CQF Exercises 3.1 Solutions

Throughout this exercise you may use assume (where appropriate) the following results without proof

$$
\begin{aligned}
d_1 &= \frac{\log(S/E) + \left(r - D + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}, \\
d_2 &= \frac{\log(S/E) + \left(r - D - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \quad \text{and} \\
N(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp(-\phi^{\ 2}/2) d\phi
\end{aligned}
$$

where $S \geq 0$ is the spot price, $t \leq T$ is the time, $E > 0$ is the strike, $T > 0$ the expiry date, $r \geq 0$ the interest rate, D is the dividend yield and σ is the volatility of S.

1. The Black–Scholes formula for a European call option $C(S,t)$ is given by

$$C(S,t) = S\exp(-D(T-t))N(d_1) - E\exp(-r(T-t))N(d_2).$$

a) By differentiating with respect to S and σ show that the delta and vega are given by

$$\Delta = e^{(-D(T-t))} N(d_1), \ and \ \ v = \sqrt{\frac{T-t}{2\pi}} S e^{(-D(T-t))} e^{\left(\frac{-d_1{}^2}{2}\right)}.$$

Note:

$$\frac{\partial d_1}{\partial S} = \frac{\partial d_2}{\partial S} \quad \text{and} \ \frac{\partial d_2}{\partial \sigma} = \frac{\partial d_1}{\partial \sigma} - \sqrt{T-t}$$

So

$$
\begin{aligned}
\Delta &= \frac{\partial C}{\partial S} \\
&= e^{(-D(T-t))} N(d_1) + S e^{(-D(T-t))} \frac{1}{\sqrt{2\pi}} e^{\left(-\frac{d_1{}^2}{2}\right)} \frac{\partial d_1}{\partial S} - E\exp(-r(T-t)) \frac{1}{\sqrt{2\pi}} e^{\left(-\frac{d_2{}^2}{2}\right)} \frac{\partial d_2}{\partial S} \\
&= e^{(-D(T-t))} N(d_1) + \frac{1}{\sqrt{2\pi}} \frac{\partial d_1}{\partial S} \underbrace{\left(S e^{(-D(T-t))} e^{\left(-\frac{d_1{}^2}{2}\right)} - E e^{(-r(T-t))} e^{\left(-\frac{d_2{}^2}{2}\right)} \right)}_{=0} \\
&= e^{(-D(T-t))} N(d_1) \quad \text{because the term in the bracket above is zero.}
\end{aligned}
$$

$$
\begin{aligned}
\upsilon &= \frac{\partial C}{\partial \sigma} \\
&= Se^{(-D(T-t))}\frac{1}{\sqrt{2\pi}}e^{\left(-\frac{{d_1}^2}{2}\right)}\frac{\partial d_1}{\partial \sigma} - Ee^{(-r(T-t))}\frac{1}{\sqrt{2\pi}}e^{\left(-\frac{{d_2}^2}{2}\right)}\frac{\partial d_2}{\partial \sigma} \\
&= Se^{(-D(T-t))}\frac{1}{\sqrt{2\pi}}e^{\left(-\frac{{d_1}^2}{2}\right)}\left(\frac{\partial d_2}{\partial \sigma} + \sqrt{T-t}\right) - \frac{1}{\sqrt{2\pi}}Ee^{(-r(T-t))}e^{\left(-\frac{{d_2}^2}{2}\right)}\frac{\partial d_2}{\partial \sigma} \\
&= \sqrt{\frac{T-t}{2\pi}}Se^{(-D(T-t))}e^{\left(-\frac{{d_1}^2}{2}\right)} + \frac{\partial d_2}{\partial \sigma}\frac{1}{\sqrt{2\pi}}\left[\underbrace{Se^{(-D(T-t))}e^{\left(-\frac{{d_1}^2}{2}\right)} - Ee^{(-r(T-t))}e^{\left(-\frac{{d_2}^2}{2}\right)}}_{=0}\right] \\
&= \sqrt{\frac{T-t}{2\pi}}Se^{(-D(T-t))}e^{\left(-\frac{{d_1}^2}{2}\right)} \quad \left(= \sqrt{\frac{T-t}{2\pi}}Ee^{(-r(T-t))}e^{\left(-\frac{{d_2}^2}{2}\right)}\right)
\end{aligned}
$$

2. The Black–Scholes Equation (BSE) in the absence of dividends is

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-D)\,S\frac{\partial V}{\partial S} - rV = 0.$$

Find all separable solutions of the form $V(S,t) = \Phi(S)\,\Psi(t)$.

See solution sheet for mod 1 exam with, only minor modification of $-D$, for the dividend.

As previously: $\Psi' = c\Psi \rightarrow \Psi = k\exp(ct)$

Secondly a 2nd order Cauchy-Euler equation:

$$\frac{1}{2}\sigma^2 S^2 \Phi'' + (r-D)\,S\Phi' + (c-r)\,\Phi = 0$$

Putting

$$\Phi(S) = S^d$$

gives a quadratic in d

$$d^{\,2} + \left(\frac{2\,(r-D)}{\sigma^2} - 1\right)d - \frac{2}{\sigma^2}\,(r-c) = 0$$

hence

$$
\begin{aligned}
d_{\pm} &= \frac{1}{2}\left(1-\frac{2(r-D)}{\sigma^2}\right) \pm \frac{1}{2}\sqrt{\left(\frac{2(r-D)}{\sigma^2}-1\right)^2+\frac{8}{\sigma^2}(r-c)} \\
d_{\pm} &= \frac{1}{2}\left(1-\frac{2(r-D)}{\sigma^2}\right) \pm \frac{1}{2}\sqrt{\frac{4(r-D)^2}{\sigma^4}+1-\frac{4(r-D)}{\sigma^2}+\frac{8(r-c)}{\sigma^2}} \\
&= \frac{1}{2}\left(1-\frac{2(r-D)}{\sigma^2}\right) \pm \sqrt{\left(\frac{r-D}{\sigma^2}+\frac{1}{2}\right)^2-\frac{2(c-D)}{\sigma^2}} \\
&\equiv \frac{1}{2}\left(1-\frac{2(r-D)}{\sigma^2}\right) \pm \sqrt{\left(\frac{r-D}{\sigma^2}-\frac{1}{2}\right)^2+\frac{2(r-c)}{\sigma^2}}
\end{aligned}
$$

<u>3 cases to consider</u>:

(1) Solution for distinct roots - $\Phi(S) = aS^{d_+} + bS^{d_-}$

$$V(S,t) = \exp(ct)\, S^{\frac{1}{2}-\frac{r-D}{\sigma^2}}\left[AS^{\overline{d}_+} + BS^{\overline{d}_-}\right] \qquad A,\ B \text{ - constants}$$

where

$$\overline{d}_+ = \sqrt{\left(\frac{r}{\sigma^2}+\frac{1}{2}\right)^2-\frac{2(c-D)}{\sigma^2}}\ ; \qquad \overline{d}_- = -\sqrt{\left(\frac{r}{\sigma^2}+\frac{1}{2}\right)^2-\frac{2(c-D)}{\sigma^2}}$$

(2) Repeated Root - $\Phi(S) = S^{\frac{1}{2}-\frac{r}{\sigma^2}}[a + b\log S]$
Now $\left(\frac{r-D}{\sigma^2}+\frac{1}{2}\right)^2 = \frac{2(c-D)}{\sigma^2} \to c = D + \frac{\sigma^2}{2}\left(\frac{r-D}{\sigma^2}+\frac{1}{2}\right)^2$ therefore

$$V(S,t) = \exp\left(D + \frac{\sigma^2}{2}\left(\frac{r-D}{\sigma^2}+\frac{1}{2}\right)^2 t\right) S^{\left(\frac{1}{2}-\frac{r-D}{\sigma^2}\right)}[\varepsilon + \zeta\log S] \qquad \varepsilon,\ \zeta \text{ - constants}$$

(3) Complex Roots i.e. $\frac{2c}{\sigma^2} > \left(\frac{r}{\sigma^2}+\frac{1}{2}\right)^2$ - $d_+ = \alpha + i\beta$; $d_- = \alpha - i\beta$

$$\Phi(S) = S^{\alpha}[A\cos(\beta\ln S) + B\sin(\beta\ln S)]$$

where

$$\alpha = \left(\frac{1}{2}-\frac{r-D}{\sigma^2}\right); \qquad \beta = \sqrt{\left|\left(\frac{r-D}{\sigma^2}+\frac{1}{2}\right)^2-\frac{2(c-D)}{\sigma^2}\right|}$$

$$V(S,t) = \exp(ct)\, S^{\left(\frac{1}{2}-\frac{r-D}{\sigma^2}\right)} \left[A\cos(\beta \ln S) + B\sin(\beta \ln S)\right]$$

3. The Black–Scholes formula for a European call option $C(S,t)$ is

$$C(S,t) = S\exp(-D(T-t))N(d_1) - E\exp(-r(T-t))N(d_2)$$

From this expression, find the Black–Scholes value of the call option in the following limits:

(a) (time tends to expiry) $t \to T^{-}$, $\sigma > 0$
$\exp(-r(T-t))$, $\exp(-D(T-t)) \to 1$

$$d_{1\ 2} \to \frac{\log(S/E)}{\sigma\sqrt{T-t}} + O\left(\sqrt{T-t}\right) \to \begin{cases} \infty & S > E \\ 0 & S = E \\ -\infty & S < E \end{cases} \qquad \text{so} \quad C \to \begin{cases} S-E & S > E \\ 0 & S = E \\ 0 & S < E \end{cases}$$

(b) (volatility tends to zero) $\sigma \to 0^{+}$, $t < T$;

$$\begin{aligned} d_{1\ 2} &\to \frac{\log(S/E) + (r-D)(T-t)}{\sigma\sqrt{T-t}} + O(\sigma) = \frac{\log(S\exp(-D(T-t))/E\exp(-r(T-t)))}{\sigma\sqrt{T-t}} + O(\sigma) \\ &\to \begin{cases} \infty & Se^{(-D(T-t))} > Ee^{(-r(T-t))} \\ 0 & Se^{(-D(T-t))} = Ee^{(-r(T-t))} \\ -\infty & Se^{(-D(T-t))} < Ee^{(-r(T-t))} \end{cases} \qquad \text{so} \quad C \to \max\left[Se^{(-D(T-t))} - Ee^{(-r(T-t))}, 0\right] \end{aligned}$$

(c) (volatility tends to infinity) $\sigma \to \infty$, $t < T$;

$$\begin{aligned} d_{1\ 2} &\to \pm\frac{1}{2}\sigma\sqrt{T-t} + O\left(\frac{1}{\sigma}\right) \to \pm\infty \\ C &\to Se^{(-D(T-t))}N(\infty) - Ee^{(-r(T-t))}N(-\infty) = Se^{(-D(T-t))} \end{aligned}$$

(d) (expiry tends to infinity) $T \to \infty$
$e^{-D(T-t)} \to 0$ if $D \neq 0$ & $e^{-r(T-t)} \to 0 \Rightarrow C \to 0$
if $D = 0$, $d_1 \to \infty$ and $C \to S$.

(e) (dividends yield tends to infinity) $D \to \infty$, $t < T$, $\sigma > 0$ and finite

$d_{1\ 2} \to -\infty \therefore N\left(d_{1\ 2}\right) \to 0,\ \ e^{-D(T-t)} \to 0\ \ \&\ C \to 0$

4. Suppose S evolves according to the stochastic differential equation (SDE)

$$dS = \mu S dt + S^{\alpha} dX$$

where μ and α are positive constants. Given that the interest rate is zero, derive the corresponding Black–Scholes partial differential equation (PDE) for the option based upon this asset S (you are not required to solve any equation). Write this PDE in terms of the Greeks.

We know that if S evolves according to the stochastic differential equation (SDE)

$$dS = a\left(S, t\right) dt + b\left(S, t\right) dX$$

and $V = V\left(S, t\right)$ then Itô gives

$$dV = \left[\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{1}{2} S^{2\alpha} \frac{\partial^2 V}{\partial S^2}\right] dt + \left[S^{\alpha} \frac{\partial V}{\partial S}\right] dX$$

Then set up a portfolio $\Pi = V - \Delta S \Rightarrow$ in one time-step (we hold Δ fixed) $d\,\Pi = dV - \Delta dS$

So

$$d\Pi = \left[\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{1}{2} S^{2\alpha} \frac{\partial^2 V}{\partial S^2}\right] dt + \left[S^{\alpha} \frac{\partial V}{\partial S}\right] dX - \Delta\left[\mu S dt + S^{\alpha} dX\right]$$

therefore we take $\Delta = \frac{\partial V}{\partial S}$ to eliminate the risk associated with the portfolio (to cancel out terms with dX), which gives

$$d\Pi = \left[\frac{\partial V}{\partial t} + \frac{1}{2} S^{2\alpha} \frac{\partial^2 V}{\partial S^2}\right] dt$$

portfolio now riskless. No arbitrage tells us that we are guaranteed return at risk free rate, so

$$\begin{aligned}
\left[\frac{\partial V}{\partial t} + \frac{1}{2} S^{2\alpha} \frac{\partial^2 V}{\partial S^2}\right] dt &= r\Pi dt \\
\left[\frac{\partial V}{\partial t} + \frac{1}{2} S^{2\alpha} \frac{\partial^2 V}{\partial S^2}\right] dt &= r\left(V - S\frac{\partial V}{\partial S}\right) dt
\end{aligned}$$

$$\frac{\partial V}{\partial t}+\frac{1}{2}S^{2\alpha}\frac{\partial^2 V}{\partial S^2}+rS\frac{\partial V}{\partial S}-rV=0.$$

Using Greeks $\frac{\partial V}{\partial t}=\Theta$ and $\frac{\partial^2 V}{\partial S^2}=\Gamma$ allows us to write this pde as

$$\Theta+\frac{1}{2}S^{2\alpha}\Gamma+rS\Delta=rV.$$

5. The call and put option values in turn are given by

$$\begin{aligned} C(S,t) &= S\exp(-D(T-t))N(d_1)-E\exp(-r(T-t))N(d_2) \\ P(S,t) &= E\exp(-r(T-t))N(-d_2)-S\exp(-D(T-t))N(-d_1). \end{aligned}$$

The put-call parity relationship for these options is

$$\begin{aligned} C-P &= (S\exp(-D(T-t))N(d_1)-E\exp(-r(T-t))N(d_2))- \\ &\quad (E\exp(-r(T-t))N(-d_2)-S\exp(-D(T-t))N(-d_1)) \end{aligned}$$

which becomes

$$S\exp(-D(T-t))\left(N(d_1)+N(-d_1)\right)-E\exp(-r(T-t))\left(N(d_2)+N(-d_2)\right).$$

We use

$$N(x)+N(-x)=1$$

giving the result

$$S\exp(-D(T-t))-E\exp(-r(T-t)).$$

6. The value of an option $V(S,t)$ satisfies the Black–Scholes equation. Write the option value in the form

$$V(S,t)=\exp(-r(T-t))q(S,t). \qquad (*)$$

Show that the function $q(S,t)$ satisfies the equation

$$\frac{\partial q}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 q}{\partial S^2}+(r-D)S\frac{\partial q}{\partial S}=0.$$

This is the backward Kolmogorov equation, used for calculating the expected value of stochastic quantities.

Substitute

$$
\begin{aligned}
\frac{\partial V}{\partial t} &= \exp(-r(T-t))\frac{\partial}{\partial t}q(S,t) + rV(S,t), \\
\frac{\partial V}{\partial S} &= \exp(-r(T-t))\frac{\partial q}{\partial S} \quad \& \\
\frac{\partial^2 V}{\partial S^2} &= \exp(-r(T-t))\frac{\partial^2 q}{\partial S^2}
\end{aligned}
$$

from (*) into the BSE, all the exponentials cancel out and the above equation is left.

Thus the value of an option can be expressed in the form

$$V(S,t) = \exp(-r(T-t))\,\mathbb{E}[\text{Payoff}(S)]$$

where $\mathbb{E}[x]$ means the expected value of x . This is not a real expectation, but taken under the risk-neutral random walk (so r replaces μ) and forms the basis of Monte Carlo methods applied to finance. More on this later.

7. A European Call option satisfies the following problem:

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{dV}{dS} - rV = 0$$

with boundary conditions

$$
\begin{aligned}
V(0,t) &= 0 \\
\lim_{S\longrightarrow\infty} V(S,t) &\sim S
\end{aligned}
$$

and final condition

$$V(S,T) = \max(S-E,0).$$

The first transformation gives us the following problem for $v(x,\tau)$:

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + (k-1)\frac{\partial v}{\partial x} - kv,$$

where $k = 2r/\sigma^2$, with boundary conditions

$$v(x,\tau) \longrightarrow 0 \text{ as } x \longrightarrow -\infty \text{ and } v(x,\tau) \sim e^x \text{ as } e^x \longrightarrow \infty,$$

and initial data

$$v(x,0) = \max\left(e^x - 1, 0\right).$$

The second transformation gives us

$$\beta u + \frac{\partial u}{\partial \tau} = \alpha^2 u + 2\alpha \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} + (k-1)\left(\alpha u + \frac{\partial u}{\partial x}\right) - ku.$$

We can eliminate the $\frac{\partial u}{\partial x}$ term by choosing α such that

$$0 = 2\alpha + (k-1),$$

and we can further eliminate the u term by choosing β such that

$$\beta = \alpha^2 + (k-1)\alpha - k.$$

Solving these equations for α and β, we obtain

$$\begin{aligned} \alpha &= -\tfrac{1}{2}(k-1) \\ \beta &= -\tfrac{1}{4}(k+1)^2. \end{aligned}$$

These choices for α, β give the following problem for the unknown function u :

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2},$$

with boundary conditions

$$u(x,\tau) \longrightarrow 0 \text{ as } x \longrightarrow -\infty \text{ and } u(x,\tau) \sim \exp\left(\tfrac{1}{2}(k+1)x\right) \text{ as } e^x \longrightarrow \infty,$$

and initial data

$$u(x,0) = \max\left(\exp\left(\tfrac{1}{2}(k+1)x\right) - \exp\left(\tfrac{1}{2}(k-1)x\right), 0\right).$$