Definition

Increasing Sequence of Sets

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Definition

Decreasing Sequence of Sets

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Theorem

De Morgan Laws

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Definition

Fields and σ-Fields

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Definition

Simple Function

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Definition

Canonical Representation of Simple Function

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Lemma

Monotone Approximation

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Definition

Splitting Measurable Functions

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Definition

Integral of Simple Functions

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Proposition

Integral for two functions $f \leq g$

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Let $A_1, A_2, \ldots$ be subsets of Ω. If $A_1 \supset A_2 \supset \ldots$ and $\cap_{n=1}^{\infty} = A$, A_n is said to form a decreasing sequence of sets with limit A.

Let $A_1, A_2, \ldots$ be subsets of Ω. If $A_1 \subset A_2 \subset \ldots$ and $\cup_{n=1}^{\infty} = A$, A_n is said to form an increasing sequence of sets with limit A.

Let Ω be a set. A collection $\mathcal{F}$ of subsets of Ω is called a field if it is closed under complementation and finite union:

- $\Omega \in \mathcal{F}$
- for each $A \in \mathcal{F}$, $A^C \in \mathcal{F}$,
- for all $A_1, A_2 \in \mathcal{F}$, $A_1 \cup A_2 \in \mathcal{F}$

From this, it follows that $\mathcal{F}$ is closed under finite intersection:

$$\cap_{i=1}^{n} A_i = \left(\cup_{i=1}^{n} A_i^c\right)^c \in \mathcal{F}$$

A field is called a σ-field if it also satisfies the condition that for every sequence $\{A_k\}_{k=1}^{\infty} \in \mathcal{F}$, we have that $\cup_{k=1}^{\infty} A_k \in \mathcal{F}$

Let $A_1, A_2, \ldots$ be subsets of Ω. We have

$$(\cup_n A_n)^c = \cap_n A_n^c$$

and

$$(\cap_n A_n)^c = \cup_n A_n^c$$

Let f be a simple function whose distinct values are $a_1, \ldots, a_n$ and let $A_i = \{\omega : f(\omega) = a_i\}$. Then $f = \sum_{i=1}^{n} a_i \mathbb{I}_{A_i}$ is called the canonical representation of f.

A measurable function that takes at most finitely many values is called a simple function.

Let f be a real-valued function. The positive part f^+ of f is defined as $f^+(\omega) = \max\{f(\omega), 0\}$. The negative part f^- of f is $f^-(\omega) = -min\{f(\omega), 0\}$. We have $f = f^+ - f^-$.

Let f be a nonnegative measurable extended real-valued function from Ω. Then there exists a sequence $\{f_n\}_{n=1}^{\infty}$ of nonnegative finite simple functions such that $f_n \leq f$ for all n and $\lim_{n\to\infty} f_n(\omega) = f(\omega)\, \forall \omega$.

If $f \leq g$ and both are nonnegative and simple, then

$$\int f d\mu \leq \int g d\mu.$$

Let $f : \Omega \to \bar{\mathbb{R}}^{+0}$ be a simple function with canonical representation $f(\omega) = \sum_{i=1}^{n} \mathbb{I}_{A_i}(\omega)$. The integral of f with respect to μ is defined to be $\sum_{i=1}^{n} a_i \mu(A_i)$. The integral is denoted variously as $\int f d\mu$, $\int f(\omega)\, \mu(d\omega)$ or $\int f(\omega)\, d\mu(\omega)$.

Definition

Integrable Function

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Definition

Integral for General Nonnegative Functions

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Definition

Integral of General Measurable Functions

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Definition

Integration over a set

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Proposition

Monotonicity of Integral

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Definition

Almost sure / almost everywhere

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Theorem

Additivity

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Theorem

Change of Variable

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Theorem

Relationship to Riemann Integral

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Definition

Expectation and Variance of Random Variables

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For nonnegative measurable f, we define the integral of f with respect to μ by

$$\int f d\mu = \sup_{\text{nonnegative finite simple } g \leq f} \int g d\mu$$

We say that f is integrable with respect to μ if $\int f d\mu$ is finite.

If $A \in \mathcal{F}$, we define

$$\int_A f d\mu = \int \mathbb{I}_A f d\mu.$$

Let f be measurable. If either f^+ orf^- is integrable with respect to μ, we define the integral of f with respect to μ to be

$$\int f^+ d\mu - \int f^- d\mu,$$

otherwise the integral does not exist.

Suppose that some statement about elements of Ω holds for all $\omega \in A^C$, where $\mu(A) = 0$. Then we say that the statement holds almost everywhere, denotes as $a.e.$ $[\mu]$. If P is a probability, then almost everywhere is often replaced by almost surely, denoted $a.s.$ $[P]$.

If $f \leq g$ and both integrals are defined, then $\int f d\mu \leq \int g d\mu$.

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let $(S, \mathcal{A})$ be a measurable space. Let $f : \Omega \mapsto S$ be a measurable function. Let ν be the measure induced on $(S, \mathcal{A})$ by f from μ. Let $g : S \mapsto \mathbb{R}$ be $A/\mathcal{B}^1$ measurable. Then

$$\int g d\nu = \int g(f) d\mu,$$

if either integral exists.

$$\int (f+g)\, d\mu = \int f d\mu + \int g d\mu,$$

whenever at least two of them are finite.

If P is a probability and X is a random variable, then $\int X dP$ is called the mean of X, expected value of X, or expectation of X, and denoted by $\mathbb{E}(X)$. If $\mathbb{E} = \mu$ is finite, then the variance of X is $\mathrm{Var}(X) = \mathbb{E}\left[(X-\mu)^2\right]$

Let f be a continuous function on a closed bounded interval $[a, b]$. Let μ be Lebesgue measure. Then the Riemann integral $\int_a^b f(x)dx$ equals $\int_{[a,b]} f d\mu$.