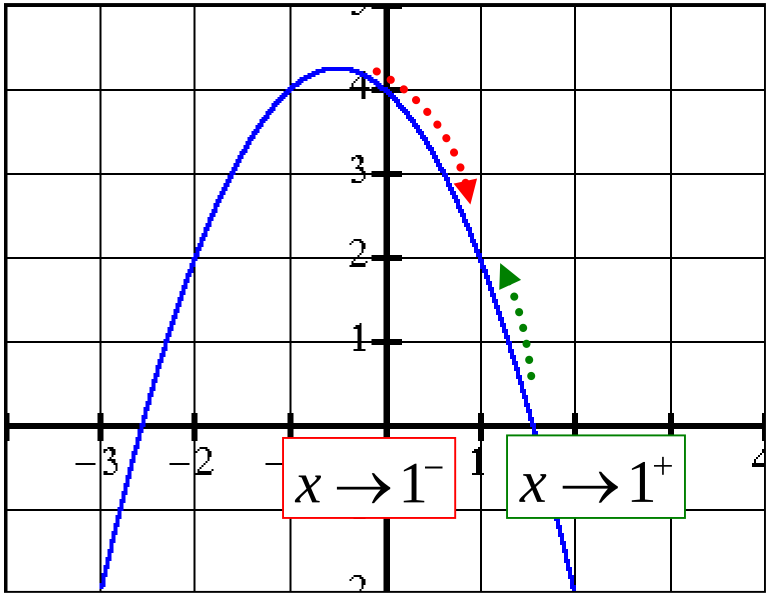
1.4 The Limit of a Function

We will see that there are other techniques of determining limits other than tables.

The Limit of a Function

The notation $\lim_{x \to a} f(x) = L$ reads "the limit of $f(x)$ as x approaches a is L". (x approaches a from either side)



$f(x) = -x^2 - x + 4$

From the graph we see that as x approaches 1 from the left or from the right the limit is 2.

$$\lim_{x \to 1} f(x) = 2$$

One-Sided Limits

Left-hand limit: $\lim\limits_{x \to a^-} f(x) = L$ limit approaching a from left.

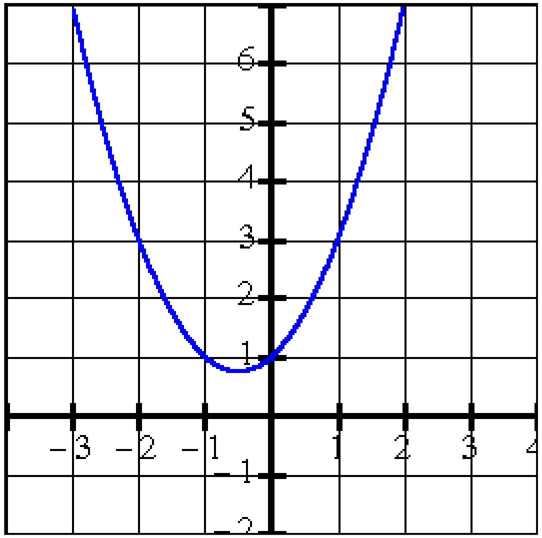
Right-hand limit: $\lim\limits_{x \to a^+} f(x) = L$ limit approaching a from right.

Two-Sided Limits

If $\lim\limits_{x \to a^-} f(x) = L$ and $\lim\limits_{x \to a^+} f(x) = L$, then $\lim\limits_{x \to a} f(x)$ exists

and is equal to L. $\lim\limits_{x \to a} f(x)$ is called a **two-sided limit**.

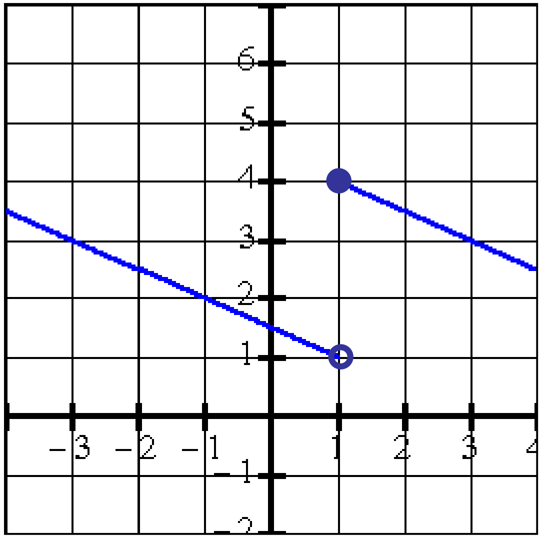
Determine: $\lim_{x \to 1} f(x)$



$$\lim_{x \to 1^-} f(x) = 3$$

$$\lim_{x \to 1^+} f(x) = 3$$

$$\therefore \lim_{x \to 1} f(x) = 3$$

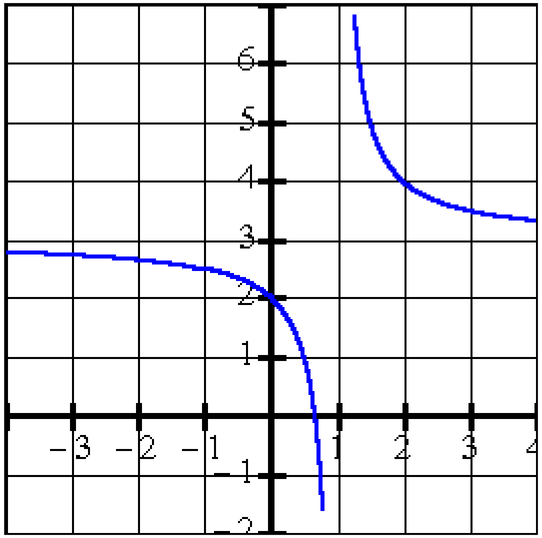


$$\lim_{x \to 1^-} f(x) = 1$$

$$\lim_{x \to 1^+} f(x) = 4$$

$$\therefore \lim_{x \to 1} f(x)$$

does not exist.



$$\lim_{x \to 1^-} f(x) = -\infty$$

$$\lim_{x \to 1^+} f(x) = \infty$$

$$\therefore \lim_{x \to 1} f(x)$$

does not exist.

If $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$ then $\lim_{x \to a} f(x)$ does not exist.

Using a table … $\lim_{x\to 1} f(x) = \dfrac{1}{x-1} + 3$

$x \to 1^-$

x	$f(x)$
0	2
0.5	1
0.9	-7
0.99	-97
0.999	-997

$x \to 1^+$

x	$f(x)$
2	4
1.5	5
1.1	13
1.01	103
1.001	1003

As x approaches 1 from the left or the right, we see that the limits are not equal.

$\therefore \lim_{x\to 1} f(x)$

does not exist.

We say that $f(x)$ is a *discontinuous function*.

Continuous Functions

A function is *continuous* at *a* if $\lim_{x \to a} f(x) = f(a)$ and

1. The value *a* is in the domain of *f*.

2. $\lim_{x \to a} f(x)$ exists

3. $\lim_{x \to a} f(x) = f(a)$

A function is *continuous* if it is continuous at *a* for all values of *a* in the domain.

A function is ***discontinuous*** at *a* when *f*(*a*) is not defined; $\lim_{x \to a} f(x)$ does not exist or $\lim_{x \to a} f(x) \neq f(a)$

Properties of Limits Involving Polynomial Functions.

Some Basic Limits

If a and c are real numbers and n is an integer, then

1. $\lim_{x \to a} c = c$ $\qquad$ $Ex\ 1.\ \lim_{x \to 1} 8 = 8$

2. $\lim_{x \to a} x = a$ $\qquad$ $Ex\ 2.\ \lim_{x \to -4} x = -4$

3. $\lim_{x \to a} x^n = a^n, \text{ if } a \neq 0 \text{ when } n < 0.$ $\qquad$ $Ex\ 3.\ \lim_{x \to 2} x^3 = 2^3$

Limit Laws

1) **Constant Law**

If $f(x) = C$ (a constant) then

$$\lim_{x\to a} f(x) = C$$

Ex. $\lim_{x\to 2}(-3) = -3$

2) **Constant Multiplier Law**

If $\lim_{x\to a} f(x) = L$ then

$$\lim_{x\to a}\left[cf(x)\right] = cL$$

Ex. $\lim_{x\to 3}(2x^2) = 2\lim_{x\to 3}(x^2)$

$= 2(3^2)$

$= 18$

3) **Sum/Difference Law**

If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$ then

$$\lim_{x\to a}\left[f(x) \pm g(x)\right] = L \pm \mathrm{M}$$

Ex. $\lim_{x\to 3}\left[x^2 - 2x\right] = \lim_{x\to 3} x^2 - \lim_{x\to 3} 2x$

$= 9 - 6 = 3$

Limits of Polynomial Functions

For any polynomial function $P(x)$, $\lim_{x \to a} P(x) = P(a)$

Example: Determine the limit of the function:

$$\lim_{x \to -3} (2x^2 + 3x - 5)$$

$2x^2 + 3x - 5$ is a polynomial function

$$\therefore \lim_{x \to -3} (2x^2 + 3x - 5) = 2(-3)^2 + 3(-3) - 5$$

$$= 18 - 9 - 5$$

$$= 4$$