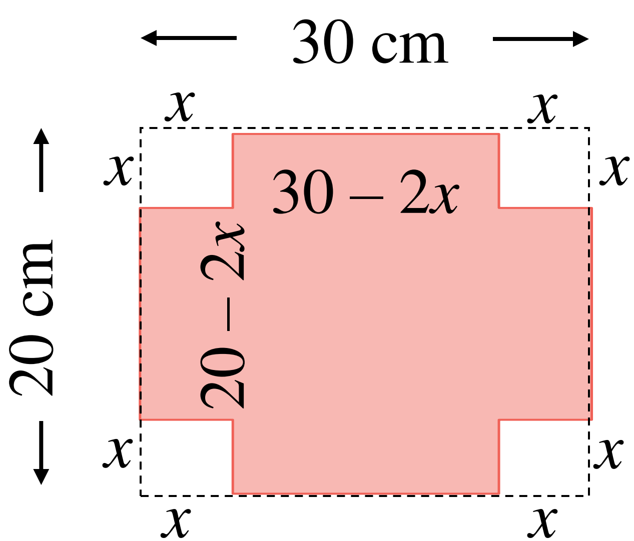
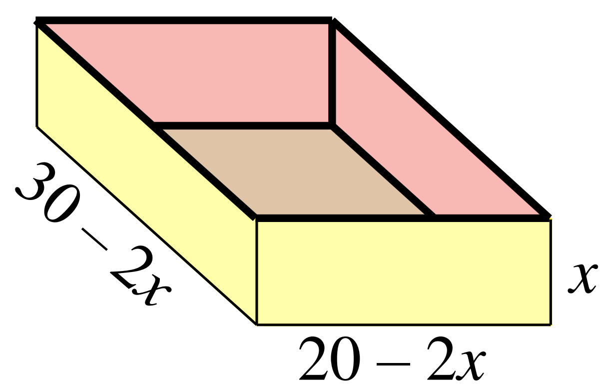
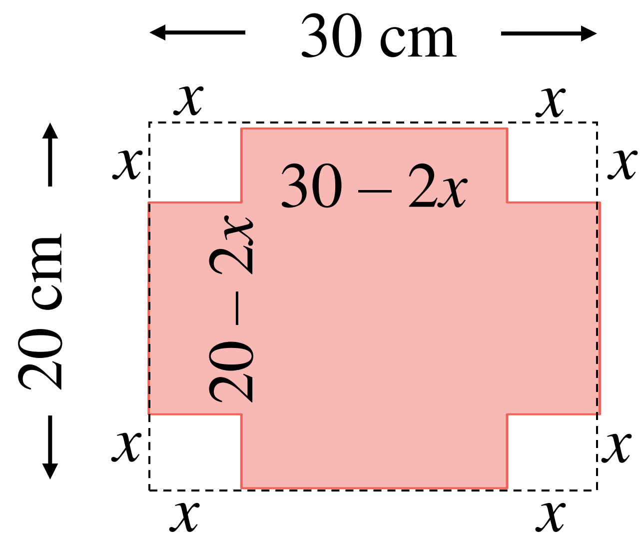
3.3 Optimization Problems

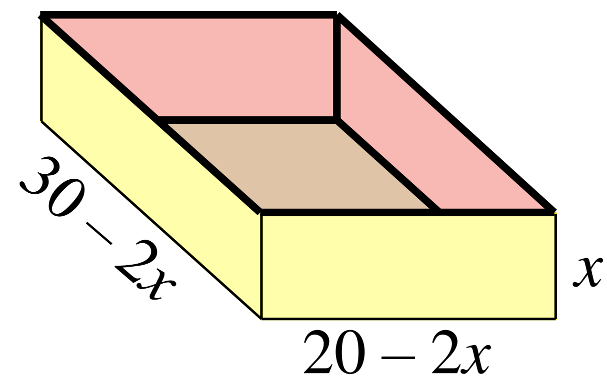
Example 1: The corners of a piece of cardboard are cut out to form a box. Determine the value of x which will produce the largest volume.





Example 1: Determine the value of x which will produce the largest volume.





$$Volume = \text{length} \times \text{width} \times \text{height}$$

$$V(x) = (30 - 2x)(20 - 2x)x \qquad 0 < x < 10$$

$$= (600 - 60x - 40x + 4x^2)x$$

$$= (600 - 100x + 4x^2)x$$

$$= 4x^3 - 100x^2 + 600x$$

$$V(x) = 4x^3 - 100x^2 + 600x$$

For extreme values, $V'(x) = 0$

$$V'(x) = 12x^2 - 200x + 600$$

$$V'(x) = 4(3x^2 - 50x + 150)$$

$$x = \frac{50 \pm \sqrt{2500 - 4(3)(150)}}{2(3)}$$

$x = 12.\ 7$ or $x = 3.9$

Since $0 < x < 10$ then the value of x that produces the maximum volume is 3.9 cm.

Example 2: Find two numbers whose difference is 15 and whose product is as small as possible.

The numbers can be represented by x and x –15.

$$P(x) = x(x - 15)$$

$$P(x) = x^2 - 15x$$

For extreme values, $P'(x) = 0$

$$P'(x) = 2x - 15$$

$$2x - 15 = 0$$

$$2x = 15$$

$$x = 7.5$$

Example 3: A cylindrical can holds 900 mL of tomatoes. Determine the measurements of the radius and the height that minimize the surface area of the can.

$$V = \pi r^2 h$$

$$900 = \pi r^2 h$$

$$\frac{900}{\pi r^2} = h$$

$$SA = 2\pi rh + 2\pi r^2$$

$$SA = 2\pi r \frac{900}{\pi r^2} + 2\pi r^2$$

$$SA(r) = \frac{1800}{r} + 2\pi r^2$$

$$SA(r) = 1800r^{-1} + 2\pi r^2$$

$$SA(r) = 1800r^{-1} + 2\pi r^2$$

$$SA'(r) = -1800r^{-2} + 4\pi r$$

$$-1800r^{-2} + 4\pi r = 0$$

$$4\pi r = \frac{1800}{r^2}$$

$$4\pi r^3 = 1800$$

$$r^3 = \frac{1800}{4\pi}$$

$$r = \sqrt[3]{\frac{1800}{4\pi}}$$

$$r = 5.23$$

$$h = \frac{900}{\pi r^2}$$

$$h = 10.46$$

Example 4: The student council sells dance tickets for \$6 and usually 250 students attend. They know that for every \$1 increase in price, 25 fewer students will attend. What price increase will maximize their revenue?

revenue = ticket price × number of students

current revenue = \$6 × 250

let x = number of \$1 price increases

	ticket price \$	# students
now	6	250
proposed	$6 + x$	$250 - 25x$

$$R(x) = (6 + x)(250 - 25x)$$

$$R(x) = (6 + x)(250 - 25x)$$

For extreme values, $R'(x) = 0$

$$R(x) = (6 + x)(250 - 25x)$$

$$= 1500 - 150x + 250x - 25x^2$$

$$= -25x^2 + 100x + 1500$$

$$R'(x) = -50x + 100$$

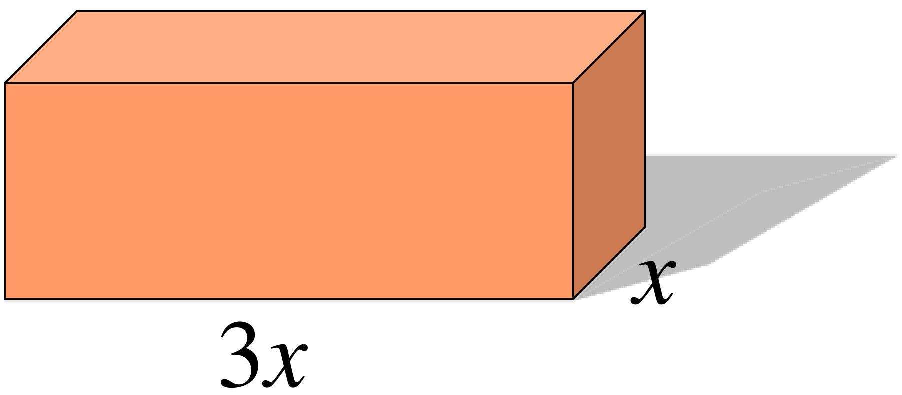
$$0 = -50x + 100$$

$$50x = 100$$

$$x = 2$$

A price increase of $2 will maximize revenue.

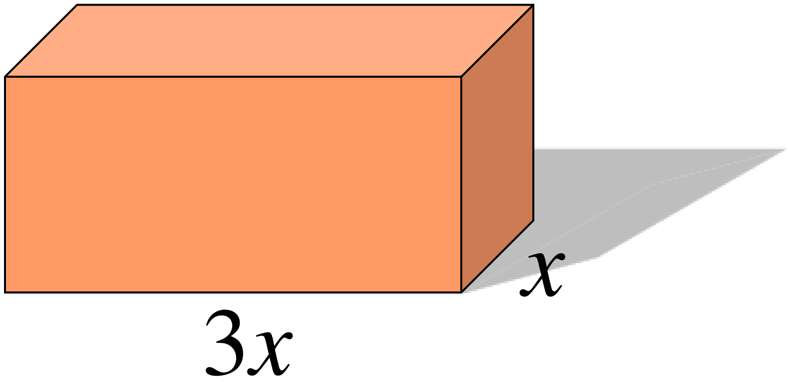
Example 5: Nicole is building a linen chest that is three times as long as it is wide. She will use oak for the front, sides and top and plywood for the bottom and back. The volume of the chest must be 0.4 m^3. Find the dimensions that minimize the cost of the wood. Oak is four times as expensive as plywood.



$$\text{volume} = 0.4\ \text{m}^3$$

$$\text{width} = x \quad \text{length} = 3x$$

$$\text{height} = \frac{0.4}{3x^2}$$



Total area of *oak* = area of front + 2 sides + top

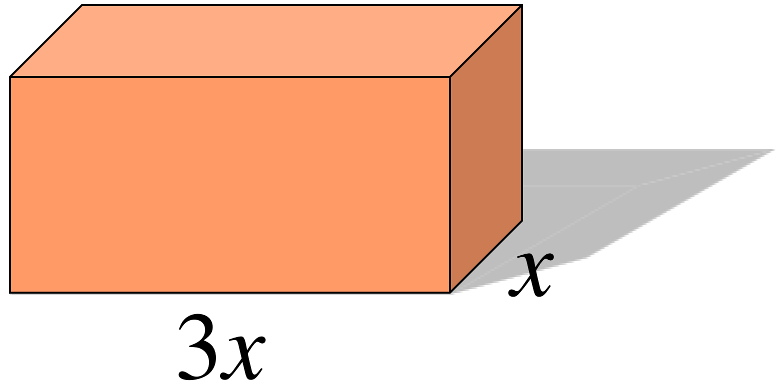
$$= 3x\left(\frac{0.4}{3x^2}\right) + 2x\left(\frac{0.4}{3x^2}\right) + 3x^2$$

$$= \frac{2}{3x} + 3x^2$$

$$\text{volume} = 0.4\ \text{m}^3$$

$$\text{width} = x \quad \text{length} = 3x$$

$$\text{height} = \frac{0.4}{3x^2}$$



$$\text{Total area of } \textit{plywood} = \text{area of back + bottom}$$

$$= 3x\left(\frac{0.4}{3x^2}\right) + 3x^2$$

$$= \frac{0.4}{x} + 3x^2$$

Total cost $= C(x)$

$$C(x) = 4c\left(\frac{2}{3x} + 3x^2\right) + c\left(\frac{0.4}{x} + 3x^2\right)$$

$$= \frac{9.2c}{3x} + 15cx^2$$

To find the minimum, set $C'(x) = 0$ and solve.

$$0 = -\frac{9.2c}{3x^2} + 30cx$$

$$\frac{9.2c}{3x^2} = 30cx$$

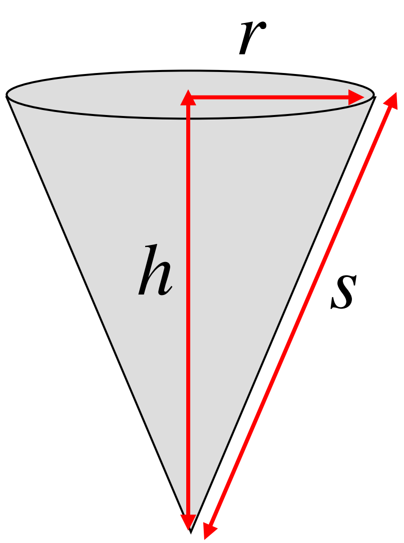
$$\frac{9.2}{90} = x^3$$

$$x \approx 0.47$$

The dimensions of the box will be 1.4 m long by 0.47 m wide by 0.61 m high.

Example 6:

A paper drinking cup in the shape of a cone has to hold 60 cm^3 of liquid. Determine radius and height of the cup that minimizes the amount of paper needed to make the cone.



$$V = \frac{1}{3}\pi r^2 h \qquad A = \pi r s \qquad s^2 = h^2 + r^2$$

$$s = \sqrt{h^2 + r^2}$$

$$V = \frac{1}{3}\pi r^2 h$$

$$60 = \frac{1}{3}\pi r^2 h$$

$$h = \frac{180}{\pi r^2}$$

$$A = \pi r s$$

$$= \pi r\sqrt{h^2 + r^2}$$

$$= \pi r\sqrt{\left(\frac{180}{\pi r^2}\right)^2 + r^2}$$

$$= \pi r\sqrt{\frac{180^2}{\pi^2 r^4} + r^2}$$

$$= \pi r\sqrt{\frac{180^2 + \pi^2 r^6}{\pi^2 r^4}}$$

$$= \frac{\pi r}{\pi r^2}\sqrt{180^2 + \pi^2 r^6}$$

$$= \frac{1}{r}\sqrt{180^2 + \pi^2 r^6}$$

$$A = \frac{1}{r}\sqrt{180^2 + \pi^2 r^6}$$

$$\frac{dA}{dr} = \frac{1}{r} \cdot \frac{6\pi^2 r^5}{2\sqrt{180^2 + \pi^2 r^6}} - \frac{\sqrt{180^2 + \pi^2 r^6}}{r^2}$$

$$= \frac{3\pi^2 r^6 - (180^2 + \pi^2 r^6)}{r^2\sqrt{180^2 + \pi^2 r^6}}$$

$$= \frac{2\pi^2 r^6 - 180^2}{r^2\sqrt{180^2 + \pi^2 r^6}}$$

The area is minimized when $\frac{dA}{dr} = 0$

$$2\pi^2 r^6 - 180^2 = 0$$

$$r^6 = \frac{180^2}{2\pi^2}$$

$$\boxed{r \approx 3.43}$$

$$\boxed{h \approx 4.86}$$

$$\boxed{r = 3.43 \text{ cm},\ h = 4.86 \text{ cm}}$$