

2.5 The Derivatives of Composite Functions

“The Chain Rule”

Given: $y = (4 - 3x)^3$ let $u = 4 - 3x$ then $y = u^3$

y is a function of u and u is a function of x is a *composite function*.

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}} \text{ provided } \frac{dy}{du} \text{ and } \frac{du}{dx} \text{ exist.}$$

Therefore if $h(x) = (f \circ g)(x)$, then $h'(x) = f'(g(x)) \cdot g'(x)$

$$\frac{d[h(x)]}{dx} = \frac{d[f(g(x))]}{d[g(x)]} \cdot \frac{d[g(x)]}{dx}$$

Example 1: Determine $\frac{dy}{dx}$ if $y = (4x + 6)^3$

$$\frac{dy}{dx} = \frac{d[(4x + 6)^3]}{dx}$$

$$\frac{dy}{dx} = \frac{d[(4x + 6)^3]}{d[(4x + 6)]} \cdot \frac{d[(4x + 6)]}{dx}$$

$$= 3(4x + 6)^2(4)$$

$$= 12(4x + 6)^2$$

If we had expanded $y = (4x + 6)^3$ and found the derivative of the product, the result would have been the same.

The Chain Rule with the Power Rule

$$\frac{d[g(x)^n]}{dx} = \frac{d[g(x)^n]}{d[g(x)]} \cdot \frac{d[g(x)]}{dx}$$

$$= n[g(x)]^{n-1} \cdot g'(x), \text{ where } n \text{ is a constant}$$

Take the derivative of the '*outer*' function multiplied by the derivative of the '*inner*' function.

Example 2: Determine $f'(s)$ if $f(s) = (2s^3 - 5)^4$

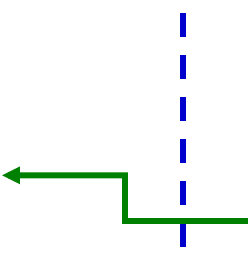
$$f'(s) = \underbrace{4(2s^3 - 5)^3}_{\text{derivative of outer}} \underbrace{(6s^2)}_{\text{derivative of inner}}$$

derivative of outer · derivative of inner

$$f'(s) = 24s^2(2s^3 - 5)^3$$

Example 3: Determine $\frac{dy}{dx}$ if $y = \sqrt{(x+3)^2 + 1}$

$$\frac{dy}{dx} = \frac{d[(x^2 + 6x + 10)^{\frac{1}{2}}]}{dx}$$



$$y = \sqrt{x^2 + 6x + 10}$$
$$y = (x^2 + 6x + 10)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{d[(x^2 + 6x + 10)^{\frac{1}{2}}]}{d[(x^2 + 6x + 10)]} \cdot \frac{d[(x^2 + 6x + 10)]}{dx}$$

$$= \frac{1}{2} (x^2 + 6x + 10)^{-\frac{1}{2}} (2x + 6)$$

$$= \frac{x + 3}{\sqrt{x^2 + 6x + 10}}$$

Differentiating a quotient with constant numerator.

Example 4: Determine $f'(t)$ if $f(t) = \frac{5}{(4t+3)^3}$

$$f(t) = 5(4t+3)^{-3}$$

$$f'(t) = \underbrace{(-3)(5)(4t+3)^{-3-1}}_{\substack{\uparrow \\ \text{derivative of outer}}} \underbrace{(4)}_{\substack{\uparrow \\ \text{derivative of inner}}}$$

$$f'(t) = (-60)(4t+3)^{-4}$$

$$= \frac{-60}{(4t+3)^4}$$

