CQF Module 1 Test

①

$$f(x) = \sum a_n (x - x_0)^n$$

$$a_n = \frac{f^{(n)}(x_0)}{n!}$$

$$\frac{x}{(1+x^2)^2}$$

$$\frac{1}{\underbrace{1-x}}$$

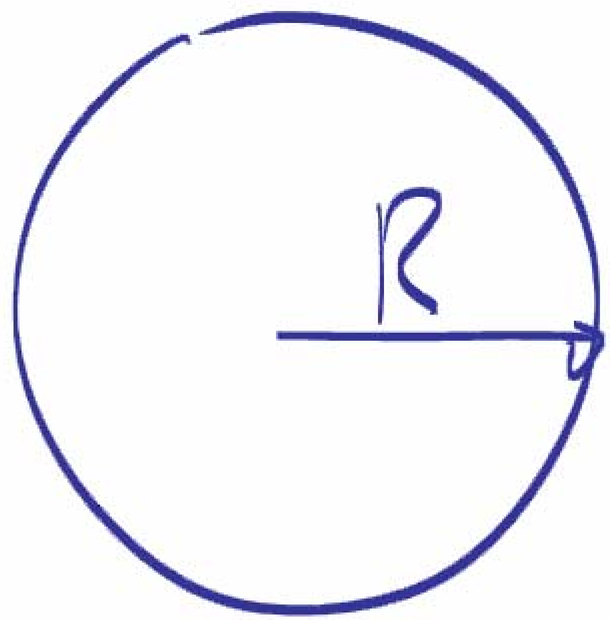
$$\frac{1}{1-(-x)} = \frac{1}{1+x}$$

$$x^2$$

$$\frac{d}{dx}\left[\frac{1}{1+x^2}\right]$$

$$(1+x^2)^{-1}$$

$$\sum a_n x^n$$



$$\log(1-x)$$

$$\int \frac{1}{1-x}$$

$$\int f(x) = \int \sum a_n (x-x_0)^n$$

(2) $\dfrac{a+ib}{c+id} \cdot \dfrac{c-id}{c-id}$ X

$\boxed{re^{i\alpha}}$

$\boxed{\dfrac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}}}$

③ $\begin{vmatrix} y-z & z-x & x-y \\ & & \\ & & \end{vmatrix}$

④ See Math,

Prob.

1) a) pdf given — work out normalising const.

$$\boxed{A=2}$$

b)

$$\mathbb{E}(X^{2n}) = \frac{n!}{2^n}$$

Say $I_n = \int_0^\infty x^n p(x)\, dx$

$n = 0$ $n = 1$

$\boxed{\text{Recursion}}$

$$\boxed{\sum_{i=1}^{12} \text{RAND()} - 6} \longrightarrow N(0,1)$$

$$\lim_{N \to \infty} \frac{\sum_{i=1}^{N} X_i - N\mu}{\sigma\sqrt{N}}$$

$$\mu \to \frac{1}{2}$$

$$\sigma \to \sqrt{\frac{1}{12}}$$

$$\mathbb{V}[\alpha X] = \alpha^2 \mathbb{V}[X]$$

Section C

2) Simply state the forward

eq^{n}

$$\frac{\partial p}{\partial t'} = \frac{1}{2} \frac{\partial^2}{\partial r'^2} \left(B^2 p \right) - \quad -$$

partial deriv $\longrightarrow$ ordinary deriv

① $G = e^{(t + a e^{x})}$

Do Itô on $G(t)$

You will need to write $a e^{x}$ in terms of G

$\ln G = t + \boxed{a e^{x}}$

b) $S_t = \underbrace{\left(A + \frac{x}{\alpha}\right)}^{\alpha}$

$\frac{dS}{dx}$, $\frac{d^2 S}{dx^2}$

Write down $\frac{dS}{dx}$ in terms of S

At the end do a comparison to set α

$$\frac{1}{2}\sigma^2 S^2 V'' + rSV' - rV = 0$$

$$\equiv \quad ax^2 y'' + bxy' + cy = 0 \longleftarrow$$

$$S^2 V'' + \frac{2r}{\sigma^2} S V' - \frac{2r}{\sigma^2} V = 0$$

B.S.E

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+rS\frac{\partial V}{\partial S}-rV=0$$

1^{st} order in time

2^{nd} order in asset price

$$\boxed{\frac{\partial u}{\partial t}=c^2\frac{\partial^2 u}{\partial x^2}}$$

1 Time condition (Final)
Payoff

$$\max(S-E, 0)$$

2 B.C.s are in S:

$S \to 0 \qquad V \to 0$

$S \to \infty \qquad V \to \infty$

2) i) $xy' = y + \sqrt{x^2 + y^2}$

homog. $\qquad y = vx$

$$\frac{1}{x^2}\left[y + \sqrt{y^2 + x^2}\right] = c$$

ii) $y' = \dfrac{2x + 9y - 20}{6x + 2y - 10}$ Reducible to homog. eqⁿ

$$x = X + h$$
$$y = Y + k$$
$\to$ homog. eqⁿ

$$Y = VX$$

(need partial fractions)

$$c = \frac{2y + x - 5}{(2x - y)^2} \quad \text{G. S.}$$

iii) $$y' = \frac{3x-4y-2}{3x-4y-3} \qquad // \text{ lines}$$

So use a substitution $u = 3x - 4y$

$$\ln|3x-4y+1| - [illegible]x + [illegible]y = K$$

(v) $2y' + y = (x-1)y^3$

$\div$ thro by y^3

$$\frac{2}{y^3}y' + \frac{1}{y^2} = (x-1)$$

$$\boxed{z = \frac{1}{y^2}}$$

$$\frac{1}{y^2}e^{-x} = xe^{-x} + c$$

$$\boxed{\frac{1}{y^2} = x + ce^{x}} \quad \text{G.S.}$$

v) $(x+3y-1)\,dx + (\cancel{x+3y}\ 3x-2y+4)\,dy = 0$

Exact eq

Sol^n: $3xy - y^2 + 4y - x + \frac{x^2}{2} = c$

3) i $(\sqrt{3} + i)^{25}$

U.S.E D.M.T r, arg

$r(\cos\theta + i\sin\theta)$

Simplify $\cos\theta$ & $\sin\theta$.

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ii) $\sin 5\theta$ & $\cos 5\theta$

$$\begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 1 & & 2 & & 1 & & & \\ & & 1 & & 3 & & 3 & & 1 & & \\ & 1 & & 4 & & 6 & & 4 & & 1 & \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \end{array}$$

$$\sin n\theta = \mathrm{Im}(\cos n\theta + i\sin n\theta)$$

$$= \mathrm{Im}(\cos\theta + i\sin\theta)^n$$

iii)

$$z^n - \frac{1}{z^n} = 2i\sin n\theta$$

Similar for $\cos n\theta$

$$z^n + \frac{1}{z^n} = 2\cos n\theta$$

(iv) Roots of $f(x) \Rightarrow$

solve $f(x) = 0$

$x^6 - 1 \longrightarrow \boxed{x^6 = 1}$ 6th roots of unity

(Wed's session on finding roots of unity).

4) Solve $\cos z = 4$

where $z = x + iy$

$\cos(x+iy) = 4 \longrightarrow$ 2 eq'ns

$\cos(iy) = \ ?$

$\sin(iy) = \ ?$

Solving a B.V.P

$$a(x)y'' + b(x)y' + c(x)y = g(x)$$

Solve over $[a, b]$

$$\left.\begin{aligned} y(x=a) &= A \\ y(x=b) &= B \end{aligned}\right\} \text{2 B.C.s}$$

$a(x)$, $b(x)$, $c(x)$ are continuous

$x - \delta x$ $\quad$ x $\quad$ $x + \delta x$

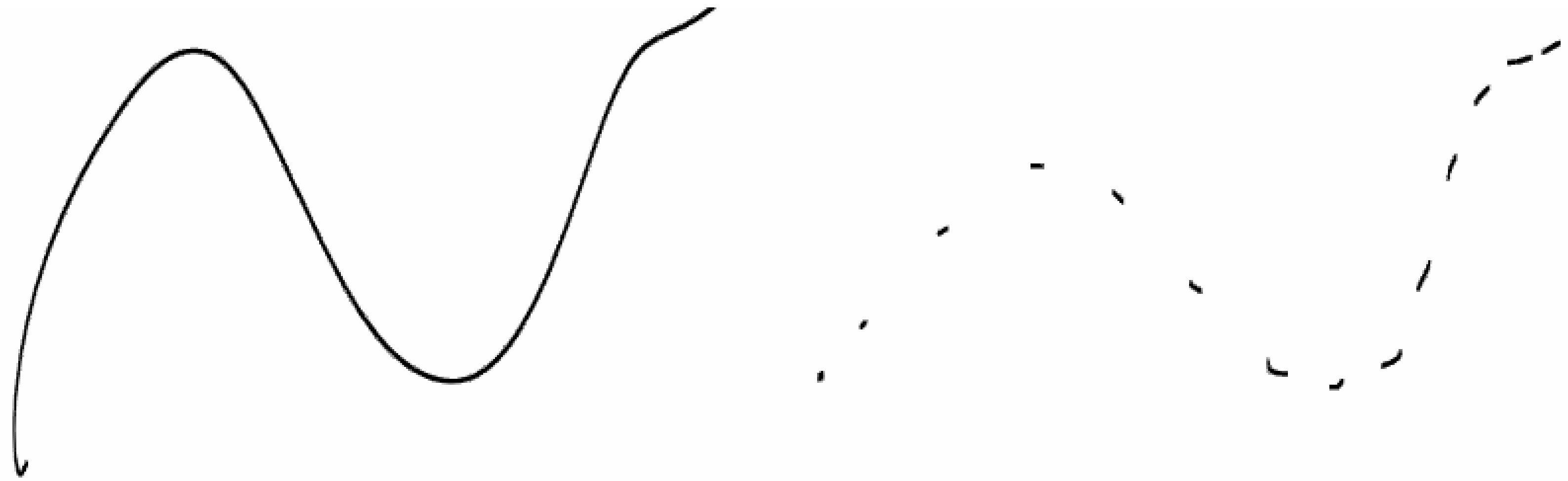
$$\delta x = \frac{e-d}{N}$$

N — no. of partitions

$N+1$ — pts.

$$x = d + i\,\delta x \qquad 0 \le i \le N$$

$$x \longrightarrow x_i \equiv x(i) \qquad y_i \equiv y(x_i)$$



Use TSE for $f(x+dx)$

$f(x-dx)$

$$y(x+\delta x) = y + y'\delta x + \frac{1}{2}y''\delta x^2 + O(\delta x^3) \quad ①$$

$$y(x-\delta x) = y - y'\delta x + \frac{1}{2}y''\delta x^2 \mp O(\delta x^3) \quad ②$$

① − ②: $y(x+\delta x) - y(x-\delta x) = 2y'\delta x + O(\delta x^3)$

Rearranging: $$y' = \frac{y(x+\delta x) - y(x-\delta x)}{2\delta x} + O(\delta x^2)$$

$$y' \sim \underline{y_{i+1} - y_{i-1}}$$

① + ② : $y(x+\delta x) + y(x-\delta x) = 2y + y'' \delta x^2 + \cdots$

$$y'' = \frac{y_{i+1} - 2y_i + y_{i-1}}{\delta x^2} + O(\delta x^2)$$

$$\boxed{y'' \sim \frac{y_{i-1} - 2y_i + y_{i+1}}{\delta x^2}}$$

Sub, t 3 & 4 in *

$$\frac{a_i}{\delta x^2}\left[s_{i-1}-2s_i+s_{i+1}\right]+\frac{b_i}{2\delta x}\left[s_{i+1}-s_i\right]+c_i s_i -$$

Collect up coeffs of s_{i-1}, s_i, s_{i+1}

s_{i-1} : $\dfrac{a_i}{\delta x^2}-\dfrac{b_i}{2\delta x}$

s_i : $-2a_i$

$$y_{i+1} : \frac{a_i}{\delta x^2} + \frac{b_i}{2\,\delta x} : \gamma_i$$

(*) is now fully discretized & can be written as

$$\boxed{\alpha_i y_{i-1} + \beta_i y_i + \gamma_i y_{i+1} = \delta_i}$$

Finite Difference Eqn

This is an Implicit scheme for the unknown terms y_i, y_{i+1}, y_{i-1}

At the boundaries $y(d) = A$

$\longrightarrow \quad y_0 = A$

$$y(e) = B \longrightarrow y_N = B$$

This is a matrix inversion problem.

$i=1: \quad \alpha_1 y_0 + \beta_1 y_1 + \gamma_1 y_2 = s_1$

$i=2: \quad \alpha_2 y_1 + \beta_2 y_2 + \gamma_2 y_3 = s_2$

$\vdots$

$i = N-1 \quad \alpha_{N-1} y_{N-2} + \beta_{N-1} y_{N-1} + \gamma_{N-1} y_N = g_{N-1}$

$i = N \quad y_N = e$

$$
\begin{bmatrix}
1 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\
\alpha_1 & \beta_1 & \gamma_1 & 0 & \cdots & \cdots & 0 \\
0 & \alpha_2 & \beta_2 & \gamma_2 & 0 & \cdots & 0 \\
\vdots & & \ddots & \ddots & \ddots & & \vdots \\
\vdots & & & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & \cdots & 0 & \alpha_{N-1} & \beta_{N-1} & \gamma_{N-1} \\
0 & \cdots & \cdots & \cdots & \cdots & 0 & 1
\end{bmatrix}
\begin{bmatrix}
y_0 \\ y_1 \\ \vdots \\ \vdots \\ \vdots \\ y_{N-1} \\ y_N
\end{bmatrix}
=
\begin{bmatrix}
A \\ g_1 \\ \\ \\ \\ g_{N-1} \\ B
\end{bmatrix}
$$

$$\boxed{Ax = g}$$