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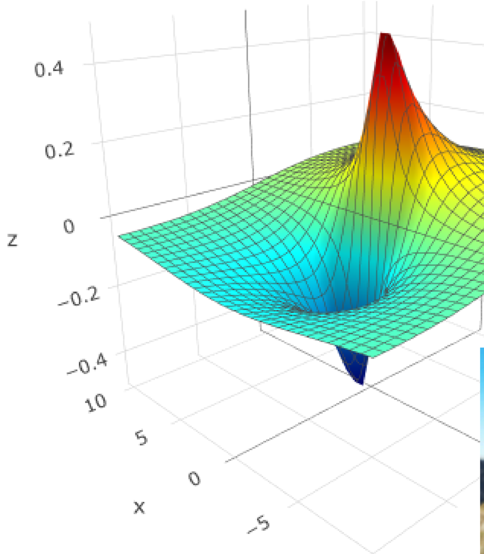
FNFN Instanton on $\mathbb{R}^8$ and its Deformations



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1. Introduction

Fairlie–Nuyts–Fubini–Nicolai (FNFN) instanton is an instanton on $\mathbb{R}^8$. Fairlie–Nuyts and Fubini–Nicolai independently came up with this instanton in their papers [3] and [4] respectively.

Let Γ_{PQ} be a generator of $SO(8)$, where $M, N = 1, \ldots, 8$. Let P^{MN}_{PQ} be the projection from the 28-dimensional vector space $\mathfrak{so}(8)$ to the 21 dimensional subspace $\mathfrak{spin}(7)$. Then

$$G_{MN} := P^{MN}_{PQ}\Gamma_{PQ}$$

is a generator of $Spin(7)$. Then the FNFN instanton is given by

$$A_M(x) = \frac{2}{3}\frac{1}{1+x^2}G_{MN}x_N \tag{1}$$

Our main objective is to investigate the space of infinitesimal deformations and the moduli space of FNFN-instanton, and finally prove the global uniqueness of FNFN instanton.

2. Deformation Theory

To investigate the space of infinitesimal deformations of FNFN instanton, we use the Deformation theory of Asymptotically conical $Spin(7)$-Instantons developed by me in [5].

A Riemannian 7-manifold Σ with $\phi \in \Omega^3_+(\Sigma)$ satisfying $d\phi = \tau_0 {*\phi}$ for some non-zero $\tau_0 \in \mathbb{R}\backslash\{0\}$ is called a *nearly G_2-manifold*.

A Riemannian 8-manifold X equipped with a 4-form $\Phi \in \Omega^4(X)$ is said to be an *$Spin(7)$-manifold* if $\nabla\Phi = 0$ (torsion free) and $\Phi = e^{0127} + e^{0347} + e^{0567} + e^{0145} + e^{0136} + e^{0235} - e^{0246} + e^{1234} + e^{1256} + e^{3456} - e^{1357} + e^{1467} + e^{2367} + e^{2457}$ in local orthonormal frame $e^0, e^1, \ldots, e^7$.

A *$Spin(7)$-cone* on Σ is $C(\Sigma) := (0,\infty) \times \Sigma$ together with the torsion free $Spin(7)$-structure $(C(\Sigma), \Phi_C)$ defined by

$$\Phi_C := r^3 dr \wedge \phi + r^4\psi$$

where $\psi = {*\phi}$ and $r \in (0,\infty)$ is the coordinate.

Let (X, g, Φ) be a non-compact $Spin(7)$-manifold. X is called an *asymptotically conical (AC) $Spin(7)$-manifold with rate* $\nu < 0$ if there exists a compact subset $K \subset X$, a compact connected nearly G_2 manifold Σ, and a constant $R > 1$ together with a diffeomorphism $h : (R,\infty) \times \Sigma \to X\backslash K$ such that

$$\left|\nabla^j_C(h^*(\Phi|_{X\backslash K}) - \Phi_C)\right|(r,p) = O(r^{\nu-j}) \quad \text{as } r \to \infty$$

for each $p \in \Sigma$, $j \in \mathbb{Z}_{\geqslant 0}$, $r \in (R,\infty)$.

A connection A on $P \to X$ is called an *asymptotically conical connection* with rate ν if there exists a connection A_Σ on $Q \to \Sigma$ (with $h^*P \cong \pi^*Q$) such that

$$\left|\nabla^j_C(h^*(A) - \pi^*(A_\Sigma))\right| = O(r^{\nu-1-j}) \quad \text{as } r \to \infty \tag{2}$$

$\nu_0 :=$ inf$\{\nu : A$ is AC with rate $\nu\}$ is called the *fastest rate of convergence of A.*

Let X be a $Spin(7)$-manifold and $P \to X$ is a principal G-bundle. Let F_A be the curvature of the connection A on the adjoint bundle $\mathfrak{g}_P := P \times_{Ad} \mathfrak{g}$. Then F_A is a *$Spin(7)$-instanton* if

$$*(\Phi \wedge F_A) = -F_A$$

The *moduli space of $Spin(7)$-instantons asymptotic to A_Σ with rate ν* is given by

$$\mathcal{M}(A_\Sigma,\nu) := \{Spin(7) \text{ instanton } A \text{ on } P \text{ satisfying (2) asymptotic to } A_\Sigma\}/\mathcal{G}_\nu$$

where $\mathcal{G}_\nu$ is the asymptotically conical gauge group.

Consider the Dirac operator

$$\mathfrak{D}^-_A : \Gamma(\$^-(X) \otimes \mathfrak{g}_P) \to \Gamma(\$^+(X) \otimes \mathfrak{g}_P)$$

For $\nu < 0$ the *space of infinitesimal deformations* is defined to be

$$\mathcal{I}(A,\nu) := \{\alpha \in \Omega^{1,k+1}_{\nu-1}(\mathfrak{g}_P) : \mathfrak{D}^-_A\alpha = 0\} = \ker \mathfrak{D}^-_A$$

The *obstruction space* is defined to be $\mathcal{O}(A,\nu)$ where,

$$\Omega^{0,k+1}_{\nu-1}(\mathfrak{g}_P) \oplus \Omega^{2,k+1}_{\nu-1}(\mathfrak{g}_P) = \mathfrak{D}^-_A\left(\Omega^{1,k+1}_{\nu-1}(\mathfrak{g}_P)\right) \oplus \mathcal{O}(A,\nu)$$

Theorem 2.1 ([5]). *Let A be an AC $Spin(7)$-instanton asymptotic to the nearly G_2-instanton A_Σ. Moreover, let ν is not a critical rate. Then there exists an open neighbourhood $\mathcal{U}(A,\nu)$ of 0 in $\mathcal{I}(A,\nu)$, and a smooth map $\kappa : \mathcal{U}(A,\nu) \to \mathcal{O}(A,\nu)$, with $\kappa(0) = 0$, such that an open neighbourhood of $0 \in \kappa^{-1}(0)$ is homeomorphic to a neighbourhood of A in $\mathcal{M}(A_\Sigma,\nu)$. Hence, the virtual dimension of the moduli space is given by* dim $\mathcal{I}(A,\nu)$ − dim $\mathcal{O}(A,\nu)$. *Moreover, $\mathcal{M}(A_\Sigma,\nu)$ is a smooth manifold if $\mathcal{O}(A,\nu) = \{0\}$.*

3. FNFN $Spin(7)$-Instanton

Let us consider $\mathbb{R}^8$ to be the asymptotically conical $Spin(7)$-manifold asymptotic to the nearly G_2 manifold $\Sigma = S^7$. We consider S^7 as a homogeneous nearly G_2 manifold $Spin(7)/G_2$. Then we have the canonical bundle $G_2 \to Spin(7) \to S^7$ (call this bundle P). Also consider the trivial bundle $Spin(7) \to Spin(7) \times S^7 \to S^7$ (call this bundle Q). Let A_{flat} be a $Spin(7)$-invariant flat connection given by $A_{\text{flat}} = A_\Sigma + a$. Let $(r,\sigma) \in (0,\infty) \times S^7$. Consider the connection

$$A(r,\sigma) = A_\Sigma(\sigma) + f(r)a(\sigma)$$

where $f(r) = \dfrac{1}{Cr^2+1}$ for $C > 0$ is a function on $\mathbb{R}^8$. This expression for $f(r)$ has been derived from extensive computation using Lie algebraic and homogeneous space techniques and using the condition that $f(r)$ satisfies the instanton equation. Since the canonical connection always satisfies the instanton equation, the connection A is in fact an instanton on $\mathbb{R}^8$. We call this the *FNFN $Spin(7)$-instanton*. Clearly FNFN $Spin(7)$-instanton A is asymptotic to the canonical connection A_Σ with fastest rate of convergence -2.

We consider the family of moduli spaces $\mathcal{M}(A_\Sigma,\nu)$ where $\nu \in (\nu_0, 0)$. Then, the deformation theory developed in [5] tells us that we must have $\nu + \frac{5}{2} \notin$ Spec $\mathfrak{D}^0_{A_{\Sigma}(-)}$. So we investigate the eigenvalues of the twisted Dirac operator on S^7 in the interval $\left(\nu_0 + \frac{5}{2}, \frac{5}{2}\right) = \left(\frac{1}{2}, \frac{5}{2}\right)$. It turns out that in this range, $-1 + \frac{5}{2} = \frac{3}{2}$ is the only eigenvalue; i.e., -1 is the only critical rate.

4. Conclusions

Theorem 4.1 ([5]). *The virtual dimension of the moduli space $\mathcal{M}(A_\Sigma,\nu)$ of the FNFN $Spin(7)$-instanton with decay rate $\nu \in (-2,0)\backslash\{-1\}$ is given by*

$$\text{virtual-dim}\,\mathcal{M}(A_\Sigma,\nu) = \begin{cases} d & \textit{if } \nu \in (-2,-1) \\ d+8 & \textit{if } \nu \in (-1,0). \end{cases}$$

5. Forthcoming Research

The forthcoming research in this project involves:

- Figure out the index of the twisted Dirac operator corresponding to the rate $\nu \in (-2,-1)$, hence figure out the virtual dimensions d and $d+8$ respectively.
- Prove a global uniqueness of FNFN-instanton by imposing a condition on the 2nd Pontryagin class of the bundle.
- Prove that FNFN instanton is unobstructed, thus establishing the moduli spaces as smooth manifolds and stating the actual dimensions.

References

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