2.4 The Quotient Rule

Let $h(x) = \dfrac{f(x)}{g(x)}$.

If both $f'(x)$ and $g'(x)$ exist, the derivative of $h(x)$ is:

$$h'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{\left[g(x)\right]^2}.$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v - uv'}{v^2}.$$

Proof

Let $h(x) = \dfrac{f(x)}{g(x)}$

$$\boxed{h'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{\left[g(x)\right]^2}}$$

$$g(x)\, h(x) = f(x)$$

$$h(x)\, g'(x) + h'(x)\, g(x) = f'(x)$$

Product Rule

$$h'(x) = \frac{f'(x) - g'(x)h(x)}{g(x)}$$

isolate $h(x)$

$$h'(x) = \frac{f'(x) - g'(x)\dfrac{f(x)}{g(x)}}{g(x)}$$

1- Find the derivative of the following:

$$y = \frac{5x-1}{4x+3} \qquad f(x) = 5x - 1 \text{ and } g(x) = 4x + 3$$

$$\frac{dy}{dx} = \frac{f'(x)g(x) - g'(x)f(x)}{[g(x)]^2}$$

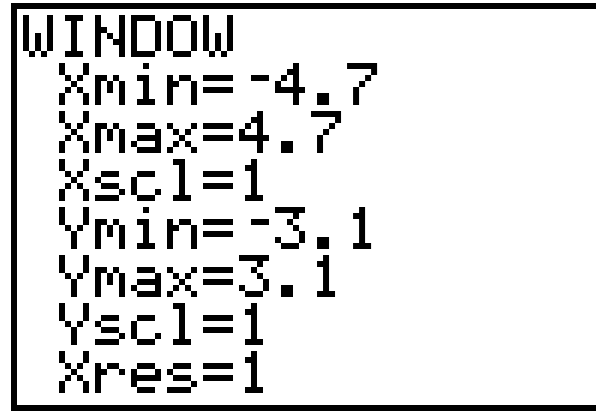
$$= \frac{5(4x+3) - 4(5x-1)}{(4x+3)^2}$$

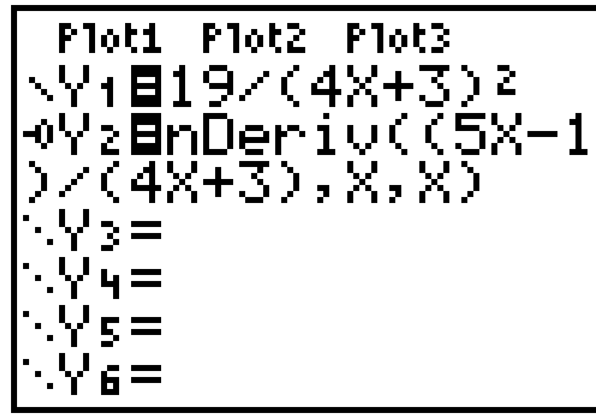
$$= \frac{20x+15-20x+4}{(4x+3)^2}$$

$$= \frac{19}{(4x+3)^2}$$

Verify using a graphing calculator:

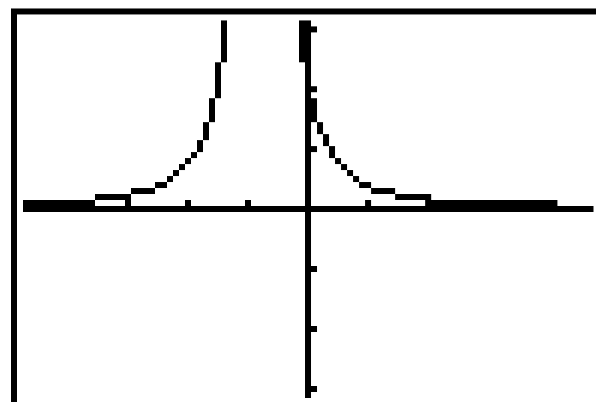
$$y = \frac{5x-1}{4x+3} \qquad y' = \frac{19}{(4x+3)^2}$$





Y_1 is what we calculated for the first derivative.

Y_2 is the derivative of the original function.



We see that both graphs are the same.

2- Find the derivative of the following:

$$y = \frac{2x^3 - 3}{x^2 + 1} \qquad f(x) = 2x^3 - 3 \text{ and } g(x) = x^2 + 1$$

$$\frac{dy}{dx} = \frac{f'(x)g(x) - g'(x)f(x)}{[g(x)]^2}$$

$$= \frac{6x^2(x^2+1) - 2x(2x^3 - 3)}{(x^2+1)^2}$$

$$= \frac{6x^4 + 6x^2 - 4x^4 + 6x}{(x^2+1)^2}$$

$$= \frac{2x^4 + 6x^2 + 6x}{(x^2+1)^2}$$

$$\boxed{= \frac{x(2x^3 + 6x + 6)}{(x^2+1)^2}}$$

Finding the Second Derivative

Example: The position of an object moving along a straight line is given as:

$$s(t) = \frac{3t}{t+1}$$

determine the position, velocity and acceleration at $t = 3$.

$$s'(t) = \frac{3(t+1) - 1(3t)}{(t+1)^2}$$

$$s'(t) = \frac{3}{(t+1)^2}$$

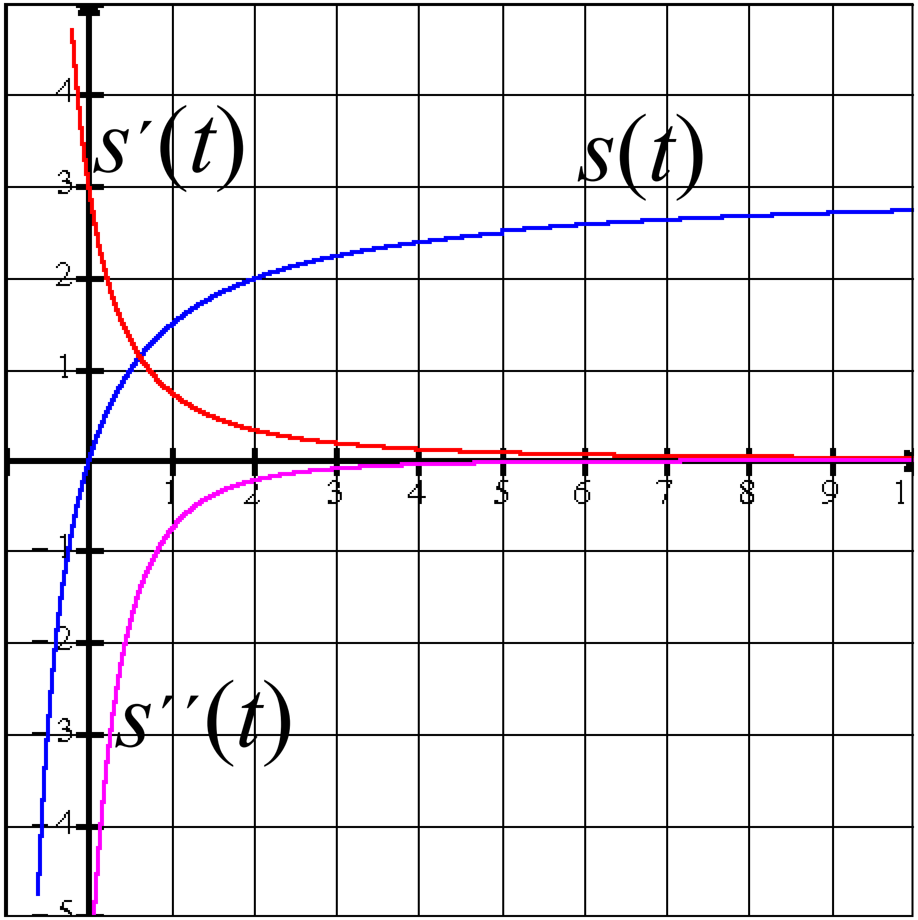
$$s''(t) = \frac{0[(t+1)^2] - 3(2t+2)}{[(t+1)^2]^2}$$

$$s''(t) = \frac{-6t-6}{[(t+1)^2]^2} = \frac{-6t-6}{(t+1)^4}$$

$$s(t) = \frac{3t}{t+1}$$

$$s'(t) = \frac{3}{(t+1)^2}$$

$$s''(t) = \frac{-6t-6}{[(t+1)^2]^2}$$



at $t = 3$

position

$s(3) = 2.25\text{m}$

velocity

$s'(3) = 0.187\text{m/s}$

acceleration

$s''(3) = -0.094\text{m/s}^2$