1.5 Properties of Limits

Finding the Limits of Rational Functions by *Substitution*

Let $h(x) = \dfrac{f(x)}{g(x)}$ be a rational function

Let a be a real number in the domain of h.

The limit of $h(x)$ as x approaches a is:

$$\lim_{x \to a} h(x) = \frac{f(a)}{g(a)}, \text{ if } g(a) \neq 0.$$

Example 1a: Determine the limit of the function.

$$\lim_{x \to 3} \frac{x^2 + x - 1}{x - 2}$$

attempt to evaluate at $x = 3$.

$$= \frac{3^2 + 3 - 1}{3 - 2}$$

$$= 11$$

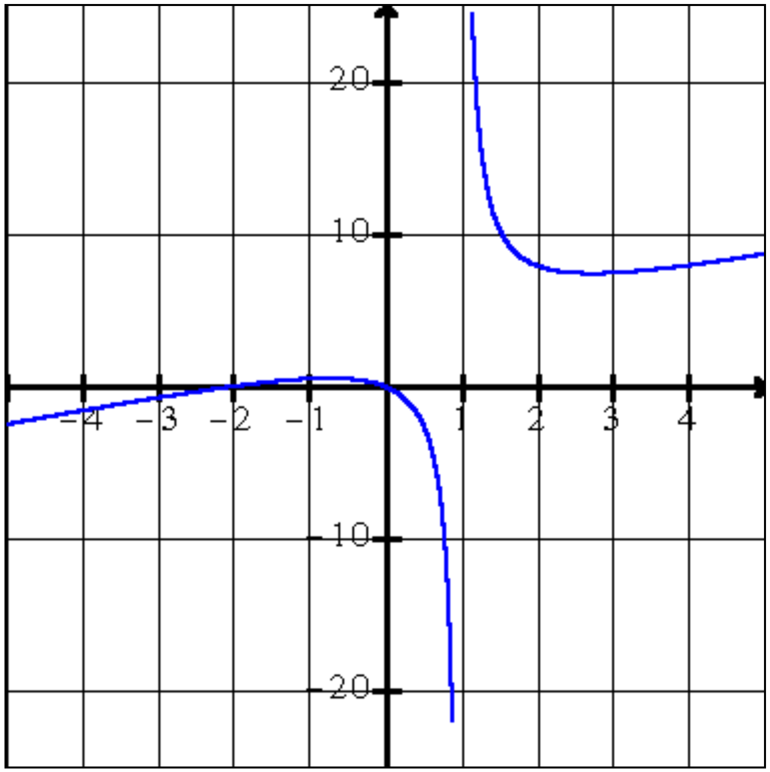
$\therefore f(x)$ does exist at $x = 3$ and $\lim_{x \to 3} \dfrac{x^2 + x - 1}{x - 2} = 11$

Example 1b: Determine the limit of the function.

$$\lim_{x \to 1} \frac{x^2 + 2x}{x - 1}$$

attempt to evaluate at $x = 1$.

$$\frac{1^2 + 2(1)}{1 - 1} = \frac{3}{0}$$



$f(x)$ is undefined at $x = 1$

$$\lim_{x \to 1^-} f(x) = -\infty$$

$$\lim_{x \to 1^+} f(x) = \infty$$

There is no limit as x approaches 1 from either side.

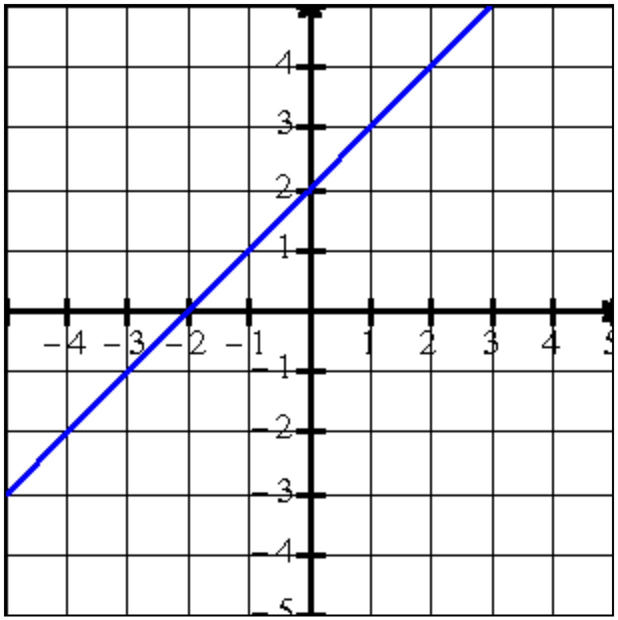
The *Indeterminate* Form of a Limit

Example 2: find $\lim_{x \to 2} \dfrac{x^2 - 4}{x - 2}$

Substitute $x = 2$ $\qquad \dfrac{2^2 - 4}{2 - 2} = \dfrac{0}{0}$

$\dfrac{0}{0}$ is called the *indeterminate* form.

1.998	3.998
1.999	3.999
2	undefined
2.001	4.001
2.002	4.002



$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$$

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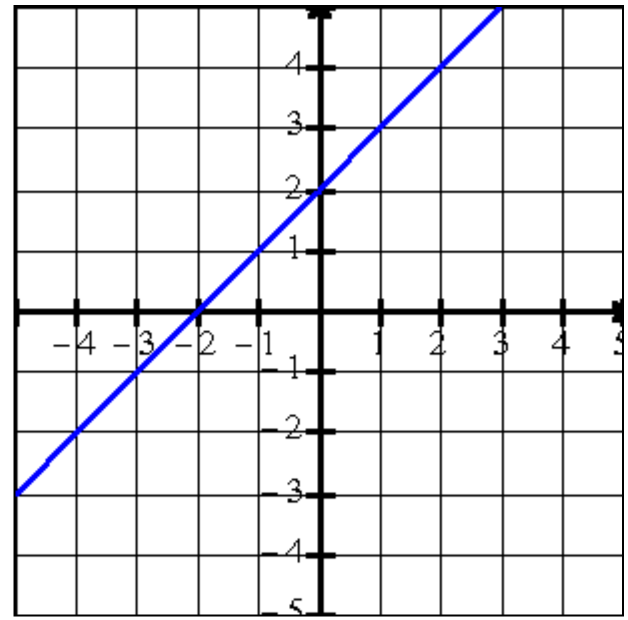


It appears that the limit as x approaches 2 is 4

Factor the original expression.

$$\lim_{x \to 2} \frac{(x+2)(x-2)}{(x-2)} = \lim_{x \to 2}(x+2)$$

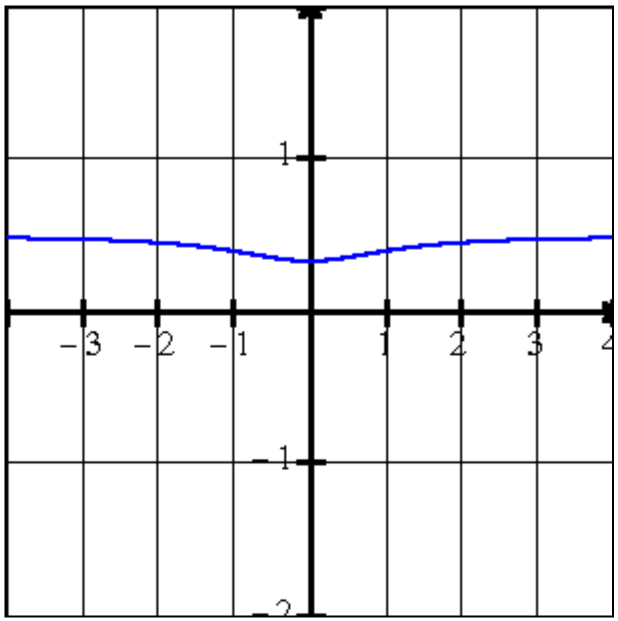
$$= 4$$

Limits at Infinity

The graph of a function will have a horizontal asymptote if the function has a finite limit L as as $x \to \pm\infty$.

Example 3: Find the equation of the horizontal asymptote.

$$\lim_{x \to \infty} \frac{x^2 + 1}{2x^2 + 3}$$



Both the numerator and denominator become large as $x \to \infty$.

$$\lim_{x\to\infty}\frac{x^2+1}{2x^2+3}$$

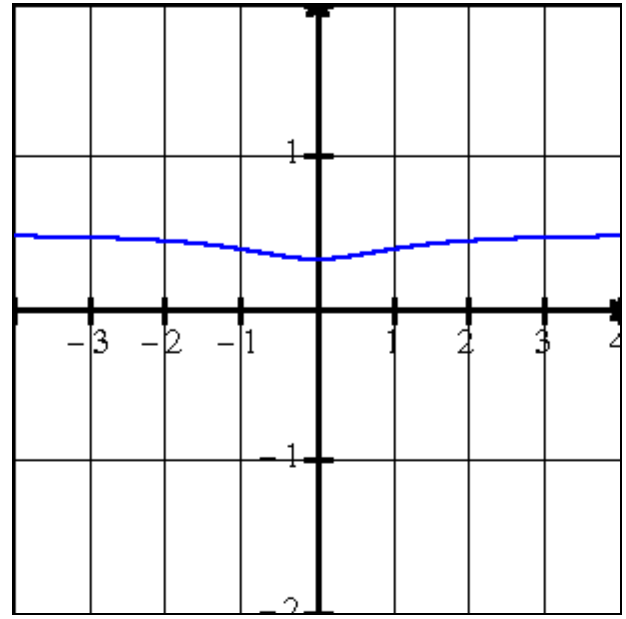
both the numerator and denominator become large as $x \to \infty$.

$\dfrac{\infty}{\infty}$ is also the *indeterminate* form.

10	0.4975369
100	0.499975
1000	0.4999998

The table indicates that the limit is 0.5 as $x \to \infty$

$$\lim_{x\to\infty}\frac{x^2+1}{2x^2+3}$$



Algebraic Method: Divide the numerator and denominator by the highest power of x in the denominator (x^2 in this case).

$$\lim_{x\to\infty}\frac{\dfrac{x^2}{x^2}+\dfrac{1}{x^2}}{\dfrac{2x^2}{x^2}+\dfrac{3}{x^2}} = \lim_{x\to\infty}\frac{1+\dfrac{1}{x^2}}{2+\dfrac{3}{x^2}}$$

terms $\to 0$ as $x \to \infty$.

$$\boxed{=\frac{1}{2}}$$

Quotient Law for Limits

If a, L and M are real numbers and

$\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$ then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$$

$$= \frac{L}{M}, M \neq 0$$

4- Evaluate: $\lim\limits_{x\to\infty} \dfrac{1-2x^2}{(4x+3)^2}$ expand denominator

$$\lim_{x\to\infty} \frac{1-2x^2}{16x^2+24x+9}$$

divide by x^2

$$\lim_{x\to\infty} \frac{\dfrac{1}{x^2}-\dfrac{2x^2}{x^2}}{\dfrac{16x^2}{x^2}+\dfrac{24x}{x^2}+\dfrac{9}{x^2}}$$

$$\lim_{x\to\infty} \frac{\dfrac{1}{x^2}-2}{16+\dfrac{24}{x}+\dfrac{9}{x^2}}$$

$$= \frac{0-2}{16+0+0}$$

$$= -\frac{1}{8}$$