4.3 Vertical and Horizontal Asymptotes

Definition of a Rational Function

A rational function has the form: $h(x) = \dfrac{f(x)}{g(x)}$

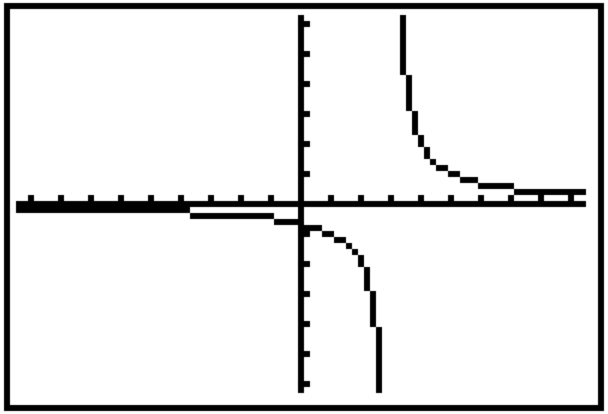
$f(x)$ and $g(x)$ are polynomials.

Domain: all real numbers except values of x where $g(x) = 0$.

The zeros of $h(x)$ are the zeros of $f(x)$ if $h(x)$ is in simplified form.

Example: Find the Domain, Range, and the Intercepts.

(a) $f(x)=\dfrac{2}{x-3}$



The function is not defined when $x-3=0$.

Domain: $\{x \mid x \neq 3, x \in \Re\}$

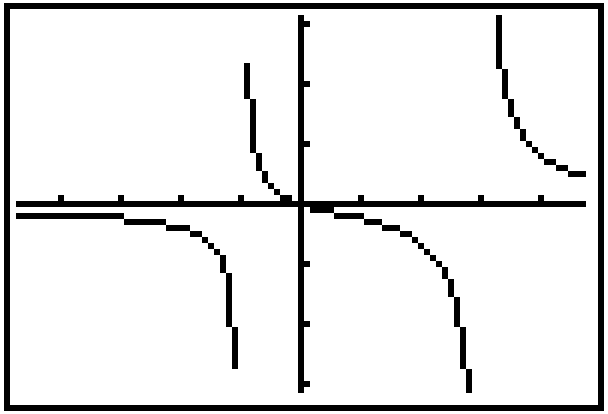
$f(0)=-\dfrac{2}{3}$ $\quad \therefore$ y-intercept is $-\dfrac{2}{3}$

Range: $\{y \mid y \neq 0, y \in \Re\}$

Example: Find the Domain and the Intercepts.

(b) $$g(x) = \frac{x}{x^2 - 2x - 3}$$

$$= \frac{x}{(x-3)(x+1)}$$



Domain: $\{x \mid x \neq 3, -1,\ x \in \Re\}$

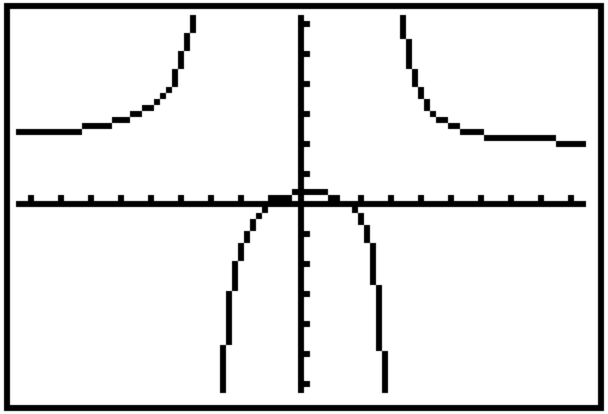
$g(0) = 0$, x-intercept is zero and the y-intercept is zero.

Range: $\{y \mid y \in \Re\}$

Example: Find the Domain and the Intercepts.

(c) $$h(x)=\frac{2x^2-x-3}{x^2-9}$$

$$=\frac{(2x-3)(x+1)}{(x-3)(x+3)}$$



Domain: $\{x \mid x \neq 3, -3,\ x \in \Re\}$

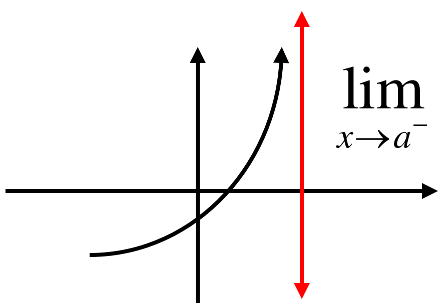
$g(0) = 1/3$, the y-intercept is $1/3$.

x-intercepts are $3/2$ and -1.

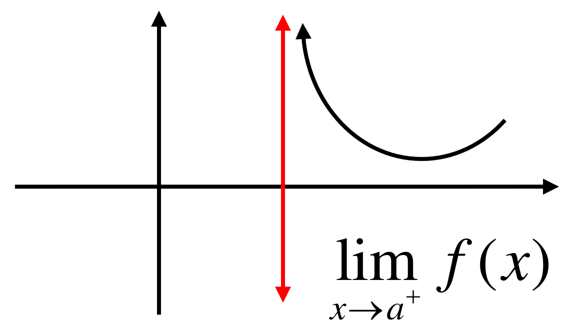
Vertical and Horizontal Asymptotes

A ***vertical asymptote*** occurs when the value of the function increases or decreases without bound as the value of x approaches a from the left or right.

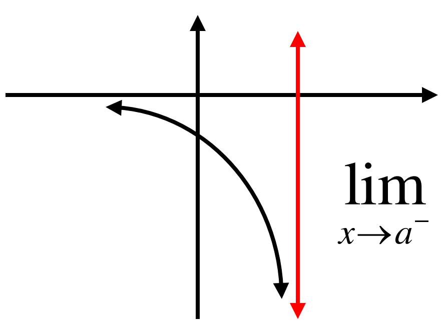
If $f(x) \to \infty$ or $f(x) \to -\infty$ as $x \to a^-$ or $x \to a^+$ then $x = a$ is a vertical asymptote.



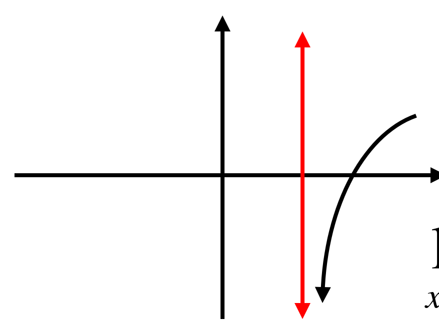
$$\lim_{x\to a^-} f(x) = \infty$$



$$\lim_{x\to a^+} f(x) = \infty$$



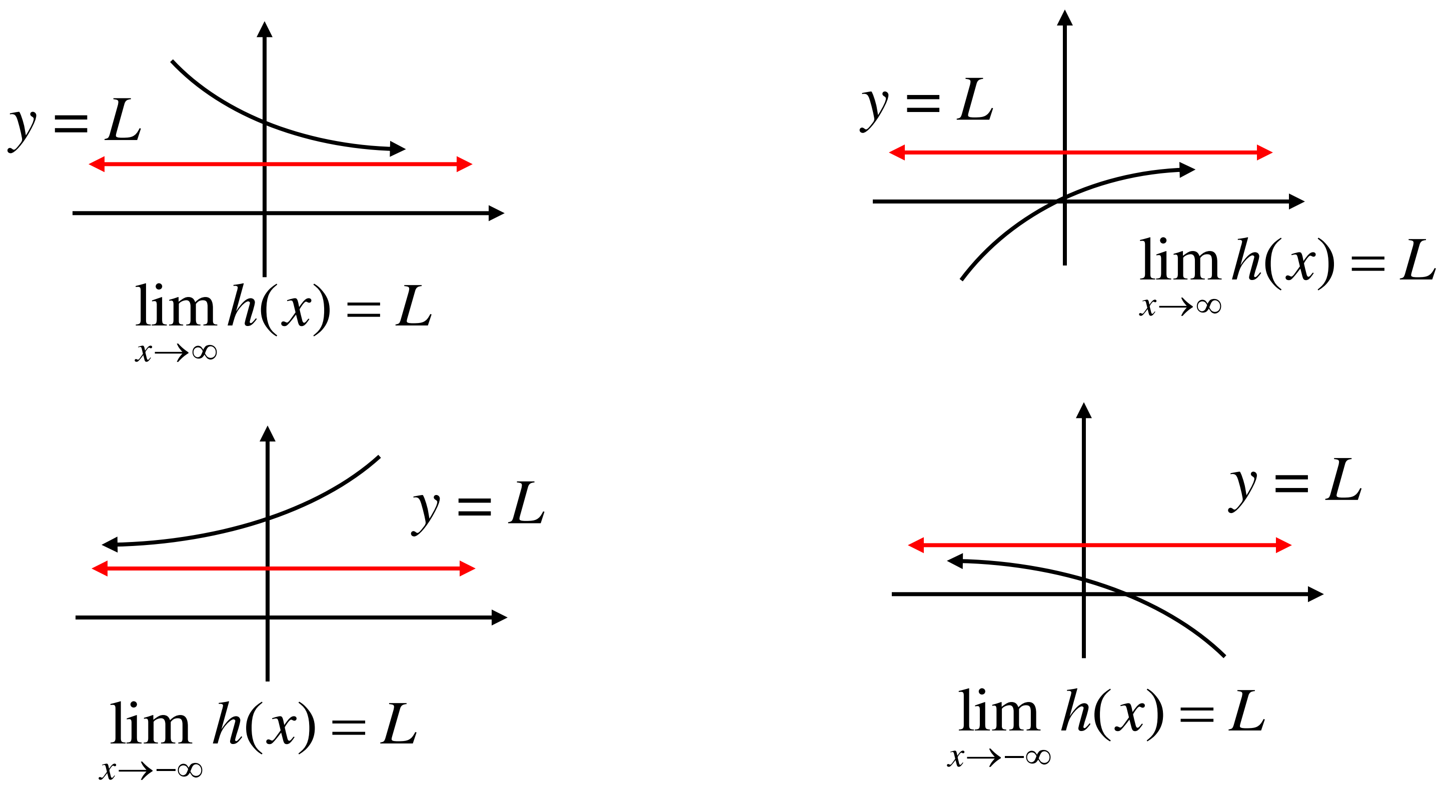
$$\lim_{x\to a^-} f(x) = -\infty$$



$$\lim_{x\to a^+} f(x) = -\infty$$

A ***horizontal asymptote*** occurs when the value of the function approaches a number L as x increases or decreases without bound.

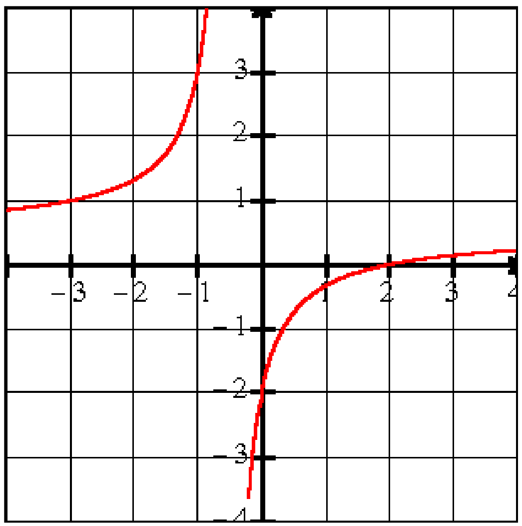
If $h(x) \to L$ as $x \to \infty$ or as $x \to -\infty$, then $y = L$ is a horizontal asymptote.



Example: Find the vertical and horizontal asymptotes.

(a) $$f(x) = \frac{x-2}{2x+1}$$

The function is undefined when $2x + 1$ equals zero.



$$x = -\frac{1}{2}$$ is a vertical asymptote.

$y = 0.5$ is a horizontal asymptote because $f(x)$ approaches 0.5 as x increases or decreases.

Example: Find the vertical and horizontal asymptotes.

(b) $h(x) = \dfrac{1-2x^2}{4-x^2}$

$= \dfrac{1-2x^2}{(2-x)(2+x)}$

is undefined for $x = \pm 2$ so $x = 2$ and $x = -2$ are vertical asymptotes.

$x \to \infty$

X	Y1	
100	2.0007	
150	2.0003	
200	2.0002	
250	2.0001	
300	2.0001	
350	2.0001	
400	2	

Y1=(1-2X²)/(4-X…

$x \to -\infty$

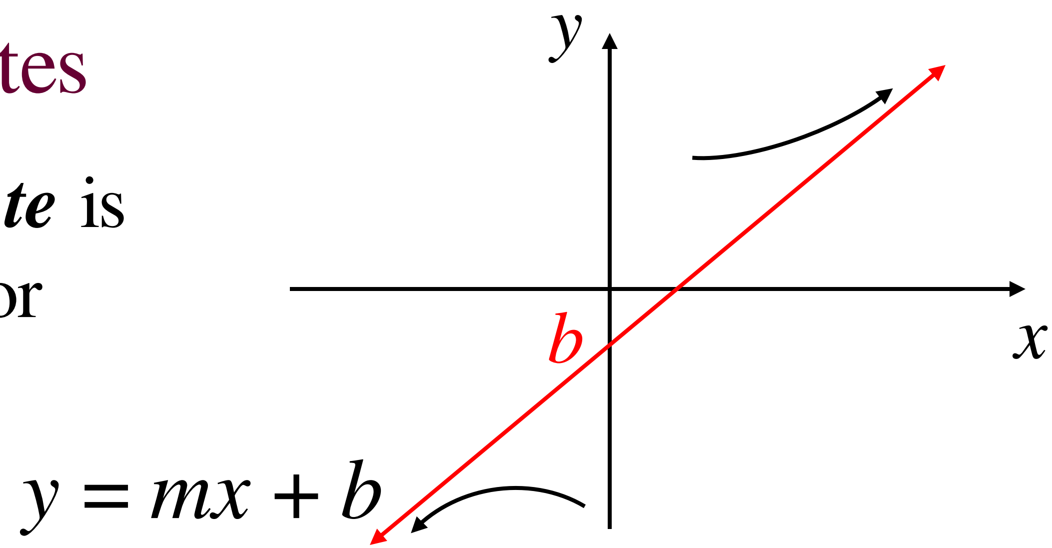
X	Y1	
-100	2.0007	
-150	2.0003	
-200	2.0002	
-250	2.0001	
-300	2.0001	
-350	2.0001	
-400	2	

Y1=(1-2X²)/(4-X…

$y = 2$ is a horizontal asymptote because $f(x)$ approaches 2 as x increases or decreases.

Oblique Asymptotes

An ***oblique asymptote*** is neither horizontal nor vertical but slanted.



The line $y = mx + b$ is an oblique asymptote if the vertical distance between the $h(x)$ and $mx + b$ approaches 0 as x increases or decreases. $\lim\limits_{x \to \pm\infty} [h(x) - (mx + b)] = 0$

$h(x) = \dfrac{f(x)}{g(x)}$ will have an oblique asymptote if the degree of $f(x)$ is one more $g(x)$.

Finding the equation of an oblique asymptote.

$$h(x) = \frac{x^2 - x - 2}{x - 3}$$

$$= \frac{(x-2)(x+1)}{x-3}$$

Since the degree of the numerator is one more than the denominator the function will have an oblique asymptote.

The x-intercepts are 2 and – 1.

$f(0) = 2/3$, so the y-intercept is 2/3.

The line $x = 3$ is the vertical asymptote.

	$x \to 3^+$	$x \to 3^-$
sign of $h(x)$	$\frac{(+)(+)}{(+)} = +$	$\frac{(+)(+)}{(-)} = -$
$h(x) \to$	∞	$-\infty$

To find the equation of the oblique asymptote, divide the numerator by the denominator.

$$h(x) = \frac{x^2 - x - 2}{x - 3}$$

$$
\begin{array}{rl}
 & \phantom{x^2 -{}} x + 2 \\ \cline{2-2}
x - 3 \,\big) & x^2 - x - 2 \\
 & -x^2 \overset{+}{-} 3x \\ \cline{2-2}
 & \phantom{x^2 - {}} 2x - 2 \\
 & -2x \overset{+}{-} 6 \\ \cline{2-2}
 & \phantom{x^2 - 2x - {}} 4 \quad \text{(remainder)}
\end{array}
$$

$$h(x) = x + 2 + \frac{4}{x - 3}$$

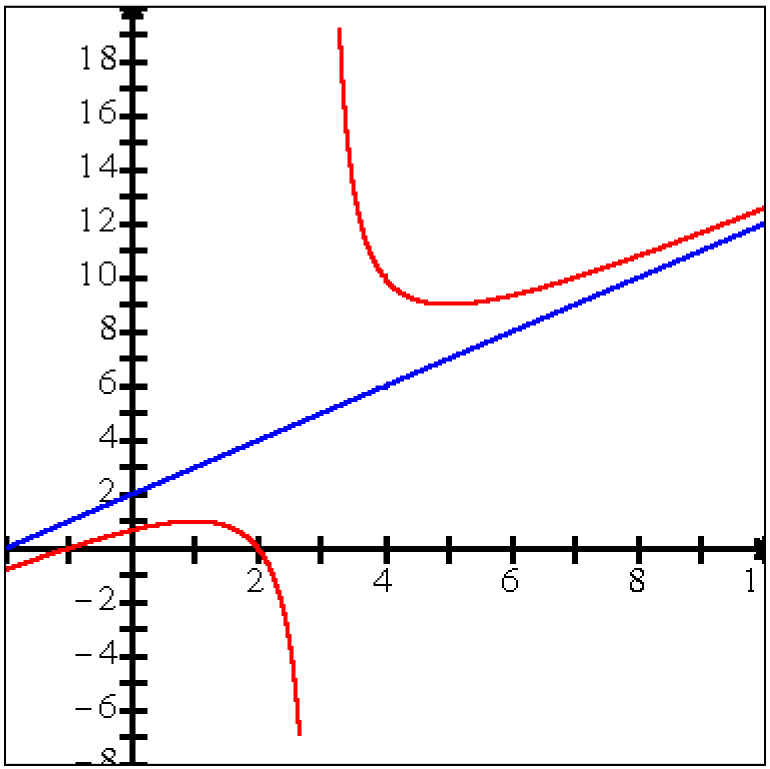
As $x \to \pm\infty$

$\frac{4}{x-3}$ becomes negligible

So $y = x + 2$ is the oblique asymptote.

$$h(x) = \frac{x^2 - x - 2}{x - 3}$$

$$y = x + 2$$

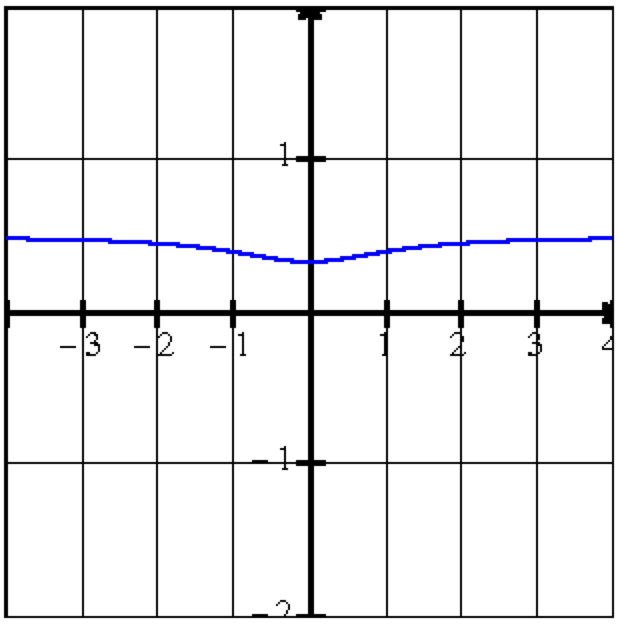


Limits at Infinity

The graph of a function will have a horizontal asymptote if the function has a finite limit L as as $x \to \pm\infty$.

Example 3: Find the equation of the horizontal asymptote.

$$\lim_{x \to \infty} \frac{x^2 + 1}{2x^2 + 3}$$



Both the numerator and denominator become large as $x \to \infty$.

$$\lim_{x \to \infty} \frac{x^2 + 1}{2x^2 + 3}$$

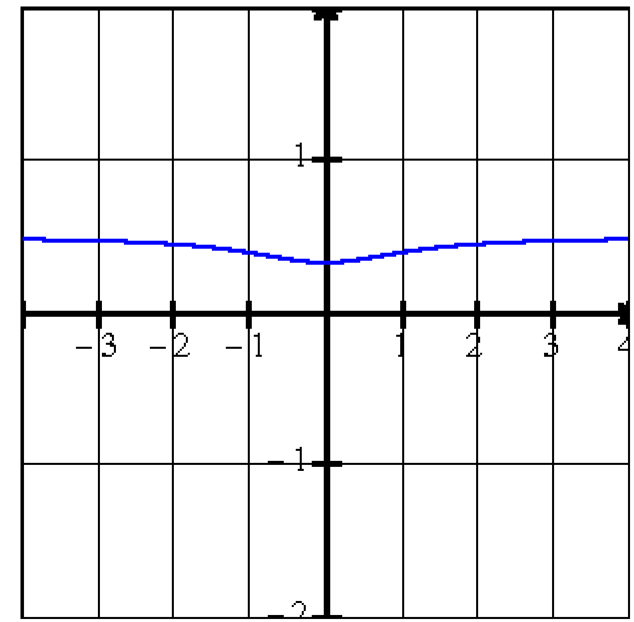
both the numerator and denominator become large as $x \to \infty$.

$\dfrac{\infty}{\infty}$ is also the *indeterminate* form.

10	0.4975369
100	0.499975
1000	0.4999998

The table indicates that the limit is 0.5 as $x \to \infty$

$$\lim_{x\to\infty}\frac{x^2+1}{2x^2+3}$$



Algebraic Method: Divide the numerator and denominator by the highest power of x in the denominator (x^2 in this case).

$$\lim_{x\to\infty}\frac{\frac{x^2}{x^2}+\frac{1}{x^2}}{\frac{2x^2}{x^2}+\frac{3}{x^2}}=\lim_{x\to\infty}\frac{1+\frac{1}{x^2}}{2+\frac{3}{x^2}}$$

terms $\to 0$ as $x\to\infty$.

$$\boxed{=\frac{1}{2}}$$

4- Evaluate: $\displaystyle\lim_{x\to\infty}\frac{1-2x^2}{(4x+3)^2}$ expand denominator

$$\lim_{x\to\infty}\frac{1-2x^2}{16x^2+24x+9}$$

divide by x^2

$$\lim_{x\to\infty}\frac{\dfrac{1}{x^2}-\dfrac{2x^2}{x^2}}{\dfrac{16x^2}{x^2}+\dfrac{24x}{x^2}+\dfrac{9}{x^2}}$$

$$\lim_{x\to\infty}\frac{\dfrac{1}{x^2}-2}{16+\dfrac{24}{x}+\dfrac{9}{x^2}}$$

$$=\frac{0-2}{16+0+0}$$

$$=-\frac{1}{8}$$