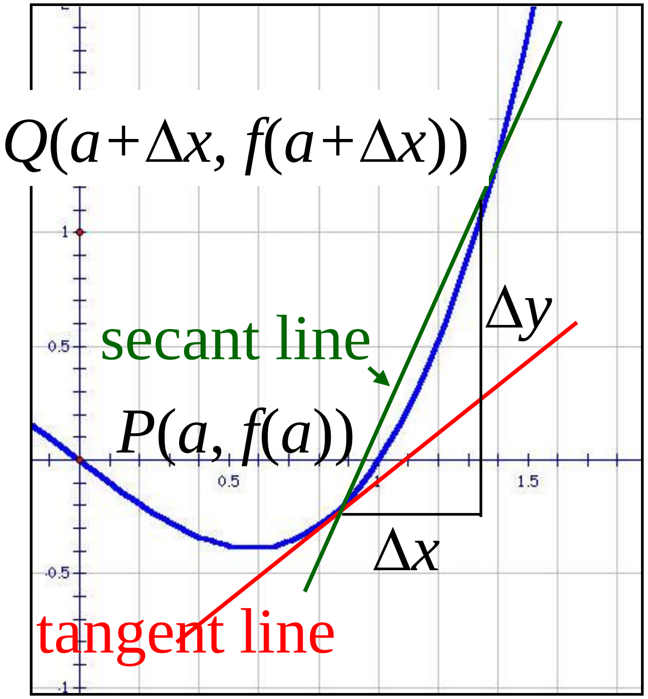
2.1 – The Derivative Function

Developing the Derivative at a Point:



slope of tangent = instantaneous rate of change

$$= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$= \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$\Delta x = x - a \qquad x = a + \Delta x$

$\Delta y = f(a+\Delta x) - f(a)$

$$\text{slope of tangent} = \lim_{\triangle x \to 0} \frac{\triangle y}{\triangle x}$$

$$= \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$= \lim_{\triangle x \to 0} \frac{f(a + \triangle x) - f(a)}{(a + \triangle x) - a}$$

$$= \lim_{\triangle x \to 0} \frac{f(a + \triangle x) - f(a)}{\triangle x}$$

let $h = \Delta x$

$$\boxed{f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}}$$

The derivative of a function f at $(a, f(a))$.

Ex.1 Using the derivative to determine the slope of a tangent. Determine the equation of the tangent to the curve $f(x) = x^2 + 2x - 3$ at point (2, 5).

$$(a, f(a)) = (2, 5)$$

$$f'(x) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h \to 0} \frac{[(2+h)^2 + 2(2+h) - 3] - 5}{h}$$

$$= \lim_{h \to 0} \frac{4 + 4h + h^2 + 4 + 2h - 3 - 5}{h}$$

$$= \lim_{h \to 0} \frac{h^2 + 6h}{h}$$

$$= \lim_{h \to 0} [h + 6]$$

$$= 6$$

The slope of the tangent line is 6.

cont …

The slope of the tangent line is 6.

$y = mx + b$

$y = 6x + b$ sub (2, 5)

$5 = 6(2) + b$

$5 = 12 + b$

$-7 = b$

$\boxed{y = 6x - 7}$

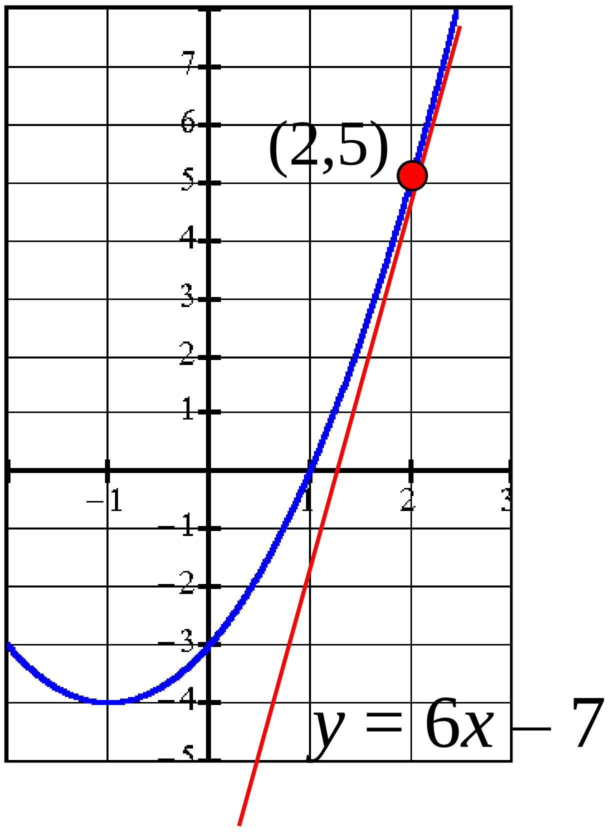
(equation of the tangent line.)

$f(x) = x^2 + 2x - 3$

(complete the square)

$f(x) = x^2 + 2x + 1 - 1 - 3$

$f(x) = (x + 1)^2 - 4$ vertex (–1,–4)



Interpretation of the Derivative $f\,'(a)$

The derivative of a function f at point (a, $f\,(a)$) can be interpreted as either:

1. The slope of the tangent line.

2. The instantaneous rate of change.

The Derivative of a Function

$$\boxed{f\,'(x) = \lim_{h \to 0} \frac{f\,(x+h) - f\,(x)}{h}}$$

for all for which the limit exists.

Differentiation: the process of finding the derivative.

Example: Determine the derivative from first principles.

$$f(x) = -3x^3 + 2x - 1$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{[-3(x+h)^3 + 2(x+h) - 1] - [-3x^3 + 2x - 1]}{h}$$

$$= \lim_{h \to 0} \frac{-3(x^3 + 3x^2h + 3xh^2 + h^3) + 2x + 2h - 1 + 3x^3 - 2x + 1}{h}$$

$$= \lim_{h \to 0} \frac{-3x^3 - 9x^2h - 9xh^2 - 3h^3 + 2h + 3x^3}{h}$$

$$= \lim_{h \to 0} \frac{-9x^2h - 9xh^2 - 3h^3 + 2h}{h}$$

$$= \lim_{h \to 0} (-9x^2 - 9xh - 3h^2 + 2)$$

$$= -9x^2 - 9x(0) - 3(0^2) + 2$$

$$\boxed{= -9x^2 + 2}$$

Differentiability

A function f is differentiable at $x = a$ if $f'(x)$ exists.

Polynomial functions are differentiable at every number in the domain.

a) Which functions have points which are not differentiable at one or more points in the domain?

b) Which properties can cause the derivative not to exist at these points?

Differentiability and Functions

A function $f(x)$ is differentiable at a if $f'(a)$ exists.

If this limit exists for all all values of a on an interval in the domain, then $f(x)$ is differentiable on this interval.

Example: Show that $f(x) = \dfrac{1}{x-1}$ is not differentiable at $x = 1$.

$$f(1) = \frac{1}{1-1}$$

$$f(1) = \frac{1}{0}$$ (undefined)

$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$\therefore$ The function is not differentiable at $x = 1$.

Differentiating a Rational Function

If $f'(x)$ exists then $\lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$ exists.

Both the one sided limits exist and are equal.

$$\lim_{h \to 0^-} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}$$

Also, the tangent line exists.

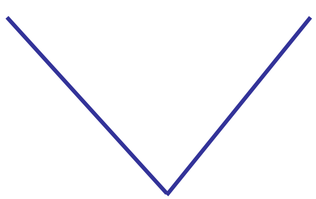
The Differentiability of a Function

A function is not differentiable at $x = a$ when

1) The graph of the function has a discontinuity at $x = a$.

2) The graph of the function has a corner or cusp.

3) The line $x = a$ is a vertical tangent.



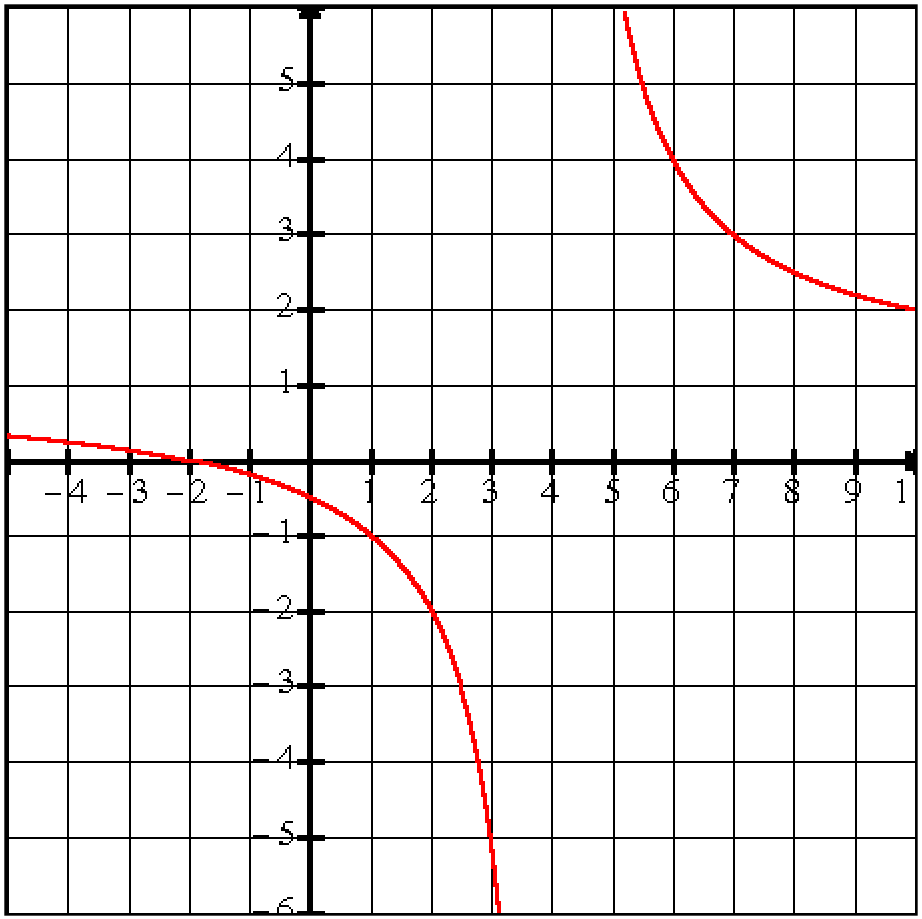
corner

cusp

$$f(x) = \frac{x+2}{x-4}$$

The function is not defined for $x = 4$.

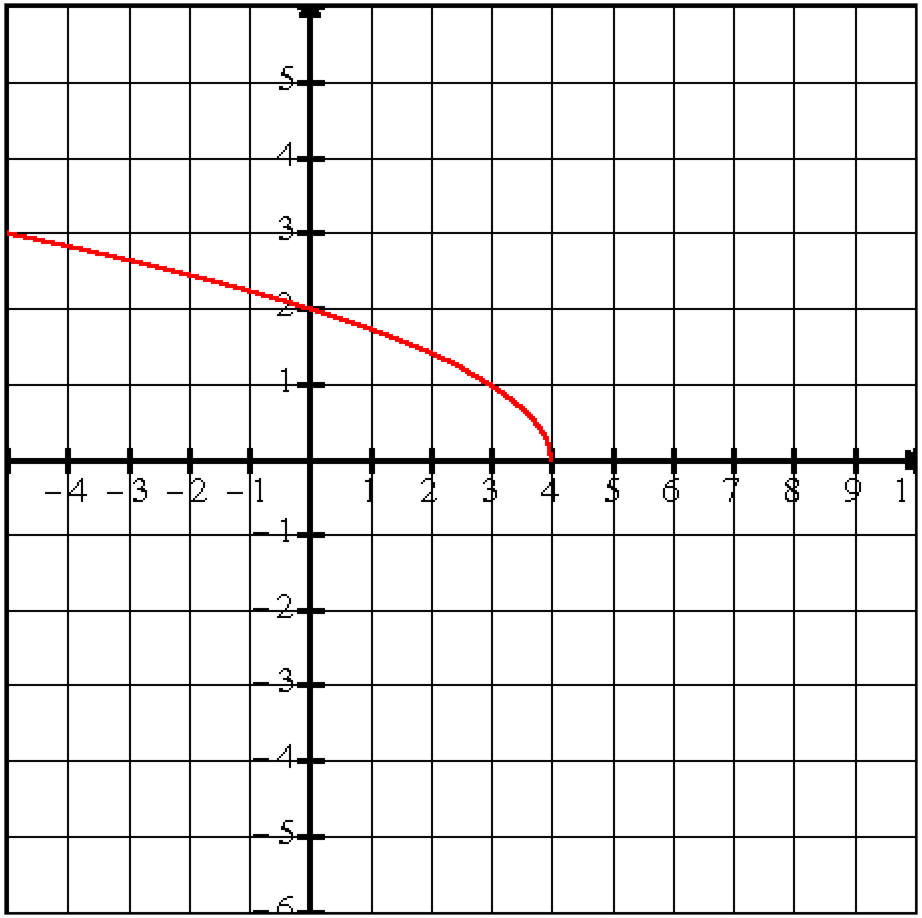
You cannot draw a tangent line at $x = 4$.



$$f(x)=\sqrt{4-x}$$

The function is not defined for $x > 4$.

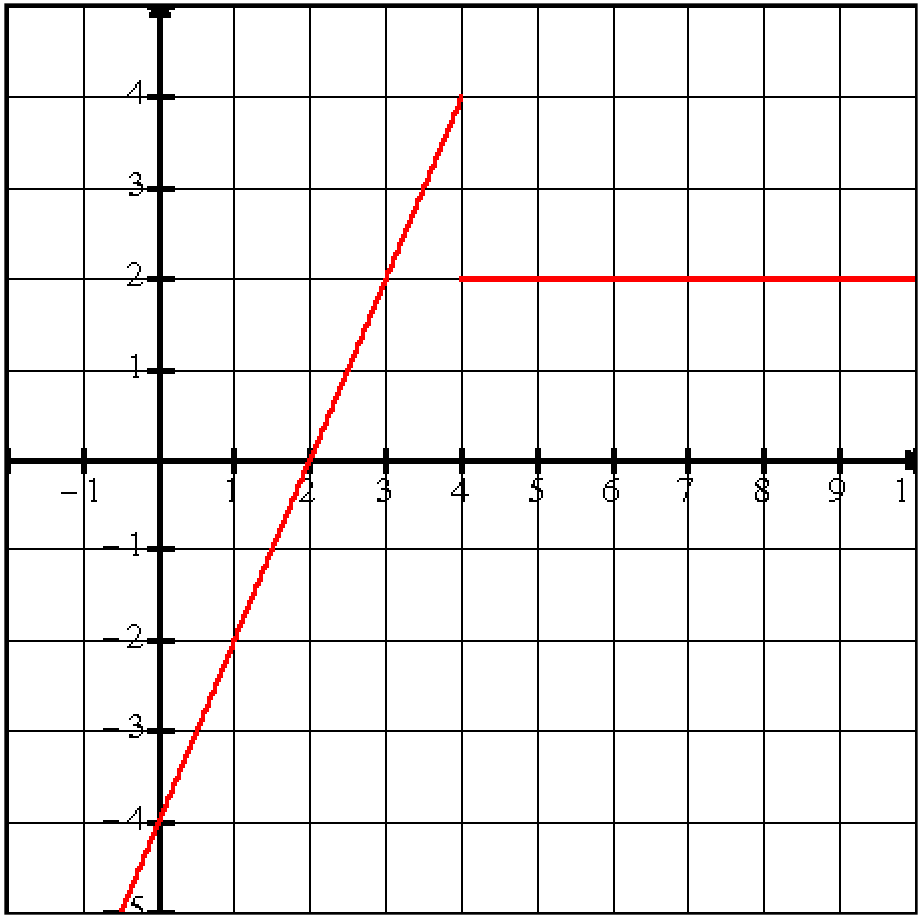
The right hand limit as $x \to 4^{+}$ does not exits.



$$\lim_{h\to 0^{+}} \frac{f(4+h)-f(4)}{h} \text{ does not exist}$$

$$\lim_{h\to 0^{-}} \frac{f(4+h)-f(4)}{h} = -\infty$$

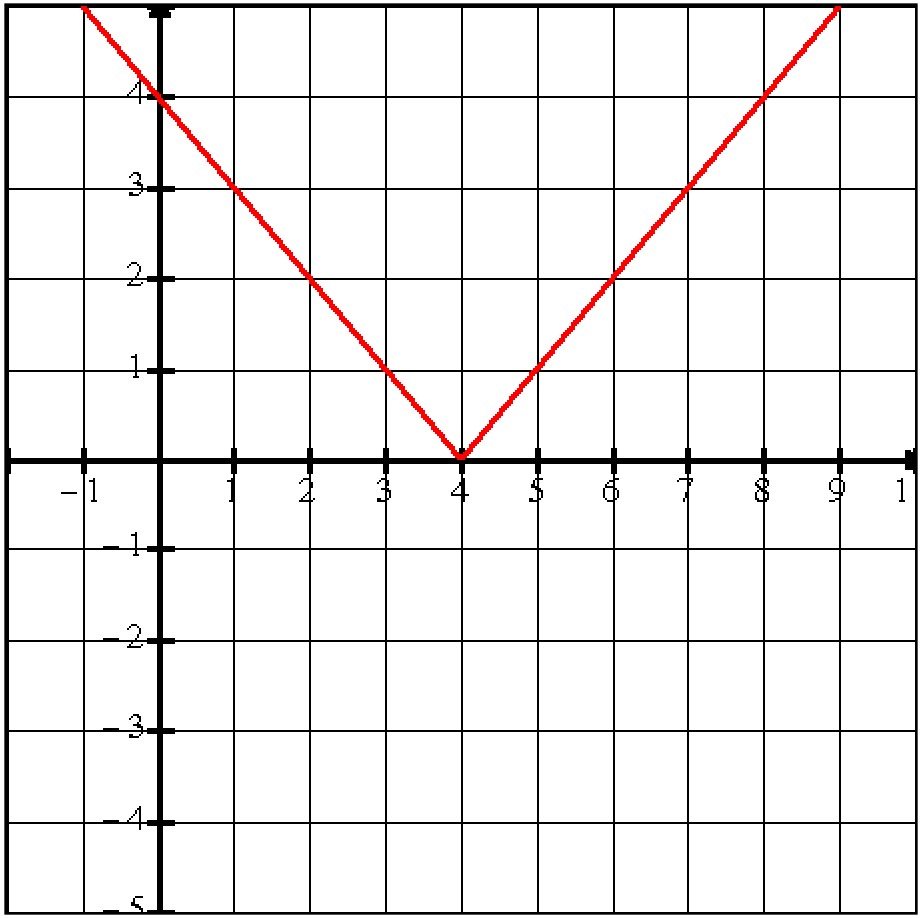
$$f(x) = \begin{cases} 4x - 2, & x \le 4 \\ 2, & x > 4 \end{cases}$$



The one-sided limits as $x \to 4$ are not the same.

$$f(x) = |x - 4|$$

x	$f(x)$
3	1
4	0
5	1

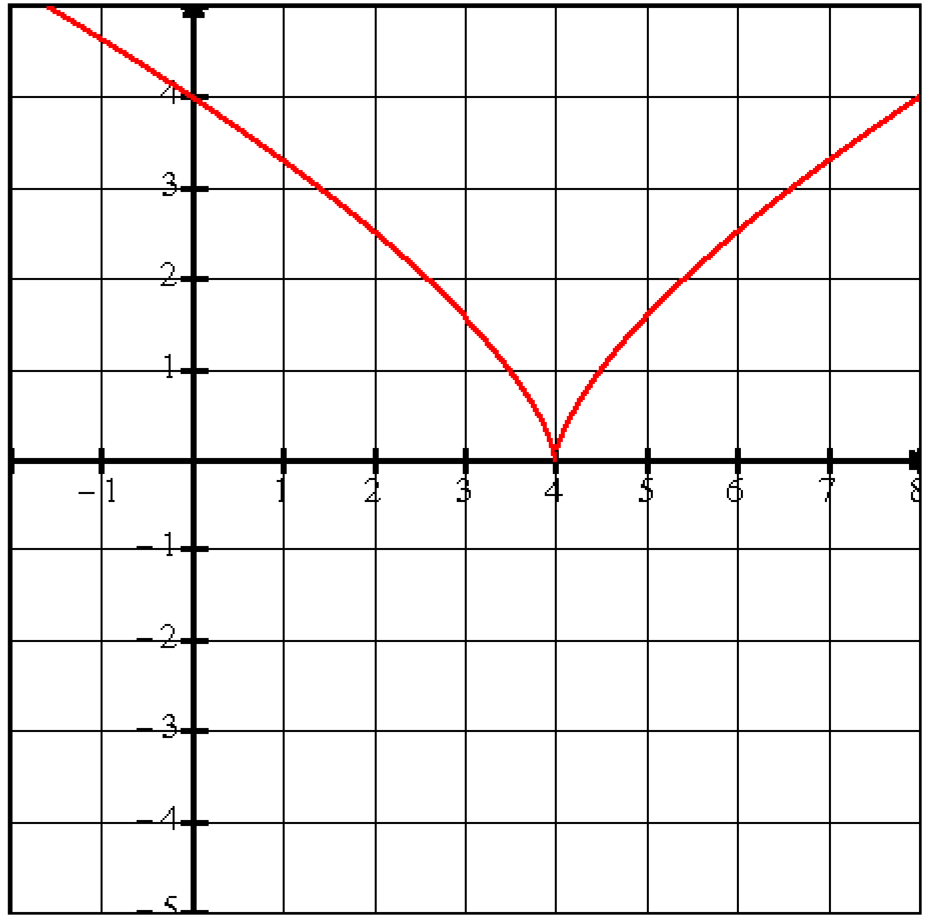


$$\lim_{h \to 0^+} \frac{f(4+h) - f(4)}{h} = 1$$

$$\lim_{h \to 0^-} \frac{f(4+h) - f(4)}{h} = -1$$

Since the limits are not the same, $f'(4)$ does not exits.

$$f(x) = (2x-8)^{\frac{2}{3}}$$



$$\lim_{h \to 0^+} \frac{f(4+h) - f(4)}{h} = \infty$$

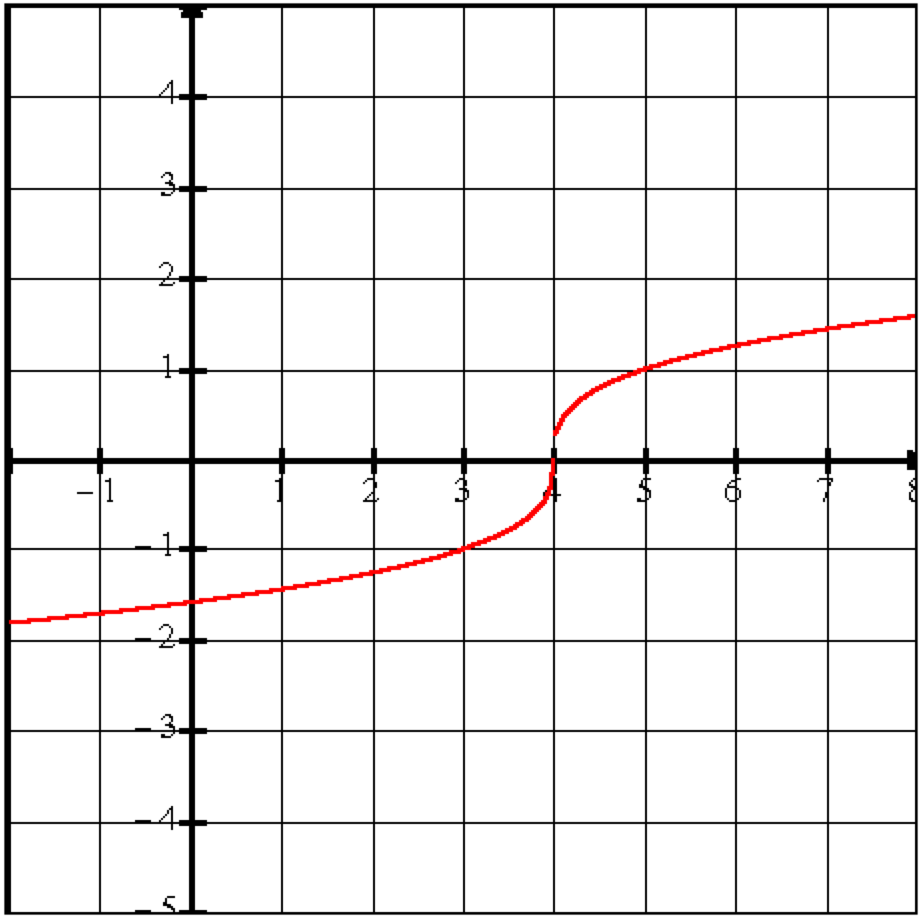
$$\lim_{h \to 0^-} \frac{f(4+h) - f(4)}{h} = -\infty$$

There is a cusp at $x = 4$

$$f(x) = \sqrt[3]{x-4}$$

The tangent line at $x = 4$ is a vertical line, so the slope is undefined.

The function is not differentiable at $x = 4$.



Using the derivative to determine an instantaneous rate of change.

Example: The height of a golf ball is given by the equation: $f(t) = -5t^2 + 12t + 2$
where $f(t)$ is in metres and t is in seconds.

Determine the instantaneous rate of change for 1 s, and 2 s.

Sol: The derivative function models the instantaneous rate of change

Find the derivative function from first principles.

$$h(t) = -5t^2 + 12t + 2$$

$$f'(t) = \lim_{h\to 0} \frac{f(t+h) - f(t)}{h}$$

$$= \lim_{h\to 0} \frac{[-5(t+h)^2 + 12(t+h) + 2] - [-5t^2 + 12t + 2]}{h}$$

$$= \lim_{h\to 0} \frac{-5(t^2 + 2th + h^2) + 12t + 12h + 2 + 5t^2 - 12t - 2}{h}$$

$$= \lim_{h\to 0} \frac{-5t^2 - 10th - 5h^2 + 12h + 5t^2}{h}$$

$$= \lim_{h\to 0} \frac{-10th - 5h^2 + 12h}{h}$$

$$= \lim_{h\to 0} (-10t - 5h + 12)$$

$$= -10t - 5(0) + 12$$

$$\boxed{f'(t) = -10t + 12}$$

$$f'(1) = -10(1) + 12$$

$$= 2 \text{ m/s}$$

$$f'(2) = -10(2) + 12$$

$$= -8 \text{ m/s}$$