Definition

Increasing Sequence of Sets

Advanced Probability

Definition

Decreasing Sequence of Sets

Advanced Probability

Theorem

De Morgan Laws

Advanced Probability

Definition

Fields and σ-Fields

Advanced Probability

Definition

Measurable Space and Measurable Sets

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Definition

σ-field generated by $\mathcal{A}$

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Theorem

Existence of $\sigma(\mathcal{A})$

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Definition

Extended Reals

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Definition

Rules of Arithmetic in $\bar{\mathbb{R}}$

Advanced Probability

Definition

Measures

Advanced Probability

Let $A_1, A_2, \ldots$ be subsets of Ω. If $A_1 \supset A_2 \supset \ldots$ and $\cap_{n=1}^{\infty} = A$, A_n is said to form a decreasing sequence of sets with limit A.

Let $A_1, A_2, \ldots$ be subsets of Ω. If $A_1 \subset A_2 \subset \ldots$ and $\cup_{n=1}^{\infty} = A$, A_n is said to form an increasing sequence of sets with limit A.

Let Ω be a set. A collection $\mathcal{F}$ of subsets of Ω is called a field if it is closed under complementation and finite union:

- $\Omega \in \mathcal{F}$
- for each $A \in \mathcal{F}$, $A^C \in \mathcal{F}$,
- for all $A_1, A_2 \in \mathcal{F}$, $A_1 \cup A_2 \in \mathcal{F}$

From this, it follows that $\mathcal{F}$ is closed under finite intersection:

$$\cap_{i=1}^{n} A_i = \left(\cup_{i=1}^{n} A_i^c\right)^c \in \mathcal{F}$$

A field is called a σ-field if it also satisfies the condition that for every sequence $\{A_k\}_{k=1}^{\infty} \in \mathcal{F}$, we have that $\cup_{k=1}^{\infty} A_k \in \mathcal{F}$

Let $A_1, A_2, \ldots$ be subsets of Ω. We have

$$\left(\cup_n A_n\right)^c = \cap_n A_n^c$$

and

$$\left(\cap_n A_n\right)^c = \cup_n A_n^c$$

Let $\mathcal{A}$ be a collection of subsets of Ω. We denote with $\sigma\left(\mathcal{A}\right)$ the smallest σ-field containing $\mathcal{A}$, which is called the σ-field generated by $\mathcal{A}$.

A set Ω together with σ-algebra $\mathcal{F}$ is called a measurable space. The elements of $\mathcal{F}$ are called measurable sets.

The extended real numbers are $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$. The positive extended realls are $\bar{\mathbb{R}}^{+} = (0, \infty]$ and the nonnegative extended reals are $\bar{\mathbb{R}}^{+0} = [0, \infty]$.

The proof has three main steps:

- Define all σ-fields containing $\mathcal{A}$ as $\mathcal{F}_i, i \in \mathcal{I}$, where $\mathcal{I}$ is some index set. Note that one such σ-field always exists since $\mathcal{A} \subset 2^{\Omega}$.
- Show that the intersection of all $\mathcal{F}_i$ is again a σ-field
- Finally, show that $\cap_{i \in \mathcal{I}} \mathcal{F}_i$ is moreover the smallest σ-field possible.

Let $(\Omega, \mathcal{F})$ be a measurable space. Let $\mu : \mathcal{F} \to \bar{\mathbb{R}}^{+0}$ satisfy:

- $\mu\left(\emptyset\right) = 0$
- For any sequence of mutually disjoint sets $\{A_n\}_{n=1}^{\infty}$ of $\mathcal{F}$ (i.e. $A_i \cap A_j = \emptyset$ for $i \neq j$), we have

$$\mu\left(\cup A_{n=1}^{\infty}\right) = \sum_{n=1}^{\infty} \mu\left(A_n\right)$$

We call $(\Omega, \mathcal{F}, \mu)$ a measure space. If $\mu\left(\Omega\right) = 1$, it is a probability space and we usually write P instead of μ.

Let $c \in \mathbb{R}$.

- $c + \infty = \infty$ and $c - \infty = -\infty$
- $\infty + \infty = \infty$ and $-\infty - \infty = -\infty$ (however, $\infty - \infty$ is NOT defined!)
- $0 \cdot \infty = 0$ and $\frac{c}{\infty} = \frac{c}{-\infty} = 0$ (however, $\frac{\infty}{\infty}$ is not defined
- $c \times \infty = \infty$ if $c > 0$ and $c \times \infty = -\infty$ if $c < 0$.

Definition **Probability Measure** Advanced Probability	Definition **Finite and σ-finite Measures** Advanced Probability
Theorem **Countable Subadditivity of Measure** Advanced Probability	Theorem **Further Properties of Measure** Advanced Probability
Definition **Counting Measure** Advanced Probability	Definition **Lebesgue Measure on $\mathbb{R}$** Advanced Probability
Definition **Monotone Sequences of Sets** Advanced Probability	Definition $\liminf A_n$ **and** $\limsup A_n$ Advanced Probability
Definition **Measurable Function** Advanced Probability	Theorem **Properties of Measurable Functions** Advanced Probability

A measure μ is finite if for all $A \in \mathcal{F}$, we have $\mu(A) < \infty$. It is called σ-finite if there exists a sequence $\{A_n\}_{n=1}^{\infty}$ such that $\mu(A_n) < \infty \, \forall n$ and $\cup_{n=1}^{\infty} A_n = \Omega$.
Observe that a finite measure is always σ-finite, but the reverse is not true: Take for example the counting measure and let $\mathcal{F}$ be the σ-field generated by the natural numbers. Then clearly $\mu(\Omega) = \mu(\mathbb{N}) = \infty$, so the measure is not finite. However, it is σ-finite since we can define $A_n = \{n\}$ with $\mu(A_n) = 1$ and $\cup_{n=1}^{\infty} A_n = \mathbb{N}$.

A measure P which satisfies $P(\Omega) = 1$ is called a probability measure. Then, the measure space $(\Omega, \mathcal{F}, P)$ is called a probability space, and the sets of $\mathcal{F}$ are called events.

- Linearity: If $\mu_1, \mu_2, \ldots$ are measures on $(\Omega, \mathcal{F})$, then $\mu = \sum_j a_j \mu_j$ is also a measure on $(\Omega, \mathcal{F})$.
- If $\mu(A_n) = 0$ for all A_n, then $\mu(\cup_{n=1}^{\infty}) = 0$.
- If $\mu(A_n) = 1$ for all A_n, then $\mu(\cap_{n=1}^{\infty} A_n) = 1$.

For an arbitrary sequence $\{A_n\}_{n=1}^{\infty}$, we have

$$\mu(\cup_{n=1}^{\infty} A_n) \leq \sum_{i=1}^{n} \mu(A_n).$$

Proof. Let $B_n = A_n - (\cup_{i<n} A_i)$. Then $\{B_n\}$ forms a disjoint sequence of sets which satisfies $\cup_{n=1}^{\infty} B_n = \cup_{n=1}^{\infty} A_n$. We thus have $\mu(\cup_{n=1}^{\infty} A_n) = \mu(\cup_{n=1}^{\infty} B_n) = \sum_{i=1}^{n} \mu(B_n)$. Since $\mu(B_n) \leq \mu(A_n) \, \forall n$, the result follows. $\square$

For the measurable space $(\mathbb{R}, \mathcal{B})$, the Lebesgue measure of any interval on the real line is the length of the interval:

$$\mu(a, b] = b - a.$$

In two- and three-dimensional space, the Lebesgue measure conincides with area and volume, respectively. It is commonly denoted as λ.

For an arbitrary Ω, let $\mathcal{F} = 2^{\Omega}$. For each finite subset A of Ω, we define $\mu(A) = |A|$ that is the measure of the set equals the number of elements in A. For all infinite subsets, we have $\mu(A) = \infty$. The measure with these properties is called the counting measure on Ω.

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. For any sequence $\{A_n\}_{n=1}^{\infty}$, we define

$$\limsup_{n\to\infty} A_n = \cap_{n=1}^{\infty} \cup_{k=n}^{\infty} A_k$$

and

$$\limsup_{n\to\infty} A_n = \cup_{n=1}^{\infty} \cap_{k=n}^{\infty} A_k.$$

An element ω is in $\limsup A_n$ only if it is in infinitely many of the A_n. Similarly, ω is in $\liminf A_n$ if it is in A_n eventually, and thus $\liminf A_n$ is sometimes called "all but finitely often".

For a measure space $(\Omega, \mathcal{F}, \mu)$, a sequence $\{A_n\}_{n=1}^{\infty}$ of elements in $\mathcal{F}$ is called monotone increasing if $A_n \subseteq A_{n+1} \forall n$. If on the other hand $A_n \supseteq A_{n+1}$, it is a monotinically decreasing sequence of sets.

Let $(\Omega, \mathcal{F})$, $(S, \mathcal{A})$ and $(T, \mathcal{B})$ be measurable spaces. Then

- If $f : \Omega \mapsto \bar{\mathbb{R}}$ and c is some constant, then cf is measurable.
- if $f : \Omega \mapsto S$ and $g : S \mapsto T$, then the composition $g \circ f = g(f) : \Omega \mapsto T$ is measurable.
- If f and f are measurable real-valued functions, so are $f + g$ and fg.

Let $(\Omega, \mathcal{F})$ and $(S, \mathcal{A})$ be measurable spaces. Then a function f which maps from Ω to S is called $\mathcal{F}/\mathcal{A}$ measurable if $f^{-1}(A) \in \mathcal{F} \, \forall A \in \mathcal{A}$, where the inverse image or pre-iamge $f^{-1}(A)$ is defined as $f^{-1}(A) = \{\omega \in \Omega : f(\omega) \in \mathcal{A}\}$. For sake of brevity, we might just say that f is measurable in this case.

Theorem **Properties of Sequence of Measurable Functions** Advanced Probability	Definition **Simple Function** Advanced Probability
Definition **Canonical Representation of Simple Function** Advanced Probability	Lemma **Monotone Approximation of Nonnegative Functions** Advanced Probability
Definition **Splitting Measurable Functions** Advanced Probability	Definition **Integral of Simple Functions** Advanced Probability
Proposition **Integral for two functions $f \leq g$** Advanced Probability	Definition **Integrable Function** Advanced Probability
Definition **Integral for General Nonnegative Functions** Advanced Probability	Definition **Integral of General Measurable Functions** Advanced Probability

A measurable function that takes at most finitely many values is called a simple function.

Let f_n be a sequence of measurable functions which satisfies $f_n : \Omega \mapsto \mathbb{R}$ for all n. Then the following are measurable:

- $\limsup_n f_n$ and $\liminf_n f_n$
- $\{\omega : \lim_{n\to\infty} f_n(\omega) \text{ exists}\}$
- $f = \begin{cases} \lim_{n\to\infty} f_n & \text{where the limit exists} \\ 0 & \text{elsewhere} \end{cases}$

Let f be a nonnegative measurable extended real-valued function from Ω. Then there exists a sequence $\{f_n\}_{n=1}^{\infty}$ of nonnegative finite simple functions such that $f_n \leq f$ for all n and $\lim_{n\to\infty} f_n(\omega) = f(\omega)\, \forall \omega$.

Let f be a simple function whose distinct values are $a_1, \ldots, a_n$ and let $A_i = \{\omega : f(\omega) = a_i\}$. Then $f = \sum_{i=1}^{n} a_i \mathbb{I}_{A_i}$ is called the canonical representation of f.

Let $f : \Omega \to \bar{\mathbb{R}}^{+0}$ be a simple function with canonical representation $f(\omega) = \sum_{i=1}^{n} \mathbb{I}_{A_i}(\omega)$. The integral of f with respect to μ is defined to be $\sum_{i=1}^{n} a_i \mu(A_i)$. The integral is denoted variously as $\int f d\mu$, $\int f(\omega)\mu(d\omega)$ or $\int f(\omega)\, d\mu(\omega)$.

Let f be a real-valued function. The positive part f^+ of f is defined as $f^+(\omega) = \max\{f(\omega), 0\}$. The negative part f^- of f is $f^-(\omega) = -\min\{f(\omega), 0\}$. We have $f = f^+ - f^-$ and $|f| = f^+ + f^-$.

We say that f is integrable with respect to μ if $\int f d\mu$ is finite.

If $f \leq g$ and both are nonnegative and simple, then

$$\int f d\mu \leq \int g d\mu.$$

Let f be measurable. If either f^+ orf^- is integrable with respect to μ, we define the integral of f with respect to μ to be

$$\int f^+ d\mu - \int f^- d\mu,$$

otherwise the integral does not exist.

For nonnegative measurable f, we define the integral of f with respect to μ by

$$\int f d\mu = \sup_{\text{nonnegative finite simple } g \leq f} \int g d\mu$$

Definition

Indicator Funtion / Characteristic Function

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Definition

Integration over a set

Advanced Probability

Proposition

Monotonicity of Integral

Advanced Probability

Definition

Almost sure / almost everywhere

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Theorem

Additivity

Advanced Probability

Theorem

Change of Variable

Advanced Probability

Theorem

Relationship to Riemann Integral

Advanced Probability

Definition

Expectation and Variance of Random Variables

Advanced Probability

Theorem

Fatou's Lemma

Advanced Probability

Theorem

Monotone convergence theorem

Advanced Probability

If $A \in \mathcal{F}$, we define

$$\int_A f d\mu = \int \mathbb{I}_A f d\mu.$$

For $(\Omega, \mathcal{F})$, we define the indicator or characteristic function $I_A : \Omega \mapsto \{0, 1\}$ for a set $A \subseteq \Omega$ as

$$\mathbb{I}_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A \end{cases}$$

Suppose that some statement about elements of Ω holds for all $\omega \in A^C$, where $\mu(A) = 0$. Then we say that the statement holds almost everywhere, denotes as $a.e.$ $[\mu]$. If P is a probability, then almost everywhere is often replaced by almost surely, denoted $a.s.$ $[P]$.

If $f \leq g$ and both integrals are defined, then $\int f d\mu \leq \int g d\mu$.

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let $(S, \mathcal{A})$ be a measurable space. Let $f : \Omega \mapsto S$ be a measurable function. Let ν be the measure induced on $(S, \mathcal{A})$ by f from μ. Let $g : S \mapsto \mathbb{R}$ be $A/\mathcal{B}^1$ measurable. Then

$$\int g d\nu = \int g(f) d\mu,$$

if either integral exists.

$$\int (f + g)\, d\mu = \int f d\mu + \int g d\mu,$$

whenever at least two of them are finite.

If P is a probability and X is a random variable, then $\int X dP$ is called the mean of X, expected value of X, or expectation of X, and denoted by $\mathbb{E}(X)$. If $\mathbb{E} = \mu$ is finite, then the variance of X is $\text{Var}(X) = \mathbb{E}\left[(X - \mu)^2\right]$

Let f be a continuous function on a closed bounded interval $[a, b]$. Let μ be Lebesgue measure. Then the Riemann integral $\int_a^b f(x)dx$ equals $\int_{[a,b]} f d\mu$.

Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of measurable nonnegative functions, and let f be a measurable function such that $f_n \leq f$ and $\lim_{n\to\infty} f_n = f$. Then

$$\lim_{n\to\infty} \int f_n d\mu = \int f d\mu$$

Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of nonnegative measurable functions. Then

$$\int \liminf_n f_n d\mu \leq \liminf_n \int f_n d\mu.$$

Theorem

Linearity of Integral

Advanced Probability

Lemma

Change of Variables

Advanced Probability

Corollary

Law of the unconsious statistician

Advanced Probability

Theorem

Density Functions

Advanced Probability

Theorem

Dominated convergence theorem

Advanced Probability

Proposition

Proposition 18

Advanced Probability

Definition

Uniform Integrability

Advanced Probability

Theorem

Properties of Integrals

Advanced Probability

Corollary

Corollary 22

Advanced Probability

Theorem

Theorem 23

Advanced Probability

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $(S, \mathcal{A})$ be a measurable space. Let $f : \Omega \mapsto S$ be a measurable funciton. Let ν be the measure induced on $(S, \mathcal{A})$ by f from μ. Let $g : S \mapsto \mathbb{R}$ be $\mathcal{A}/\mathcal{B}^1$ measurable. Then $$\int g d\nu = \int g(f) d\mu,$$ if either integral exists.	If $\int f d\mu$ and $\int g d\mu$ are defined and they are not both infinite and of opposite signs, then $\int [f + g]\, d\mu = \int f d\mu + \int g d\mu$.
Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, and let $f : \Omega \mapsto \bar{\mathbb{R}}^{+0}$ be measurable. Then $\nu(A) = \int_A f d\mu$ is a measure on $(\Omega, \mathcal{F})$. The function f is called the density of ν with respect to μ. Integrals with respect to ν can be computed as $\int g d\nu = \int f g d\mu$, if either exists.	If $X : \Omega \mapsto S$ is a random quantity with distribution μ_X and if $f : S \mapsto \mathbb{R}$ is measurable, then $\mathbb{E}[f(X)] = \int f d\mu_X$.
Let $\{f_n\}_{n=1}^{\infty}$, $\{g_n\}_{n=1}^{\infty}$ be sequences of measurable functions such that $\lvert f_n \rvert \leq g_n$, a.e. $[\mu]$. Let f and g be measurable functions such that $\lim_{n\to\infty} f_m = f$ and $\lim_{n\to\infty} g_n = n$, a.e$[\mu]$. Suppose that $\lim_{n\to\infty} = \int g d\mu < \infty$. Then, $\lim_{n\to\infty} \int f_n d\mu = \int f d\mu$.	Let $\{_n\}_{n=1}^{\infty}$ be a sequence of measurable functions, and let f and g be measurabe functions such that $f_n \to f$ a.e. $[\mu]$, $\lvert f_n \rvert \leq g$ a.e. $[\mu]$ and $\int g d\mu < \infty$. Then $$\lim_{n\to\infty} \int f_n d\mu = \int f d\mu.$$
Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Let f and g be measurable extended real-valued functions. • If f is nonnegative and $\mu(\{\omega : f(\omega) > 0\})$, then $\int f d\mu > 0$. • If f and g are integrable and if $\int_A f d\mu = \int_A g d\mu$ for all $A \in \mathcal{F}$, then $f = g$ a.e. $[\mu]$. • If μ is σ-finite and if $\int_A f d\mu = \int_A g d\mu$ for all $A \in \mathcal{F}$, then $f = g$ a.e. $[\mu]$. • Let Π be a π-system that generates $\mathcal{F}$. Suppose that Ω is a finite or countable union of elements of Π. If f and g are integrable and if $\int_A f d\mu = \int_A g d\mu$ for all $A \in \Pi$, then $f = g$ a.e $[\mu]$.	A sequence of integrable functions $\{f_n\}_{n=1}^{\infty}$ is uniformly integrable (with respect to μ) if $\lim_{c\to\infty} \sup_n \int_{\{\omega : \lvert f_n(\omega) \rvert > c\}} \lvert f_n \rvert d\mu = 0$.
Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Then μ is σ-finite if and only if there exists a strictly positive integrable function.	If μ is σ-finite and ν is related to μ as in Theorem 10, then the density of ν with respect to μ is unique, a.e. $[\mu]$.