Definition **Increasing Sequence of Sets** Advanced Probability	Definition **Decreasing Sequence of Sets** Advanced Probability
Theorem **De Morgan Laws** Advanced Probability	Definition **Fields and σ-Fields** Advanced Probability
Definition **Measurable Space and Measurable Sets** Advanced Probability	Definition **σ-field generated by $\mathcal{A}$** Advanced Probability
Theorem **Existence of $\sigma(\mathcal{A})$** Advanced Probability	Definition **Extended Reals** Advanced Probability
Definition **Rules of Arithmetic in $\bar{\mathbb{R}}$** Advanced Probability	Definition **Measures** Advanced Probability

Let $A_1, A_2, \ldots$ be subsets of Ω. If $A_1 \supset A_2 \supset \ldots$ and $\cap_{n=1}^{\infty} = A$, A_n is said to form a decreasing sequence of sets with limit A.

Let $A_1, A_2, \ldots$ be subsets of Ω. If $A_1 \subset A_2 \subset \ldots$ and $\cup_{n=1}^{\infty} = A$, A_n is said to form an increasing sequence of sets with limit A.

Let Ω be a set. A collection $\mathcal{F}$ of subsets of Ω is called a field if it is closed under complementation and finite union:

- $\Omega \in \mathcal{F}$
- for each $A \in \mathcal{F}$, $A^C \in \mathcal{F}$,
- for all $A_1, A_2 \in \mathcal{F}$, $A_1 \cup A_2 \in \mathcal{F}$

From this, it follows that $\mathcal{F}$ is closed under finite intersection:

$$\cap_{i=1}^{n} A_i = \left(\cup_{i=1}^{n} A_i^c\right)^c \in \mathcal{F}$$

A field is called a σ-field if it also satisfies the condition that for every sequence $\{A_k\}_{k=1}^{\infty} \in \mathcal{F}$, we have that $\cup_{k=1}^{\infty} A_k \in \mathcal{F}$

Let $A_1, A_2, \ldots$ be subsets of Ω. We have

$$(\cup_n A_n)^c = \cap_n A_n^c$$

and

$$(\cap_n A_n)^c = \cup_n A_n^c$$

Let $\mathcal{A}$ be a collection of subsets of Ω. We denote with $\sigma(\mathcal{A})$ the smallest σ-field containing $\mathcal{A}$, which is called the σ-field generated by $\mathcal{A}$.

A set Ω together with σ-algebra $\mathcal{F}$ is called a measurable space. The elements of $\mathcal{F}$ are called measurable sets.

The extended real numbers are $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$. The positive extended realls are $\bar{\mathbb{R}}^{+} = (0, \infty]$ and the nonnegative extended reals are $\bar{\mathbb{R}}^{+0} = [0, \infty]$.

The proof has three main steps:

- Define all σ-fields containing $\mathcal{A}$ as $\mathcal{F}_i, i \in \mathcal{I}$, where $\mathcal{I}$ is some index set. Note that one such σ-field always exists since $\mathcal{A} \subset 2^{\Omega}$.
- Show that the intersection of all $\mathcal{F}_i$ is again a σ-field
- Finally, show that $\cap_{i \in \mathcal{I}} \mathcal{F}_i$ is moreover the smallest σ-field possible.

Let $(\Omega, \mathcal{F})$ be a measurable space. Let $\mu : \mathcal{F} \to \bar{\mathbb{R}}^{+0}$ satisfy:

- $\mu(\emptyset) = 0$
- For any sequence of mutually disjoint sets $\{A_n\}_{n=1}^{\infty}$ of $\mathcal{F}$ (i.e. $A_i \cap A_j = \emptyset$ for $i \neq j$), we have

$$\mu\left(\cup A_{n=1}^{\infty}\right) = \sum_{n=1}^{\infty} \mu(A_n)$$

We call $(\Omega, \mathcal{F}, \mu)$ a measure space. If $\mu(\Omega) = 1$, it is a probability space and we usually write P instead of μ.

Let $c \in \mathbb{R}$.

- $c + \infty = \infty$ and $c - \infty = -\infty$
- $\infty + \infty = \infty$ and $-\infty - \infty = -\infty$ (however, $\infty - \infty$ is NOT defined!)
- $0 \cdot \infty = 0$ and $\frac{c}{\infty} = \frac{c}{-\infty} = 0$ (however, $\frac{\infty}{\infty}$ is not defined
- $c \times \infty = \infty$ if $c > 0$ and $c \times \infty = -\infty$ if $c < 0$.

Definition

Probability Measure

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Definition

Finite and σ-finite Measures

Advanced Probability

Theorem

Countable Subadditivity of Measure

Advanced Probability

Theorem

Further Properties of Measure

Advanced Probability

Definition

Monotone Sequences of Sets

Advanced Probability

Definition

Measurable Function

Advanced Probability

Theorem

Properties of Measurable Functions

Advanced Probability

Theorem

Properties of Sequence of Measurable Functions

Advanced Probability

Definition

Simple Function

Advanced Probability

Definition

Canonical Representation of Simple Function

Advanced Probability

A measure μ is finite if for all $A \in \mathcal{F}$, we have $\mu(A) < \infty$. It is called σ-finite if there exists a sequence $\{A_n\}_{n=1}^{\infty}$ such that $\mu(A_n) < \infty \, \forall n$ and $\cup_{n=1}^{\infty} A_n = \Omega$.
Observe that a finite measure is always σ-finite, but the reverse is not true: Take for example the counting measure and let $\mathcal{F}$ be the σ-field generated by the natural numbers. Then clearly $\mu(\Omega) = \mu(\mathbb{N}) = \infty$, so the measure is not finite. However, it is σ-finite since we can define $A_n = \{n\}$ with $\mu(A_n) = 1$ and $\cup_{n=1}^{\infty} A_n = \mathbb{N}$.

A measure P which satisfies $P(\Omega) = 1$ is called a probability measure. Then, the measure space $(\Omega, \mathcal{F}, P)$ is called a probability space, and the sets of $\mathcal{F}$ are called events.

- Linearity: If $\mu_1, \mu_2, \ldots$ are measures on $(\Omega, \mathcal{F})$, then $\mu = \sum_j a_j \mu_j$ is also a measure on $(\Omega, \mathcal{F})$.
- If $\mu(A_n) = 0$ for all A_n, then $\mu(\cup_{n=1}^{\infty}) = 0$.
- If $\mu(A_n) = 1$ for all A_n, then $\mu(\cap_{n=1}^{\infty} A_n) = 1$.

For an arbitrary sequence $\{A_n\}_{n=1}^{\infty}$, we have

$$\mu(\cup_{n=1}^{\infty} A_n) \leq \sum_{i=1}^{n} \mu(A_n).$$

Proof. Let $B_n = A_n - (\cup_{i<n} A_i)$. Then $\{B_n\}$ forms a disjoint sequence of sets which satisfies $\cup_{n=1}^{\infty} B_n = \cup_{n=1}^{\infty} A_n$. We thus have $\mu(\cup_{n=1}^{\infty} A_n) = \mu(\cup_{n=1}^{\infty} B_n) = \sum_{i=1}^{n} \mu(B_n)$. Since $\mu(B_n) \leq \mu(A_n) \, \forall n$, the result follows. $\square$

Let $(\Omega, \mathcal{F})$ and $(S, \mathcal{A})$ be measurable spaces. Then a function f which maps from Ω to S is called $\mathcal{F}/\mathcal{A}$ measurable if $f^{-1}(A) \in \mathcal{F} \, \forall A \in \mathcal{A}$, where the inverse image or pre-iamge $f^{-1}(A)$ is defined as $f^{-1}(A) = \{\omega \in \Omega : f(\omega) \in \mathcal{A}\}$. For sake of brevity, we might just say that f is measurable in this case.

For a measure space $(\Omega, \mathcal{F}, \mu)$, a sequence $\{A_n\}_{n=1}^{\infty}$ of elements in $\mathcal{F}$ is called monotone increasing if $A_n \subseteq A_{n+1} \forall n$. If on the other hand $A_n \supseteq A_{n+1}$, it is a monotinically decreasing sequence of sets.

Let f_n be a sequence of measurable functions which satisfies $f_n : \Omega \mapsto \mathbb{R}$ for all n. Then the following are measurable:

- $\limsup_n f_n$ and $\liminf_n f_n$
- $\{\omega : \lim_{n\to\infty} f_n(\omega) \text{ exists}\}$
- $f = \begin{cases} \lim_{n\to\infty} f_n & \text{where the limit exists} \\ 0 & \text{elsewhere} \end{cases}$

Let $(\Omega, \mathcal{F})$, $(S, \mathcal{A})$ and $(T, \mathcal{B})$ be measurable spaces. Then

- If $f : \Omega \mapsto \bar{\mathbb{R}}$ and c is some constant, then cf is measurable.
- if $f : \Omega \mapsto S$ and $g : S \mapsto T$, then the composition $g \circ f = g(f) : \Omega \mapsto T$ is measurable.
- If f and f are measurable real-valued functions, so are $f + g$ and fg.

Let f be a simple function whose distinct values are $a_1, \ldots, a_n$ and let $A_i = \{\omega : f(\omega) = a_i\}$. Then $f = \sum_{i=1}^{n} a_i \mathbb{I}_{A_i}$ is called the canonical representation of f.

A measurable function that takes at most finitely many values is called a simple function.

Lemma

Monotone Approximation

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Definition

Splitting Measurable Functions

Advanced Probability

Definition

Integral of Simple Functions

Advanced Probability

Proposition

Integral for two functions $f \leq g$

Advanced Probability

Definition

Integrable Function

Advanced Probability

Definition

Integral for General Nonnegative Functions

Advanced Probability

Definition

Integral of General Measurable Functions

Advanced Probability

Definition

Integration over a set

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Proposition

Monotonicity of Integral

Advanced Probability

Definition

Almost sure / almost everywhere

Advanced Probability

Let f be a real-valued function. The positive part f^+ of f is defined as $f^+(\omega) = \max\{f(\omega), 0\}$. The negative part f^- of f is $f^-(\omega) = -\min\{f(\omega), 0\}$. We have $f = f^+ - f^-$ and $|f| = f^+ + f^-$.

Let f be a nonnegative measurable extended real-valued function from Ω. Then there exists a sequence $\{f_n\}_{n=1}^{\infty}$ of nonnegative finite simple functions such that $f_n \leq f$ for all n and $\lim_{n\to\infty} f_n(\omega) = f(\omega)\,\forall\omega$.

If $f \leq g$ and both are nonnegative and simple, then

$$\int f d\mu \leq \int g d\mu.$$

Let $f : \Omega \to \bar{\mathbb{R}}^{+0}$ be a simple function with canonical representation $f(\omega) = \sum_{i=1}^{n} \mathbb{I}_{A_i}(\omega)$. The integral of f with respect to μ is defined to be $\sum_{i=1}^{n} a_i \mu(A_i)$. The integral is denoted variously as $\int f d\mu$, $\int f(\omega)\,\mu(d\omega)$ or $\int f(\omega)\,d\mu(\omega)$.

For nonnegative measurable f, we define the integral of f with respect to μ by

$$\int f d\mu = \sup_{\text{nonnegative finite simple } g \leq f} \int g d\mu$$

We say that f is integrable with respect to μ if $\int f d\mu$ is finite.

If $A \in \mathcal{F}$, we define

$$\int_A f d\mu = \int \mathbb{I}_A f d\mu.$$

Let f be measurable. If either f^+ orf^- is integrable with respect to μ, we define the integral of f with respect to μ to be

$$\int f^+ d\mu - \int f^- d\mu,$$

otherwise the integral does not exist.

Suppose that some statement about elements of Ω holds for all $\omega \in A^C$, where $\mu(A) = 0$. Then we say that the statement holds almost everywhere, denotes as $a.e.\,[\mu]$. If P is a probability, then almost everywhere is often replaced by almost surely, denoted $a.s.\,[P]$.

If $f \leq g$ and both integrals are defined, then $\int f d\mu \leq \int g d\mu$.

Theorem

Additivity

Advanced Probability

Theorem

Change of Variable

Advanced Probability

Theorem

Relationship to Riemann Integral

Advanced Probability

Definition

Expectation and Variance of Random Variables

Advanced Probability

Theorem

Fatou's Lemma

Advanced Probability

Theorem

Monotone convergence theorem

Advanced Probability

Theorem

Linearity of Integral

Advanced Probability

Lemma

Change of Variables

Advanced Probability

Corollary

Law of the unconsious statistician

Advanced Probability

Theorem

Density Functions

Advanced Probability

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let $(S, \mathcal{A})$ be a measurable space. Let $f : \Omega \mapsto S$ be a measurable function. Let ν be the measure induced on $(S, \mathcal{A})$ by f from μ. Let $g : S \mapsto \mathbb{R}$ be $A/\mathcal{B}^1$ measurable. Then

$$\int g d\nu = \int g(f) d\mu,$$

if either integral exists.

$$\int (f+g)\, d\mu = \int f d\mu + \int g d\mu,$$

whenever at least two of them are finite.

If P is a probability and X is a random variable, then $\int X dP$ is called the mean of X, expected value of X, or expectation of X, and denoted by $\mathbb{E}(X)$. If $\mathbb{E} = \mu$ is finite, then the variance of X is $\operatorname{Var}(X) = \mathbb{E}\left[(X-\mu)^2\right]$

Let f be a continuous function on a closed bounded interval $[a, b]$. Let μ be Lebesgue measure. Then the Riemann integral $\int_a^b f(x) dx$ equals $\int_{[a,b]} f d\mu$.

Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of measurable nonnegative functions, and let f be a measurable function such that $f_n \leq f$ and $\lim_{n\to\infty} f_n = f$. Then

$$\lim_{n\to\infty} \int f_n d\mu = \int f d\mu$$

Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of nonnegative measurable functions. Then

$$\int \liminf_n f_n d\mu \leq \liminf_n \int f_n d\mu.$$

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $(S, \mathcal{A})$ be a measurable space. Let $f : \Omega \mapsto S$ be a measurable funciton. Let ν be the measure induced on $(S, \mathcal{A})$ by f from μ. Let $g : S \mapsto \mathbb{R}$ be $\mathcal{A}/\mathcal{B}^1$ measurable. Then

$$\int g d\nu = \int g(f) d\mu,$$

if either integral exists.

If $\int f d\mu$ and $\int g d\mu$ are defined and they are not both infinite and of opposite signs, then $\int [f+g]\, d\mu = \int f d\mu + \int g d\mu$.

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, and let $f : \Omega \mapsto \bar{\mathbb{R}}^{+0}$ be measurable. Then $\nu(A) = \int_A f d\mu$ is a measure on $(\Omega, \mathcal{F})$. The function f is called the density of ν with respect to μ. Integrals with respect to ν can be computed as $\int g d\nu = \int f g d\mu$, if either exists.

If $X : \Omega \mapsto S$ is a random quantity with distribution μ_X and if $f : S \mapsto \mathbb{R}$ is measurable, then $\mathbb{E}[f(X)] = \int f d\mu_X$.

Theorem

Dominated convergence theorem

Advanced Probability

Proposition

Proposition 18

Advanced Probability

Definition

Uniform Integrability

Advanced Probability

Theorem

Properties of Integrals

Advanced Probability

Corollary

Corollary 22

Advanced Probability

Theorem

Theorem 23

Advanced Probability

Let $\{f_n\}_{n=1}^{\infty}$, $\{g_n\}_{n=1}^{\infty}$ be sequences of measurable functions such that $|f_n| \leq g_n$, a.e. $[\mu]$. Let f and g be measurable functions such that $\lim_{n\to\infty} f_m = f$ and $\lim_{n\to\infty} g_n = n$, a.e$[\mu]$. Suppose that $\lim_{n\to\infty} = \int g d\mu < \infty$. Then, $\lim_{n\to\infty} \int f_n d\mu = \int f d\mu$.

Let $\{_n\}_{n=1}^{\infty}$ be a sequence of measurable functions, and let f and g be measurabe functions such that $f_n \to f$ a.e. $[\mu]$, $|f_n| \leq g$ a.e. $[\mu]$ and $\int g d\mu < \infty$. Then

$$\lim_{n\to\infty} \int f_n d\mu = \int f d\mu.$$

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Let f and g be measurable extended real-valued functions.

- If f is nonnegative and $\mu(\{\omega : f(\omega) > 0\})$, then $\int f d\mu > 0$.
- If f and g are integrable and if $\int_A f d\mu = \int_A g d\mu$ for all $A \in \mathcal{F}$, then $f = g$ a.e. $[\mu]$.
- If μ is σ-finite and if $\int_A f d\mu = \int_A g d\mu$ for all $A \in \mathcal{F}$, then $f = g$ a.e. $[\mu]$.
- Let Π be a π-system that generates $\mathcal{F}$. Suppose that Ω is a finite or countable union of elements of Π. If f and g are integrable and if $\int_A f d\mu = \int_A g d\mu$ for all $A \in \Pi$, then $f = g$ a.e $[\mu]$.

A sequence of integrable functions $\{f_n\}_{n=1}^{\infty}$ is uniformly integrable (with respect to μ) if $\lim_{c\to\infty} \sup_n \int_{\{\omega:|f_n(\omega)|>c\}} |f_n| d\mu = 0$.

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Then μ is σ-finite if and only if there exists a strictly positive integrable function.

If μ is σ-finite and ν is related to μ as in Theorem 10, then the density of ν with respect to μ is unique, a.e. $[\mu]$.