# CQF Module 3.2 The Greeks

1. Use put-call parity to find the relationships bewteen the deltas($\Delta$), gammas($\Gamma$), vegas($vega$), thetas($\Theta$), rhos($\rho$) of European call and put options.

   **Solution:** Starting with put-call parity

   $$C\left(S,t\right) - P\left(S,t\right) = S - Ee^{-r(T-t)} \qquad (*)$$

   Differentiating (*) wrt $S$ gives

   $$\frac{\partial C}{\partial S} - \frac{\partial P}{\partial S} = 1$$

   i.e.
   $$\Delta_C = 1 + \Delta_P \qquad (1)$$

   Now differentiate (*) for a 2nd time (i.e differentiate (1) once)

   $$\frac{\partial^2 C}{\partial S^2} - \frac{\partial^2 P}{\partial S^2} = 0,$$

   i.e.
   $$\Gamma_C = \Gamma_P.$$

   We follow a similar approach for the other greeks.

   Diff. (*) wrt $\sigma$ to obtain
   $$\frac{\partial C}{\partial \sigma} = \frac{\partial P}{\partial \sigma},$$

   and wrt $t$ gives
   $$\frac{\partial C}{\partial t} - \frac{\partial P}{\partial t} = -rEe^{-r(T-t)}$$
   $$\Rightarrow \Theta_C = \Theta_P - rEe^{-r(T-t)}.$$

   Finally, differentiate (*) wrt $r$

   $$\frac{\partial C}{\partial r} - \frac{\partial P}{\partial r} = E\left(T - t\right)e^{-r(T-t)},$$

   i.e.
   $$\rho_C = \rho_P + E\left(T - t\right)e^{-r(T-t)}.$$

2. Show that for a delta-neutral portfolio of options on a non-dividend paying stock, $\Pi$,

$$\Theta + \frac{1}{2}\sigma^2 S^2 \Gamma = r\Pi.$$

**Solution:** The portfolio satisfies the Black-Scholes equation

$$\frac{\partial \Pi}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 \Pi}{\partial S^2} + rS\frac{\partial \Pi}{\partial S} - r\Pi = 0$$

and so

$$\Theta + rs\Delta + \frac{1}{2}\sigma^2 S^2 \Gamma = r\Pi$$

If the portfolio is is delta neutral, then $\Delta = 0$ and hence

$$\Theta + \frac{1}{2}\sigma^2 S^2 \Gamma = r\Pi$$

3. Show that

$$\frac{\partial \Delta}{\partial \sigma} = \frac{\partial\ vega}{\partial S}, \quad \frac{\partial \Gamma}{\partial \sigma} = \frac{\partial^2\ vega}{\partial S^2}, \quad \frac{\partial \Theta}{\partial \sigma} = \frac{\partial\ vega}{\partial t}, \quad \frac{\partial \Delta}{\partial r} = \frac{\partial \rho}{\partial S}.$$

**Solution:** We use the same method as in the previous question to change the derivatives, i.e.

$$\frac{\partial \Delta}{\partial \sigma} = \frac{\partial}{\partial \sigma}\left(\frac{\partial V}{\partial S}\right) = \frac{\partial}{\partial S}\left(\frac{\partial V}{\partial \sigma}\right) = \frac{\partial\ vega}{\partial S}$$

$$\frac{\partial \Gamma}{\partial \sigma} = \frac{\partial}{\partial \sigma}\left(\frac{\partial^2 V}{\partial S^2}\right) = \frac{\partial^2\ vega}{\partial S^2}$$

$$\frac{\partial \Theta}{\partial \sigma} = \frac{\partial}{\partial \sigma}\left(V_t\right) = \frac{\partial\ vega}{\partial t}$$

$$\frac{\partial \Delta}{\partial r} = \frac{\partial}{\partial r}\left(\frac{\partial V}{\partial S}\right) = \frac{\partial \rho}{\partial S}$$

4. The Black–Scholes formula for a European call option $C\left(S, t\right)$ is given by

$$C\left(S,t\right) = S\exp(-D(T-t))N(d_1) - E\exp(-r(T-t))N(d_2).$$

Show that the Speed of this option $\left(\dfrac{\partial \Gamma}{\partial S}\right)$ is given by

$$\text{Speed} = \frac{\partial^3 C}{\partial S^3} = -\frac{\Gamma}{S}\left(1 + \frac{d_1}{\sigma\sqrt{T-t}}\right)$$

**You do not need to prove the result for $\Gamma$.**

**Solution:** We know that gamma for this option (standard result) is

$$\Gamma = \frac{\exp(-D(T-t))N'(d_1)}{\sigma S\sqrt{T-t}} \quad \text{where } N'(d_1) = \frac{1}{\sqrt{2\pi}}\exp(-d_1{}^2/2)$$

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\begin{aligned}
\frac{\partial \Gamma}{\partial S} &= \frac{\partial}{\partial S}\left\{\frac{\exp(-D(T-t))}{\sigma S\sqrt{T-t}}\exp(-d_1{}^2/2)\right\} \quad \text{there is also a factor } \frac{1}{\sqrt{2\pi}} \text{ through out}\\
&= \frac{\exp(-D(T-t))}{\sigma S\sqrt{T-t}}\frac{\partial}{\partial S}\left\{\exp(-d_1{}^2/2)\right\} + \exp(-d_1{}^2/2)\frac{\partial}{\partial S}\left\{\frac{\exp(-D(T-t))}{\sigma S\sqrt{T-t}}\right\}\\
&= \frac{\exp(-D(T-t))}{\sigma S\sqrt{T-t}}\exp(-d_1{}^2/2)\left\{\frac{-d_1}{\sigma S\sqrt{T-t}}\right\} - \frac{1}{S}\left\{\frac{\exp(-D(T-t))}{\sigma S\sqrt{T-t}}\exp(-d_1{}^2/2)\right\}\\
&= \Gamma\left[\frac{-d_1}{\sigma S\sqrt{T-t}} - \frac{1}{S}\right] = -\frac{1}{S}\Gamma\left[1 + \frac{d_1}{\sigma\sqrt{T-t}}\right].
\end{aligned}
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5. Consider a delta-neutral portfolio of derivatives, $\Pi$. For a small change in the price of the underlying asset, $\delta S$, over a short time interval, $\delta t$, show that the change in the portfolio value, $\delta\Pi$, satisfies

$$\delta\Pi = \Theta\delta t + \frac{1}{2}\Gamma\delta S^2$$

where $\Theta = \dfrac{\partial \Pi}{\partial t}$ and $\Gamma = \dfrac{\partial^2 \Pi}{\partial S^2}$.

**Solution:** Applying Itô's lemma to the value of the portfolio, $\Pi$ :

$$\delta\Pi = \frac{\partial \Pi}{\partial S}\delta S + \frac{\partial \Pi}{\partial t}\delta t + \frac{1}{2}\frac{\partial^2 \Pi}{\partial S^2}\delta S^2$$

to order $\delta t$. If the portfolio is delta-neutral, then $\Delta = \dfrac{\partial \Pi}{\partial S} = 0$, and so

$$\delta\Pi = \frac{\partial \Pi}{\partial t}\delta t + \frac{1}{2}\frac{\partial^2 \Pi}{\partial S^2}\delta S^2 = \Theta\delta t + \frac{1}{2}\Gamma\delta S^2.$$

6. (a) By differentiating the Black–Scholes equation with respect to $\sigma$, show that the vega of an option, $vega$, satisfies the differential equation

$$\frac{\partial vega}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 vega}{\partial S^2} + rS\frac{\partial vega}{\partial S} - rvega + \sigma S^2 \Gamma = 0$$

where $\Gamma = \partial^2 V/\partial S^2$. What is the final condition?

$$\frac{\partial}{\partial\sigma}\left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0\right) \quad\rightarrow$$

$$\frac{\partial}{\partial t}\left(\frac{\partial V}{\partial\sigma}\right) + \left\{\frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2}\left(\frac{\partial V}{\partial\sigma}\right) + \sigma S^2 \frac{\partial^2 V}{\partial S^2}\right\} + rS\frac{\partial}{\partial S}\frac{\partial V}{\partial\sigma} - r\frac{\partial V}{\partial\sigma} \quad = \quad 0$$

and we know $vega = \frac{\partial V}{\partial\sigma}$ & $\Gamma = \frac{\partial^2 V}{\partial S^2}$, hence

$$\frac{\partial vega}{\partial t} + \left\{\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 vega}{\partial S^2} + \sigma S^2 \Gamma\right\} + rS\frac{\partial vega}{\partial S} - rvega = 0$$

at $t = T$ $V(S,T) = \max(S - E)$, we want a condition for $v$ so

$$vega(S,T) = \frac{\partial}{\partial\sigma}V(S,T) = \frac{\partial}{\partial\sigma}\max(S - E) = 0$$

So final condition is $vega(S,T) = 0$.

(b) Similarly, find the PDE satisfied by $\rho$, the sensitivity of the option value to the interest rate.

We differentiate the BSE wrt $r$ using the same method as above, in terms of switching around the derivative terms to obtain

$$\frac{\partial\rho}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2\rho}{\partial S^2} + rS\frac{\partial\rho}{\partial S} - r\rho + S\frac{\partial V}{\partial S} - V = 0$$

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