Example: Zero-coupon bond options (for Vasicek process)

- A zero-coupon bond pays \$1 at maturity T.

$$P(r(t),t,T) = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r(u)du}\right]$$

- Assume a one-factor *affine* interest rate process under $\mathbb{Q}$:

$$dr(t) = \kappa(\theta - r(t))dt + \sigma dW^{\mathbb{Q}}(t)$$

- The bond satisfied the following PDE with boundary condition:

$$\begin{aligned} \frac{1}{2}P_{rr}\sigma^2 + P_r\,\kappa(\theta - r(t)) + P_t - r\,P &= 0 \\ P(r,T,T) &= 1 \end{aligned}$$

- Here $P(r(t),t,T) = e^{A(T-t)-B(T-t)r(t)}$ where $A(T-t)$ and $B(T-t)$ solve a system of ODEs with boundary conditions $A(0) = B(0) = 0$.

- The solution is:

$$B(\tau) = \frac{1-e^{-\kappa\tau}}{\kappa} \quad \text{and} \quad A(\tau) = \int_0^\tau \left(\frac{1}{2}B^2(u)\sigma - B(u)\kappa\theta\right) du$$

Example: Zero-coupon bond options (for Vasicek process)

- The bond with maturity T, $P(r,t,T)$, is a tradable asset, thus:

$$\frac{dP(r,t,T)}{P(r,t,T)} = r(t)dt + \sigma_P(t,T)dW^{\mathbb{Q}}(t)$$

  The volatility of the bond's price process is $\sigma_P(t,T) = -\sigma\,B(T-t)$. (Why?)

  Note that the volatility goes to zero as time approaches maturity T.

- Consider a European call option with maturity T_0 over a zero-coupon bond with maturity $T_1 > T_0$.

$$\begin{aligned} C(r,t,T_0) &= \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^{T_0} r(u)du}\max[P(T_0,T_1)-K,0]\right] \\ &= \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^{T_0} r(u)du}P(T_0,T_1)\mathbf{1}_{\{P(T_0,T_1)>K\}}\right] \\ &\qquad -K\,\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^{T_0} r(u)du}\mathbf{1}_{\{P(T_0,T_1)>K\}}\right] \\ &= \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^{T_1} r(u)du}\mathbf{1}_{\{P(T_0,T_1)>K\}}\right] - K\,\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^{T_0} r(u)du}\mathbf{1}_{\{P(T_0,T_1)>K\}}\right] \end{aligned}$$

Example: Zero-coupon bond options (for Vasicek process)

- At maturity T_i, $P(T_i, T_i) = 1$ for $i = 0, 1$.

- We can change numeraire to solve for $X_i(t, T_0, T_1) = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^{T_i} r(u)du}\mathbf{1}_{\{P(T_0,T_1)>K\}}\right]$.

$$\frac{X_i(t, T_0, T_1)}{P(t, T_i)} = \frac{\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^{T_i} r(u)du} P(T_i, T_i)\mathbf{1}_{\{P(T_0,T_1)>K\}}\right]}{P(t, T_i)} = \mathbb{E}_t^{\mathbb{R}(T_i)}\left[\mathbf{1}_{\{P(T_0,T_1)>K\}}\right]$$

- The measure $\mathbb{R}(T_i)$ ($\sim \mathbb{Q}$) is known as the T_i-*forward measure* and defined by:

$$\frac{d\mathbb{R}(T_i)}{d\mathbb{Q}} = Z_i(T) = \frac{e^{-\int_0^T r(u)du} P(T, T_i)}{P(0, T_i)}$$

- Note that $\mathbb{E}^{\mathbb{Q}}[Z_i(T)] = Z_i(0) = 1$ and $Z_i(t) > 0$ for all $0 \le t \le T_i$.

- $W^{\mathbb{R}}(t)$ is a standard Brownian motion under the measure $\mathbb{R}(T_i)$

$$W^{\mathbb{R}}(t) = W^{\mathbb{Q}}(t) - \int_0^t \sigma_P(u, T_i)du$$

Example: Zero-coupon bond options (for Vasicek process)

- The price of the zero-coupon bond call option is:

$$\begin{aligned} C(t, T_0) &= X_1(t, T_0, T_1) - K\, X_0(t, T_0, T_1) \\ &= P(t, T_1)\, N(d_1) - P(t, T_0)\, K\, N(d_2) \end{aligned}$$

where

$$\begin{aligned} d_1 &= \frac{\ln \frac{P(t,T_1)}{K\,P(t,T_0)} + \frac{1}{2}\Sigma(t, T_0)^2}{\Sigma(t, T_0)} \\ d_2 &= \frac{\ln \frac{P(t,T_1)}{K\,P(t,T_0)} - \frac{1}{2}\Sigma(t, T_0)^2}{\Sigma(t, T_0)} \\ \Sigma(t, T_0)^2 &= \int_t^{T_0} (\sigma_P(u, T_1) - \sigma_P(u, T_0))^2 du \end{aligned}$$