scholes (1973) by implementing the concept of self-financing portfolio. This portfolio is also constructed by buying the option and selling the replicating portfolio or vice versa. The condition on the self-financing portfolio is

$$\Pi_A = c - \left(\frac{\partial c}{\partial S}\right) S - Q_c = 0$$

where S refers to the self-financing portfolio.

The omitted error term in the above analysis, $\mathrm{d}e(t)$, can reflect a replication error, a hedging error or an arbitrage error. It can have different interpretations. The term $\mathrm{d}e(t)$ is neglected or omitted when the revision of the portfolio is done to allow for the replicating portfolio to be self-financing.

When this term is positive, this may refer to additional cash that must be put in the portfolio. When it is negative, a withdrawal of cash from the portfolio is possible.

9.3.2. *An Extension to Account for Information Costs in Option Pricing Theory*

In the same way, the previous analysis can be extended to account for information costs. In this context, we must have:

$$c = Q_S S + Q_c \quad \text{or} \quad c = \left(\frac{\partial c}{\partial S}\right) S + \frac{1}{(r+\lambda)}\left[\left(\frac{\partial c}{\partial t}\right) + \frac{1}{2}\left(\frac{\partial^2 c}{\partial^2 S}\right)\sigma^2 S^2\right]$$

and

$$\frac{1}{2}\left(\frac{\partial^2 c}{\partial^2 S}\sigma^2 S^2\right) + (r+\lambda)c - (r+\lambda)\left(\frac{\partial c}{\partial S}\right) S + \left(\frac{\partial c}{\partial t}\right) = 0.$$

This equation corresponds to an extended version of the well-known Black–Scholes–Merton partial differential equation accounting for the effect of information costs. For the sake of simplicity, we assume that information costs are equal in both markets: the option market and the underlying asset market. Or in practice, institutions and investors support these costs on both markets. Therefore, a more suitable analysis must account for two costs: an information cost λ_c on the option market and an information cost λ_S on the underlying asset market. In this case, we obtain the following more general equation as in Bellalah and Jacquillat (1995) and Bellalah (1999):

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 c}{\partial^2 S} + (r+\lambda_c)c - (r+\lambda_S)\frac{\partial c}{\partial S} S + \frac{\partial c}{\partial t} = 0.$$

10. Differential Equation for a Derivative Security on a Spot Asset in the Presence of a Continuous Dividend Yield and Information Costs

We denote by V the price of a derivative security on a stock with a continuous dividend yield q. The dynamics of the underlying asset are given by: $\mathrm{d}S = \mu S\,\mathrm{d}t + \sigma S\,\mathrm{d}z$, where the drift term μ and the volatility σ are constants.

Using Ito's lemma for the function $V(S, t)$ gives

$$\mathrm{d}V = \left(\frac{\partial V}{\mathrm{d}S}\mu S + \frac{\mathrm{d}V}{\mathrm{d}t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2\right)\mathrm{d}t + \frac{\partial V}{\partial S}\sigma S\,\mathrm{d}z.$$

It is possible to construct a portfolio Π by holding a position in the derivative security and a certain number of units of the underlying asset $\Pi = -V + (\partial V/dS)S$. Over a short time interval, the change in the portfolio value can be written as

$$\Delta\Pi = \left(-\frac{\partial V}{\partial t} - \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2\right)\Delta t.$$

Over the same time interval, dividends are given by $qS(\partial V/\partial S)\Delta t$. We denote by ΔW the change in the wealth of the portfolio holder. In this case, we have

$$\Delta W = \left(-\frac{\partial V}{\partial t} - \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 + qS\frac{\partial V}{dS}\right)\Delta t.$$

Since this change is independent of the Wiener process, the portfolio is instantaneously risk-less and must earn the risk-free rate plus information costs or

$$\left(-\frac{\partial V}{\partial t} - \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 + qS\frac{\partial V}{\partial S}\right)\Delta t = -(r + \lambda_V)V\Delta t + (r + \lambda_S)S\frac{\partial V}{\partial S}\Delta t.$$

This gives

$$\frac{\partial V}{\partial t} + (r + \lambda_S - q)S\frac{\partial V}{\partial S} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 = (r + \lambda_V)V.$$

This equation must be satisfied by the derivative security in the presence of information costs and a continuous dividend yield.

11. The Valuation of Securities Dependent on Several Variables in the Presence of Incomplete Information: A General Method

When a variable does not indicate the price of a traded security, the pricing of derivatives must account for the market price of risk. The market price of risk γ for a traded security is given by

$$\gamma = \frac{\mu - r - \lambda}{\sigma} \tag{10}$$

where μ indicates the expected return from the security.

This equation can also be written as

$$\mu - r - \lambda = \gamma\sigma. \tag{11}$$

The excess return over the risk-free rate in the presence of shadow costs on a security corresponds to its market price of risk multiplied by its volatility.

When $\gamma > 0$, the expected return on an asset is higher than the risk-free rate plus information costs.

When $\gamma = 0$, the expected return on an asset is exactly the risk-free rate plus information costs.

When $\gamma < 0$, the expected return on an asset is less than the risk-free rate plus information costs.

When a variable does not indicate the price of a traded security, its market price of risk corresponds to the market price of risk of a traded security whose price is a function only on the value of the variable and time. The value of the market price of risk of the variable is the same at each instant of time. In fact, we can show as in Hull (2000) that two traded securities depending on the same asset must have the same price of risk, i.e. that Eq. (10) must be verified.

Consider the following dynamics for a variable θ which is not a tradable asset:

$$\frac{\mathrm{d}\theta}{\theta} = \mu(\theta, t)\mathrm{d}t + s(\theta, t)\mathrm{d}z.$$

We denote by V_1 and V_2, respectively, the prices of two derivative securities as a function of θ and t. The dynamics of these derivatives can be written as

$$\frac{\mathrm{d}V_1}{V_1} = \mu_1\,\mathrm{d}t + \sigma_1\,\mathrm{d}z$$

$$\frac{\mathrm{d}V_2}{V_2} = \mu_1\,\mathrm{d}t + \sigma_2\,\mathrm{d}z.$$

These two processes can be written in discrete time as

$$\Delta V_1 = \mu_1 V_1 \Delta t + \sigma_1 V_1 \Delta z \tag{12}$$

$$\Delta V_2 = \mu_2 V_2 \Delta t + \sigma_2 V_2 \Delta z. \tag{13}$$

It is possible to construct a portfolio Π which is risk-free using $\sigma_2 V_2$ of the first derivative security and $-\sigma_1 V_1$ of the second derivative security:

$$\Pi = \sigma_2 V_2 V_1 - \sigma_1 V_1 V_2.$$

The change in the value of this portfolio can be written as

$$\Delta\Pi = \sigma_2 V_2 \Delta V_1 - \sigma_1 V_1 \Delta V_2.$$

Using Eqs. (12) and (13), the change in the portfolio value can be written as

$$\Delta\Pi = \mu_1\sigma_2 V_1 V_2 - \mu_2\sigma_1 V_1 V_2 \Delta t.$$

Since the portfolio Π is instantaneously risk-less, it must earn the risk-free rate plus information costs on both markets. Hence, we must have

$$\mu_1\sigma_2 - \mu_2\sigma_1 = (r+\lambda_1)\sigma_2 - (r+\lambda_2)\sigma_1$$

or

$$\frac{\mu_1 - (r+\lambda_1)}{\sigma_1} = \frac{\mu_2 - (r+\lambda_2)}{\sigma_2}. \tag{14}$$

The term $(\mu - (r+\lambda))/\sigma$ must be the same for all securities that depend on time and the variable θ. It is also possible to show that V_1 and V_2 must depend positively on θ.

Since the volatility of V_1 is σ_1, it is possible to use Ito's lemma for σ_1 to obtain: $\sigma_1 V_1 = s\theta(\partial V_1/\partial\theta)$.

Hence, when V_1 is positively related to the variable θ, the σ_1 is positive and corresponds to the volatility of V_1. But, when f_1 is negatively related to the variable θ, σ_1 is negative and the equation for V_1 can be written as

$$\frac{\mathrm{d}V_1}{V_1} = \mu_1\,\mathrm{d}t + (-\sigma_1)(-\mathrm{d}z).$$

This indicates that the volatility is $-\sigma_1$ rather than σ_1.

The result in Eq. (14) can be generalized to n state variables.

Consider n variables which are assumed to follow Ito diffusion processes where for each state variable i between 1 and n, we have $\mathrm{d}\theta_i = m_i\theta_i\,\mathrm{d}t + s_i\theta_i\,\mathrm{d}z_i$ where $\mathrm{d}z_i$ are Wiener processes. The terms m_i and s_i correspond to the expected growth rate and the volatility of the θ_i with $i = 1, \ldots, n$.

The price process for a derivative security that depends on the variables θ_i can be written as:

$$\frac{\mathrm{d}V}{V} = \mu\,\mathrm{d}t + \sum_{i=1}^{n}\sigma_i\,\mathrm{d}z_i$$

where μ corresponds to the expected return from the security and σ_i is its volatility. The volatility of V is σ_i when all the underlying variables except θ_i are kept fixed. This result is obtained directly using an extension of the generalized version of Ito's lemma in its discrete form. We show in Appendix 4 that

$$\mu - r - \lambda_V = \sum_{i=1}^{n} \gamma_i \sigma_i \tag{15}$$

where γ_i indicates the market price of risk for the variable θ_i. Equation (15) shows that the expected excess return on the security (option) in the presence of shadow costs depends on γ_i and σ_i.

When $\gamma_i \sigma_i > 0$, a higher return is required by investors to get compensated for the risk arising from the variables θ_i.

When $\gamma_i \sigma_i < 0$, a lower return is required by investors to get compensated for the risk arising from θ_i.

12. The General Differential Equation for the Pricing of Derivatives

We denote by:

— θ_i: value of ith state variable,
— m_i: expected growth in ith state variable,
— γ_i: market price of risk of ith state variable,
— s_i: volatility of ith state variable,
— r: instantaneous risk-free rate,
— λ_i: shadow cost of incomplete information of ith state variable,

where i takes the values from 1 to n.

Garman (1976) and Cox, Ingersoll and Ross (1985) have shown that the price of any contingent claim must satisfy the following partial differential equation:

$$\frac{\partial V}{\partial t} + \sum_i \theta_i \frac{\partial V}{\partial \theta_i}(m_i - \gamma_i s_i) + \frac{1}{2}\sum_{i,k} \rho_{i,k} s_i s_k \theta_i \theta_k \frac{\partial^2 f}{\partial \theta_i \partial \theta_k} = rf$$

where $\rho_{i,j}$ stands for the correlation coefficient between the variables θ_i and θ_k.

We show in Appendix 4 how to obtain a similar equation in the presence of incomplete information. In this context, the equation becomes

$$\frac{\partial V}{\partial t} + \sum_i \theta_i \frac{\partial V}{\partial \theta_i}(m_i - \gamma_i s_i) + \frac{1}{2}\sum_{i,k} \rho_{i,k} s_i s_k \theta_i \theta_k \frac{\partial^2 V}{\partial \theta_i \partial \theta_k} = (r + \lambda)V. \tag{16}$$

In the presence of a single state variable, θ, the equation becomes

$$\frac{\partial V}{\partial t} + \theta \frac{\partial V}{\partial \theta}(m - \gamma s) + \frac{1}{2} s^2 \theta^2 \frac{\partial^2 V}{\partial^2 \theta} = (r + \lambda) V. \tag{17}$$

*/ For a non-dividend paying security, the expected return and volatility must satisfy

$$m - r - \lambda = \gamma s$$
$$m - \gamma s = r + \lambda.$$

In this case, Eq. (17) becomes the extended Black–Scholes equation in the presence of information costs.

*/ For a dividend-paying security at a rate q, we have

$$q + m - r - \lambda = \gamma s$$

or

$$m - \gamma s = r + \lambda - q.$$

In this case, Eq. (17) becomes

$$\frac{\partial V}{\partial t} + (r + \lambda_s - q) S \frac{\partial V}{\partial S} + \frac{1}{2} s^2 S^2 \frac{\partial^2 V}{\partial^2 S} = (r + \lambda_V) V.$$

13. Extension of the Risk-Neutral Argument in the Presence of Information Costs

We know that the market price of risk is given by

$$\gamma = \frac{\mu - r - \lambda}{\sigma},$$

or

$$\mu - r - \lambda = \gamma \sigma.$$

Appendix 5 shows how to price a derivative as if the world were risk neutral. This is possible when the expected growth rate of each state variable is $(m_i - \gamma_i s_i)$ rather than m_i.

For the case of a non-dividend paying traded asset, we have

$$m_i - r - \lambda_i = \gamma_i s_i$$

or

$$m_i - \gamma_i si = r + \lambda_i.$$

This result shows that a change in the expected growth rate of the state variable from θ_i to $(m_i - \gamma_i s_i)$ is equivalent to using an expected return from the security equal to the risk-less rate plus shadow costs of incomplete information.

For the case of a dividend paying traded asset, we have

$$q_i + m_i - r - \lambda_i = \gamma_i s_i$$

or

$$m_i - \gamma_i s_i = r + \lambda_i - q_i.$$

This result shows that a change in the expected growth rate of the state variable from θ_i to $(m_i - \gamma_i s_i)$ is equivalent to using an expected return (including continuous dividends at a rate q) from the security equal to the risk-less rate plus shadow costs of incomplete information.

This analysis allows the pricing of any derivative security as the value of its expected payoff discounted to the present at the risk-free rate plus the information cost on that security or

$$f = \mathrm{e}^{-(r+\lambda_V)(T-t)} \hat{E}[f_T]$$

where V_T corresponds to the security's payoff at maturity T and $\hat{E}$ to the expectation operator in a risk-neutral economy. This refers to an economy where the drift rate in θ_i corresponds to $(m_i - \gamma_i s_i)$. When the interest rate r is stochastic, it is considered as the other underlying state variables. In this case, the drift rate in r becomes $\gamma_r s_r$, where γ_r refers to the market price of risk related to r. The term s_r indicates its volatility. In this case, the pricing of a derivative is given by the discounting of its terminal payoff at the average value of r as:

$$V = \hat{E}[\mathrm{e}^{-(\bar{r}+\lambda_V)(T-t)} V_T]$$

where $\bar{r}$ corresponds to the average risk-free rate between current time t and maturity T.

14. Extension to Commodity Futures Prices within Incomplete Information

Consider the pricing of a long position in a commodity forward contract with delivery price K and maturity T.

14.1. *Commodity Futures Prices*

We denote by S the spot price of the commodity, F the futures price, μ the growth rate of the commodity price, σ its volatility and Γ its market price of risk. Using the extension of the risk-neutral valuation principle, the price is given by

$$V = \mathrm{e}^{-(r+\lambda_V)(T-t)}\hat{E}[S_T - K]$$

or

$$f = \mathrm{e}^{-(r+\lambda_f)(T-t)}(\hat{E}[S_T] - K)$$

where $\hat{E}(.)$ refers to the expected value in a risk-neutral economy. The forward or futures price F corresponds to the value of K that makes the value of the contract f equal to zero in this last equation. So, we have

$$F = \hat{E}[S_T]. \tag{18}$$

This equation shows that the futures price corresponds to the expected spot price in a risk-neutral world.

If $(\gamma\sigma)$ is constant and the drift μ is a function of time, then

$$\hat{E}[S_T] = E[S_T]\mathrm{e}^{-\gamma\sigma(T-t)}$$

where E corresponds to the real expectations or the expectations in the real world. Using Eq. (18), we have

$$F = E[S_T]\mathrm{e}^{-\gamma\sigma(T-t)}. \tag{19}$$

If $\Gamma = 0$, the futures price is an unbiased estimate of the expected spot price. However, if $\gamma > 0$, the futures price corresponds to a downward-biased estimate of the expected spot price.

If $\gamma < 0$, the futures price is a downward-biased estimate of the expected spot price.

14.2. Convenience Yields

When the drift or the expected growth in the commodity price is constant, we have $E(S_T) = Se^{\mu(T-t)}$.

Using Eq. (19), we have $F = Se^{(\mu-\gamma\sigma)(T-t)}$. This last equation is consistent with the cost of carry model, when the convenience yield y satisfies the following relationship:

$$\mu - \gamma\sigma = r + \lambda + g - y.$$

This equation shows that the commodity can be assimilated to a traded security paying a continuous dividend yield equal to the convenience yield y less the storage costs u. Hence, the convenience yield must satisfy the following relationship:

$$y = g + r + \lambda - \mu + \gamma\sigma.$$

When the convenience yield is zero, we have

$$\mu - \gamma\sigma = r + \lambda + g.$$

This result shows that some commodities can be assimilated to traded securities paying negative dividend yields which are equal to storage costs.

Summary

"Because options are specialised and relatively unimportant financial securities, the amount of time and space devoted to the development of a pricing theory might be questioned", Professor Robert Merton (1973), *Bell Journal of Economics and Management Science*.

Thirty years ago, no one could have imagined the changes that were about to occur in finance theory and the financial industry. The seeds of change were contained in option theory being conducted by the Nobel Laureates Fisher Black, Myron Scholes and Robert Merton.

Valuing claims to future income streams is one of the central problems in finance. The first known attempt to value options appears in Bachelier (1900) doctoral dissertation using an arithmetic Brownian motion. This process amounts to negative asset prices. Sprenkle (1961) and Samuelson (1965) used a geometric Brownian motion that eliminates the occurrence of negative asset prices.

Samuelson and Merton (1969) proposed a theory of option valuation by treating the option price as a function of the stock price. They advanced the theory by realizing that the discount rate must be determined in part by the requirement that investors hold all the amounts of stocks and the option. Their final formula depends on the utility function assumed for a "typical" investor.

Several discussions are done with Robert Merton (1973) who was also working on option valuation. Merton (1973) pointed out that assuming continuous trading in the option or its underlying asset can preserve a hedged portfolio between the option and its underlying asset. Merton was able to prove that in the presence of a non-constant interest rate, a discount bond maturing at the option expiration date must be used.

Black–Scholes (1973) and Merton (1973) showed that the construction of a riskless hedge between the option and its underlying asset, allows the derivation of an option pricing formula regardless of investors risk preferences.

The main attractions of the Black–Scholes model are that their formula is a function of "observable" variables and that the model can be extended to the pricing of any type of option.

Using some assumptions similar to those used in deriving the original B–S option formula, Black (1976) presented a model for the pricing of commodity options and forward contracts.

Black (1976) showed that in the absence of interest rate uncertainty, a European commodity option on a futures (or a forward) contract can be priced using a minor modification of the Black and Scholes (1973) option pricing formula. In deriving expressions for the behavior of the futures price, they assumed that both taxes and transaction costs are zero and that the CAPM applies at each instant of time.

This analysis was extended by several authors to account for other observable variables.

This chapter presented in detail the basic concepts and techniques underlying rational derivative asset pricing in the context of analytical European models along the lines of Black–Scholes (1973), Black (1976) and Merton (1973).

First, an overview of the analytical models proposed by the precursors is given.

Second, the simple model of Black–Scholes (1973) is derived in detail for the valuation of options on spot assets and some of its applications are presented.

Third, the Black model, which is an extension of the Black–Scholes model for the valuation of futures contracts and commodity options is analysed. Also, applications of the model are proposed.

Fourth, the basic limitations of the Black–Scholes–Merton theory are studied and the models are applied to the valuation of several financial contracts. The Black–Scholes hedge works in the real, discrete, frictionful world when the hedger uses the correct volatility of the prices at which they actually trade and when the asset prices do not jump too much.

The Black–Scholes formula gives a rough approximation to the formula investors would use if they knew how to account for the above factors. Modifications of the Black–Scholes formula can move it to the hypothetical perfect formula.

Fifth, we derive the differential equation for a derivative security on a spot asset in the presence of a continuous dividend yield and information costs.

Sixth, we provide the valuation of securities dependent on several variables in the presence of incomplete information.

Seventh, we propose the general differential equation in the same context.

Finally, we show how to extend the risk-neutral argument and the theory to the valuation of commodity futures contracts within incomplete information.

Questions

1. What is wrong in Bachelier's formula?
2. What is wrong in Sprenkle's formula?
3. What is wrong in Boness's formula?
4. What is wrong in Samuelson's formula?
5. What are the main differences between the Black and Scholes model and the precursors models?
6. How can we obtain the put–call parity relationship for options on spot assets?
7. How can we obtain the put–call parity relationship for options on futures contracts?
8. Is the Black model appropriate for the valuation of derivative assets whose values depend on interest rates? Justify your answer.
9. Is there any difference between a futures price and the value of a futures contract?
10. What are the holes in the Black–Scholes–Merton theory?
11. What are information costs?
12. What are the main results in the models of Merton (1973) and Barone-Adesi and Whaley (1987) model for the pricing of European commodity and commodity futures options?

13. What are option price sensitivities?
14. Describe Ito's lemma and some of its applications.
15. Explain Taylor series, Ito's theorem and the replication argument.
16. How can we derive the differential equation for a derivative security on a spot asset in the presence of a continuous dividend yield and information costs?
17. How can we develop a general context for the valuation of securities dependent on several variables in the presence of incomplete information?
18. How can we develop the general differential equation for the pricing of derivatives?
19. How can one extend the risk-neutral argument in the presence of information costs?
20. How can one extend the analysis to commodity futures prices within incomplete information?

Appendix 1: Greek Letter Risk Measures in Analytical Models

A.1 *Black–Scholes Model*

Call sensitivity parameters

$$\Delta_c = N(d_1), \quad \Gamma_c = \frac{\partial \Delta_c}{\partial S} = \frac{1}{S\sigma\sqrt{T}} n(d_1),$$
$$\Theta_c = \frac{\partial c}{\partial T} = \frac{S\sigma n(d_1)}{2\sqrt{T}} - rK\mathrm{e}^{-rT} N(d_2),$$
$$\upsilon_c = \frac{\partial c}{\partial \sigma} = S\sqrt{T} n(d_1), \quad \rho_c = \frac{\partial_c}{\partial r} = KT\mathrm{e}^{-rT} N(d_2).$$

Put sensitivity parameters

$$\Delta_p = \Delta_c - 1, \quad \Gamma_p = \frac{\partial \Delta_p}{\partial S} = \frac{1}{S\sigma\sqrt{T}} n(d_1),$$
$$\Theta_p = \frac{\partial p}{\partial T} = -\frac{S\sigma n(d_1)}{2\sqrt{T}} + rK\mathrm{e}^{-rT} N(-d_2),$$
$$\upsilon_p = \frac{\partial p}{\partial \sigma} = S\sqrt{T} n(d_1), \quad \rho_p = \frac{\partial p}{\partial r} = -KT\mathrm{e}^{-rT} N(-d_2).$$

A.2 Black's Model

The option sensitivity parameters in the Black's Model are presented as follows:

Call sensitivity parameters

$$\Delta_c = \mathrm{e}^{-rT}N(d_1), \quad \Gamma_c = \frac{\partial \Delta_c}{\partial S} = \frac{\mathrm{e}^{-rT}}{S\sigma\sqrt{T}}n(d_1),$$
$$\Theta_c = \frac{\partial c}{\partial T} = -\frac{S\mathrm{e}^{-rT}\sigma n(d_1)}{2\sqrt{T}} + rS\mathrm{e}^{-rT}N(d_1) - rK\mathrm{e}^{-rT}N(d_2),$$
$$v_c = \frac{\partial c}{\partial \sigma} = S\mathrm{e}^{-rt}\sqrt{T}n(d_1).$$

Put sensitivity parameters

$$\Delta_p = \Delta_c - \mathrm{e}^{-rT}, \quad \Gamma_p = \frac{\partial \Delta_p}{\partial S} = \frac{1}{S\sigma\sqrt{T}}n(d_1),$$
$$\Theta_p = -\frac{\partial p}{\partial T} = \frac{S\sigma\mathrm{e}^{-rT}n(d_1)}{2\sqrt{T}} - rS\mathrm{e}^{-rT}N(-d_1) + rK\mathrm{e}^{-rT}N(-d_2),$$
$$v_p = \frac{\partial p}{\partial \sigma} = S\sqrt{T}n(d_1).$$

A.3 Garman and Kohlhagen's Model

Call sensitivity parameters

$$\Delta_c = \mathrm{e}^{-r^*T}N(d_1), \quad \Gamma_c = \frac{\partial \Delta_c}{\partial S} = \frac{\mathrm{e}^{-r^*T}}{S\sigma\sqrt{T}}n(d_1),$$
$$\Theta_c = \frac{\partial c}{\partial T} = r^*S\mathrm{e}^{-r^*T}N(d_1) - rK\mathrm{e}^{-rT}N(d_2) - \frac{S\mathrm{e}^{-r^*}\sigma n(d_1)}{2\sqrt{T}},$$
$$\rho_c = \frac{\partial c}{\partial r^*} = -TS\mathrm{e}^{-r^*T}N(d_1),$$
$$v_c = \frac{\partial c}{\partial \sigma} = S\mathrm{e}^{-r^*T}\sqrt{T}n(d_1).$$

Put sensitivity parameters

$$\Delta_p = \mathrm{e}^{-r^*T}[N(d_1 - 1], \quad \Gamma_p = \frac{\partial \Delta_p}{\partial S} = \frac{\mathrm{e}^{-r^*T}}{S\sigma\sqrt{T}}n(d_1),$$

$$\Theta_p = \frac{\partial p}{\partial T} = -r^* S\mathrm{e}^{-r^*T} N(-d_1) + rK\mathrm{e}^{-rT} N(-d_2) - \frac{S\sigma \mathrm{e}^{-r^*T} n(d_1)}{2\sqrt{T}},$$

$$\upsilon_p = \frac{\partial p}{\partial \sigma} = S\mathrm{e}^{-r^*T}\sqrt{T} n(d_1), \quad \rho = \frac{\partial p}{\partial r^*} = TS\mathrm{e}^{-r^*T} N(-d_1).$$

A.4 *Merton's and Barone-Adesi and Whaley's Model*

Call sensitivity parameters

$$\Delta = \mathrm{e}^{(b-r)} N(d_1), \quad \Gamma_c = \frac{\partial \Delta_c}{\partial S} = \frac{\mathrm{e}^{(b-r)}}{S\sigma\sqrt{T}} n(d_1),$$

$$\Theta_c = \frac{\partial p}{\partial T} = (r-b) S\mathrm{e}^{(b-r)T} N(d_1) - rK\mathrm{e}^{-rT} N(d_2) - \frac{S\mathrm{e}^{(b-r)T} n(d_1)}{2\sqrt{T}},$$

$$\upsilon_c = \frac{\partial c}{\partial \sigma} = S\mathrm{e}^{-r^*T}\sqrt{T} n(d_1), \quad \rho_c = \frac{\partial c}{\partial b} = TS\mathrm{e}^{(b-r)T} N(d_1).$$

Put sensitivity parameters

$$\Delta_p = -\mathrm{e}^{(b-r)T}[N(-d_1) + 1], \quad \Gamma_p = \frac{\partial \Delta_p}{\partial S} = \frac{\mathrm{e}^{(b-r)T}}{S\sigma\sqrt{T}} n(d_1),$$

$$\Theta_p = \frac{\partial p}{\partial T} = S\mathrm{e}^{(b-r)} N(-d_1) + rK\mathrm{e}^{rT} N(d_2) \frac{S\mathrm{e}^{(b-r)}\sigma N(d_2)}{2\sqrt{T}},$$

$$\upsilon_p = \frac{\partial p}{\partial \sigma} = S\mathrm{e}^{(b-r)T}\sqrt{T} n(d_1), \quad \rho = \frac{\partial p}{\partial b} = -TS\mathrm{e}^{(b-r)T} N(-d_2).$$

Appendix 2: The Relationship Between Hedging Parameters

Using the definitions of the delta, gamma and theta, the B–S equation can be written as

$$\frac{\partial c(S,t)}{\partial t} = rc(S,t) - rS\left[\frac{\partial c(S,t)}{\partial S}\right] - \frac{1}{2}\sigma^2 S^2 \left[\frac{\partial^2 c(S,t)}{\partial S^2}\right]$$

or

$$-\Theta = -rc(S,t) + rS\Delta + \frac{1}{2}\sigma^2 S^2 \Gamma$$

or

$$rc(S, t) = \Theta + rS\Delta + \frac{1}{2}\sigma^2 S^2 \Gamma.$$

For a delta-neutral position, the following relationship applies

$$rc(S, t) = \Theta + \frac{1}{2}\sigma^2 S^2 \Gamma.$$

Using the definitions of the hedging parameters, the Black equation can be written as

$$\frac{\partial c(F, t)}{\partial t} = rc(F, t) - \frac{1}{2}\sigma^2 F^2 \left[\frac{\partial^2 c(F, t)}{\partial S^2}\right]$$

or

$$\Theta = rc(F, t) - \frac{1}{2}\sigma^2 F^2 \Gamma.$$

Using the definitions of the delta, gamma and theta, the Merton and BAW (1987) equation can be written as

$$\frac{1}{2}\sigma^2 S^2 \left[\frac{\partial^2 c(S, t)}{\partial S^2}\right] + bS\left[\frac{\partial c(S, t)}{\partial S}\right] - rc(S, t) + \Theta = 0$$

or

$$rc(S, t) = \Theta + bS\Delta + \frac{1}{2}S^2\sigma^2\Gamma.$$

For a delta-neutral position, the following relationship applies: $rc(S, T) = \frac{1}{2}S^2\sigma^2\Gamma + \Theta$.

Appendix 3: On the Valuation of Options and Information Costs

An important question in financial economics is how frictions affect equilibrium in capital markets since in a world of costly information, some investors will have incomplete information. Merton (1987) advanced the investor recognition hypothesis in a mean–variance model.

A.1 *Incomplete Information, Costly Arbitrage and Asset Pricing*

Shapiro (2000) examines equilibrium in a dynamic pure-exchange economy under a generalization of Merton's (1987) investor recognition hypothesis IRH). The premise in Merton's (1987) model and Shapiro's (2000) extension is that the costs of gathering and processing data lead some investors to focus on stocks with high visibility and also to entrust a portion of their wealth to money managers employed by pension plans. In this context, a trading strategy shaped by real-world information costs should incorporate an investment in well-known, visible stocks, and an investment delegated to professional money managers. In this theory, an investor considers only the stocks visible to him, i.e. those about which he has sufficient information to implement optimal portfolio rebalancing. In general, information about larger firms is likely to be available at a lower cost. The claim that large firms are more widely known is consistent with the empirical evidence that large firms have more shareholders as in Merton (1987). As documented by Falkenstein (1996), large firms present in general longer listing histories. Falkenstein (1996) documents that both size and age of a firm are positively correlated with the number of new stories in major newspapers about the firm.

For these reasons, it is important to account for information costs in the pricing of assets and derivatives. Most of the option pricing models are based on an arbitrage argument. While most traders are aware of the Black–Scholes (1973) theory, the arbitrage mechanism assumed cannot work in a real options market in the same way that it does in a supposed frictionless market. From a theoretical standpoint, it is possible to account for some imperfections in the derivation of option pricing models. However, the mathematical problems raised by treating real market conditions are often too complex to be tractable in theory.

From an empirically standpoint, it is possible to simulate trading strategies using historical option prices to see how much options arbitrage is affected by market imperfections. In this case, the transaction cost structure varies considerably among investors, traders and market makers which complicates the empirical tests. In practice, an arbitrageur follows a strategy that limits both trading costs (transaction costs) and risk. The first main point here is that hedging in not implemented in a continuous time framework.

The second point is that transaction costs are different from the costs of collecting information or information costs.

The third point, is that in less liquid markets, it is not always possible to implement an arbitrage strategy as described in the Black–Scholes theory.

The fourth point is that the appropriate hedge must account for some of the costs of arbitrage.

In a standard Black–Scholes approach in which the hedge is implemented instantaneously, the force of arbitrage drives the option price to its theoretical value.

A.2 The Intuition of the Derivation of Option Pricing Models with Costly Arbitrage

As it appears in the work of Scholes (1998), the option pricing technology was adopted because it reduces transaction costs. The wave of financial innovation might be explained by the reduction in the cost of computer and communications technology. This lower-cost technology plays a significant role in the globalization of products and financial markets. Advances in financial theory allowed financial services firms to meet the complex needs of the clients around the world at lower cost than was previously possible.[2] The arbitrage argument in Modigliani and Miller (see Miller, 1988) provided a general model of corporate finance by showing that the value of the firm is independent of how it financed its activities.[3]

Our analysis is based mainly on the use of capital asset pricing models for the valuation of derivatives. The work of Markowitz (1952), Sharpe (1964) and Lintner (1965) on the capital asset pricing model provided the general equilibrium model of asset prices under uncertainty. This model represents a fundamental tool in measuring the risk of a security under uncertainty. The first work of Black and Scholes was to test the standard CAPM by developing the concept of a zero-beta portfolio. A zero-beta-minimum variance portfolio can be implemented by buying low beta stocks and selling high beta stocks. If the realized returns on this portfolio are different from the interest rate, this would be a violation of the predictions of the original

[2]Financial service firms are the leaders in using derivatives in their own risk-management programs. In fact, using the available information and the option pricing technology, financial services firms can value their commitments and decide what risks to transfer and what risks to retain. As it appears in the tables reported in Scholes (1998), the OTC market in derivatives has grown much faster than the exchange market during the last 10 years. Therefore, it is expected that clients would find it cheaper to execute a program through their financial service than to execute it themselves in the exchange market.

[3]The generalization of the Modigliani and Miller analysis in the presence of shadow costs of incomplete information is done in our papers (Bellalah, 2000a,b).

CAPM.[4] We justify the use of Merton's (1987) model by the increasing empirical support for the implications of that simple model of capital market equilibrium with incomplete information. As it appears in Merton's (1998) paper, his main contribution to the Black–Scholes option pricing theory was to demontrate the following result: in the limit of continuous trading, the Black–Scholes dynamic trading strategy designed to offset the risk exposure of an option would provide a perfect hedge. Hence, when trading is done without cost, the Black–Scholes dynamic strategy using the option's underlying asset and a risk-free bond would exactly replicate the option's payoff.[5] This approach is used in the original Black–Scholes model who derived their formula using the standard CAPM.

Appendix 4: A General Equation for Derivative Securities

Consider a derivative security whose price depends on n state variables and time t. The security can be priced under the standard Black–Scholes assumptions. The state variables are assumed to follow Ito diffusion processes where for each state variable i between 1 and n, we have

$$\mathrm{d}\theta_i = m_i \theta_i \,\mathrm{d}t + s_i \theta_i \,\mathrm{d}z_i$$

where the growth rate m_i and the volatility s_i can be functions of any of the n variables and time.

Let us denote, respectively, by

— V_j: price of the jth traded security for j between 1 and $n+1$,
— r: risk-free rate,

[4]As it appears in Scholes (1998), his first work on option valuation was to apply the capital asset pricing model to value the warrants. The expected return of the warrant could not be constant for each time period if the beta of the stock was constant each period. This leads to the use of the CAPM to establish a zero-beta portfolio of common stocks and warrants. The portfolio is implemented by selling enough shares of common stock per each warrant held each period in order to create a zero-beta portfolio. In the context of the CAPM, the expected return on the net investment in the zero-beta portfolio would be equal to the riskless rate of interest.

[5]In the absence of a continuous trading, which represents an idealized prospect, replication with discrete trading intervals is at best only approximate. Merton (1976) study a mixture of jump and diffusion processes to capture the prospect of nonlocal movements in the underlying asset's return process.

Replication is not possible when the sample path of the underlying asset is not continuous. In this case, the derivation of an option pricing model is completed by using an equilibrium asset pricing model.

— λ_j: information cost for the jth traded security for j between 1 and $n+1$,
— $\rho_{i,k}$: correlation coefficient between dz_i and dz_k.

Since the $(n+1)$ traded securities depend on θ_i, then using Ito's lemma, we have

$$df_j = \mu_j V_j \, dt + \sum_i \sigma_{i,j} V_j \, dz_i \tag{A.1}$$

where

$$\mu_j V_j = \frac{\partial V_j}{\partial t} + \sum_i \frac{\partial V_j}{\partial \theta_i} m_i \theta_i + \frac{1}{2} \sum_{i,k} \rho_{i,k} s_i s_j \theta_i \theta_j \frac{\partial^2 f}{\partial \theta_i} \partial \theta_k \tag{A.2}$$

$$\sigma_{ij} f_j = \frac{\partial f_j}{\partial \theta_i} s_i \theta_i. \tag{A.3}$$

In this context, it is possible to construct a portfolio using the $(n+1)$ traded securities.

We denote by a_j the amount of the jth security in the portfolio Π so that

$$\Pi = \sum_j a_j V_j$$

where the a_j are chosen in a way to eliminate the stochastic components of the returns. Using Eq. (A.1), we have

$$\sum_j a_j \sigma_{ij} f_j = 0 \tag{A.4}$$

for i between 1 and n. The instantaneous return from this portfolio can be written as

$$d\Pi = \sum_j a_j \mu_j V_j \, dt$$

where the cost of constructing this portfolio is $\sum_j a_j V_j$.

If this portfolio is riskless, it must earn the riskless rate plus information costs corresponding to each asset in the portfolio

$$\sum_j a_j \mu_j f_j = \sum_j a_j f_j (r + \lambda_j) \tag{A.5}$$

which is equivalent to

$$\sum_j a_j f_j(\mu_j - r - \lambda_j) = 0. \tag{A.6}$$

Equations (A.4) and (A.6) are consistent, only if

$$f_j(\mu_j - r + \lambda_j) = \sum_i \gamma_i \sigma_{ij} f_j \tag{A.7}$$

or

$$\mu_j - r - \lambda_j = \sum_i \gamma_i \sigma_{ij} \tag{A.8}$$

where for γ_i, i is between 1 and n.

Using Eqs. (A.2) and (A.3) and replacing into Eq. (A.7) gives the following equation:

$$\frac{\partial V_j}{\partial t} + \sum_i \frac{\partial V_j}{\partial \theta_i} m_i \theta_i + \frac{1}{2} \sum_{i,k} \rho_{ik} s_i s_k \theta_i \theta_k \frac{\partial^2 V_j}{\partial \theta_i \partial \theta_k} - (r + \lambda_j) V_j = \sum_i \gamma_i \frac{\partial V_j}{\partial \theta_i} s_i \theta_i.$$

This last equation reduces to

$$\frac{\partial V_j}{\partial t} + \sum_i \theta_i \frac{\partial V_j}{\partial \theta_i}(m_i - \gamma_i s_i) + \frac{1}{2} \sum_{i,k} \rho_{ik} s_i s_k \theta_i \theta_k \frac{\partial^2 V_j}{\partial \theta_i \partial \theta_k} = (r + \lambda_j) V_j.$$

Hence, any security f contingent on the state variables θ_i and time must satisfy the following second order differential equation:

$$\frac{\partial V}{\partial t} + \sum_i \theta_i \frac{\partial V_j}{\partial \theta_i}(m_i - \gamma_i s_i) + \frac{1}{2} \sum_{i,k} \rho_{ik} s_i s_k \theta_i \theta_k \frac{\partial^2 V}{\partial \theta_i \partial \theta_k} = (r + \lambda) V. \tag{A.9}$$

Appendix 5: Extension to the Risk-Neutral Valuation Argument

When the variable θ_i is not a traded asset, it is possible to assume the existence of a traded asset $\tilde{\theta}_i$ paying a continuous dividend $\hat{q}$ where

$$\hat{q} = r + \lambda_i - m_i + \gamma_i s_i.$$

It is important to note that this term corresponding to the dividend yield allows the conversion of θ_i to a tradeable security. The values of $\hat{\theta}_i$ and θ_i must be equal and the following differential equation must be verified:

$$\frac{\partial V}{\partial t} + \sum_i \hat{\theta}_i \frac{\partial V}{\partial \hat{\theta}_i}(r + \lambda_i - \hat{q}_i) + \frac{1}{2}\sum_{i,k} \rho_{i,k} s_i s_k \hat{\theta}_i \hat{\theta}_k \frac{\partial^2 V}{\partial \hat{\theta}_i \partial \hat{\theta}_k} = (r + \lambda)V.$$

This equation is independent of risk preferences.

Since $(r + \lambda_i - \tilde{q}) = m_i - \gamma_i s_i$, the derivative security can be valued in a risk-neutral economy if the drift term in θ_i is modified from m_i to $m_i - \gamma_i s_i$.

Appendix 6: The Cumulative Normal Distribution Function

The following approximation of the cumulative normal distribution function $N(x)$ produces values to within four decimal place accuracy:

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp(-z^2/2)\mathrm{d}z$$

$$\begin{cases} N(x) = 1 - n(x)(a_1 k + a_2 k^2 + a_3 k^3), & \text{when} \quad x \geq 0 \\ 1 - n(-x), & \text{when} \quad x < 0 \end{cases}$$

where $k = \frac{1}{1+0.33267x}$, $a_1 = 0.4361836$, $a_2 = -0.1201676$, $a_3 = 0.9372980$, $n(x) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-x^2/2}$.

The next approximation provides the values of $N(x)$ within six decimal places of the true value:

$$\begin{cases} N(x) = 1 - n(x)(a_1 k + a_2 k^2 + a_3 k^3 + a_4 k^4 + a_5 k^5), & \text{when } x \geq 0 \\ 1 - n(-x), & \text{when } x < 0 \end{cases}$$

where

$$k = \frac{1}{1 + 0.2316419x},$$

$$a_1 = 0.319381530, \quad a_2 = -0.356563782, \quad a_3 = 1.781477937,$$
$$a_4 = -1.821255978, \quad a_5 = 1.330274429.$$

Appendix 7: The Bivariate Normal Density Function

$$F(x, y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2\pi(1-\rho^2)}(x^2 - 2\rho xy + y^2)\right].$$

A.1 *The Cumulative Bivariate Normal Density Function*

The standardized cumulative normal function gives the probability that a specified random variable is less than a and that another random variable is less than b when the correlation between the two variables is ρ. It is given by:

$$M(a, b; \rho) = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{a} \int_{-\infty}^{b} \exp\left[-\frac{x^2 - 2\rho xy + y^2}{2\pi(1-\rho^2)}\right].$$

This following approximation produces values of $M(a, b; \rho)$ to within six decimal places accuracy:

$$\phi(a, b; \rho) = \frac{\sqrt{1-\rho^2}}{\pi} \sum_{i=1}^{5} \sum_{j=1}^{5} x_i x_j f(y_i, y_j),$$

where

$$f(y_i, y_j) = \exp[a_1(2y_i - a_1) + b_1(2y_j - b_1) + 2\rho(y_i - a_1)(y_j - b_1)],$$

$$a_1 = \frac{a}{\sqrt{2(1-\rho^2)}}, \quad b_1 = \frac{b}{\sqrt{2(1-\rho^2)}},$$

$$\begin{aligned}
x_1 &= 0.24840615 & y_1 &= 0.10024215 \\
x_2 &= 0.39233107 & y_2 &= 0.48281397 \\
x_3 &= 0.21141819 & y_3 &= 1.0609498 \\
x_4 &= 0.033246660 & y_4 &= 1.7797294 \\
x_5 &= 0.00082485334 & y_5 &= 2.6697604.
\end{aligned}$$

If the product of a, b, and ρ is nonpositive, we must compute the cumulative bivariate normal probability by applying the following rules:

1. If $a \preceq 0$, $b \preceq 0$, and $\rho \preceq 0$, then
$$M(a, b; \rho) = \phi(a, b; \rho).$$

2. If $a \preceq 0$, $b \succeq 0$, and $\rho \succeq 0$, then
$$M(a, b; \rho) = N(a) - \phi(a, -b; -\rho).$$

3. If $a \succeq 0$, $b \preceq 0$, and $\rho \succeq 0$, then
$$M(a, b; \rho) = N(b) - \phi(-a, b; -\rho).$$

4. If $a \succeq 0$, $b \succeq 0$, and $\rho \preceq 0$, then
$$M(a, b; \rho) = N(a) + N(b) - 1 + \phi(-a, -b; \rho).$$

In cases where the product of a, b, and ρ is positive, compute the cumulative bivariate normal function as

$$M(a, b; \rho) = M(a, 0; \rho_1) + M(b, 0; \rho_2) - \delta$$

where $M(a, 0; \rho_1)$ and $M(a, 0; \rho_2)$ are computed from the rules where the product of a, b, and ρ is negative, and

$$\rho_1 = \frac{(\rho a - b)\,\mathrm{Sign}(a)}{\sqrt{a^2 - 2\rho ab + b^2}}, \quad \rho_2 = \frac{(\rho b - a)\,\mathrm{Sign}(b)}{\sqrt{a^2 - 2\rho ab + b^2}},$$

$$\delta = \frac{1 - \mathrm{Sign}(a) \times \mathrm{Sign}(b)}{4}, \quad \mathrm{Sign}(x) = \begin{cases} +1, & \text{when } x \geq 0 \\ -1, & \text{when } x < 0 \end{cases}.$$

Bibliography

Ayres, HF (1963). Risk aversion in the warrants market. *Industrial Management Review*, 4 (Fall), 497–505.

Bachelier, L (1900). Théorie de la spéculation. Annales de l'école normale supérieure, 17.

Barone-Adesi, G and RE Whaley (1987). Efficient analytic approximation of American option values. *Journal of Finance*, XLII, 2(81), 303–320.

Bellalah, M (1999). The valuation of futures and commodity options with information costs. *Journal of Futures Markets*, September.

Bellalah, M (2000a). A risk management approach to options trading on the Paris bourse. *Derivatives Strategy*, 5(6), 31–33.

Bellalah, M (2000b). Valuation of American CAC 40 index options and wildcard options. *International Review of Economics and Finance*, to appear.

Bellalah, M (2001). Market imperfections, information costs and the valuation of derivatives: some general results. *International Journal of Finance*, 13(3), 1895–1927.

Bellalah, M and B Jacquillat (1995). Option valuation with information costs: theory and tests. *The Financial Review*, 30(3), August, 617–635.

Bellalah, M, Ma Bellalah and R Portait (2001). The cost of capital in international finance. *International Journal of Finance*, 13(3), 1958–1973.

Bellalah, M and JL Prigent (2001). Pricing standard and exotic options in the presence of a finite mixture of Gaussian Distributions. *International Journal of Finance*, 13(3), 1975–2000.

Bellalah, M, JL Prigent and C Villa (2001). Skew without Skewness: asymmetric smiles; information costs and stochastic volatilities. *International Journal of Finance*, 13(2), 1826–1837.

Bellalah, M and F Selmi (2001). On the quadratic criteria for hedging under transaction costs. *International Journal of Finance*, 13(3), 2001–2020.

Black, F (1976). The pricing of commodity contracts. *Journal of Financial Economics*, 79(3), 167–179.

Black, F (1989). How we came up with the option formula. *Journal of Portfolio Management*, Winter, 4–8.

Black, F and M Scholes (1973). The pricing of options and corporate liabilities. *Journal of Political Economy*, 81, 637–659.

Boness, AJ (1964). Elements of a theory of stock option value. *Journal of Political Economy*, 72 (April), 163–175.

Chen, A (1970). A model of warrant pricing in a dynamic market. *Journal of Finance*, 25, 1041–1060.

Churchill, RV (1963). *Fourier Series and Boundary Value Problems*, 2nd ed. New York, McGraw-Hill.

Cox, JC, JE Ingersoll and SA Ross (1985). A theory of the term structure of interest rates. *Econometrica*, 53, 385–407.

Cox, JC and SA Ross (1976). The valuation of options for alternative stochastic processes. *Journal of Financial Economics*, 3, 145–166.

Dusak, K (1973). Futures trading and investor returns: an investigation of commodity market risk premiums. *Journal of Political Economy*, 1387–1404.

Falkenstein, S (1996). Preferences for stock characteristics as revealed by mutual fund portfolio holdings. *Journal of Finance*, 51(1), 111–135.

Garman, M (1976). *A General Theory of Asset Valuation under Diffusion State Processes*, W.P. N 50, University of California, Berkeley.

Garman, M and S Kohlhagen (1983). Foreign currency option values. *Journal of International Money and Finance*, 2, 231–237.

Hull, J (2000). *Options, Futures and Other Derivatives*, Englewood Cliffs, New Jersey, Prentice Hall, Inc.

Lintner, J (1965). Security prices, risk and maximal gains from diversification. *Journal of Finance*, 587–516.

Markowitz, HM (1952). Portfolio selection. *Journal of Finance*, 7(1), 77–91.

Merton, R (1973). Theory of rational option pricing. *Bell Journal of Economics and Management Science*, 4, 141–183.

Merton, RC (1987). A simple model of capital market equilibrium with incomplete information. *Journal of Finance*, 42, 483–510.

Merton, RC (1998). Applications of option pricing theory: twenty-five years later. *American Economic Review*, 2, 323–348.

Miller, M (1988). The Modigliani–Miller propositions after thirty years. *Journal of Economic Perspectives*, 99–120.

Omberg, (1991). On the theory of perfect hedging. *Advances in Futures and Options Research*, 5.

Samuelson, PA (1965). Rational theory of warrant pricing. *Industrial Management Review*, 6, 13–31.

Samuelson, P and RC Merton (1969). A complete model of warrant pricing that maximises utility. *Industrial Management Review*, 10, 17–46.

Scholes, M (1998). Derivatives in a dynamic environment. *American Economic Review*, 2, 350–370.

Shapiro, A (2000). The investor recognition hypothesis in a dynamic general equilibrium: theory and evidence. Working Paper, Stern School of Business, New York University.

Sharpe, WF (1964). Capital asset prices: a theory of market equilibrium under conditions of risk. *Journal of Finance*, 425–442.

Smith, CW (1976). Option pricing: a review. *Journal of Financial Economics*, 3 (Jan/March 1976), 3–52.

Smithson, C (1991). Wonderful life. *Risk*, 4 (9) (October), 50–51.

Sprenkle, C (1961). Warrant prices as indications of expectations. *Yale Economic Essays*, 1, 179–232.

Thorp, E and S Kassouf (1967). *Beat the Market*, New York, Random House.