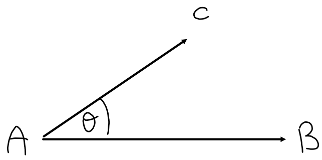
## Section 7.3 - The Dot Product of Two Geometric Vectors

The dot product for any two vectors is defined as the product of their magnitudes multiplied by the cosine of the angle between the vectors when placed tail to tail.




$\overrightarrow{AC} \cdot \overrightarrow{AB} = |\overrightarrow{AC}||\overrightarrow{AB}| \cos\theta, \quad 0^\circ \le \theta \le 180^\circ$

Since the result of the dot product is a scalar, the dot product is also known as the scalar product.

Properties of Dot Product:

① If $0^\circ \le \theta < 90^\circ$, $\cos\theta > 0$, so $\vec{a} \cdot \vec{b} > 0$

② If $\theta = 90^\circ$, $\cos\theta = 0$, so $\vec{a} \cdot \vec{b} = 0$

$\therefore$ when two non-zero vectors are $\perp$, their dot product is always 0.

③ If $90^\circ < \theta \le 180^\circ$, $\cos\theta < 0$, so $\vec{a} \cdot \vec{b} < 0$

ex: Two vectors $\vec{a}$ and $\vec{b}$ placed tail to tail have magnitudes of 4 and 7. The angle between the vectors is $60^\circ$. Calculate $\vec{a} \cdot \vec{b}$

Solution: $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos 60$
$= (4)(7)\left(\frac{1}{2}\right)$
$= 14$

④ $\vec{p} \cdot \vec{q} = |\vec{p}||\vec{q}|\cos\theta = |\vec{q}||\vec{p}|\cos\theta = \vec{q} \cdot \vec{p}$
"commutative property"

⑤ Calculate the dot product between a vector and itself. Since the angle is $0^\circ$,

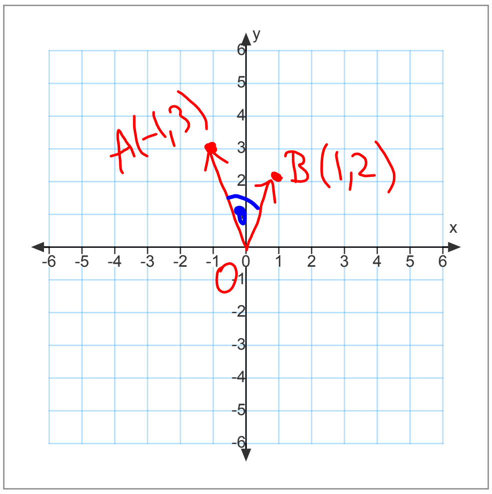
$\vec{p} \cdot \vec{p} = |\vec{p}||\vec{p}| \cos 0^\circ$
$= |\vec{p}||\vec{p}|(1)$
$= |\vec{p}|^2$ "magnitudes property"

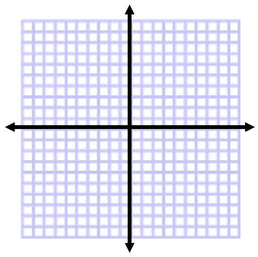
⑥ $\vec{p} \cdot (\vec{q} + \vec{r}) = \vec{p} \cdot \vec{q} + \vec{p} \cdot \vec{r}$
distributive property

⑦ $(k\vec{p}) \cdot \vec{q} = \vec{p} \cdot (k\vec{q}) = k(\vec{p} \cdot \vec{q})$
"associative property with a scalar"

ex: $(\vec{a} + 5\vec{b}) \cdot (2\vec{a} - 3\vec{b})$
$= \vec{a} \cdot 2\vec{a} - 3\vec{a} \cdot \vec{b} + 10\vec{a} \cdot \vec{b} - 15\vec{b} \cdot \vec{b}$
$= 2|\vec{a}|^2 + 7\vec{a} \cdot \vec{b} - 15|\vec{b}|^2$

May 3-8:15 AM



What is $\vec{OA} \cdot \vec{OB}$ ?

$|\vec{OA}| = \sqrt{10}$

$|\vec{OB}| = \sqrt{5}$

$\vec{AB} = (2,-1)$

$|\vec{AB}| = \sqrt{5}$

$$|\vec{AB}|^2 = |\vec{OA}|^2 + |\vec{OB}|^2 - 2|\vec{OA}||\vec{OB}|\cos\theta$$
$$5 = 10 + 5 - 2(\sqrt{10})(\sqrt{5})\cos\theta$$
$$5 = 15 - 2\sqrt{50}\cos\theta$$
$$\cos\theta = \frac{5}{\sqrt{50}} = \frac{5}{5\sqrt{2}} = \frac{1}{\sqrt{2}}$$
$$\theta = 45^\circ$$

$$\vec{OA} \cdot \vec{OB} = |\vec{OA}||\vec{OB}|\cos\theta$$
$$= (\sqrt{10})(\sqrt{5})\left(\frac{1}{\sqrt{2}}\right)$$
$$= \frac{\sqrt{50}}{\sqrt{2}} = \sqrt{\frac{50}{2}} = \sqrt{25}$$
$$= 5$$

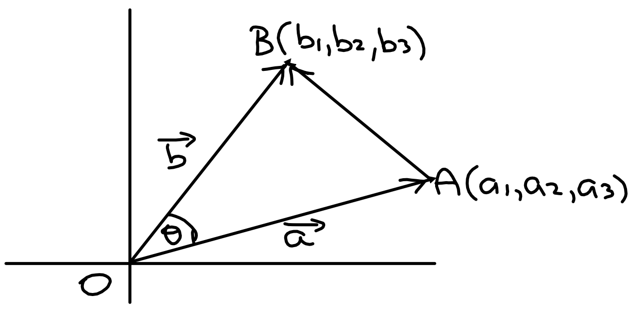
Algebraic Method:

$$\vec{OA} \cdot \vec{OB} = (-1,3) \cdot (1,2)$$
$$= (-1)(1) + (3)(2)$$
$$= -1 + 6$$
$$= 5$$

# The Dot Product of Algebraic Vectors

**Proof:** In $R^3$ if $\vec{a} = (a_1, a_2, a_3)$
and $\vec{b} = (b_1, b_2, b_3)$ then

$$\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$$




In $\triangle OAB$,

$$|\vec{AB}|^2 = |\vec{OA}|^2 + |\vec{OB}|^2 - 2|\vec{OA}||\vec{OB}|\cos\theta$$

and $\vec{AB} = (b_1 - a_1, b_2 - a_2, b_3 - a_3)$

$$\therefore |\vec{AB}|^2 = (b_1 - a_1)^2 + (b_2 - a_2)^2 + (b_3 - a_3)^2$$

we also know that

$$|\vec{OA}|^2 = a_1^2 + a_2^2 + a_3^2$$

and

$$|\vec{OB}|^2 = b_1^2 + b_2^2 + b_3^2$$

and lastly,

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

So using a lot of Substitution we have,

$$(b_1-a_1)^2 + (b_2-a_2)^2 + (b_3-a_3)^2 = a_1^2 + a_2^2 + a_3^2 + b_1^2 + b_2^2 + b_3^2 - 2\vec{a}\cdot\vec{b}$$

Through expansion,

$$\cancel{b_1^2} - 2a_1b_1 + \cancel{a_1^2} + \cancel{b_2^2} - 2a_2b_2 + \cancel{a_2^2} + \cancel{b_3^2} - 2a_3b_3 + \cancel{a_3^2}$$
$$= \cancel{a_1^2} + \cancel{a_2^2} + \cancel{a_3^2} + \cancel{b_1^2} + \cancel{b_2^2} + \cancel{b_3^2} - 2\vec{a}\cdot\vec{b}$$

$$\therefore -2a_1b_1 - 2a_2b_2 - 2a_3b_3 = -2\vec{a}\cdot\vec{b}$$

OR

$$a_1b_1 + a_2b_2 + a_3b_3 = \vec{a}\cdot\vec{b}$$

$$\therefore \vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3 \text{ in } \mathbb{R}^3$$

and

$$\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 \text{ in } \mathbb{R}^2$$

May 10-9:19 AM

ex1: If $\vec{a}=(-2,1)$ and $\vec{b}=(1,2)$ calculate $\vec{a}\cdot\vec{b}$ and state whether the angle is acute, obtuse or $90^\circ$.

$$\vec{a}\cdot\vec{b}=(-2,1)\cdot(1,2)$$
$$=(-2)(1)+(1)(2)$$
$$=-2+2$$
$$=0$$

Since dot product is 0, the vectors are $\perp$. $\therefore 90^\circ$ angle

ex2: $\vec{a}=(2,3,-1)$ and $\vec{b}=(4,3,-17)$

$$\vec{a}\cdot\vec{b}=(2)(4)+(3)(3)+(-1)(-17)$$
$$=8+9+17$$
$$=34$$

Since 34 is positive, $\cos\theta>0$, $0\leq\theta<90$. $\therefore$ angle is acute

ex3: For what values of K are the vectors $\vec{x}=(3K,14K,2)$ and $\vec{y}=(K,-1,4)$ perpendicular

Solution:

$$\vec{x}\cdot\vec{y}=0$$
$$(3K,14K,2)\cdot(K,-1,4)=0$$
$$3K^2-14K+8=0$$
$$(3K-2)(K-4)=0$$

$\therefore 3K=2$, $K=\frac{2}{3}$ ; $K-4=0$, $K=4$

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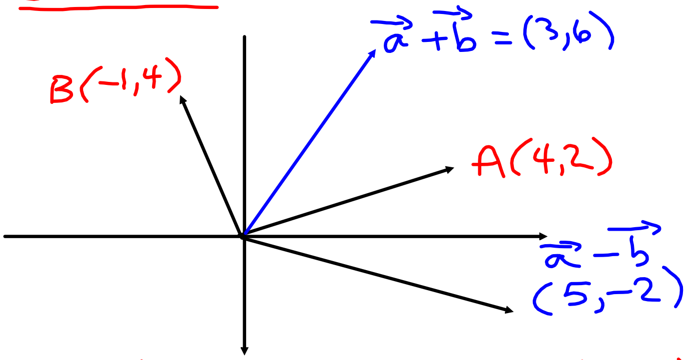
ex: A parallelogram has its sides determined by $\vec{a}=(4,2)$ & $\vec{b}=(-1,4)$. Determine the angle between the diagonals of the parallelogram formed by the vectors.

Solution:




$\vec{a}+\vec{b} = (4,2)+(-1,4) = (3,6)$

$\vec{a}-\vec{b} = (4,2)-(-1,4) = (5,-2)$

(3,6) and (5,−2) are the components of the diagonals. (position vectors)

$\therefore \vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

OR

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$$

$$\cos\theta = \frac{(3,6)\cdot(5,-2)}{|(3,6)|\ |(5,-2)|}$$

$$\cos\theta = \frac{(3)(5)+(6)(-2)}{\sqrt{3^2+6^2}\ \sqrt{5^2+(-2)^2}}$$

$$\cos\theta = \frac{15-12}{\sqrt{45}\sqrt{29}}$$

$$\cos\theta = 0.083$$

$$\theta = 85.2^\circ$$

$\therefore$ the angle between the diagonals is $85.2^\circ$. The supplementary angle of $94.8^\circ$ is also correct.

May 8-12:22 PM

ex: Find a vector (or vectors) perpendicular to each of the vectors $\vec{a}(2,5,1)$ and $\vec{b}(-1,1,2)$

Solution:

Let vector $\vec{x} = (x,y,z)$.
Since it is perpendicular,
$(x,y,z)\cdot(2,5,1) = 0$ and $(x,y,z)\cdot(-1,1,2) = 0$

$\therefore$ ① $2x + 5y + z = 0$
② $-x + y + 2z = 0$

① $2x + 5y + z = 0$
② $-2x + 2y + 4z = 0$

$7y + 5z = 0$
$5z = -7y$
$\boxed{z = -\frac{7}{5}y}$

now sub $z = -\frac{7}{5}y$ into ①

$2x + 5y + \left(-\frac{7}{5}y\right) = 0$

$2x + \frac{18}{5}y = 0$

$2x = -\frac{18}{5}y$

$x = -\frac{18}{10}y$

$\boxed{x = -\frac{9}{5}y}$

$\therefore \vec{x} = \left(-\frac{9}{5}y, y, -\frac{7}{5}y\right)$

or sub in parameter $t$ for $y$

$\therefore \vec{x} = \left(-\frac{9}{5}t, t, -\frac{7}{5}t\right)$

$\vec{x} = t\left(-\frac{9}{5}, 1, -\frac{7}{5}\right)$

Check:

$\left(-\frac{9}{5}, 1, -\frac{7}{5}\right)\cdot(2,5,1)$
$= -\frac{18}{5} + 5 - \frac{7}{5}$
$= 0$ ✓

$\left(-\frac{9}{5}, 1, -\frac{7}{5}\right)\cdot(-1,1,2)$
$= \frac{9}{5} + 1 - \frac{14}{5}$
$= 0$ ✓

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