Balancing
Benefits & Costs

1. Maximizing benefits – costs
2. Margin
3. Sunk costs & Decision making
4. Constrained optimization

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Maximizing benefits – costs

Let’s see a car repair decision.

Consider first the benefits of repairing your car. The following table shows how the price you’ll get for your car depends on the number of hours your mechanic works on it. Notice that the more repair time, the more your car is worth. For example, your car will be worth $1,150 more if your mechanic works on it for two hours, and $1,975 more if she works on it for four hours.

Second, think about the costs. You’ll have to pay for your mechanic’s time and the parts he uses. The following Table shows that the cost of two hours of your mechanic’s time and parts is $350. In addition, you won’t be able to work at your pizza delivery job while your car is being repaired. For example, the following Table shows that those two hours of repair work will cost you $30 in lost wages. Your total cost (also shown in Table) is the sum of these two costs.

Benefits of Repairing Your Car

Repair Time (Hours)	Total Benefit
0	$0
1	615
2	1,150
3	1,600
4	1,975
5	2,270
6	2,485

Costs of Repairing Your Car

Repair Time (Hours)	Cost of Mechanic and Parts	Lost Earnings from Pizza Delivery Job	Total Cost
0	$0	$0	$0
1	135	15	150
2	350	30	380
3	645	45	690
4	1,020	60	1,080
5	1,475	75	1,550
6	2,010	90	2,100

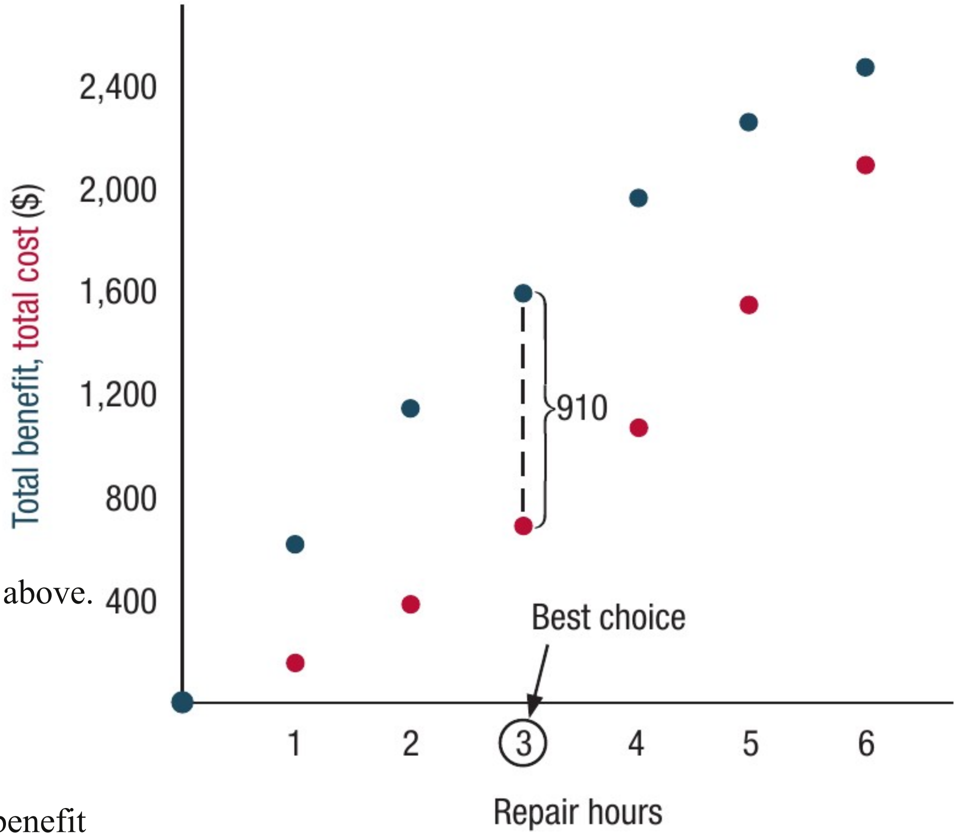
The costs of repairing your car are of two different types. When you hire your mechanic and buy parts, you incur an out-of-pocket cost. But when repairing your car forces you to skip your pizza delivery job, no money comes out of your pocket; instead, you forgo the opportunity to have others hand money to you. That type of cost is known as an **opportunity cost**—the cost associated with forgoing the opportunity to employ a resource in its best alternative use.

Maximizing benefits – costs

To make the right decision, you need to find the number of repair hours that maximizes your **net benefit**—the total benefit you derive less/minus the total cost you incur. The following Table shows the total benefit, total cost, and net benefit for each of your possible choices. The best choice is three hours, which has a net benefit of $910.

Total Benefit and Total Cost of Repairing Your Car

Repair Time (Hours)	Total Benefit	Total Cost	Net Benefit
0	$0	$0	$0
1	615	150	465
2	1,150	380	770
3	1,600	690	910
4	1,975	1,080	895
5	2,270	1,550	720
6	2,485	2,100	385



The Figure on the right graphs the total benefits and total costs from the Table above.

The horizontal axis measures the number of repair hours and the vertical axis measures total benefit and total cost. The figure shows the total benefit for each number of hours in blue and the total cost in red.

At the best choice of three hours, the vertical distance between the blue total benefit point and red total cost point is larger than at any other number of hours.

Maximizing benefits – costs

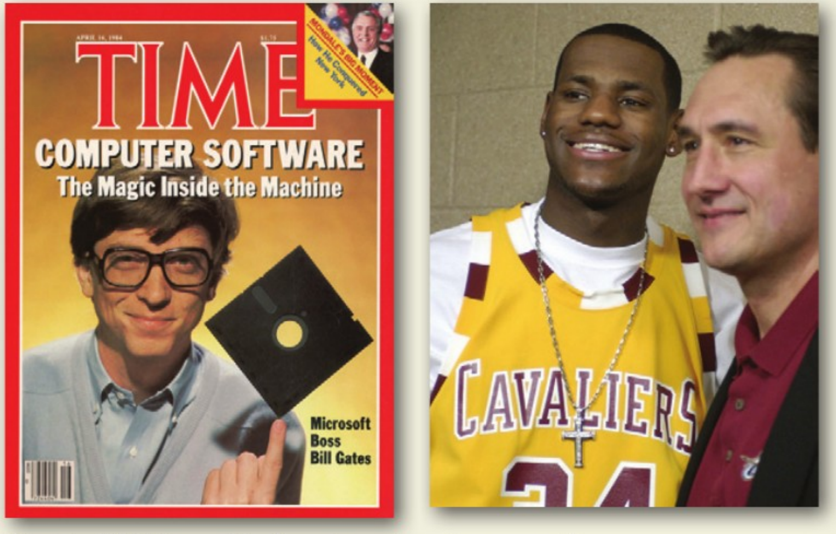
Application: Benefits and Costs of a College Degree

What do Bill Gates and LeBron James in common, aside from being very, very rich? The answer is that: neither earned a college degree.

Each year millions of high school students decide to go to college. *Is a college degree a good investment for the typical student*? The answer involves a comparison of the benefits and costs. While a college education has many benefits (for example, you will become a coder and be hired by Facebook), we'll look only at its effects on lifetime earnings.

Table on the right shows the average earnings of high school and college graduates in the years 2000–2009. Clearly college graduates earn quite a bit more than high school graduates. Over a lifetime, the extra earnings are worth about $670,000 before taxes and about $470,000 after taxes. But *what about the costs of attending college*? One obvious cost is tuition, which varies tremendously. Public (state) colleges cost much less than private schools. For example, in 2011, a year at the University of Texas at Austin cost a Texas resident about $9,000, while a year at a private college can cost over $40,000. Overall, over 65 percent of four-year-college students attended colleges where tuition was less than $15,000. In addition, many students paid reduced tuition through their college's financial aid programs. On top of tuition, books cost them roughly $1,000 per year.

But tuition and books are not the only costs of going to college. There's also an important opportunity cost: while you're in college you probably can't hold a full-time job. According to the right Table, for an average high school graduate of age 25 to 29, this cost is roughly $30,000 per year, which comes to about $21,000 per year after taxes—exceeding tuition costs for most students. (For college students younger than 25, the lost earnings are probably a little less than this.) On an after-tax basis, this opportunity cost raises the cost of a college education for those paying full tuition to roughly $150,000, or $250,000 for the most expensive private colleges. With a total benefit of over $450,000 and a total cost below $250,000, going to college looks like a good decision!



What do Bill Gates and LeBron James have in common?

Average Annual Earnings of High School and College Graduates, 2000–2009

Age	High School Graduates	College Graduates
25–29	$30,029	$46,051
30–34	33,388	59,812
35–39	36,740	71,633
40–44	38,909	77,617
45–49	40,419	80,070
50–54	40,754	78,106
55–59	39,858	73,633
60–64	38,872	70,928

Maximizing benefits – costs

Application: Benefits and Costs of a College Degree

What about Bill Gates and LeBron James? In some cases, the opportunity costs of going to college are so great that getting a degree doesn't make sense, at least not from the perspective of maximizing lifetime earnings. In 1975, Bill Gates dropped out of Harvard to start Microsoft. Had he stayed in school, he would probably have missed the opportunity to found the world's largest, most profitable software company. When LeBron James decided to skip college and go directly to the NBA, he signed a three-year $12.96 million contract with the Cleveland Cavaliers, along with a highly lucrative endorsement deal with Nike. For both Bill Gates and LeBron James, the opportunity costs of a college degree were just too high relative to the benefits.

* These figures in the right Table come from work reported on in
Anthony P. Carnevale, Stephen J. Rose, and Ban Cheah,
"*The College Payoff: Education, Occupations, Lifetime Earnings*,"
Georgetown University Center on Education and the Workforce.

**Gates's future earnings from his decision to drop out of college were, however, uncertain.
We'll discuss how to evaluate uncertain returns in the future.

What do Bill Gates and LeBron James have in common?

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Maximizing benefits – costs

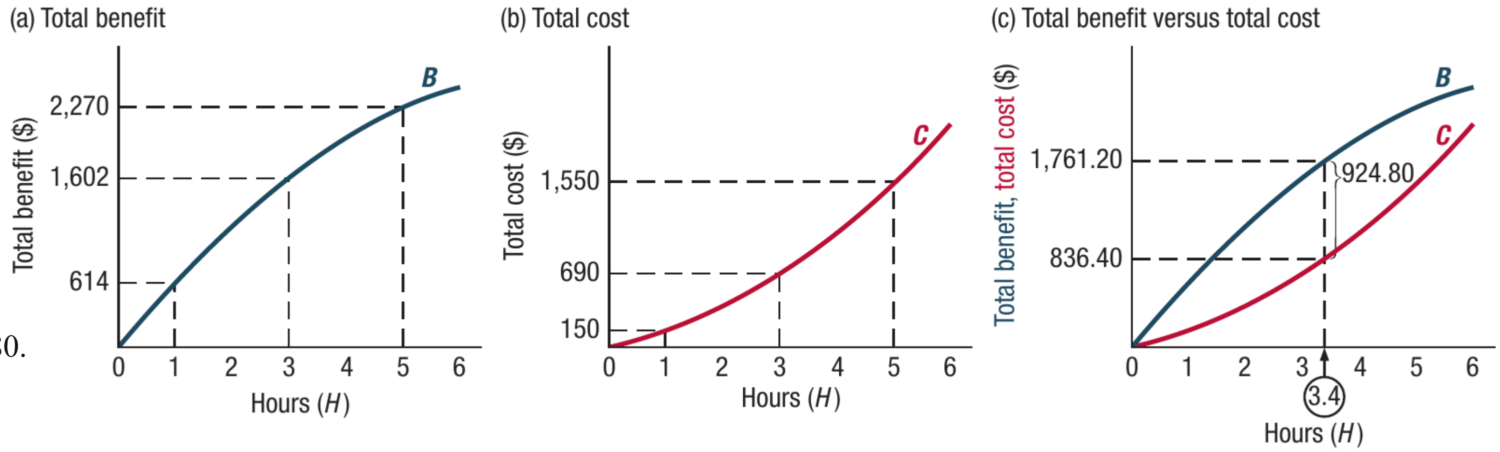
Maximizing Net Benefits with Finely Divisible Actions

Many quantities, such as time, can be divided into very tiny units (such as minutes,seconds, etc.). When this is so, it is often a convenient approximation to treat that quantity as if it is “**finely divisible**,” so that the decision maker can adjust his choice in arbitrarily small increments.

Let us go back to the car repair decision. If repair time is finely divisible, we can use total benefit and total cost curves to find the best choice. The following Figures (a) and (b) show the total benefit and total cost curves, respectively, for your car repair decision. The horizontal axes measure hours of your mechanic’s time. The vertical axes measure in dollars either the total benefit [in (a)] or the total cost [in (b)] of the possible amounts of repair work. To draw these figures, we assume that the total benefit with H hours of repair work is described by the function B(H) and the total cost is described by the function C(H) which are specified below. Both the total benefit and total cost increase as the amount of repair work rises from zero to six hours.

Figure (c) combines these two curves and identifies the best choice, which gives you the largest possible net benefit (total benefit less total cost). Graphically, it is the number of hours at which the vertical distance between the total benefit and total cost curves is largest. In Figure (c), your best choice is 3.4 hours, at which point total benefit is $1,761.20 and total cost is $836.40. The difference between total benefit and total cost at 3.4 hours is $924.80.

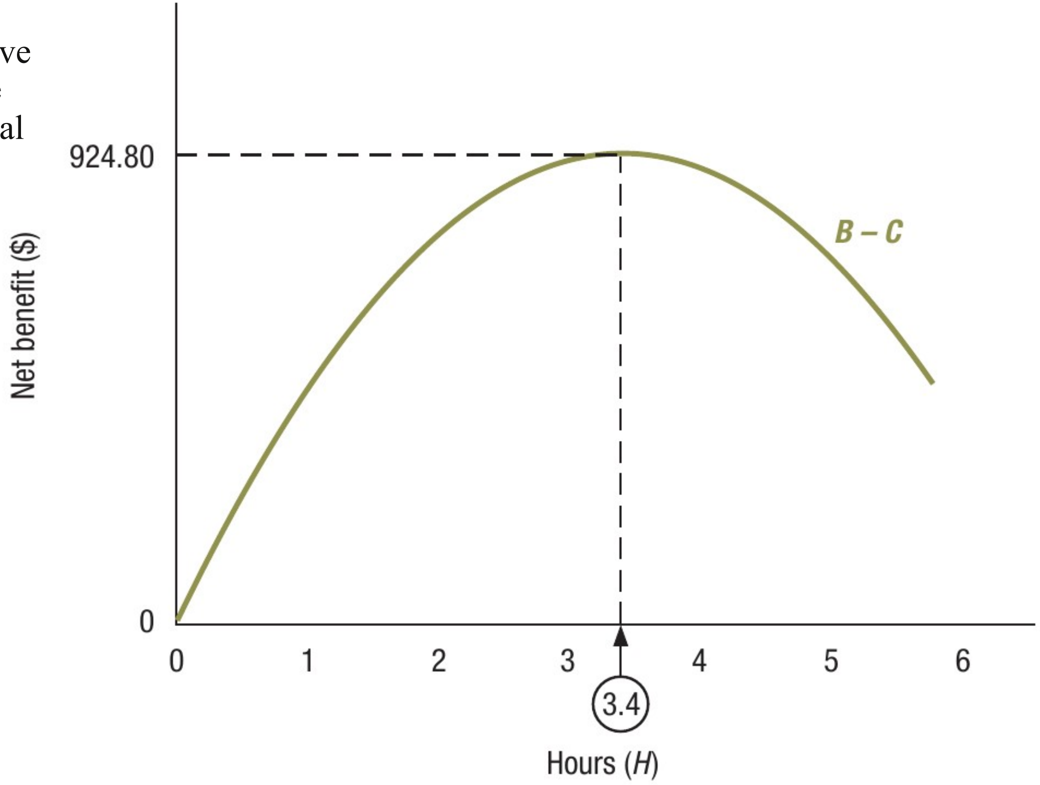
Total Benefit and Total Cost Curves When Repair Time Is Finely Divisible. The blue curve in (a) shows the total benefit from repairing your car for each amount of repair work between zero and six hours. It represents the total benefit function $B(H) = 654H - 40H^2$. The red curve in (b) shows the total cost of repairing your car for each amount of repair work between zero and six hours. It represents the total cost function $C(H) = 110H + 40H^2$. The two curves are combined in (c), which shows that the net benefit (which equals the vertical distance between the two curves at any given number of hours) is largest at a choice of 3.4 hours. In this case, total benefit is $1,761.20, total cost is $836.40, and net benefit is $924.80.



Maximizing benefits – costs

Maximizing Net Benefits with Finely Divisible Actions (continued...)

The Figure on the right shows the **net benefit**, *B* - *C*, in the form of a curve (the horizontal axis again measures the number of repair hours, while the vertical axis now measures the net benefit, the difference between the total benefit and total cost). The best choice of 3.4 hours leads to the highest point on the curve.



Check by yourself!

Margin

Another way to approach the maximization of net benefits is the **marginal effects**. When economists think about a decision, they focus on its **marginal benefits** and **marginal costs**. These concepts capture the way total benefit and cost change as the activity changes a little bit.

Consider again your car repair decision. Marginal benefit and marginal cost measure how much your benefits and costs change due to a small change in repair time—the smallest change that is possible.

Marginal units are the last ΔX units, where ΔX is the smallest amount you can add or subtract.

Marginal Cost

Marginal cost (*MC*) measures the additional cost you incur because of the last ΔH hours of repair time (the marginal units of repair time). Specifically, suppose that $C(H)$ is the total cost of H hours of repair work. [For example, in previous Table, the total cost of 3 hours of repair work is $C(3) = \$690$.] When you pay for H hours of repair work, the extra cost due to the last ΔH hours is $\Delta C = C(H) - C(H - \Delta H)$, which is the cost of H hours of repair time less the cost of $H - \Delta H$ hours.

In addition, economists measure marginal cost on a *per unit* basis, i.e., we divide the extra cost by the number of marginal hours, ΔH:

$$MC = \frac{\Delta C}{\Delta H} = \frac{C(H) - C(H - \Delta H)}{\Delta H}$$

This expression tells us how much marginal repair time costs per hour.

More generally, the **marginal cost of an action,** at an activity level of X units, equals the extra cost incurred due to the **marginal units**, $C(X) - C(X - \Delta X)$, divided by the number of marginal units, ΔX. That is, $MC = [C(X) - C(X - \Delta X)]/\Delta X$.

Right table shows how the total cost and marginal cost of repair work depend on the number of hours. Since you can vary repair work in one-hour increments ($\Delta H = 1$), above formula in this case simplifies t $MC = C(H) - C(H - 1)$. So the entries in the marginal cost column are just the change in cost due to the last hour of time purchased (represented graphically by the dashed arrows). For example, your marginal cost when you choose three hours of repair work (H = 3) is C(3) - C(2) = 690 - 380 = \$310 per hour.

Total Cost and Marginal Cost of Repairing Your Car

Repair Time (Hours)	Total Cost ($)	Marginal Cost (MC) ($/hour)
0	$0	—
1	150	$150
2	380	230
3	690	310
4	1,080	390
5	1,550	470
6	2,100	550

Margin

Marginal Benefit

We can find the marginal benefit (MB) of repair work in a similar way. Suppose that B(H) is the total benefit of H hours of repair work. Then when you choose H hours of repair work, the extra benefit from the last ΔH hours is $\Delta B = B(H) - B(H - \Delta H)$. Dividing by ΔH expresses this change on a per-hour basis, which gives the marginal benefit:

$$MB = \frac{\Delta B}{\Delta H} = \frac{B(H) - B(H - \Delta H)}{\Delta H}$$

This expression tells us how much marginal repair time benefits per hour.

More generally, the **marginal benefit of an action** at an activity level of X units equals the extra benefit produced by the ΔX marginal units, measured on a per-unit basis. That is, $MB = \Delta B/\Delta X = [B(X) - B(X - \Delta X)]/\Delta X$.

The right Table shows the total and marginal benefits for your car repair decision.
The marginal benefit corresponds to above formula where $\Delta H = 1$, so $MB = B(H) - B(H - 1)$.

Total Benefit and Marginal Benefit of Repairing Your Car

Repair Time (Hours)	Total Benefit ($)	Marginal Benefit (MB) ($/hour)
0	$0	—
1	615	$615
2	1,150	535
3	1,600	450
4	1,975	375
5	2,270	295
6	2,485	215

Margin

Best Choices and Marginal Analysis

By comparing marginal benefits and marginal costs, we can determine whether an increase or a decrease in the level of an activity raises or lowers the net benefit. For an illustration, look at Table below, which lists together the marginal benefits and marginal costs.

According to the table, the marginal benefit of the first hour is $615, while the marginal cost is $150. Since 615 is larger than 150, one hour of repair work is better than none—you come out ahead by $465 (that is, $615 - $150). Next, suppose you're thinking about hiring your mechanic for five hours. Is the last hour worthwhile? According to the table, the marginal benefit of the fifth hour is $295, which is less than the marginal cost ($470). So it's better to hire your mechanic for four hours than for five—you lose $175 ($470 - $295) on the fifth hour.
If you are making a best choice, then by definition neither a small increase nor a small decrease in your choice can increase your net benefit. Thus, at a best choice, the marginal benefit of the last unit must be at least as large as its marginal cost, and the marginal benefit of the next unit must be no greater than its marginal cost. That is, if X* is a best choice, then $MB \geq MC$ at X*, and $MB \leq MC$ at $X^* + \Delta X$. We call this the **No Marginal Improvement Principle**.

Marginal Benefit and Marginal Cost of Repairing Your Car

Repair Time (Hours)	Marginal Benefit (MB) ($/hour)		Marginal Cost (MC) ($/hour)
0	—		—
1	$615	>	$150
2	535	>	230
3	450	>	310
4	375	<	390
5	295	<	470
6	215	<	550

The No Marginal Improvement Principle:

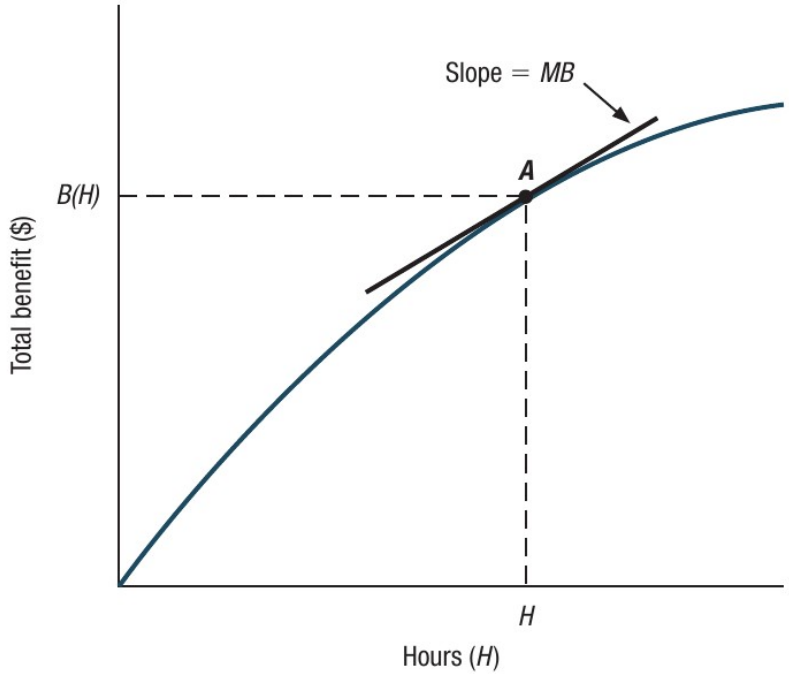
At a best choice, the marginal benefit of the last unit must be at least as large as its marginal cost, and the marginal benefit of the next unit must be no greater than its marginal cost.

Margin

Marginal Benefit and Marginal Cost with Finely Divisible Actions

Now let’s see how to define MB and MC when actions are finely divisible. The following Figure shows again the total benefit curve from previous Figure (a) on page 6.

Suppose we choose any point on the curve, such as the point A associated with H hours (see the right figure).



The marginal benefit associated with H hours of repair time equals the slope of the total benefit curve at point A. This slope is shown by the black line drawn tangent to the total benefit curve at that point (the tangent line at point A touches but does not cross the curve at that point). The slope of that line measures $\Delta B/\Delta H$ for very small changes in hours starting at H hours.

This relationship between MB and the slope of the total benefit curve is summarized as follows:

When actions are finely divisible, the marginal benefit when choosing action X equals the slope of the total benefit curve at X.

Similarly, for the marginal cost and the total cost curve, we have:

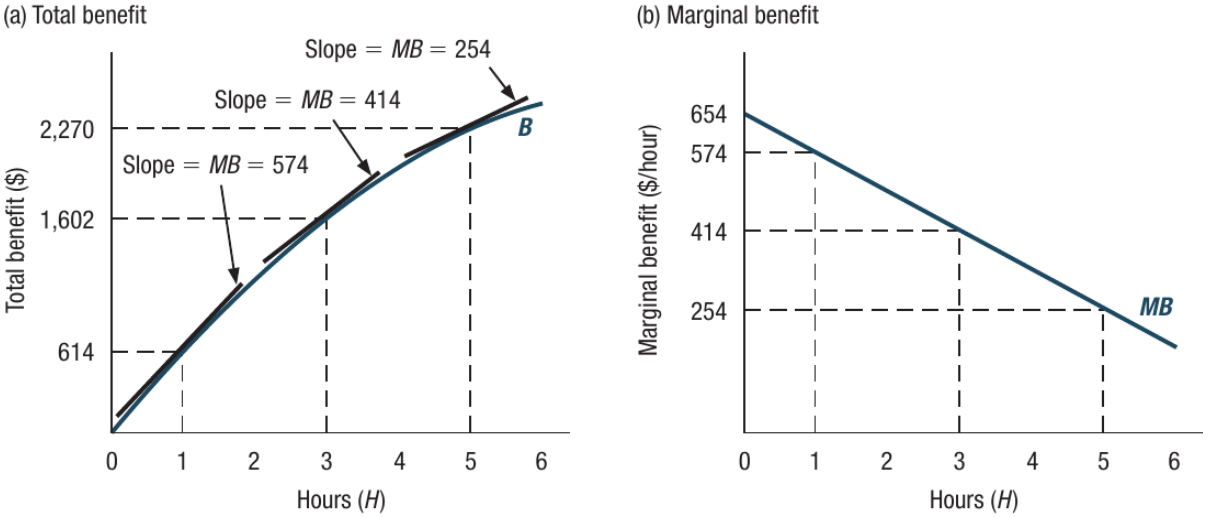
When actions are finely divisible, the marginal cost when choosing action X equals the slope of the total cost curve at X.

Margin

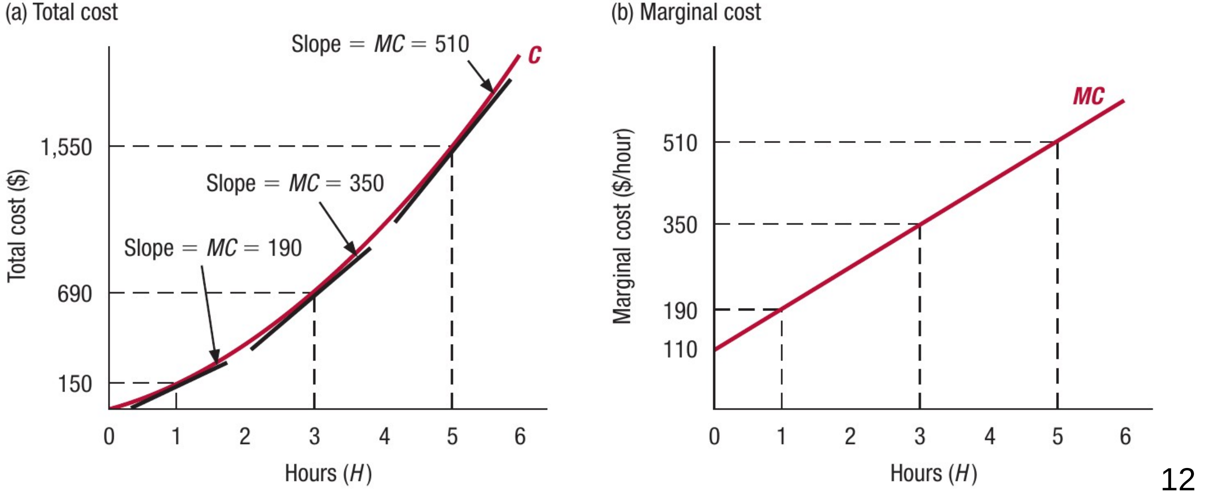
Marginal Benefit and Marginal Cost with Finely Divisible Actions

<u>Marginal Benefit and Marginal Cost Curves</u>

By computing the marginal benefit at many different levels of H, we can graph the marginal benefit curve. Figure (b) on the right shows the marginal benefit curve for the total benefit curve in in Figure (a), we’ve drawn straight lines tangent to the total benefit curve at three different numbers of hours: H = 1, H = 3, and H = 5. The slope of each line equals the marginal benefit at each of those levels. Figure (b) plots the marginal benefit at those levels and others. Note that the marginal benefit curve slopes downward, i.e., marginal benefit shrinks as the amount of repair work increases. This reflects the fact that the tangent lines in Figure (a) become flatter as we move from left to right along the total benefit curve.



Similarly, for the marginal cost curve. In the Figure (a) on the right is the total cost curve, with the addition of tangent lines at H = 1, H = 3, and H = 5. Figure (b) on the right shows the marginal cost at those levels and others. Note that the marginal cost curve slopes upward, i.e., the marginal cost grows larger as the amount of repair work increases, reflecting the fact that the tangent lines in Figure (a) become steeper as we move from left to right along the total cost curve.



Margin

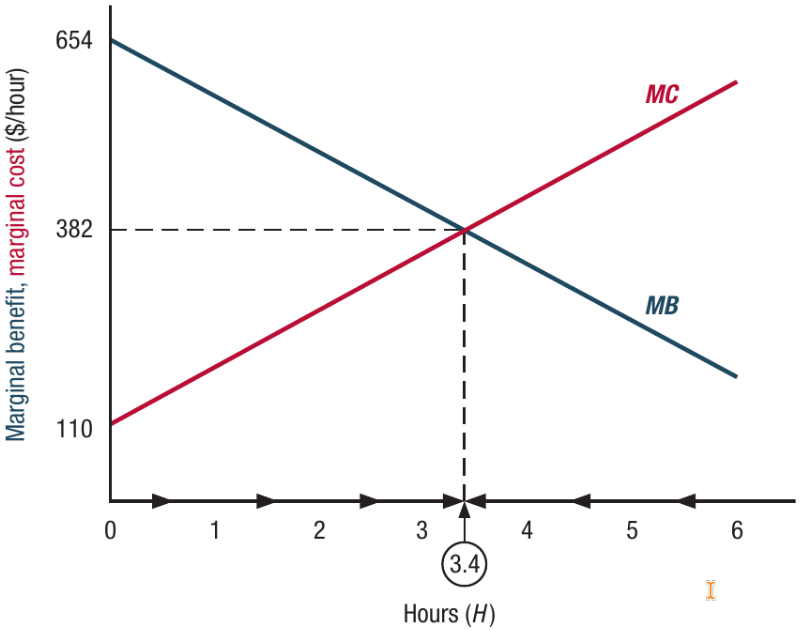
Best Choices and Marginal Analysis with Finely Divisible Action

When actions are finely divisible we have: **The No Marginal Improvement Principle (for Finely Divisible Actions)**

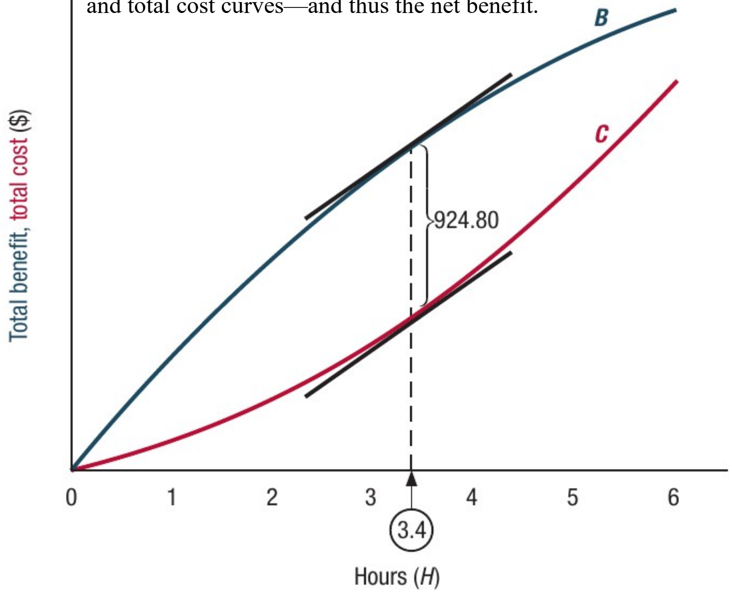
If actions are finely divisible, then marginal benefit equals marginal cost (MB = MC) at any best choice, provided that it is possible to both increase and decrease the level of the activity a little bit.

Proof: Suppose that action X* is the best choice. If MB > MC at X*, then a small increase in the activity level would increase the net benefit, which violates the best choice, then we must have MB ≤ MC at X*. Likewise, if MB < MC at X *, then a small decrease in the activity level would increase the net benefit, which violates the best choice, then we must have MB ≥ MC at X*. Thus, we must have MB = MC at X*.

In the right Figure, at the best choice of 3.4 hours, the No Marginal Improvement Principle holds, so MB = MC. At any number of hours below 3.4, marginal benefit exceeds marginal cost, so that a small increase in repair time will improve the net benefit (as indicated by the rightward Pointing arrows on the horizontal axis).
At any number of hours above 3.4, marginal cost exceeds marginal benefit, so that a small decrease in repair time will improve net benefit (as indicated by the leftward pointing arrows on the horizontal axis.)



The tangent lines to the two curves at 3.4 hours must be parallel, as shown in the figure. If that were not the case, then by varying the number of hours in one direction or the other we could increase the distance between the total benefit and total cost curves—and thus the net benefit.



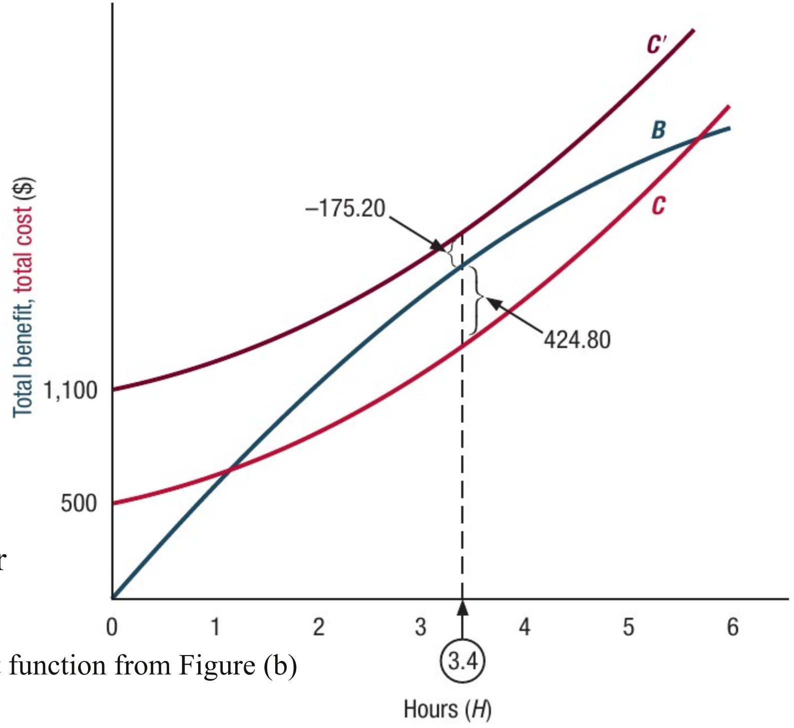
Sunk Costs & Decision Making

Sometimes a decision is associated with costs that the decision maker has already incurred or to which she has previously committed. At the time she makes her choice, they are unavoidable regardless of what she does. These are called **sunk costs**.

For example, suppose that, in addition to the charges discussed earlier, your mechanic has already purchased $500 worth of parts for your car that she can't return or use elsewhere. If you are responsible for the $500 regardless of what you have her do, then it's a sunk cost.

A sunk cost *has no effect on your best choice*, even though it increases total costs.

Right Figure illustrates this point. It shows the same benefit curve as in Figure (a) on page 6. In addition, the light red total cost curve, labeled C, adds a $500 sunk cost to the cost curve from Figure (b) on page 6. The sunk cost shifts the total cost curve up by $500. However, as the figure shows, the vertical distance between the total benefit curve and the light red total cost curve is still largest at H = 3.4. Including the $500 sunk cost, the net benefit is $424.80 ($500 less than before).



The same point also follows from thinking on the margin.

Because you must pay the $500 sunk cost regardless of the number of hours of repair work you choose, it has no effect on marginal costs. The marginal benefit and marginal cost curves are exactly the same as they were in previous Figure without the $500 sunk cost, so your best choice doesn't change.

The same conclusion would hold even if the sunk cost was large enough to make your total cost exceed your total benefit.

For example, the dark red cost function labeled C' in right Figure adds a $1,100 sunk cost to the cost function from Figure (b) on page 6. The net benefit from choosing 3.4 hours of repair work is now -$175.20.

Implication: a decision maker can always make the correct decision by simply ignoring sunk costs—that is, by pretending they are zero.

Constrained Optimization

Many economic problems we'll study have the feature that a decision maker faces a constraint that affects several decisions, requiring that she make **trade-offs** among them.

> For example, you can't spend more than the amount in your bank account is a constraint that affects both where you go for spring break and whether you buy a new smartphone. Likewise, consider a consumer who has to decide how much food and clothing to buy, but has a limited budget to spend. This budget constraint implies that the more she spends on food, the less of her budget is available for clothing.

Consider again your car repair decision, but suppose now that you have to decide how much repair time to devote to the engine, and how much to the car's body. The total benefit of doing H_E hours of engine work is $B_E(H_E)$, while the total benefit of H_B hours of body work is $B_B(H_B)$. For simplicity, suppose that all repair work costs \$100 per hour, and that you have a total budget of \$1,000 available to spend, so you can afford 10 hours in total.

The problem you face is one of choosing the number of hours to devote to each type of repair work, H_E and H_B, to maximize your total benefits, subject to the constraint that the total number of hours equals 10. This is an example of a **constrained optimization** problem. We can represent this problem as follows:

$$\max_{H_E, H_B} B_E(H_E) + B_B(H_B) \text{, subject to } \quad H_E + H_B = 10$$

objective function **constraint**

At least 2 methods to solve above problem: **substitution method** and **Lagrange multipliers method**.

1st method:
Substituting $H_B = 10 - H_E$ into the objective function for H_B, we can change the constrained optimization problem into the following unconstrained problem:

$$\max_{H_E} B_E(H_E) + B_B(10 - H_E)$$

The No Marginal Improvement Principle (for Finely Divisible Actions) tells us: at a best choice for H_E, the marginal benefit from an increase in H_E must equal its marginal cost. The marginal benefit is $MB_E(H_E)$. <u>The marginal cost is an opportunity cost</u>–the benefits you lose because you can't afford as much body work. Because you get one less hour of body work for each hour of additional engine work, this marginal cost equals $MB_B(10 - H_E)$. So the best choice of HE satisfies the condition: $MB_E(H_E) = MB_B(10 - H_E)$.

Constrained Optimization (Advanced Topic)

Proposition:
Let f and g be functions of two variables defined on a set S that are continuously differentiable on the interior of S, let c be a number, and suppose that (x^*, y^*) is an interior point of S that solves the problem

$$\max_{x,y} f(x, y) \text{ subject to } g(x, y) = c,$$

or the problem

$$\min_{x,y} f(x, y) \text{ subject to } g(x, y) = c$$

Suppose that $g'_1(x^*(c), y^*(c)) \neq 0$ or $g'_2(x^*(c), y^*(c)) \neq 0$, then there is a unique number λ such that (x^*, y^*) is a stationary point of the Lagrangean

$$L(x, y) = f(x, y) - \lambda(g(x, y) - c).$$

That is, (x^*, y^*) satisfies the first-order conditions

$$\begin{aligned} L'_1(x^*, y^*) &= f'_1(x^*, y^*) - \lambda g'_1(x^*, y^*) = 0 \\ L'_2(x^*, y^*) &= f'_2(x^*, y^*) - \lambda g'_2(x^*, y^*) = 0. \end{aligned}$$

In addition, $g(x^*, y^*) = c$.

Constrained Optimization (Advanced Topic)

Example:
Consider the problem:

$$\max_{x,y} x^a y^b \text{ subject to } px + y = m,$$

where $a > 0$, $b > 0$, $p > 0$, and $m > 0$, and the objective and constraint functions are defined on the set of all points (x, y) with $x \geq 0$ and $y \geq 0$ (and are continuously differentiable on the interior of this set).

The Lagrangean is

$$L(x, y) = x^a y^b - \lambda(px + y - m)$$

so the first-order conditions are

$$\begin{aligned} ax^{a-1}y^b - \lambda p &= 0 \\ bx^a y^{b-1} - \lambda &= 0 \end{aligned}$$

and the constraint is $px + y = m$. From the first two conditions we have $ay = pbx$. Substituting into the constraint we obtain

$$x = am/((a + b)p) \text{ and } y = bm/(a + b),$$

so that (x, y) is an interior point of the domain of the objective function and

$$\lambda = [a^a b^b/(a+b)^{a+b-1}][m^{a+b-1}/p^a].$$

The value of the objective function at this point is $[am/((a + b)p)]^a[bm/(a + b)]^b$, which is positive.

We have $g'_1(x, y) = p$ and $g'_2(x, y) = 1$, so there are no values of (x, y) for which $g'_1(x, y) = g'_2(x, y) = 0$.

The boundary of the set on which the objective function is defined is the set of points (x, y) with $x = 0$ or $y = 0$. At every such point the value of the objective function is 0.

We conclude that if the problem has a solution, it is $(x, y) = (am/((a + b)p), bm/(a + b))$.