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Interest rate options close to expiry

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Abstract. We use an asymptotic expansion to study the behavior of American-style interest rate caplets and floorlets close to expiry, under the assumption that interest rates obey a mean-reverting random walk as given by the Vasicek model. Series solutions are obtained for the location of the free boundary and the price of the option for both the caplet and floorlet.

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§1. Introduction

Over the past thirty years, there has been a revolution in quantitative finance and mathematicians have used powerful mathematical tools to model countless diverse assets such as equity options, interest rate swaps, and electricity futures. Amongst options, closed form expressions have been found for many European-style options, meaning options that can only be exercised at expiry. The most well-known of these closed form expressions is of course the Black-Scholes option pricing formula [9, 21] for equity options, but a number of such solutions are also known for interest rate options, with a selection of these given in for example [13].

American-style options, which can be exercised at any time up to and including expiry, are harder to price analytically, and except for a few special cases, closed form pricing formulas for American-style options have remained elusive. This is in large part because the American-style early exercise feature often leads to a free boundary problem somewhat similar to the Stefan problems that arise in physical problems such as melting and solidification, and in order to price American-style options, it is necessary to first locate the free boundary. Because of this, analytical research on American options has taken a different path. Over the past several years, a number of papers have appeared on vanilla American equity puts and calls, which are contracts where the holder can exchange the option at or before expiry for the amount $\max(S-E,0)$ for a call and $\max(E-S,0)$ for a put, where S is the equity price at the time of exercise and E is the strike price of the option. One approach has involved using techniques such as integral transforms to reformulate the problem as an integral equation [16, 15, 12, 14, 25], while another has been to use a technique due to Tao [26]-[34] to find series solutions for both the value of the option and the location of the free boundary close to expiry [11, 2, 3, 18].

While these approaches have been used successfully for American-style vanilla and exotic equity options [4, 6, 7], few if any such studies have looked at American-style interest rate options. In the present study, we will consider interest rate caplets and floorlets. A caplet is the interest rate counterpart of a call, and an American caplet pays the amount $\max(r-E,0)$ at or before expiry, where r is the underlying spot interest rate and E is the strike, while an American floorlet is the counterpart of a put and pays the amount $\max(E-r,0)$ at or before expiry. Typically, a caplet might be purchased by an investor who has to make a stream of payments based on a floating interest rate such as LIBOR, the London InterBank Offer Rate, and who wishes to protect himself against sharp increases in interest rates, while a floorlet might be purchased by an investor receiving such a payment stream who wishes to protect himself against sharp decreases. Thus it follows that a caplet is an insurance against high interest rates, whilst a floorlet is an insurance against low rates. These interest rate derivatives can be used individually, as envisaged in the present study, or combined into portfolios: a portfolio of caplets with payoffs on a series of different dates is known as a cap, with a similar portfolio of floorlets being known as a floor. The market for caplets and floorlets, and caps and floors, is OTC (over the counter) rather than exchange traded, and according to [23], the market makers for these types of OTC interest rate options tend to be the large investment banks and commercial banks, but there are fewer market makers and generally wider spreads than in the markets for options on either mortgages or treasury securities. The end use buyers tend to be institutions with risks they need to cover. For example, for caplets and caps, buyers include institutions that lend money on a long term basis but are funded by short term deposits and businesses that fund by rolling over short term debt; both categories face losses if short-term interest rates rise and caplets or caps can protect against the risk of such losses. Buyers of a floorlet and floors tend to be firms that face losses if short-term rates fall. The sellers of caplets and floorlets are quite varied, and include outright sellers who wish to generate premium income, hedgers who are seeking to smooth out the cash flows on other fixed income securities, and even buyers of capped floating rate notes (FRN) who are if effect buying an uncapped FRN and selling a cap.

In this study, we will use Tao's method, originally formulated in the context of physical Stefan problems, to find series solutions both for the prices of American caplets and floorlets close to expiry, and also for the location of the associated free boundaries. This approach involves expanding the solution and the location of the free boundary as a series in the time remaining until expiry, which is treated as a small parameter. This method has previously been used for both vanilla and exotic equity options [11, 2, 3, 18, 4, 6, 7], and the analysis here will follow the same lines as those studies, in part because in order to use Tao's method, we must first use a change of variables to transform the governing equation into the nonhomogeneous diffusion equation. This transformation is straightforward for equity options, where the price obeys the Black-Scholes-Merton partial differential equation [9, 21], and is discussed in standard texts such as [36]. However, while the Black-Scholes-Merton partial differential equation is widely accepted for equity options, a variety of different models are used for interest rate derivatives. In the present work, we will use the Vasicek model, which is a mean reverting model popular amongst academic practitioners. The main reason for choosing the Vasicek model is precisely because it is also straightforward to transform the governing equation for this model into the nonhomogeneous diffusion equation [5]. The details of this model will be given in the next section, where we present our analysis.

§2. Analysis

In this section, we will use the method of Tao [26]-[34] to study the behavior of American caplets and floorlets close to expiry. These are the interest rate counterparts of vanilla American call and put equity options. If held to expiry, an American caplet pays an amount $\max(r-E,0)$ and an American floorlet pays $\max(E-r,0)$, where r is the interest rate and E is the strike. Because these options are American, they can be exercised at any time prior to expiry, paying at exercise $\max(r-E,0)$ for a caplet and $\max(E-r,0)$ for a floorlet.

To price interest rate derivatives, it is necessary to model the behavior of interest rates. It is usual to assume that the spot interest rate r obeys the stochastic differential equation,

$$dr = u(r,t)dt + w(r,t)dX, \tag{2.1}$$

where dX is normally distributed with zero mean and variance dt and w is the volatility. The random walk described by (2.1) can be somewhat different to the lognormal random walk usually assumed for equity prices,

$$dS = \mu S dt + \sigma S dX, \tag{2.2}$$

which leads to the celebrated Black-Scholes option pricing model [9, 21] for equity options. Returning to the equation (2.1) for interest rates, constructing a risk neutral portfolio leads us to the following partial differential equation (PDE) for the price $V(r,t)$ of an interest rate derivative,

$$\frac{\partial V}{\partial t}+\frac{w^2}{2}\frac{\partial^2 V}{\partial r^2}+(u-\lambda w)\frac{\partial V}{\partial r}-rV \quad = \quad 0, \tag{2.3}$$

where $\lambda(r,t)$ is the market price of interest rate risk, and $u-\lambda w$ is the risk adjusted drift. This equation is valid for times $t \leq T$, where T is the expiry of the derivative. The derivation of (2.3) can be found in for example [36], and this equation governs the behavior of all interest rate derivatives: the boundary and initial conditions rather than the PDE differentiate amongst them [22].

There are a number of popular interest rate models, and several of these are special cases of the general affine model, for which $u-\lambda w=a(t)-b(t)r$ and $w=(c(t)r-d(t))^{1/2}$; a table of these special cases can be found in §46.2 of [36]. One popular model is the Vasicek model [35], for which $u-\lambda w=a-br$ and $w=\sigma$, with a, b and σ constants rather than functions of time, so that (2.3) becomes

$$\frac{\partial V}{\partial t}+\frac{\sigma^2}{2}\frac{\partial^2 V}{\partial r^2}+(a-br)\frac{\partial V}{\partial r}-rV \quad = \quad 0\ . \tag{2.4}$$

This model is mean-reverting to a constant level, which is a desirable property for interest rates, and is popular amongst academic practitioners because it is highly tractable and it is possible to find closed form expressions for many interest rate derivatives using this model, and it has also been used to model the interest rate element of convertible securities [19, 17]. This equation must be solved together with the pay-off at expiry of $V(r,T)=\max(r-E,0)$ for a caplet and $\max(E-r,0)$ for a floorlet.

Because we are considering American-style derivatives, we can exercise them at any time at or before expiry, and this leads to the constraint that the price of the derivatives cannot fall below the pay-off from immediate exercise, which is $\max(r-E,0)$ for a caplet and $\max(E-r,0)$ for a floorlet. The possibility of early exercise leads to a free boundary problem similar to that for American options, and also to the Stefan problems which occur in the physical processes of melting and solidification. On the free boundary, which we label $r=r_f(t)$, we exchange the option for the pay-off, and this leads to the condition that the value of the option and it's derivative with respect to r must be continuous across the boundary. For a caplet, we require $V=r-E$ and $(\partial V/\partial r)=1$ at the free boundary, while for a floorlet, we require $V=E-r$ and $(\partial V/\partial r)=-1$ there. These are essentially the same conditions as for American equity options, although of course for the equity options the conditions will involve the derivatives with respect to the stock price rather than the interest rate; the condition on the derivative is known as the high contact condition [24].

Having presented the governing PDE (2.4) and associated boundary and initial conditions, we will now present our analysis. To use Tao's method [26]-[34], we will follow the approach taken for equity options [11, 2, 3, 18], and transform the PDE (2.4) into the nonhomogeneous diffusion equation. To do so, we make the transformation [5]

$$\begin{aligned} V(r,t) &= \exp\left[\left(\frac{\sigma^2}{2b^2}-\frac{a}{b}\right)(T-t)-\frac{r}{b}+\frac{a}{b^2}-\frac{\sigma^2}{b^3}+\frac{\sigma x_0}{2b^{3/2}}\right]v\left(x,\tau\right) \\ &\quad +V_0(r), \end{aligned} \tag{2.5}$$

where $V_0(r)=r-E$ for the caplet and $E-r$ for the floorlet, and we have introduced the new variables

$$\begin{aligned} \tau &= 1-e^{-2b(T-t)} \\ x &= \frac{2\sqrt{b}}{\sigma}\left[r-\frac{a}{b}+\frac{\sigma^2}{b^2}\right]e^{-b(T-t)}, \end{aligned} \tag{2.6}$$

which we can invert,

$$\begin{aligned} r &= \frac{a}{b}-\frac{\sigma^2}{b^2}+\frac{\sigma x}{2\sqrt{b\left(1-\tau\right)}} \\ t &= T+\frac{\ln(1-\tau)}{2b} \end{aligned} \tag{2.7}$$

This leads to the nonhomogeneous diffusion equation

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2}+f(x,\tau), \tag{2.8}$$

where the nonhomogeneous term $f(x,\tau)=g(x,\tau)$ for the caplet and $-g(x,\tau)$ for the floorlet, with

$$\begin{aligned} g(x,\tau) &= \frac{\left[a+r\left(E-b\right)-r^2\right]}{2b} \\ &\quad\times\exp\left[\frac{r}{b}+\frac{\left(4b^3+2ab-\sigma^2\right)(T-t)}{2b^2}-\frac{a}{b^2}-\frac{\sigma^2}{b^3}+\frac{\sigma x_0}{2b^{3/2}}\right] \\ &= \frac{1}{2b^2}\exp\left[\frac{\sigma}{2b^{3/2}}\left(\frac{x}{\sqrt{1-\tau}}-x_0\right)\right](1-\tau)^{\frac{\sigma^2-2ab-4b^3}{4b^3}} \\ &\quad\times\left[aE+\sigma^2-\frac{a^2+E\sigma^2}{b}+\frac{2a\sigma^2}{b^2}-\frac{\sigma^4}{b^3}\right. \\ &\qquad\left.+\frac{\sigma\left(2\sigma^2+Eb^2-2ab-b^3\right)}{2b^{3/2}\sqrt{1-\tau}}-\frac{\sigma^2x^2}{4\left(1-\tau\right)}\right]. \end{aligned} \tag{2.9}$$

The equation (2.8) must be solved together with the boundary condition that $v = (\partial v/\partial x) = 0$ on the free boundary $x = x_f(\tau)$, and the pay-off at expiry. For the caplet, this pay-off is

$$v(x,0) = \begin{cases} 0 & x > x_* \\ -\frac{\sigma(x-x_*)}{2\sqrt{b}} \exp\left[\frac{\sigma(x-x_0)}{2b^{3/2}}\right] & x < x_* \end{cases}, \tag{2.10}$$

where $x_* = 2\sqrt{b}\left(E - a/b + \sigma^2/b^2\right)/\sigma$. For the floorlet, we have

$$v(x,0) = \begin{cases} 0 & x < x_* \\ \frac{\sigma(x-x_*)}{2\sqrt{b}} \exp\left[\frac{\sigma(x-x_0)}{2b^{3/2}}\right] & x > x_* \end{cases}. \tag{2.11}$$

It is possible to deduce the location of the free boundary in the limit $\tau \to \infty$ by considering perpetual options, which do not expire. These obey the time-independent version of (2.4), which has a solution,

$$\begin{aligned} V_\infty(r) &= \sqrt{\frac{\sigma b^{3/2}}{b^2 r + \sigma^2 - ab}} \exp\left[\frac{r\,(br - 2a)}{2\sigma^2}\right] \\ &\times \left(A_1 W\left[\frac{b^3 + \sigma^2 - 2ab}{4b^3}, \frac{1}{4}; \left(\frac{b^2 r + \sigma^2 - ab}{\sigma b^{3/2}}\right)^2\right] \right. \\ &\left. + A_2 M\left[\frac{b^3 + \sigma^2 - 2ab}{4b^3}, \frac{1}{4}; \left(\frac{b^2 r + \sigma^2 - ab}{\sigma b^{3/2}}\right)^2\right]\right), \end{aligned} \tag{2.12}$$

where W and M are Whittaker functions [1]. The constants A_1 and A_2 in this expression and also the location r_∞ of the free boundary can be found by applying the conditions on the free boundary.

We can also deduce the location of the free boundary at expiry by substituting the pay-off $V(r,T)$ into the PDE (2.4) to calculate $(\partial V/\partial t)$ at expiry: if $(\partial V/\partial t) > 0$, then the value of the option will drop below the pay-off from immediate exercise as we move backwards in time from expiry. For the caplet, this yields

$$\frac{\partial V}{\partial t}(r,T) = \begin{cases} r^2 + r(b-E) - a & r > E \\ 0 & r < E \end{cases}, \tag{2.13}$$

so that if $a \le bE$ the free boundary at expiry is situated at $r_0 = E$ or $x_0 = 2b^{1/2}E/\sigma - 2a/(\sigma b^{1/2}) + 2\sigma/b^{3/2}$, while for $a > bE$, it is situated at the root of $a + r_0\,(E-b) - r_0^2 = 0$ or equivalently the root of

$$\begin{aligned} & x_0^2 + x_0\left[\frac{2b^{3/2} - 2b^{1/2}E}{\sigma} + \frac{4a}{\sigma b^{1/2}} - \frac{4\sigma}{b^{3/2}}\right] \\ & -4 + \frac{4a^2}{\sigma^2 b} - \frac{8a}{b^2} - \frac{4aE}{\sigma^2} + \frac{4\sigma^2}{b^3} + \frac{4E}{b} = 0, \end{aligned} \tag{2.14}$$

so that

$$
\begin{aligned}
r_0 &= \frac{1}{2}\left[E - b + \sqrt{(E-b)^2 + 4a}\right] \\
x_0 &= \frac{2\sigma}{b^{3/2}} - \frac{2a}{\sigma b^{1/2}} + \frac{b^{1/2}}{\sigma}\left[E - b + \sqrt{(E-b)^2 + 4a}\right].
\end{aligned}
\tag{2.15}
$$

Similarly, for the floorlet, we find

$$
\frac{\partial V}{\partial t}(r,T) = \begin{cases} -r^2 - r(b-E) + a & r < E \\ 0 & r > E \end{cases},
\tag{2.16}
$$

so that if $a \geq bE$ the free boundary at expiry is situated at $r_0 = E$ or $x_0 = 2b^{1/2}E/\sigma - 2a(\sigma b^{1/2}) + 2\sigma/b^{3/2}$, while for $a < bE$, it is situated at

$$
\begin{aligned}
r_0 &= \frac{1}{2}\left[E - b + \sqrt{(E-b)^2 + 4a}\right] \\
x_0 &= \frac{2\sigma}{b^{3/2}} - \frac{2a}{\sigma b^{1/2}} + \frac{b^{1/2}}{\sigma}\left[E - b + \sqrt{(E-b)^2 + 4a}\right].
\end{aligned}
\tag{2.17}
$$

The location of the free boundary at expiry tells us that we must consider the cases $a < bE$, $a = bE$ and $a > bE$ separately, and we must also consider the caplet and floorlet separately.

2.1. Caplet with $a > bE$

For this case, the free boundary starts from

$$
x_0 = \frac{2\sigma}{b^{3/2}} - \frac{2a}{\sigma b^{1/2}} + \frac{b^{1/2}}{\sigma}\left[E - b + \sqrt{(E-b)^2 + 4a}\right]
\tag{2.18}
$$

at expiry. We will follow [11, 2] and pose an expansion

$$
\begin{aligned}
v(x,\tau) &\sim \sum_{n=3}^{\infty} \tau^{n/2} V_n^{(0)}(\xi), \\
x_f(\tau) &\sim \sum_{n=0}^{\infty} x_n \tau^{n/2},
\end{aligned}
\tag{2.19}
$$

where the similarity variable $\xi = (x - x_0)/(2\sqrt{t})$. This expansion is essentially the approach due to Tao [26]-[34]. To simplify the analysis, we introduce the operator

$$
\mathcal{L}_n \equiv \frac{1}{4}\frac{d^2}{d\xi^2} + \frac{\xi}{2}\frac{d}{d\xi} - \frac{n}{2}.
\tag{2.20}
$$

If we substitute the expansion for $v(x,\tau)$ into the PDE (2.8), at the first few orders, we find

$$\begin{aligned}
\mathcal{L}_3 V_3^{(0)} &= \left[\frac{1}{2b^{1/2}} + \frac{x_0\sigma}{2b^2} - \frac{a}{b^{5/2}} + \frac{E}{2b^{3/2}} + \frac{\sigma^2}{b^{7/2}}\right]\sigma\xi, \\
\mathcal{L}_4 V_4^{(0)} &= \frac{\sigma^2 + aE}{2b^2} - \frac{E\sigma^2 + a^2}{2b^3} + \frac{a\sigma^2}{b^4} - \frac{\sigma^4}{2b^5} \\
&+ \frac{x_0\sigma}{4b^{1/2}}\left(\frac{E}{2b} - \frac{a}{b^2} + \frac{\sigma^2}{b^3} - \frac{1}{2}\right) + \left[1 - \frac{E}{2b} + \frac{a}{b^2} - \frac{\sigma^2}{b^3} + \frac{x_0\sigma}{2b^{3/2}}\right]\frac{\sigma^2\xi^2}{b^2}, \\
\mathcal{L}_5 V_5^{(0)} &= \left[\frac{x_0\sigma}{2b^5} + \frac{1}{b^{7/2}}\left(\frac{3}{2} - \frac{E}{2b} + \frac{a}{b^2} - \frac{\sigma^2}{b^3}\right)\right]\frac{\sigma^3\xi^3}{2} \\
&+ \left[\frac{x_0\sigma}{b^{3/2}}\left(3 + \frac{E}{b} - \frac{a}{b^2} + \frac{3\sigma^2}{2b^3}\right) + 3 - \frac{3E}{b} + \frac{7a}{b^2} + \frac{6Ea - 5\sigma^2}{2b^3}\right. \\
&\left. - \frac{4a^2 - 7E\sigma^2}{2b^4} + \frac{5a\sigma^2}{b^5} - \frac{3\sigma^4}{b^6}\right]\frac{\sigma\xi}{4b^{1/2}}.
\end{aligned} \tag{2.21}$$

It is straightforward to find solutions to (2.21) that satisfy the condition at $\tau = 0$,

$$\begin{aligned}
V_3^{(0)} &= C_3^{(0)}\left[\left(3\xi + 2\xi^3\right)\operatorname{erfc}(-\xi) + \frac{2\left(1+\xi^2\right)e^{-\xi^2}}{\sqrt{\pi}}\right] \\
&+ \left[\frac{E}{2b} + \frac{\sigma^2}{b^3} - \frac{1}{2} - \frac{x_0\sigma}{2b^{3/2}} - \frac{a}{b^2}\right]\frac{\sigma\xi}{b^{1/2}}, \\
V_4^{(0)} &= C_4^{(0)}\left[\left(3 + 12\xi^2 + 4\xi^4\right)\operatorname{erfc}(-\xi) + \frac{2\left(5+2\xi^2\right)e^{-\xi^2}}{\sqrt{\pi}}\right] \\
&+ \left[-1 + \frac{E}{2b} - \frac{a}{b^2} + \frac{\sigma^2}{b^3} - \frac{x_0\sigma}{2b^{3/2}}\right]\frac{\sigma^2\xi^2}{b^2} \\
&+ \frac{x_0\sigma}{4b^{1/2}}\left[\frac{1}{4} - \frac{E}{4b} + \frac{a}{2b^2} - \frac{\sigma^2}{b^3}\right] \\
&- \frac{2\sigma^2 + aE}{4b^2} + \frac{3E\sigma^2 + 2a^2}{8b^3} - \frac{3a\sigma^2}{4b^4} + \frac{\sigma^4}{2b^5}, \\
V_5^{(0)} &= C_5^{(0)}\left[\left(15\xi + 20\xi^3 + 4\xi^5\right)\operatorname{erfc}(-\xi) + \frac{2\left(4 + 9\xi^2 + 2\xi^4\right)e^{-\xi^2}}{\sqrt{\pi}}\right] \\
&+ \left[\frac{E}{2b} - \frac{a}{b^2} - \frac{\sigma x_0}{2b^{3/2}} + \frac{\sigma^2}{b^3} - \frac{3}{2}\right]\frac{\sigma^3\xi^3}{2b^{7/2}} \\
&+ \left[\frac{a}{b^2} - \frac{E}{b} - 3 - \frac{3\sigma^2}{b^3}\right]\frac{x_0\sigma^2\xi}{8b^2}
\end{aligned}$$

$$(2.22) + \left[\frac{3E}{2b} - \frac{3}{2} - \frac{7a}{2b^2} - \frac{2\sigma^2 + 3Ea}{2b^3} + \frac{2a^2 + 5E\sigma^2}{2b^4} - \frac{4a\sigma^2}{b^5} + \frac{3\sigma^4}{b^6}\right]\frac{\sigma\xi}{4b^{1/2}}.$$

We should note that in deriving this solution, we have only imposed the condition at $\tau = 0$ for $E < r < r_0$, where $v(x, 0) = 0$. For $r > r_0$ the caplet would already have been exercised, so the condition does not apply. To impose the condition for $r < E$, it would be necessary to pose a second expansion about $r = E$ and match that expansion to the present one; since the main goal of this study is to find the location of the free boundary close to expiry, we do not need to do that, just as Dewynne [11] did not need to do it for the American put.

If we apply the conditions on the free boundary by substituting the assumed form (2.19) for $x_f(\tau)$ into the solution (2.22), at leading order we get the pair of equations,

$$\begin{aligned}
C_3^{(0)}\left[\frac{6x_1 + x_1^3}{4}\mathrm{erfc}\left(-\frac{x_1}{2}\right) + \frac{(4 + x_1^2)\, e^{-x_1^2/4}}{2\sqrt{\pi}}\right]& \\
+\frac{\sigma x_1\left(2\sigma^2 - 2ab + Eb^2 - b^3 - x_0\sigma b^{3/2}\right)}{4b^{7/2}} &= 0, \\
3C_3^{(0)}\left[\left(1 + \frac{x_1^2}{2}\right)\mathrm{erfc}\left(-\frac{x_1}{2}\right) + \frac{x_1 e^{-x_1^2/4}}{\sqrt{\pi}}\right]& \\
+\frac{\sigma\left(2\sigma^2 - 2ab + Eb^2 - b^3 - x_0\sigma b^{3/2}\right)}{2b^{7/2}} &= 0, \qquad (2.23)
\end{aligned}$$

so that x_1 is the solution of

$$x_1^3\mathrm{erfc}\left(-\frac{x_1}{2}\right) + \frac{(2x_1^2 - 4)\, e^{-x_1^2/4}}{\sqrt{\pi}} = 0, \qquad (2.24)$$

or $x_1 = 0.90344659785$, while

$$C_3^{(0)} = -\frac{\sigma x_1^3\sqrt{\pi}}{24b^{7/2}e^{-x_1^2/4}}\left(2\sigma^2 - 2ab + Eb^2 - b^3 - x_0\sigma b^{3/2}\right). \qquad (2.25)$$

At the next order, we get another pair of equations,

$$\begin{aligned}
&C_4^{(0)}\left[\frac{(12 + 12x_1^2 + x_1^4)}{4}\mathrm{erfc}\left(-\frac{x_1}{2}\right) + \frac{(10x_1 + x_1^3)\, e^{-x_1^2/4}}{2\sqrt{\pi}}\right] \\
+\;& 3C_3^{(0)}x_2\left[\frac{2 + x_1^2}{4}\mathrm{erfc}\left(-\frac{x_1}{2}\right) + \frac{x_1 e^{-x_1^2/4}}{2\sqrt{\pi}}\right] \\
+\;& \frac{\sigma x_2}{2b^{1/2}}\left[\frac{E}{2b} - \frac{1}{2} - \frac{a}{b^2} + \frac{\sigma^2}{b^3} - \frac{\sigma x_0}{2b^{3/2}}\right]
\end{aligned}$$

$$+ \frac{\sigma^2 x_1^2}{4b^2}\left[\frac{E}{2b} - 1 - \frac{a}{b^2} + \frac{\sigma^2}{b^3} - \frac{\sigma x_0}{2b^{3/2}}\right] + \frac{\sigma x_0}{4b^{1/2}}\left[\frac{1}{4} - \frac{E}{4b} - \frac{a}{2b^2} + \frac{\sigma^2}{b^3}\right]$$

$$- \frac{2\sigma^2 + aE}{4b^2} + \frac{2a^2 + 3E\sigma^2}{8b^3} - \frac{3a\sigma^2}{4b^4} + \frac{\sigma^4}{2b^5} = 0, \tag{2.26}$$

and

$$2C_4^{(0)}\left[\left(6x_1 + x_1^3\right)\operatorname{erfc}\left(-\frac{x_1}{2}\right) + \frac{2\left(4 + x_1^2\right)e^{-x_1^2/4}}{\sqrt{\pi}}\right]$$
$$+ 3C_3^{(0)}x_2\left[x_1\operatorname{erfc}\left(-\frac{x_1}{2}\right) + \frac{2e^{-x_1^2/4}}{\sqrt{\pi}}\right]$$
$$+ \frac{\sigma^2 x_1}{b^2}\left[\frac{E}{2b} - 1 - \frac{a}{b^2} + \frac{\sigma^2}{b^3} - \frac{\sigma x_0}{2b^{3/2}}\right] = 0, \tag{2.27}$$

which have a solution

$$x_2 = \frac{\sigma x_1^2\left[2\sigma^2 - b\left(E^2 + 6a\right) + 3Eb^2 - 2b^3 - \sigma x_0 b^{3/2}\right]}{b^{5/2}\left(2 + x_1^2\right)\left[(E-b)^2 + 4a\right]} - \frac{x_0}{2 + x_1^2} \tag{2.28}$$

and

$$\begin{aligned} C_4^{(0)} &= \frac{C_3^{(0)}}{4\sigma b^{3/2}\left(2 + x_1^2\right)\left(2\sigma^2 - 2ab + Eb^2 - b^3 - x_0\sigma b^{3/2}\right)} \\ &\times \Big[2\sigma^2 x_1^2\left(2\sigma^2 - 2ab + Eb^2 - 2b^3 - x_0\sigma b^{3/2}\right) \\ &+ x_0\sigma b^{3/2}\left(b^3 + 2ab - Eb^2 - 4\sigma^2\right) \\ &+ 8\sigma^4 - 12ab\sigma^2 + b^2\left(4a^2 + 6E\sigma^2\right) - 8\sigma^2 b^3 - 4Eab^3\Big]. \end{aligned} \tag{2.29}$$

Hence for the caplet with $a > bE$, the free boundary close to expiry is of the form

$$x_f(\tau) \sim \sum_{n=0}^{\infty} x_n \tau^{n/2}, \tag{2.30}$$

with x_0, x_1 and x_2 as given above.

2.2. Caplet with $a < bE$

For this case, the free boundary starts from

$$x_0 = \frac{2b^{1/2}E}{\sigma} - \frac{2a}{\sigma b^{1/2}} + \frac{2\sigma}{b^{3/2}}, \tag{2.31}$$

and the initial condition is $v(x,0) = -(\sigma(x-x_0))/(2\sqrt{b})$ for $x < x_0$. Initially, we will try an expansion of the form

$$\begin{aligned} v(x,\tau) &\sim \sum_{n=1}^{\infty} \tau^{n/2} V_n^{(0)}(\xi), \\ x_f(\tau) &\sim \sum_{n=0}^{\infty} x_n \tau^{n/2}. \end{aligned} \tag{2.32}$$

If we substitute the expansion for $v(x,\tau)$ into the PDE (2.8), at the first few orders, we find

$$\begin{aligned} \mathcal{L}_1 V_1^{(0)} &= 0, \\ \mathcal{L}_2 V_2^{(0)} &= \frac{Eb-a}{2b}, \\ \mathcal{L}_3 V_3^{(0)} &= \frac{\left(b^2+2bE-a\right)\sigma\xi}{2b^{5/2}}, \\ \mathcal{L}_4 V_4^{(0)} &= \frac{\left(4b^2+3bE-a\right)\sigma^2\xi^2}{4b^4} \\ &+ \frac{3E}{4} + \frac{2E^2-3a}{4b} + \frac{\sigma^2-2aE}{4b^2} + \frac{3\sigma^2 E}{8b^3} - \frac{a\sigma^2}{8b^4}. \end{aligned} \tag{2.33}$$

It is straightforward to find solutions to (2.33) that satisfy the condition at $\tau = 0$,

$$\begin{aligned} V_1^{(0)} &= C_1^{(0)}\left[\xi\operatorname{erfc}(-\xi) + \frac{e^{-\xi^2}}{\sqrt{\pi}}\right] - \frac{\sigma\xi}{b^{1/2}}, \\ V_2^{(0)} &= C_2^{(0)}\left[\left(2\xi^2+1\right)\operatorname{erfc}(-\xi) + \frac{2\xi e^{-\xi^2}}{\sqrt{\pi}}\right] + \frac{a-bE}{2b}, \\ V_3^{(0)} &= C_3^{(0)}\left[\left(2\xi^3+3\xi\right)\operatorname{erfc}(-\xi) + \frac{2\left(\xi^2+1\right)e^{-\xi^2}}{\sqrt{\pi}}\right] + \frac{\left(a-b^2-2bE\right)\sigma\xi}{2b^{5/2}}, \\ V_4^{(0)} &= C_4^{(0)}\left[\left(4\xi^2+12\xi^2+3\right)\operatorname{erfc}(-\xi) + \frac{2\left(2\xi^3+5\xi\right)e^{-\xi^2}}{\sqrt{\pi}}\right] \\ &+ \frac{\left(a-4b^2-3bE\right)\sigma^2\xi^2}{4b^4} \\ &- \frac{3E}{8} + \frac{3a-2E^2}{8b} + \frac{2aE-3\sigma^2}{8b^2} - \frac{3\sigma^2 E}{8b^3} + \frac{a\sigma^2}{8b^4}. \end{aligned} \tag{2.34}$$

If we attempt to apply the conditions on the free boundary by substituting the assumed form (2.32) for $x_f(\tau)$ into the solution (2.34), at leading order we get the pair of equations,

$$\frac{x_1}{2}\left[C_1^{(0)}\operatorname{erfc}\left(-\frac{x_1}{2}\right) - \frac{\sigma}{b^{1/2}}\right] + \frac{C_1^{(0)} e^{-x_1^2/4}}{\sqrt{\pi}} = 0,$$

$$C_1^{(0)}\mathrm{erfc}\left(-\frac{x_1}{2}\right)-\frac{\sigma}{b^{1/2}} \quad = \quad 0, \tag{2.35}$$

so $C_1^{(0)}=\sigma/(2b^{1/2})$ and $e^{-x_1^2/4}=0$ and $\mathrm{erfc}(-x_1/2)=0$ or $x_1=\infty$. The fact that we require $x_1=\infty$ is a problem, and in a moment, we will see that in our analysis, where we have grouped terms in powers of τ, the statement "$e^{-x_1^2/4}=\mathrm{erfc}(x_1/2)=0$" actually means that the terms $e^{-x_1^2/4}$ and $\mathrm{erfc}(-x_1/2)$ are $O\left(\tau^{1/2}\right)$, so they vanish at this order but re-appear at a later order in the analysis. This same situation occurs for the American equity put with a dividend yield less than the risk-free rate, which we have previously studied using the same techniques [18].

Returning to the boundary conditions, at the next power of τ, we find

$$\begin{aligned} C_2^{(0)}\left(x_1^2+2\right)+\frac{a/b-E}{2} \quad &= \quad 0, \\ 4C_2^{(0)}x_1 \quad &= \quad 0. \end{aligned} \tag{2.36}$$

The second of these has a solution $C_2^{(0)}=0$, but the first then becomes $a/b-E=0$ which has no solution, except for the special case $a=bE$ which we will consider separately later. It is to deal with this inconsistency that we require $e^{-x_1^2/4}$ and $\mathrm{erfc}(x_1/2)$ to be $O\left(\tau^{1/2}\right)$, so that they enter into this equation and remove the inconsistency. To accomplish this, the expansion for $x_f(\tau)$ must be of the form

$$x_f(\tau)\sim\sum_{n=1}^{\infty}\tau^{n/2}f_n(-\ln\tau), \tag{2.37}$$

where

$$f_n(-\ln\tau)\sim(-\ln\tau)^{a_n}\sum_{m=0}^{\infty}x_n^{(m)}(-\ln\tau)^{-m}. \tag{2.38}$$

The presence of logs in the series (2.37,2.38) for $x_f(\tau)$ and the functions f_n necessitate the presence of logs in the series (2.32) for $v(x,\tau)$, which will be of the form

$$v(x,\tau)=\tau^{1/2}V_1^{(0)}(\xi)+\sum_{n=2}^{\infty}\sum_{m=0}^{\infty}\tau^{n/2}\left(-\ln\tau\right)^{-m}V_n^{(m)}(\xi). \tag{2.39}$$

With this expression for x_f, on the free boundary we have

$$\begin{aligned} e^{-\xi^2} \quad &= \quad \exp\left[-\frac{x_f^2}{4\tau}\right] \\ &\sim \quad e^{-f_1^2/4}\left[1-\frac{1}{2}f_1f_2\tau^{1/2}+\left(\frac{1}{8}f_1^2f_2^2-\frac{1}{2}f_1f_3-\frac{1}{4}f_2^2\right)\tau+\cdots\right]. \end{aligned} \tag{2.40}$$

At leading order in this expression, we require that $e^{-f_1^2/4} \sim O\left(\tau^{1/2}\right)$, so that $\exp\left[-\frac{x_1^{(0)2}}{4}(-\ln\tau)^{2a_1}\right] \sim \tau^{1/2}$ or $-\frac{x_1^{(0)2}}{4}(-\ln\tau)^{2a_1} \sim \frac{1}{2}\ln\tau$, which means that $a_1 = 1/2$ and $x_1^{(0)} = \sqrt{2}$, and hence

$$e^{-f_1^2/4} \sim \tau^{1/2}e^{-x_1^{(2)}/\sqrt{2}}\left[1-\left(\frac{x_1^{(2)}}{\sqrt{2}}+\frac{x_1^{(1)2}}{4}\right)(-\ln\tau)^{-1}+\cdots\right]. \tag{2.41}$$

Similarly, we can show that

$$\begin{aligned}\operatorname{erfc}(\xi) &= \operatorname{erfc}\left[\frac{x_f}{2\sqrt{\tau}}\right]\\ &\sim \operatorname{erfc}\left[\frac{f_1}{2}\right]-\frac{e^{-f_1^2/4}}{\sqrt{\pi}}\left[f_2\tau^{1/2}+\left(f_3-\frac{1}{4}f_1f_2^2\right)\tau\cdots\right],\end{aligned} \tag{2.42}$$

and we can use the result that as $\zeta\to\infty$ [1],

$$\operatorname{erfc}(\zeta) \sim \frac{e^{-\zeta^2}}{\zeta\sqrt{\pi}}\left[1+\sum_{m=1}^{\infty}\frac{1\times3\times\cdots\times(2m-1)}{(-2\zeta^2)^m}\right] \tag{2.43}$$

to give

$$\begin{aligned}\operatorname{erfc}\left[\frac{f_1}{2}\right] &\sim \tau^{1/2}(-\ln\tau)^{-1/2}\pi^{-1/2}e^{-x_1^{(2)}/\sqrt{2}}\\ &\times \left[\sqrt{2}-\left(x_1^{(1)}+x_1^{(2)}+\sqrt{2}+\frac{x_1^{(2)2}}{2\sqrt{2}}\right)(-\ln\tau)^{-1}+\cdots\right].\end{aligned} \tag{2.44}$$

Before we can compute the coefficients in the series (2.37,2.38) for the location of the free boundary, it is necessary to solve for some of the terms involving logs in the series (2.39) for $v(x,\tau)$. We note first that the terms in this series not involving logs are as given above in (2.34), together with the coefficients found above, so that

$$\begin{aligned}V_1^{(0)} &= \frac{\sigma\xi}{b^{1/2}}\left[-\xi\operatorname{erfc}(\xi)+\frac{e^{-\xi^2}}{\sqrt{\pi}}\right],\\ V_2^{(0)} &= \frac{a-bE}{2b}.\end{aligned} \tag{2.45}$$

Considering the terms at $O\left(\tau^{n/2}(-\ln\tau)^{-1}\right)$, at successive orders we find

$$\begin{aligned}\mathcal{L}_2V_2^{(1)} &= 0,\\ \mathcal{L}_3V_3^{(1)} &= 0.\end{aligned} \tag{2.46}$$

The solutions at the first few orders are given by

$$\begin{aligned} V_2^{(1)} &= C_2^{(1)} \left[\left(2\xi^2 + 1 \right) \operatorname{erfc}(-\xi) + \frac{2\xi e^{-\xi^2}}{\sqrt{\pi}} \right], \\ V_3^{(1)} &= C_3^{(1)} \left[\left(2\xi^3 + 3\xi \right) \operatorname{erfc}(-\xi) + \frac{2\left(\xi^2+1\right) e^{-\xi^2}}{\sqrt{\pi}} \right]. \end{aligned} \tag{2.47}$$

The conditions on the free boundary yield at leading order in τ,

$$\begin{aligned} 2C_2^{(1)} + \frac{a/b - E}{2} &= \mathcal{O}\left([-\ln\tau]^{-1} \right), \\ \left[2^{5/2} C_2^{(1)} - \frac{\sigma e^{-x_1^{(1)}/\sqrt{2}}}{\sqrt{2b\pi}} \right] [-\ln\tau]^{-1/2} &= \mathcal{O}\left([-\ln\tau]^{-3/2} \right), \end{aligned} \tag{2.48}$$

which have a solution

$$\begin{aligned} C_2^{(1)} &= \frac{E - a/b}{4}, \\ x_1^{(1)} &= -\sqrt{2} \ln \left[\frac{2b^{1/2}\sqrt{\pi}\,(a/b - E)}{\sigma} \right]. \end{aligned} \tag{2.49}$$

At the next power of τ, we get the pair of equations,

$$\begin{aligned} & \sqrt{2} C_3^{(0)} [-\ln\tau]^{3/2} \\ + & \left[\frac{\sigma\left(a - 2bE - b^2\right)}{2^{3/2} b^{5/2}} + 3C_3^{(0)} \left(\sqrt{2} + x_1^{(1)} \right) + \sqrt{2} C_3^{(1)} \right] [-\ln\tau]^{1/2} \\ = & \ \mathcal{O}\left([-\ln\tau]^{-1/2} \right), \\ & 6C_3^{(0)} [-\ln\tau]^{1} \\ + & \ \frac{\sigma\left(a - 2bE - b^2\right)}{2b^{5/2}} + 6C_3^{(0)} \left(1 + \sqrt{2} x_1^{(1)} \right) + 6C_3^{(1)} + x_2^{(0)} \left(E - \frac{a}{b} \right) \\ = & \ \mathcal{O}\left([-\ln\tau]^{-1} \right), \end{aligned} \tag{2.50}$$

so that $C_3^{(0)} = 0$ and

$$\begin{aligned} C_3^{(1)} &= \frac{\sigma\left(b^2 + 2bE - a\right)}{4b^{5/2}}, \\ x_2^{(0)} &= \frac{\sigma\left(b^2 + 2bE - a\right)}{b^{3/2}\left(a - bE\right)}. \end{aligned} \tag{2.51}$$

Hence for the caplet with $a < bE$, the free boundary close to expiry is of the form

$$\begin{aligned} x_f(\tau) &\sim x_0 + \sqrt{-\tau\ln\tau} \left[\sqrt{2} + x_1^{(1)} (-\ln\tau)^{-1} + \cdots \right] \\ &+ \tau \left[x_2^{(0)} + \cdots \right] + \cdots, \end{aligned} \tag{2.52}$$

with $x_1^{(1)}$ and $x_2^{(0)}$ as given above.

2.3. Caplet with $a = bE$

For this case, the free boundary starts from $x_0 = 2\sigma/b^{3/2}$. This case was touched on briefly earlier, when we mentioned that (2.36) had a solution for this case but not for $a < bE$. Once again, the initial condition is $v(x,0) = -(\sigma(x-x_0))/(2\sqrt{b})$ for $x < x_0$. As with the case $a < bE$, we will try an expansion of the form (2.32). If we substitute the expansion for $v(x,\tau)$ into the PDE (2.8), at the first few orders, we recover the equations (2.33) with solutions (2.34), but with a replaced by bE. If we attempt to apply the conditions on the free boundary by substituting the assumed form (2.32) for $x_f(\tau)$ into the solution (2.34), at leading order we get the pair of equations,

$$\begin{aligned} \frac{x_1}{2}\left[C_1^{(0)}\mathrm{erfc}\left(-\frac{x_1}{2}\right) - \frac{\sigma}{b^{1/2}}\right] + \frac{C_1^{(0)}e^{-x_1^2/4}}{\sqrt{\pi}} &= 0, \\ C_1^{(0)}\mathrm{erfc}\left(-\frac{x_1}{2}\right) - \frac{\sigma}{b^{1/2}} &= 0, \end{aligned} \tag{2.53}$$

so that $C_1^{(0)} = \sigma/(2b^{1/2})$ and $e^{-x_1^2/4} = 0$ and $\mathrm{erfc}\,(-x_1/2) = 2$ or $x_1 = \infty$, which is a similar problem to that encountered when $a < bE$. At the next order, we find

$$\begin{aligned} C_2^{(0)}\left(x_1^2 + 2\right) &= 0, \\ 4C_2^{(0)}x_1 &= 0, \end{aligned} \tag{2.54}$$

so that $C_2^{(0)} = 0$. At the next order, we find the pair of equations,

$$\begin{aligned} x_1\left[3C_3^{(0)} - \frac{\sigma\,(E+b)}{4b^{3/2}}\right] + \frac{1}{2}C_3^{(0)}x_1^3 &= 0, \\ 2\left[3C_3^{(0)} - \frac{\sigma\,(E+b)}{4b^{3/2}}\right] + 3C_3^{(0)}x_1^2 &= 0, \end{aligned} \tag{2.55}$$

which has no solution. The $\mathrm{erfc}(-\xi)$ and $e^{-\xi^2}$ terms from $V_1^{(0)}$ must be added to (2.55) to rectify this. To do this, if we suppose that $x_f(\tau)$ is of the form (2.37), then we require $e^{-f_1^2/4} \sim \tau f_1$, as opposed to the relation $e^{-f_1^2/4} \sim \tau^{1/2}$ for the case $a < bE$, so that

$$f_1(\tau) \sim \left[2W_L\left(\frac{1}{2\tau^2}\right)\right]^{1/2} \tag{2.56}$$

where W_L is a special function, the Lambert W function, which is defined to be the solution to the equation $W_L(x)e^{W_L(x)} = x$. It follows that

$$\begin{aligned} f_1(\tau) &\sim \left[2W_L\left(\frac{\tau^{-2}}{2}\right)\right]^{1/2} \sum_{m=0}^{\infty} x_1^{(m)} \left[2W_L\left(\frac{\tau^{-2}}{2}\right)\right]^{-m} \\ f_n(\tau) &\sim \left[2W_L\left(\frac{\tau^{-2}}{2}\right)\right]^{a_n} \sum_{m=0}^{\infty} x_n^{(m)} \left[2W_L\left(\frac{\tau^{-2}}{2}\right)\right]^{-m}, \end{aligned} \tag{2.57}$$

with $x_1^{(0)} = 1$. This means that our series for $v(x,\tau)$ must be of the form

$$\begin{aligned} v(x,\tau) &= \tau^{1/2} V_1^{(0)}(\xi) + \tau V_2^{(0)}(\xi) \\ &+ \sum_{n=3}^{\infty}\sum_{m=0}^{\infty} \tau^{n/2} \left[2W_L\left(\frac{\tau^{-2}}{2}\right)\right]^{-m} V_n^{(m)}(\xi), \end{aligned} \tag{2.58}$$

with

$$\begin{aligned} V_1^{(0)} &= \frac{\sigma\xi}{b^{1/2}} \left[-\xi \operatorname{erfc}(\xi) + \frac{e^{-\xi^2}}{\sqrt{\pi}}\right], \\ V_2^{(0)} &= 0, \end{aligned} \tag{2.59}$$

and $V_3^{(0)}$ is given in (2.34) with a set equal to bE. For the $\tau^{n/2}\left[2W_L\left(\frac{\tau^{-2}}{2}\right)\right]^{-1}$ terms we have

$$\mathcal{L}_3 V_3^{(1)} = 0, \tag{2.60}$$

with a solution

$$V_3^{(1)} = C_3^{(1)} \left[\left(2\xi^3 + 3\xi\right) \operatorname{erfc}(-\xi) + \frac{2\left(\xi^2+1\right)e^{-\xi^2}}{\sqrt{\pi}}\right]. \tag{2.61}$$

The conditions on the free boundary yield at leading order in τ,

$$\begin{aligned} &\frac{C_3^{(0)}}{2}\left[2W_L\left(\frac{\tau^{-2}}{2}\right)\right]^{-3/2} \\ &+ \left[3C_3^{(0)}\left(1 + \frac{x_1^{(1)}}{2}\right) + \frac{C_3^{(1)}}{2} - \frac{\sigma(b+E)}{4b^{3/2}}\right]\left[2W_L\left(\frac{\tau^{-2}}{2}\right)\right]^{-1/2} \\ &= \mathcal{O}\left(\left[2W_L\left(\frac{\tau^{-2}}{2}\right)\right]^{1/2}\right) \end{aligned} \tag{2.62}$$

and

$$
\begin{aligned}
& 3C_3^{(0)}\left[2W_L\left(\frac{\tau^{-2}}{2}\right)\right]^{-1} \\
+ \quad & 6C_3^{(0)}\left(1+x_1^{(1)}\right)+3C_3^{(1)}-\frac{\sigma}{2b^{3/2}}\left[E+b+\frac{2be^{-x_1^{(1)}/2}}{\sqrt{\pi}}\right] \\
= \quad & \mathcal{O}\left(\left[2W_L\left(\frac{\tau^{-2}}{2}\right)\right]^{1}\right),
\end{aligned}
\tag{2.63}
$$

which have a solution $C_3^{(0)}=0$ and

$$
\begin{aligned}
C_3^{(1)} &= \frac{\sigma\,(E+b)}{2b^{3/2}}, \\
x_1^{(1)} &= -2\ln\left[\frac{\sqrt{\pi}\,(E+b)}{b}\right].
\end{aligned}
\tag{2.64}
$$

Hence for the caplet with $a=bE$, the free boundary close to expiry is of the form

$$
x_f \sim x_0+\sqrt{2\tau W_L\left(\frac{\tau^{-2}}{2}\right)}\left[1+x_1^{(1)}\left[2W_L\left(\frac{\tau^{-2}}{2}\right)\right]^{-1}\right]+\cdots, \tag{2.65}
$$

with $x_1^{(1)}$ as given above.

2.4. Floorlet with $a<bE$

This case is very similar to the caplet with $a>bE$. The free boundary starts from

$$
x_0 = \frac{2\sigma}{b^{3/2}}-\frac{2a}{\sigma b^{1/2}}+\frac{b^{1/2}}{\sigma}\left[E-b+\sqrt{(E-b)^2+4a}\right] \tag{2.66}
$$

at expiry. We will use an expansion of the same form as for the caplet with $a>bE$, that is (2.19). If we substitute the expansion for $v(x,\tau)$ into the PDE (2.8), at the first few orders we find,

$$
\begin{aligned}
\mathcal{L}_3V_3^{(0)} &= -\left[\frac{1}{2b^{1/2}}+\frac{x_0\sigma}{2b^2}-\frac{a}{b^{5/2}}+\frac{E}{2b^{3/2}}+\frac{\sigma^2}{b^{7/2}}\right]\sigma\xi, \\
\mathcal{L}_4V_4^{(0)} &= -\frac{\sigma^2+aE}{2b^2}+\frac{E\sigma^2+a^2}{2b^3}-\frac{a\sigma^2}{b^4}+\frac{\sigma^4}{2b^5}
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{x_0\sigma}{4b^{1/2}}\left(\frac{E}{2b} - \frac{a}{b^2} + \frac{\sigma^2}{b^3} - \frac{1}{2}\right) - \left[1 - \frac{E}{2b} + \frac{a}{b^2} - \frac{\sigma^2}{b^3} + \frac{x_0\sigma}{2b^{3/2}}\right]\frac{\sigma^2\xi^2}{b^2},\\
\mathcal{L}_5 V_5^{(0)} &= -\left[\frac{x_0\sigma}{2b^5} + \frac{1}{b^{7/2}}\left(\frac{3}{2} - \frac{E}{2b} + \frac{a}{b^2} - \frac{\sigma^2}{b^3}\right)\right]\frac{\sigma^3\xi^3}{2}\\
&- \left[\frac{x_0\sigma}{b^{3/2}}\left(3 + \frac{E}{b} - \frac{a}{b^2} + \frac{3\sigma^2}{2b^3}\right) + 3 - \frac{3E}{b} + \frac{7a}{b^2} + \frac{6Ea - 5\sigma^2}{2b^3}\right.\\
&\left. - \frac{4a^2 - 7E\sigma^2}{2b^4} + \frac{5a\sigma^2}{b^5} - \frac{3\sigma^4}{b^6}\right]\frac{\sigma\xi}{4b^{1/2}}. \qquad (2.67)
\end{aligned}
$$

It is straightforward to find solutions to (2.67) that satisfy the condition at $\tau = 0$,

$$
\begin{aligned}
V_3^{(0)} &= C_3^{(0)}\left[-\left(3\xi + 2\xi^3\right)\mathrm{erfc}(\xi) + \frac{2\left(1+\xi^2\right)e^{-\xi^2}}{\sqrt{\pi}}\right]\\
&- \left[\frac{E}{2b} + \frac{\sigma^2}{b^3} - \frac{1}{2} - \frac{x_0\sigma}{2b^{3/2}} - \frac{a}{b^2}\right]\frac{\sigma\xi}{b^{1/2}},\\
V_4^{(0)} &= C_4^{(0)}\left[-\left(3 + 12\xi^2 + 4\xi^4\right)\mathrm{erfc}(\xi) + \frac{2\left(5+2\xi^2\right)e^{-\xi^2}}{\sqrt{\pi}}\right]\\
&- \left[-1 + \frac{E}{2b} - \frac{a}{b^2} + \frac{\sigma^2}{b^3} - \frac{x_0\sigma}{2b^{3/2}}\right]\frac{\sigma^2\xi^2}{b^2} - \frac{x_0\sigma}{4b^{1/2}}\left[\frac{1}{4} - \frac{E}{4b} + \frac{a}{2b^2} - \frac{\sigma^2}{b^3}\right]\\
&+ \frac{2\sigma^2 + aE}{4b^2} - \frac{3E\sigma^2 + 2a^2}{8b^3} + \frac{3a\sigma^2}{4b^4} - \frac{\sigma^4}{2b^5},\\
V_5^{(0)} &= C_5^{(0)}\left[-\left(15\xi + 20\xi^3 + 4\xi^5\right)\mathrm{erfc}(\xi) + \frac{2\left(4 + 9\xi^2 + 2\xi^4\right)e^{-\xi^2}}{\sqrt{\pi}}\right]\\
&- \left[\frac{E}{2b} - \frac{a}{b^2} - \frac{\sigma x_0}{2b^{3/2}} + \frac{\sigma^2}{b^3} - \frac{3}{2}\right]\frac{\sigma^3\xi^3}{2b^{7/2}} - \left[\frac{a}{b^2} - \frac{E}{b} - 3 - \frac{3\sigma^2}{b^3}\right]\frac{x_0\sigma^2\xi}{8b^2}\\
&- \left[\frac{3E}{2b} - \frac{3}{2} - \frac{7a}{2b^2} - \frac{2\sigma^2 + 3Ea}{2b^3} + \frac{2a^2 + 5E\sigma^2}{2b^4} - \frac{4a\sigma^2}{b^5} + \frac{3\sigma^4}{b^6}\right]\frac{\sigma\xi}{4b^{1/2}}. \qquad (2.68)
\end{aligned}
$$

If we apply the conditions on the free boundary by substituting the assumed form (2.19) for $x_f(\tau)$ into the solution (2.68), at leading order we get the pair of equations,

$$
\begin{aligned}
C_3^{(0)}\left[\frac{\left(4+\xi^2\right)e^{-x_1^2/4}}{2\sqrt{\pi}} - \frac{6x_1 + x_1^3}{4}\mathrm{erfc}\left(\frac{x_1}{2}\right)\right]&\\
-\frac{\sigma x_1\left(2\sigma^2 - 2ab + Eb^2 - b^3 - x_0\sigma b^{3/2}\right)}{4b^{7/2}} &= 0,
\end{aligned}
$$

$$3C_3^{(0)}\left[\frac{x_1 e^{-x_1^2/4}}{\sqrt{\pi}} - \left(1+\frac{x_1^2}{2}\right)\mathrm{erfc}\left(\frac{x_1}{2}\right)\right]$$

$$-\frac{\sigma\left(2\sigma^2 - 2ab + Eb^2 - b^3 - x_0\sigma b^{3/2}\right)}{2b^{7/2}} = 0, \tag{2.69}$$

so that x_1 is the solution of

$$x_1^3\mathrm{erfc}\left(\frac{x_1}{2}\right) - \frac{\left(2x_1^2-4\right)e^{-x_1^2/4}}{\sqrt{\pi}} = 0, \tag{2.70}$$

or $x_1 = -0.90344659785$, while

$$C_3^{(0)} = \frac{\sigma x_1^3\sqrt{\pi}}{24b^{7/2}e^{-x_1^2/4}}\left(2\sigma^2 - 2ab + Eb^2 - b^3 - x_0\sigma b^{3/2}\right). \tag{2.71}$$

At the next order, we get another pair of equations,

$$C_4^{(0)}\left[-\frac{\left(12+12x_1^2+x_1^4\right)}{4}\mathrm{erfc}\left(\frac{x_1}{2}\right) + \frac{\left(10x_1+x_1^3\right)e^{-x_1^2/4}}{2\sqrt{\pi}}\right]$$

$$+\ 3C_3^{(0)}x_2\left[-\frac{2+x_1^2}{4}\mathrm{erfc}\left(\frac{x_1}{2}\right) + \frac{x_1 e^{-x_1^2/4}}{2\sqrt{\pi}}\right]$$

$$+\ \frac{\sigma x_2}{2b^{1/2}}\left[-\frac{E}{2b} + \frac{1}{2} + \frac{a}{b^2} - \frac{\sigma^2}{b^3} + \frac{\sigma x_0}{2b^{3/2}}\right]$$

$$+\ \frac{\sigma^2 x_1^2}{4b^2}\left[-\frac{E}{2b} + 1 + \frac{a}{b^2} - \frac{\sigma^2}{b^3} + \frac{\sigma x_0}{2b^{3/2}}\right]$$

$$+\ \frac{\sigma x_0}{4b^{1/2}}\left[-\frac{1}{4} + \frac{E}{4b} - \frac{a}{2b^2} + \frac{\sigma^2}{b^3}\right]$$

$$+\ \frac{2\sigma^2+aE}{4b^2} - \frac{2a^2+3E\sigma^2}{8b^3} + \frac{3a\sigma^2}{4b^4} - \frac{\sigma^4}{2b^5} = 0, \tag{2.72}$$

and

$$2C_4^{(0)}\left[-\left(6x_1+x_1^3\right)\mathrm{erfc}\left(\frac{x_1}{2}\right) + \frac{2\left(4+x_1^2\right)e^{-x_1^2/4}}{\sqrt{\pi}}\right]$$

$$+\ 3C_3^{(0)}x_2\left[-x_1\mathrm{erfc}\left(\frac{x_1}{2}\right) + \frac{2e^{-x_1^2/4}}{\sqrt{\pi}}\right]$$

$$+\ \frac{\sigma^2 x_1}{b^2}\left[-\frac{E}{2b} + 1 + \frac{a}{b^2} - \frac{\sigma^2}{b^3} + \frac{\sigma x_0}{2b^{3/2}}\right] = 0, \tag{2.73}$$

which have a solution,

$$x_2 = \frac{\sigma x_1^2\left[2\sigma^2 - b\left(E^2+6a\right) + 3Eb^2 - 2b^3 - \sigma x_0 b^{3/2}\right]}{b^{5/2}\left(2+x_1^2\right)\left[(E-b)^2+4a\right]} - \frac{x_0}{2+x_1^2} \tag{2.74}$$

and

$$
\begin{aligned}
C_4^{(0)} &= \frac{C_3^{(0)}}{4\sigma b^{3/2}\left(2+x_1^2\right)\left(2\sigma^2-2ab+Eb^2-b^3-x_0\sigma b^{3/2}\right)} \\
&\times \left[2\sigma^2 x_1^2\left(2\sigma^2-2ab+Eb^2-2b^3-x_0\sigma b^{3/2}\right)\right. \\
&+x_0\sigma b^{3/2}\left(b^3+2ab-Eb^2-4\sigma^2\right) \\
&\left.+8\sigma^4-12ab\sigma^2+b^2\left(4a^2+6E\sigma^2\right)-8\sigma^2 b^3-4Eab^3\right]. \qquad (2.75)
\end{aligned}
$$

Hence for the floorlet with $a < bE$, the free boundary close to expiry is of the form

$$
x_f(\tau) \sim \sum_{n=0}^{\infty} x_n \tau^{n/2}, \qquad (2.76)
$$

with x_0, x_1 and x_2 as given above.

2.5. Floorlet with $a > bE$

This case is very similar to the caplet with $a < bE$. The free boundary starts from

$$
x_0 = \frac{2b^{1/2}E}{\sigma} - \frac{2a}{\sigma b^{1/2}} + \frac{2\sigma}{b^{3/2}}, \qquad (2.77)
$$

and the initial condition is $v(x,0) = (\sigma(x-x_0))/(2\sqrt{b})$ for $x > x_0$. Initially, we will try the same form of expansion as (2.32). If we substitute the expansion for $v(x,\tau)$ into the PDE (2.8), at the first few orders, we find

$$
\begin{aligned}
\mathcal{L}_1 V_1^{(0)} &= 0, \\
\mathcal{L}_2 V_2^{(0)} &= -\frac{Eb-a}{2b}, \\
\mathcal{L}_3 V_3^{(0)} &= -\frac{\left(b^2+2bE-a\right)\sigma\xi}{2b^{5/2}}, \\
\mathcal{L}_4 V_4^{(0)} &= -\frac{\left(4b^2+3bE-a\right)\sigma^2\xi^2}{4b^4} \\
&- \frac{3E}{4} - \frac{2E^2-3a}{4b} - \frac{\sigma^2-2aE}{4b^2} - \frac{3\sigma^2 E}{8b^3} + \frac{a\sigma^2}{8b^4}. \qquad (2.78)
\end{aligned}
$$

It is straightforward to find solutions to (2.78) that satisfy the condition at $\tau = 0$,

$$
V_1^{(0)} = C_1^{(0)}\left[-\xi\,\mathrm{erfc}(\xi) + \frac{e^{-\xi^2}}{\sqrt{\pi}}\right] + \frac{\sigma\xi}{b^{1/2}},
$$

$$
\begin{aligned}
V_2^{(0)} &= C_2^{(0)}\left[-\left(2\xi^2+1\right)\operatorname{erfc}(\xi)+\frac{2\xi e^{-\xi^2}}{\sqrt{\pi}}\right]-\frac{a-bE}{2b},\\
V_3^{(0)} &= C_3^{(0)}\left[-\left(2\xi^3+3\xi\right)\operatorname{erfc}(\xi)+\frac{2\left(\xi^2+1\right)e^{-\xi^2}}{\sqrt{\pi}}\right]-\frac{\left(a-b^2-2bE\right)\sigma\xi}{2b^{5/2}},\\
V_4^{(0)} &= C_4^{(0)}\left[-\left(4\xi^2+12\xi^2+3\right)\operatorname{erfc}(\xi)+\frac{2\left(2\xi^3+5\xi\right)e^{-\xi^2}}{\sqrt{\pi}}\right]\\
&- \frac{\left(a-4b^2-3bE\right)\sigma^2\xi^2}{4b^4}\\
&+ \frac{3E}{8}-\frac{3a-2E^2}{8b}-\frac{2aE-3\sigma^2}{8b^2}+\frac{3\sigma^2E}{8b^3}-\frac{a\sigma^2}{8b^4}.
\end{aligned}
\tag{2.79}
$$

If we attempt to apply the conditions on the free boundary by substituting the assumed form (2.32) for $x_f(\tau)$ into the solution (2.79), at leading order we get the pair of equations,

$$
\begin{aligned}
\frac{x_1}{2}\left[\frac{\sigma}{b^{1/2}}-C_1^{(0)}\operatorname{erfc}\left(\frac{x_1}{2}\right)\right]+\frac{C_1^{(0)}e^{-x_1^2/4}}{\sqrt{\pi}} &= 0,\\
\frac{\sigma}{b^{1/2}}-C_1^{(0)}\operatorname{erfc}\left(\frac{x_1}{2}\right) &= 0,
\end{aligned}
\tag{2.80}
$$

so $C_1^{(0)}=\sigma/(2b^{1/2})$ and $e^{-x_1^2/4}=0$ and $\operatorname{erfc}(x_1/2)=2$ or $x_1=-\infty$. The fact that we require $x_1=-\infty$ is a problem, and we must take a similar approach to that used for the caplet with $a<bE$, and once again introduce logs.

Returning to the boundary conditions, at the next power of τ, we find

$$
\begin{aligned}
-C_2^{(0)}\left(x_1^2+2\right)+\frac{E-a/b}{2} &= 0,\\
-4C_2^{(0)}x_1 &= 0.
\end{aligned}
\tag{2.81}
$$

The second of these has a solution $C_2^{(0)}=0$, but the first then becomes $E-a/b=0$ which has no solution, except for the special case $a=bE$ which we will consider separately. To deal with this inconsistency, we require $e^{-x_1^2/4}$ and $\operatorname{erfc}(-x_1/2)$ to be $\mathcal{O}\left(\tau^{1/2}\right)$, so that they enter into this equation and remove the inconsistency. To accomplish this, the expansion for $x_f(\tau)$ must be of the same form as (2.37,2.38) for the caplet with $a<bE$. The scaling arguments used here are very similar to those for the caplet with $a<bE$, except we now require $\operatorname{erfc}(-x_1/2)$ rather than $\operatorname{erfc}(x_1/2)$ to be $\mathcal{O}\left(\tau^{1/2}\right)$, so that once again $a_1=1/2$ but now $x_1^{(0)}=\sqrt{2}$. The presence of logs in the series (2.37,2.38) for $x_f(\tau)$ and the functions f_n once again necessitate the presence of logs in the series (2.32) for $v(x,\tau)$, which will be of the form (2.39). Before we can compute the coefficients in the series (2.37,2.38) for the location of the

free boundary, it is necessary to solve for some of the terms involving logs in the series (2.39) for $v(x,\tau)$. Once again, the terms in the series not involving logs are as given above in (2.79), together with the coefficients found above, so that

$$\begin{aligned} V_1^{(0)} &= \frac{\sigma\xi}{b^{1/2}}\left[\xi\operatorname{erfc}(-\xi)+\frac{e^{-\xi^2}}{\sqrt{\pi}}\right], \\ V_2^{(0)} &= \frac{bE-a}{2b}. \end{aligned} \tag{2.82}$$

Considering the terms at $\mathcal{O}\left(\tau^{n/2}(-\ln\tau)^{-1}\right)$, at successive orders we find

$$\begin{aligned} \mathcal{L}_2 V_2^{(1)} &= 0, \\ \mathcal{L}_3 V_3^{(1)} &= 0. \end{aligned} \tag{2.83}$$

The solutions at the first few orders are given by solution

$$\begin{aligned} V_2^{(1)} &= C_2^{(1)}\left[-\left(2\xi^2+1\right)\operatorname{erfc}(\xi)+\frac{2\xi e^{-\xi^2}}{\sqrt{\pi}}\right], \\ V_3^{(1)} &= C_3^{(1)}\left[-\left(2\xi^3+3\xi\right)\operatorname{erfc}(\xi)+\frac{2\left(\xi^2+1\right)e^{-\xi^2}}{\sqrt{\pi}}\right]. \end{aligned} \tag{2.84}$$

The conditions on the free boundary yield at leading order in τ, leading order

$$\begin{aligned} -2C_2^{(1)}+\frac{E-a/b}{2} &= \mathcal{O}\left([-\ln\tau]^{-1}\right), \\ \left[2^{5/2}C_2^{(1)}+\frac{\sigma e^{x_1^{(1)}/\sqrt{2}}}{b^{1/2}\sqrt{2\pi}}\right][-\ln\tau]^{-1/2} &= \mathcal{O}\left([-\ln\tau]^{-3/2}\right), \end{aligned} \tag{2.85}$$

which have a solution

$$\begin{aligned} C_2^{(1)} &= \frac{E-a/b}{4}, \\ x_1^{(1)} &= \sqrt{2}\ln\left[\frac{2b^{1/2}\sqrt{\pi}\,(a/b-E)}{\sigma}\right]. \end{aligned} \tag{2.86}$$

At the next power of τ, we get the pair of equations,

$$\begin{aligned} &\sqrt{2}C_3^{(0)}[-\ln\tau]^{3/2} \\ +&\left[\frac{\sigma\left(b^2+2bE-a\right)}{2^{3/2}b^{5/2}}+3C_3^{(0)}\left(\sqrt{2}-x_1^{(1)}\right)+\sqrt{2}C_3^{(1)}\right][-\ln\tau]^{1/2} \\ =&\ \mathcal{O}\left([-\ln\tau]^{-1/2}\right), \end{aligned}$$

$$
\begin{aligned}
& -6C_3^{(0)}\left[-\ln\tau\right]^{1} \\
+ \quad & \frac{\sigma\left(b^2+2bE-a\right)}{2b^{5/2}} - 6C_3^{(0)}\left(1-\sqrt{2}x_1^{(1)}\right) - 6C_3^{(1)} + x_2^{(0)}\left(\frac{a}{b}-E\right) \\
(2.87) = \quad & \mathcal{O}\left(\left[-\ln\tau\right]^{-1}\right),
\end{aligned}
$$

so that $C_3^{(0)} = 0$ and

$$
\begin{aligned}
C_3^{(1)} &= \frac{\sigma\left(b^2+2bE-a\right)}{4b^{5/2}} \\
x_2^{(0)} &= \frac{\sigma\left(b^2+2bE-a\right)}{b^{3/2}\left(a-bE\right)}.
\end{aligned}
\tag{2.88}
$$

Hence for the floorlet with $a > bE$, the free boundary close to expiry is of the form

$$
\begin{aligned}
x_f(\tau) &\sim x_0 + \sqrt{-\tau\ln\tau}\left[-\sqrt{2} + x_1^{(1)}\left(-\ln\tau\right)^{-1} + \cdots\right] \\
&+ \tau\left[x_2^{(0)} + \cdots\right] + \cdots,
\end{aligned}
\tag{2.89}
$$

with $x_1^{(1)}$ and $x_2^{(0)}$ as given above.

2.6. Floorlet with $a = bE$

This case is similar to the caplet with $a = bE$, and was touched on briefly when we considered the floorlet with $a > bE$, when we mentioned that (2.81) had a solution for this case but not for $a > bE$. The free boundary starts from $x_0 = 2\sigma/(b^{3/2})$, and the initial condition is $v(x,0) = (\sigma(x-x_0))/(2\sqrt{b})$ for $x > x_0$. As for the case $a < bE$, we will try an expansion of the form (2.32). If we substitute the expansion for $v(x,\tau)$ into the PDE (2.8), at the first few orders, we recover the equations (2.78) with solutions (2.79), but with a replaced by bE. If we attempt to apply the conditions on the free boundary by substituting the assumed form (2.32) for $x_f(\tau)$ into the solution (2.79), at leading order we get the pair of equations,

$$
\begin{aligned}
\frac{x_1}{2}\left[-C_1^{(0)}\operatorname{erfc}\left(\frac{x_1}{2}\right) + \frac{\sigma}{b^{1/2}}\right] + \frac{C_1^{(0)}e^{-x_1^2/4}}{\sqrt{\pi}} &= 0, \\
-C_1^{(0)}\operatorname{erfc}\left(\frac{x_1}{2}\right) + \frac{\sigma}{b^{1/2}} &= 0,
\end{aligned}
\tag{2.90}
$$

so that $C_1^{(0)} = \sigma/(2b^{1/2})$ and $e^{-x_1^2/4} = 0$ and $\operatorname{erfc}(x_1/2) = 2$ or $x_1 = -\infty$, which is a similar problem to that encountered when $a > bE$. At the next order, we find

$$\begin{aligned} -C_2^{(0)}\left(x_1^2+2\right) &= 0, \\ -4C_2^{(0)}x_1 &= 0, \end{aligned} \tag{2.91}$$

so that $C_2^{(0)}=0$. At the next order, we get the pair of equations,

$$\begin{aligned} x_1\left[-3C_3^{(0)}+\frac{\sigma\left(E+b\right)}{4b^{3/2}}\right]-\frac{1}{2}C_3^{(0)}x_1^3 &= 0, \\ 2\left[-3C_3^{(0)}+\frac{\sigma\left(E+b\right)}{4b^{3/2}}\right]-3C_3^{(0)}x_1^2 &= 0, \end{aligned} \tag{2.92}$$

which has no solution. The $\operatorname{erfc}(\xi)$ and $e^{-\xi^2}$ terms from $V_1^{(0)}$ must be added to (2.92) to rectify this. To do this, we must proceed as for the caplet with $a=bE$. If we suppose that $x_f(\tau)$ is of the form (2.37), then we require once again that $e^{-f_1^2/4}\sim\tau f_1$, so that

$$f_1(\tau)\sim\left[2W_L\left(\frac{1}{2\tau^2}\right)\right]^{1/2} \tag{2.93}$$

where W_L is the Lambert W function. It follows that $f_1(\tau)$ and the general term $f_n(\tau)$ are as given by (2.57), with $x_1^{(0)}=-1$. As for the caplet with $a=bE$, the series for $v(x,\tau)$ must be of the form (2.58) with

$$\begin{aligned} V_1^{(0)} &= \frac{\sigma\xi}{b^{1/2}}\left[\xi\operatorname{erfc}\left(-\xi\right)+\frac{e^{-\xi^2}}{\sqrt{\pi}}\right], \\ V_2^{(0)} &= 0, \end{aligned} \tag{2.94}$$

and $V_3^{(0)}$ given in (2.79) with a set equal to bE. For the $\tau^{n/2}\left[2W_L\left(\frac{\tau^{-2}}{2}\right)\right]^{-1}$ terms, we have

$$\mathcal{L}_3V_3^{(1)} = 0, \tag{2.95}$$

with a solution

$$V_3^{(1)} = C_3^{(1)}\left[-\left(2\xi^3+3\xi\right)\operatorname{erfc}\left(\xi\right)+\frac{2\left(\xi^2+1\right)e^{-\xi^2}}{\sqrt{\pi}}\right]. \tag{2.96}$$

The conditions on the boundary yield at leading order in τ,

$$\begin{aligned} &\frac{C_3^{(0)}}{2}\left[2W_L\left(\frac{\tau^{-2}}{2}\right)\right]^{-3/2} \\ &+\left[3C_3^{(0)}\left(1-\frac{x_1^{(1)}}{2}\right)+\frac{C_3^{(1)}}{2}+\frac{\sigma\left(b+E\right)}{4b^{3/2}}\right]\left[2W_L\left(\frac{\tau^{-2}}{2}\right)\right]^{-1/2} \\ &= \mathcal{O}\left(\left[2W_L\left(\frac{\tau^{-2}}{2}\right)\right]^{1/2}\right) \end{aligned} \tag{2.97}$$

and

$$
\begin{aligned}
& -3C_3^{(0)}\left[2W_L\left(\frac{\tau^{-2}}{2}\right)\right]^{-1} \\
& - \quad 6C_3^{(0)}\left(1-x_1^{(1)}\right)-3C_3^{(1)}+\frac{\sigma}{2b^{3/2}}\left(E+b-\frac{2be^{x_1^{(1)}/2}}{\sqrt{\pi}}\right) \\
& = \quad \mathcal{O}\left(\left[2W_L\left(\frac{\tau^{-2}}{2}\right)\right]^{1}\right),
\end{aligned}
\tag{2.98}
$$

which have a solution $C_3^{(0)}=0$ and

$$
\begin{aligned}
C_3^{(1)} &= -\frac{\sigma\,(E+b)}{2b^{3/2}} \\
x_1^{(1)} &= 2\ln\left[\frac{\sqrt{\pi}\,(E+b)}{b}\right].
\end{aligned}
\tag{2.99}
$$

Hence for the caplet with $a=bE$, the free boundary close to expiry is of the form

$$
\begin{aligned}
x_f(\tau) \quad \sim \quad & x_0 \\
& +\sqrt{2\tau W_L\left(\frac{\tau^{-2}}{2}\right)}\left[-1+x_1^{(1)}\left[2W_L\left(\frac{\tau^{-2}}{2}\right)\right]^{-1}\right]+\cdots,
\end{aligned}
\tag{2.100}
$$

with $x_1^{(1)}$ as given above.

§3. Discussion

In the previous section, we considered the behavior of American caplets and floorlets close to expiry; these are the interest rate options whose equity counterparts are American put and call options. In our analysis, we assumed that the spot interest rate r obeyed a mean-reverting random walk described by the Vasicek model [5]. In our analysis, we used a change of variables [5] to transform the governing PDE into the nonhomogeneous diffusion equation, which enabled us to use Tao's method [26]-[34] to find series solutions. We found that there were three possible behaviors for the free boundary close to expiry. Writing this free boundary as $x_f(\tau)$, where τ is the transformed time remaining until expiry, these three behaviors were

$$
x_f \quad \sim \quad x_0
$$

$$
+ \left\{ \begin{array}{c} x_1\tau^{1/2} + x_2\tau + x_3\tau^{3/2} + \cdots \\ \sqrt{-\tau\ln\tau}\left[\pm\sqrt{2} + x_1^{(1)}\left(-\ln\tau\right)^{-1} + \cdots\right] + \tau\left[x_2^{(0)} + \cdots\right] + \cdots \\ \sqrt{2\tau W_L\left(\frac{\tau^{-2}}{2}\right)}\left[\pm 1 + x_1^{(1)}\left[2W_L\left(\frac{\tau^{-2}}{2}\right)\right]^{-1}\right] + \tau\left[x_2^{(0)} + \cdots\right] + \cdots \end{array} \right. . \tag{3.1}
$$

These same three behaviors occur for American equity put and call options [2, 3, 8, 11, 12, 14, 15, 18, 25]. In one sense, this is surprising because interest rates obey a rather different random walk to equity prices. In another sense, this is not surprising as, in this and other problems [11, 18, 4, 6, 7], it appears that the first of three behaviors given in (3.1), namely the $\tau^{1/2}$ behavior, prevails when both V and $(\partial V/\partial r)$ (or V and $(\partial V/\partial S)$ for equity options) are continuous at the free boundary at expiry, while the second form, the $\sqrt{\tau\ln\tau}$ behavior, prevails when $(\partial V/\partial r)$ or $(\partial V/\partial S)$ are discontinuous there, and the third form, the $\sqrt{\tau W_L\left(\tau^{-2}/2\right)}$ behavior, occurs on the boundary between the other two cases. Although the behaviors in (3.1) were found both here for interest rate caplets and floorlets and in equity options with American-style features [11, 18, 4, 7, 6], it should be recalled that to use Tao's method, it was necessary to use a change of variables to transform the governing PDE into the nonhomogeneous diffusion equation. For the Vasicek model, this was accomplished using (2.5-2.7), but of course a slightly different transformation was used for equity options [11, 2, 3, 18], and in the original variables, the free boundary for interest rate caplets and floorlets will of course look somewhat different to that for American call and put equity options.

In closing, we note that in the previous section, the results for the floorlet and caplet were very similar. It would seem probable that some sort of symmetry exists between floorlets and caplets, perhaps along the same as that between American put and call options [10, 20], and it would be interesting to know the exact form of that symmetry.

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