Implicit Differentiation

a) $2x + 3y = 8$ is an equation that defines a function, although it is not in explicit form.

Solving for y, we have the function in explicit form. $$y = -\frac{2}{3}x + \frac{8}{3}$$

b) $y^2 + x = 4$ is an equation that defines two functions.

The explicit forms are:

$y = -\sqrt{4 - x}$ and $y = \sqrt{4 - x}$

It is not always convenient or desirable to express the function in explicit form.

To find dy/dx when y is defined as an implicit function of x, we differentiate each term with respect to x, regarding y as a differentiable function of x.

We then solve for dy/dx.

Example 1: Find dy/dx if $y^2 + 3x^2 = 8$

Find the derivative of each term and solve for dy/dx.

$$\frac{d(y^2)}{dx} + \frac{d(3x^2)}{dx} = \frac{d(8)}{dx}$$

$$2y\frac{dy}{dx} + 6x\frac{dx}{dx} = 0$$

$$2y\frac{dy}{dx} + 6x = 0$$

$$\boxed{\frac{dy}{dx} = -\frac{3x}{y}}$$

Example 2: Find dy/dx if $3y^4 + xy^2 + 2x^2 - 8 = 0$

use the product rule:

$$\frac{d(3y^4)}{dx} + \frac{d(xy^2)}{dx} + \frac{d(2x^2)}{dx} - \frac{d(8)}{dx} = \frac{d(0)}{dx}$$

$$12y^3\frac{dy}{dx} + \left[x\left(2y\frac{dy}{dx}\right) + y^2(1)\right] + 4x - 0 = 0$$

$$12y^3\frac{dy}{dx} + 2xy\frac{dy}{dx} + y^2 + 4x = 0$$

$$\left(12y^3 + 2xy\right)\frac{dy}{dx} = -y^2 - 4x$$

$$\boxed{\frac{dy}{dx} = \frac{-y^2 - 4x}{12y^3 + 2xy}}$$

Example 3: Find $\left.\frac{dy}{dx}\right|_{(1,2)}$ if $3x^3y^2 - 2y^3 = -4$

$$\frac{d(3x^3y^2)}{dx} - \frac{d(2y^3)}{dx} = \frac{d(-4)}{dx}$$

$$9x^2y^2 + 3x^3(2y)\frac{dy}{dx} - 6y^2\frac{dy}{dx} = 0$$

$$(6x^3y - 6y^2)\frac{dy}{dx} = -9x^2y^2$$

sub (1, 2) ➔

$$\frac{dy}{dx} = \frac{-9x^2y^2}{6x^3y - 6y^2}$$

$$= \frac{-9(1)^2(2)^2}{6(1)^3(2) - 6(2)^2} = \frac{-36}{-12}$$

$$= 3$$

Example 4: Find $\frac{dy}{dx}$ if $2x^2y + (y^2 + x)^3 = x^4$

$$\frac{d(2x^2y)}{dx} + \frac{d(y^2+x)^3}{dx} = \frac{d(x^4)}{dx}$$

$$2x^2\frac{d(y)}{dx} + y(4x) + 3(y^2+x)^2\left(2y\frac{dy}{dx} + 1\right) = 4x^3$$

$$2x^2\frac{d(y)}{dx} + y(4x) + 3(y^2+x)^2\left(2y\frac{dy}{dx}\right) + 3(y^2+x)^2 = 4x^3$$

$$\left[2x^2 + 6y(y^2+x)^2\right]\frac{dy}{dx} = 4x^3 - 4xy - 3(y^2+x)^2$$

$$\boxed{\frac{dy}{dx} = \frac{4x^3 - 4xy - 3(y^2+x)^2}{2x^2 + 6y(y^2+x)^2}}$$