Chapter 5
Derivatives of Exponential and Trigonometric Function

5.1 Introducing a Special Number, *e*

Leonard Euler (1707-1783) first introduced the symbol π and *i* to represent imaginary numbers.

He also introduced the number *e*.

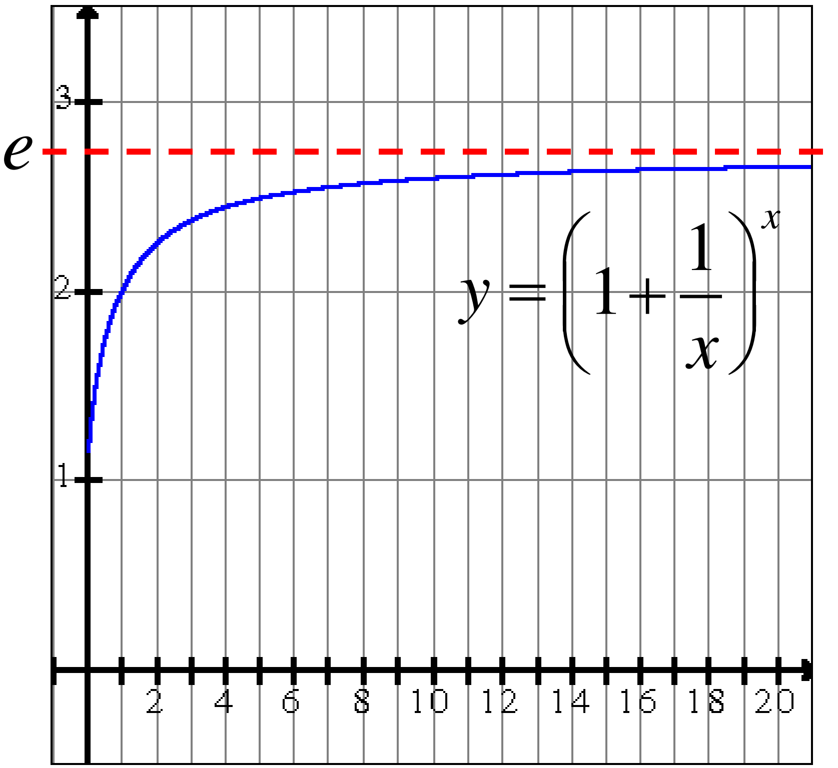
$e = 2.718\ 281\ 828 \ldots$

$$e = \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$$

Fundamental Limit of Calculus

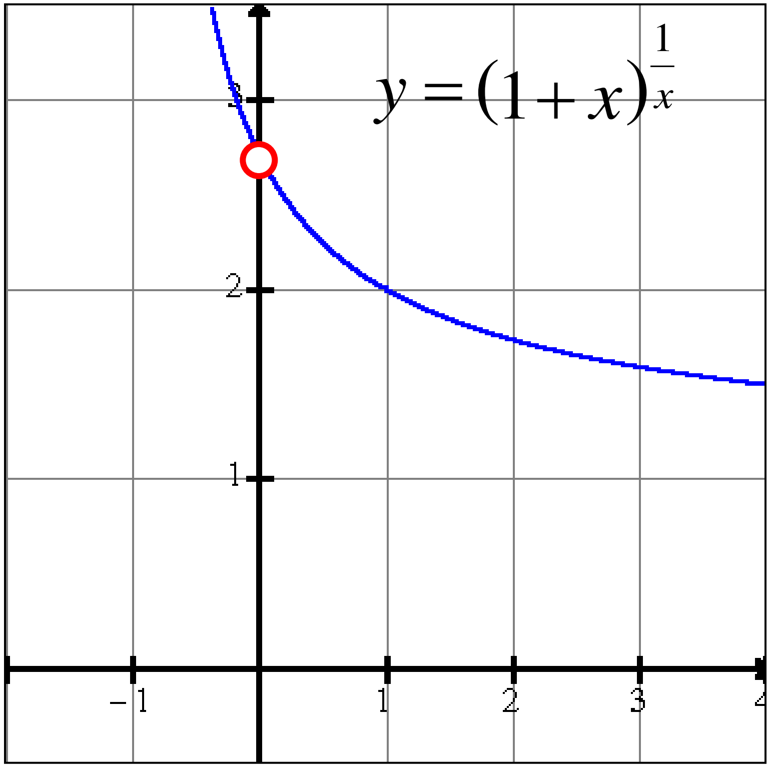
From the table we see that as x gets larger, the value of $(1+1/x)^x$ approaches the number e.

x	$y=\left(1+\frac{1}{x}\right)^x$
1	2
2	2.25
3	2.37037037
100	2.704813829
1000	2.716923932



A second limit for defining e is: $e = \lim_{x \to 0} (1+x)^{\frac{1}{x}}$

x	$y = (1+x)^{\frac{1}{x}}$
-0.1	2.867971991
-0.01	2.731999026
-0.001	2.719642216
0	undefined
0.001	2.716923932
0.01	2.704813829
0.1	2.59374246



e often appears as a base for exponential growth and decay.

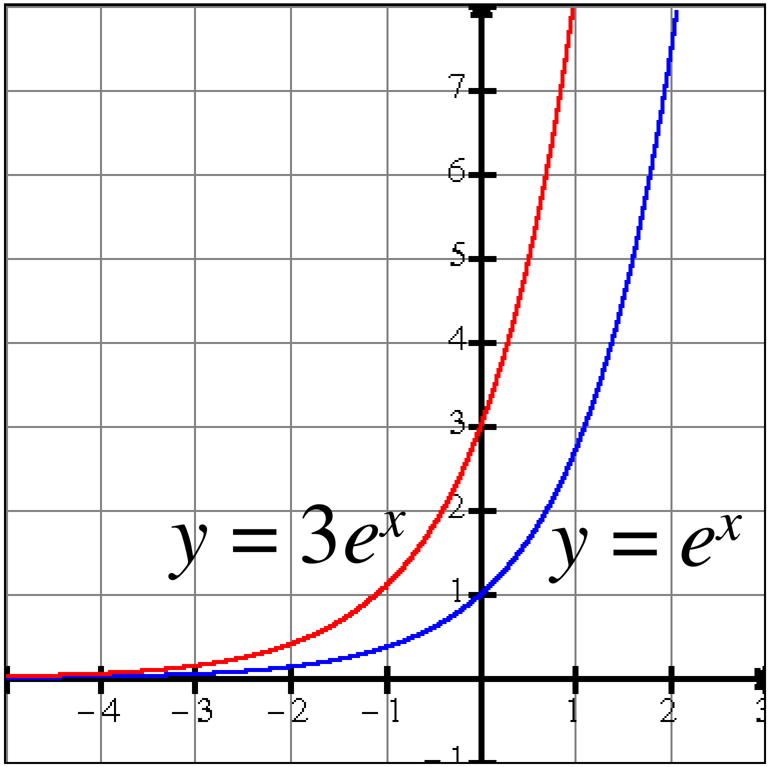
The graph of $y = e^x$

(a) $y = e^x$

The horizontal asymptote is $y = 0$ and the y-intercept is 1.

(b) $y = 3e^x$

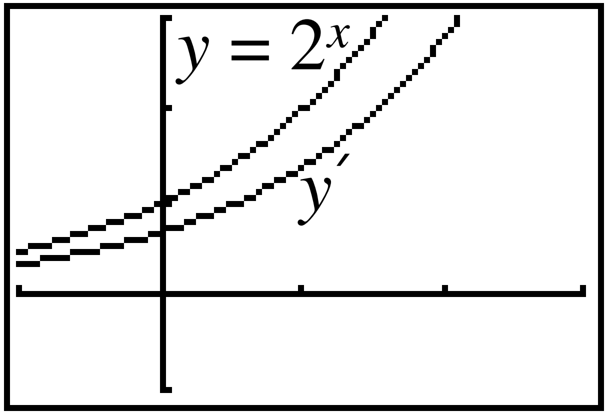
The horizontal asymptote is $y = 0$ and the y-intercept is 3.

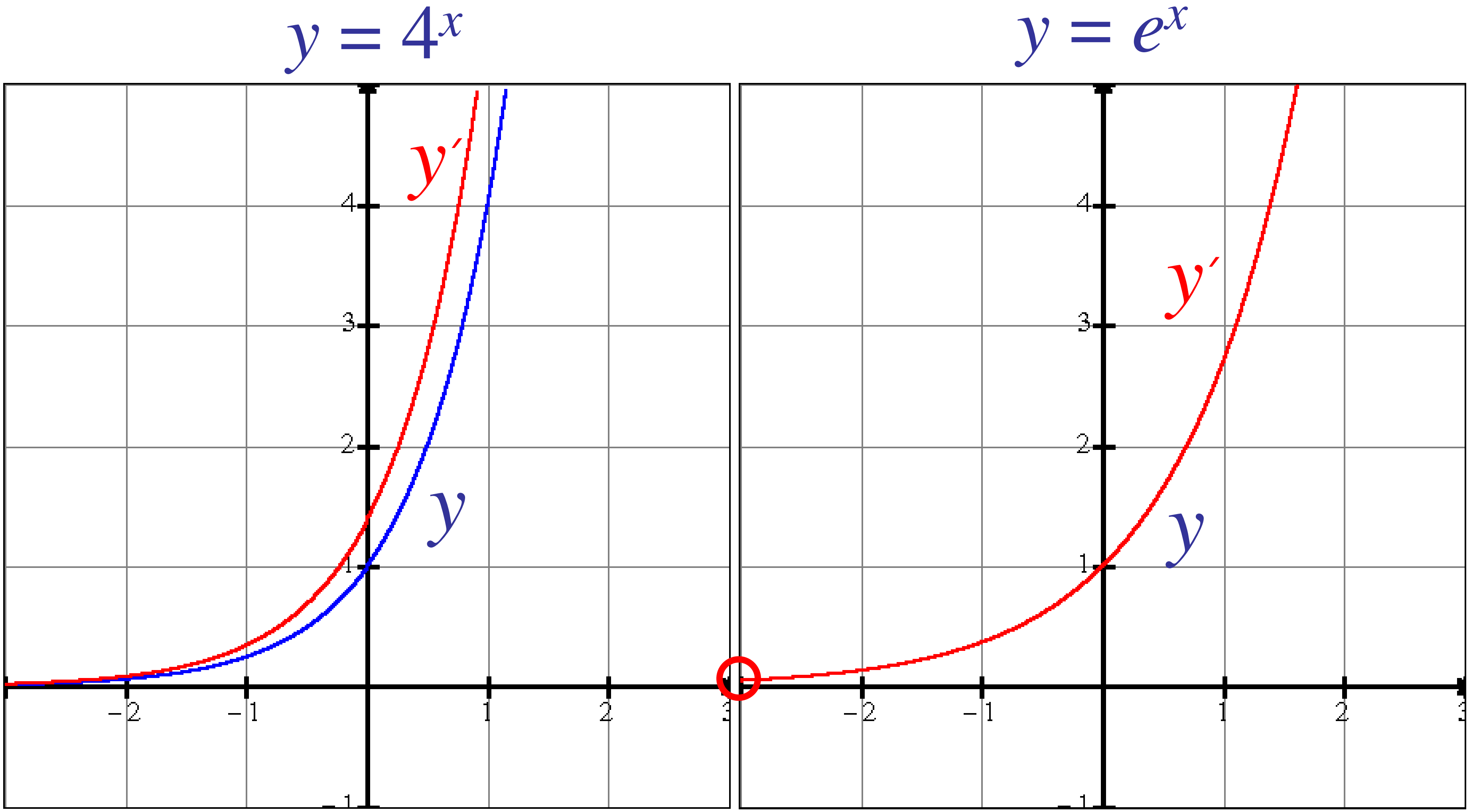


The Derivative of $y = e^x$

Investigating exponential functions and their derivatives.

The derivative also appears to be an exponential.





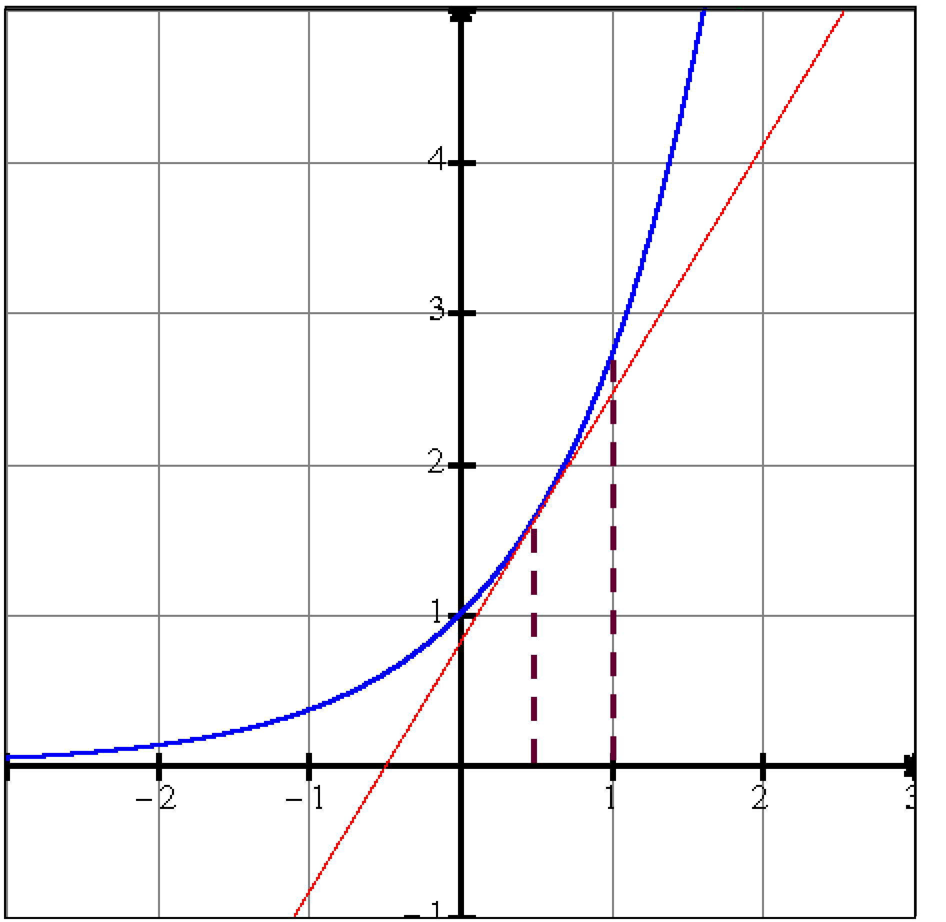
$$\boxed{\frac{d(e^x)}{dx} = e^x}$$

The slope of the tangent at any point is the value of the function at that point.

The slope of the tangent at point $(1, e^1)$ is e^1.

The slope of the tangent at point $(0.5, e^{0.5})$ is $e^{0.5}$.

$y = e^x$



$m = e^{0.5}$ $m = e^1$

Example: Find the derivative of other functions involving e.

a) $y = x^3 e^x$

Use the product rule

$$\frac{dy}{dx} = x^3 \frac{d(e^x)}{dx} + e^x \frac{d(x^3)}{dx}$$

$$= x^3 e^x + 3x^2 e^x$$

$$= x^2 e^x (x + 3)$$

b) $y = e^{4x}$

Use the chain rule

Let $u = 4x$

$$y = e^u$$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

$$= e^u \frac{d(4x)}{dx}$$

$$= 4e^{4x}$$

Finding the Equation of a Tangent to an Exponential Function

Determine the equation of the tangent to the graph of $y = e^{-3x}$ at the point where $x = 0.5$

First find: $\dfrac{dy}{dx}$

$$\frac{dy}{dx} = e^{-3x}\frac{d(-3x)}{dx}$$

$$= -3e^{-3x}$$

Where $x = 0.5$

$$\frac{dy}{dx} = -3e^{-3(0.5)}$$

$$= -3e^{-1.5} \quad \text{(slope)}$$

We also need to know the value of y when $x = 0.5$

➔

Determine $y = e^{-3x}$ at the point where $x = 0.5$

$$y = e^{-3(0.5)}$$

$$= e^{-1.5}$$

We can now find the equation of the tangent at point $(0.5, e^{-1.5})$

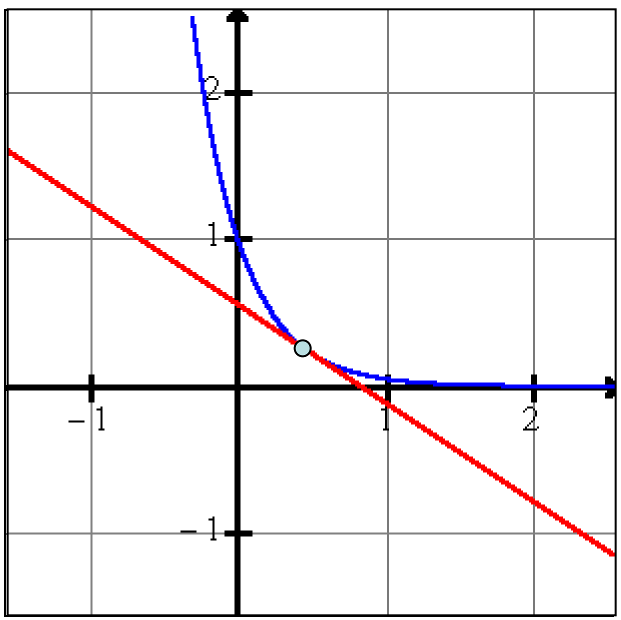
Equation of a line is $y - y_1 = m(x - x_1)$

$$y - e^{-1.5} = -3e^{-1.5}(x - 0.5)$$

$$y = -3e^{-1.5}x + 1.5e^{-1.5} + e^{-1.5}$$

$$= -3e^{-1.5}x + 2.5e^{-1.5}$$

$$= -0.669x + 0.558$$



Graphing Exponential Functions

Graph the function $y = xe^x$

Since $y = x$ and $y = e^x$ are both defined for all values of x, the domain of $y = xe^x$ is $x \in [-\infty, \infty]$.

as $x \to \infty, \quad y \to \infty$

as $x \to -\infty, \quad y \to 0$

y-intercept $(x = 0)$ is 0.

x-intercepts $(y = 0)$

$$0 = xe^x$$

$$x = 0, \ (e^x > 0)$$

X	Y1	
0	0	
-25	-3E-10	
-50	-1E-20	
-75	-2E-31	
-100	-4E-42	
-125	-6E-53	
-150	-1E-63	

X=0

Next determine any local extrema.

$$\frac{dy}{dx} = \frac{d(xe^x)}{dx}$$

$$= x\frac{d(e^x)}{dx} + e^x \frac{d(x)}{dx}$$

$$= xe^x + e^x$$

To find critical numbers,

let $\dfrac{dy}{dx} = 0$

$$xe^x + e^x = 0$$

$$x + 1 = 0$$

$$x = -1$$

Find the second derivative to determine if a max. or min. occurs at the critical number.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(xe^x + e^x)$$

$$= \frac{d}{dx}(xe^x) + \frac{d}{dx}(e^x)$$

$$= xe^x + e^x + e^x$$

$$= xe^x + 2e^x$$

Substitute $x = -1$

Substitute $x = -1$

$$\frac{d^2y}{dx^2} = xe^x + 2e^x)$$

$$\frac{d^2y}{dx^2} = -1e^{-1} + 2e^{-1}$$

$$\frac{d^2y}{dx^2} > 0$$

$\therefore$ a minimum occurs at $x = -1$

Points of Inflection

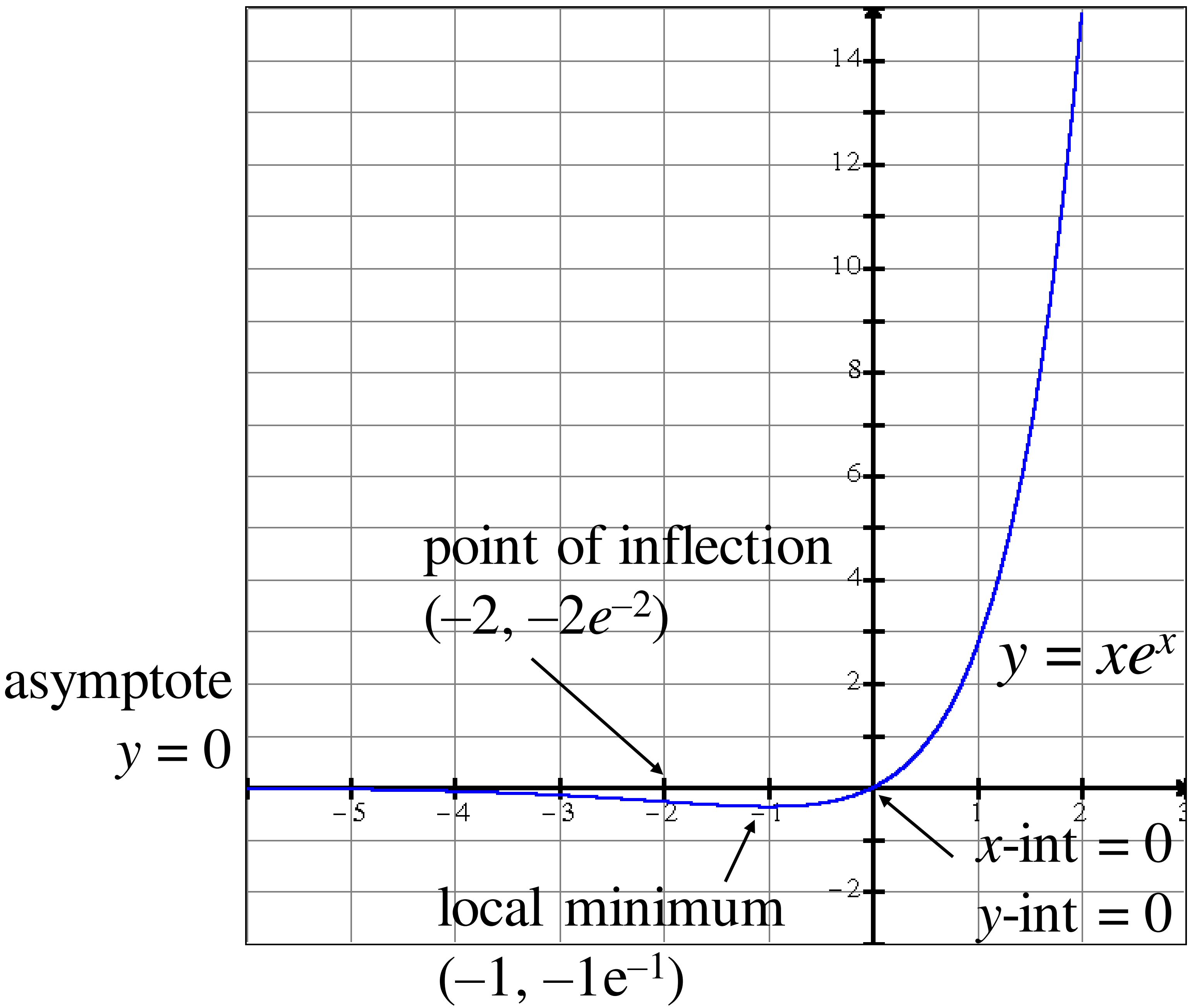
$$\frac{d^2y}{dx^2} = 0$$

$$xe^x + 2e^x = 0$$

$$e^x(x + 2) = 0$$

$$x = -2$$

The y-value at the point of inflection is $-2e^{-2}$, or -0.271.



point of inflection
$(-2, -2e^{-2})$
asymptote
$y = 0$
$y = xe^x$
local minimum
$(-1, -1e^{-1})$
x-int = 0
y-int = 0
14
12
10
8
6
4
2
−2
−5
−4
−3
−2
−1
1
2
3

Determine the domain and range of:

$$y = e^x$$

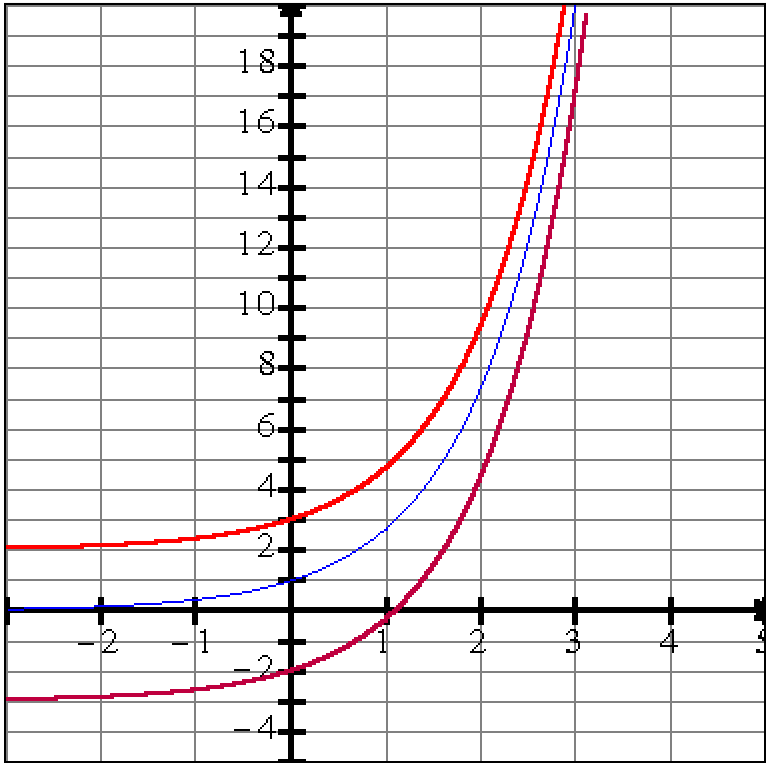
Domain: $x \in \mathrm{R}$

Range: $y > 0$

$$y = e^x + 2$$

Domain: $x \in \mathrm{R}$

Range: $y > 2$



$$y = e^x - 3$$

Domain: $x \in \mathrm{R}$ Range: $y > -3$