MCV4U - Unit 6 Test - Intro to Vectors
Mark: /44

Problem

1. Vector $\overrightarrow{AB}$ goes from (1, 0) to (2, 1). Vector $\overrightarrow{CD}$ starts at (4, 2).
   a. Calculate the magnitude of $\overrightarrow{AB}$. [2]
   b. If $\overrightarrow{AB} = \overrightarrow{CD}$, determine the endpoint of $\overrightarrow{CD}$. [1]

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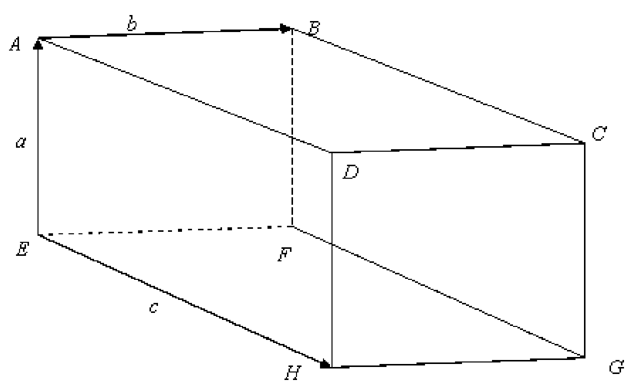
2. Using the prism, answer the following questions.

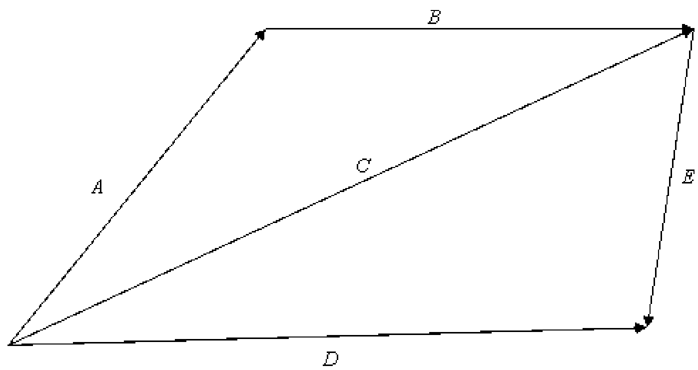


a. Write $\overrightarrow{EC}$ in terms of $\vec{a}$, $\vec{b}$, and $\vec{c}$. [1]

b. Write $\overrightarrow{CE}$ in terms of $\vec{a}$, $\vec{b}$, and $\vec{c}$. [1]

c. What type of vectors are $\overrightarrow{EC}$ and $\overrightarrow{CE}$ in relation to each other? Explain. [2]

3. If $\left|\overrightarrow{A}\right| = 7$, $\left|\overrightarrow{B}\right| = 3$, $\left|\overrightarrow{D}\right| = 9$ the angle between $\overrightarrow{A}$ and $\overrightarrow{B}$ is 100° and the angle between $\overrightarrow{C}$ and $\overrightarrow{D}$ is 35°, what is the magnitude of $\overrightarrow{E}$? What is $\overrightarrow{E}$ in terms of $\overrightarrow{A}$, $\overrightarrow{B}$, and $\overrightarrow{D}$? [5]



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4. $\overrightarrow{A}$ and $\overrightarrow{B}$ are unit vectors with an angle of 120° between them. Determine $\left|3\overrightarrow{A} + 2\overrightarrow{B}\right|$. [5]

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5. If $\vec{x} = \vec{i} - \vec{j} + 2\vec{k}$, $\vec{y} = 3\vec{i} + \vec{j} - 7\vec{k}$ and $\vec{z} = 3\vec{j} + \vec{k}$, simplify $2(\vec{x} - \vec{y}) + \vec{z}$? [3]

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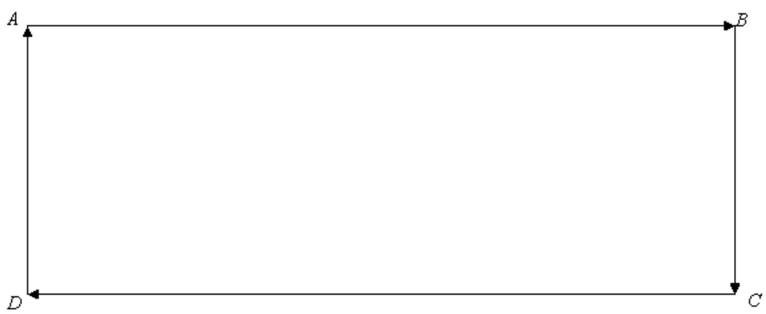
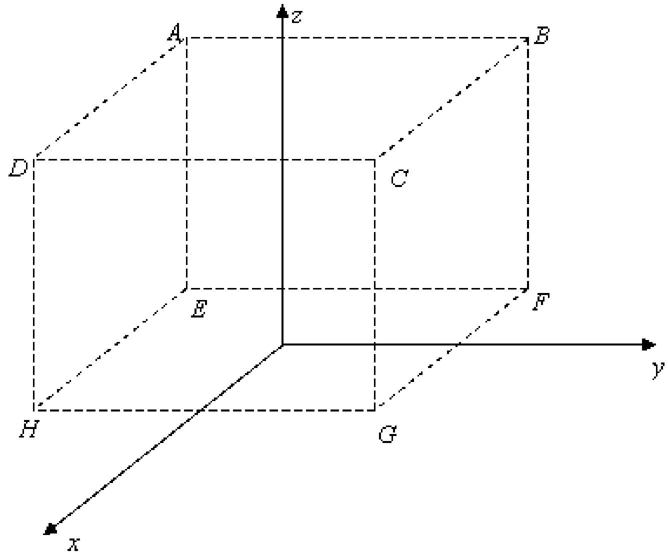
6. In rectangle *ABCD*, the diagonals meet at *X*. Determine an expression for $\overrightarrow{AX}$ in terms of $\overrightarrow{AB}$ and $\overrightarrow{BC}$. Explain. [2]



7. The prism is bisected by the *xz*-plane and the *yz*-plane. The point $C = (2, 3, 5)$. Determine $\left|\overrightarrow{CE}\right|$. [4]



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8. A triangle in R^2 has two sides represented by the vectors $\overrightarrow{OA} = (2, –3)$ and $\overrightarrow{OB} = (3, 1)$. Determine the measures of the angles of the triangle. [5]

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9. A rectangle is formed in R^2 by the vectors $\overrightarrow{OA} = (1, 2)$ and $\overrightarrow{OB} = (-6, 3)$. [5]

a. Determine its perimeter.

b. Determine its area.

c. Determine the length of its diagonals.

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10. Determine 3 vectors that form the rectangular prism that satisfies the following conditions. [3]

1. Volume of 60 cubic units
2. Length in the *x*-direction of 10 units
3. Height in the *y*-direction of 3 units
4. A vertex at the origin.

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11. If $a(0, -1, 3) + b(1, 1, 1) - c(1, 2, 5) = (-2, 3, -8)$, determine a, b and c. [5]

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Answer Section

PROBLEM

1. ANS:
   a. Use the distance formula:

   $$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(2-1)^2 + (1-0)^2} = \sqrt{2}$$

   b. Determine the slope of $\overrightarrow{AB}$ to determine its direction:

   $$\frac{y_2 - y_1}{x_2 - x_1} = \frac{1}{1}$$

   Use the slope to determine the possible endpoint, (5, 3). Check by determining the magnitude of $\overrightarrow{CD}$ with endpoint (5, 3):

   $$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(5-4)^2 + (3-2)^2} = \sqrt{2}$$

   PTS: 1 REF: Thinking OBJ: 6.1 - An Introduction to Vectors

2. ANS:
   a. $\vec{a} + \vec{b}$ gets us to point *B*. Then we add $\vec{c}$ to get to point *C*. So, $\vec{a} + \vec{b} + \vec{c} = \overrightarrow{EC}$.
   b. $-\vec{a} - \vec{b}$ gets us to point *H*. Then we subtract $\vec{c}$ to get to point *E*. So, $-\vec{a} - \vec{b} - \vec{c} = \overrightarrow{CE}$.
   c. They are opposite vectors; they have the same magnitudes in opposite directions.

   PTS: 1 REF: Communication OBJ: 6.2 - Vector Addition

3. ANS:
   First determine $\vec{C}$ using the cosine law:

   $$\vec{C}^{\,2} = \left|\vec{A}\right|^2 + \left|\vec{B}\right|^2 - 2\left|\vec{A}\right|\left|\vec{B}\right|\cos 100^\circ = 65.29$$

   Then taking the square root shows:

   $$\vec{C} = 8.08$$

   Now that we have $\vec{C}$, we can determine $\vec{E}$ using the cosine law again:

   $$\vec{E} = \sqrt{\left|\vec{C}\right|^2 + \left|\vec{D}\right|^2 - 2\left|\vec{C}\right|\left|\vec{D}\right|\cos 35^\circ} = 4.51$$

   $\vec{E}$ is the difference of $\vec{D}$ and $\vec{C}$. But we need to use $\vec{A}$ and $\vec{B}$, so $\vec{C}$ is the sum of $\vec{A}$ and $\vec{B}$. Thus, $\vec{E} = \vec{D} - \vec{C} = \vec{D} - \vec{A} - \vec{B}$

   PTS: 1 REF: Thinking OBJ: 6.2 - Vector Addition

4. ANS:

If you arrange the vectors $3\overrightarrow{A}$ and $2\overrightarrow{B}$ from head to tail, $3\overrightarrow{A} + 2\overrightarrow{B}$ is the vector from the tail of $3\overrightarrow{A}$ to the head of $2\overrightarrow{B}$. The angle opposite of $3\overrightarrow{A} + 2\overrightarrow{B}$ is 60° by using a drawing and the given information. $\left|3\overrightarrow{A}\right|$ = 3 since a the magnitude of a unit vector is 1, so 3 unit vectors has a magnitude of 3. The same reasoning results in $\left|2\overrightarrow{B}\right| = 2$. Now the cosine law must be used to determine $\left|3\overrightarrow{A} + 2\overrightarrow{B}\right|$:

$$\begin{aligned}\left|3\overrightarrow{A} + 2\overrightarrow{B}\right| &= \sqrt{\left|3\overrightarrow{A}\right|^2 + \left|2\overrightarrow{B}\right|^2 - 2\left|3\overrightarrow{A}\right|\left|2\overrightarrow{B}\right|\cos(60)} \\ &= \sqrt{9+4-6} \\ &= \sqrt{7} \text{ or } 2.65\end{aligned}$$

PTS: 1 REF: Thinking OBJ: 6.3 - Multiplication of a Vector by a Scalar

5. ANS:

$$\begin{aligned}2(\vec{x}-\vec{y})+\vec{z} &= 2\vec{x}-2\vec{y}+\vec{z} \\ &= 2(\vec{i}-\vec{j}+2\vec{k})-2(3\vec{i}+\vec{j}-7\vec{k})+(3\vec{j}+\vec{k}) \\ &= (2\vec{i}-2\vec{j}+4\vec{k})-(6\vec{i}+2\vec{j}-14\vec{k})+(3\vec{j}+\vec{k}) \\ &= (2\vec{i}-6\vec{i})+(-2\vec{j}-2\vec{j}+3\vec{j})+(4\vec{k}+14\vec{k}+\vec{k}) \\ &= -4\vec{i}-\vec{j}+19\vec{k}\end{aligned}$$

PTS: 1 REF: Thinking OBJ: 6.4 - Properties of Vectors

6. ANS:

$\overrightarrow{AB} + \overrightarrow{BC}$ equals one of the diagonals, $\overrightarrow{AC}$. Knowing the diagonals of a rectangle bisect each other, we can conclude that $\overrightarrow{AX} = \frac{1}{2}(\overrightarrow{AB}+\overrightarrow{BC})$.

PTS: 1 REF: Communication OBJ: 6.4 - Properties of Vectors

7. ANS:

We need to use the properties of right triangles to determine $\left|\overrightarrow{CE}\right|$. The vectors $\overrightarrow{CE}$, $\overrightarrow{AE}$ and $\overrightarrow{AC}$ form a right triangle. So, we need to determine $\left|\overrightarrow{AE}\right|$ and $\left|\overrightarrow{AC}\right|$.

We can easily determine $\left|\overrightarrow{AE}\right|$ with the given information about the point *C*. Since $\overrightarrow{AE}$ goes from the height of 5 to 0, we can see that its magnitude is 5.

For $\left|\overrightarrow{AC}\right|$, we see that $\overrightarrow{AC}$, $\overrightarrow{AB}$ and $\overrightarrow{BC}$ forms a right triangle. In a similar way to determining $\left|\overrightarrow{AE}\right|$, we see $\left|\overrightarrow{AB}\right| = 6$ and $\left|\overrightarrow{BC}\right| = 4$. Using the Pythagorean theorem, $\sqrt{\left|\overrightarrow{AB}\right|^2 + \left|\overrightarrow{BC}\right|^2} = \left|\overrightarrow{AC}\right|$. So $\left|\overrightarrow{AC}\right| = 2\sqrt{13}$.

Now that we know $\left|\overrightarrow{AE}\right|$ and $\left|\overrightarrow{AC}\right|$, we can determine $\left|\overrightarrow{CE}\right|$ with the Pythagorean theorem.

$$\left|\overrightarrow{CE}\right| = \sqrt{\left|\overrightarrow{AE}\right|^2 + \left|\overrightarrow{AC}\right|^2} = \sqrt{77}$$

PTS: 1 REF: Thinking OBJ: 6.5 - Vectors in R^2 and R^3

8. ANS:

To determine the angle formed by $\overrightarrow{OA}$ and $\overrightarrow{OB}$ at the origin, we will need the cosine law. We will need the magnitudes of the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OA} - \overrightarrow{OB}$.

$\left|\overrightarrow{OA}\right| = \sqrt{2^2 + (-3)^2} = \sqrt{13}$

$\left|\overrightarrow{OB}\right| = \sqrt{3^2 + 1^2} = \sqrt{10}$

$\overrightarrow{OA} - \overrightarrow{OB}$ = (2 – 3, –3 – 1) = (–1, –4)

$\left|\overrightarrow{OA} - \overrightarrow{OB}\right| = \sqrt{(-1)^2 + (-4)^2} = \sqrt{17}$

Using the cosine law:

$$\theta = \cos^{-1}\left(\frac{\left|\overrightarrow{OA}\right|^2 + \left|\overrightarrow{OB}\right|^2 - \left|\overrightarrow{OA} - \overrightarrow{OB}\right|}{2\left|\overrightarrow{OA}\right|\left|\overrightarrow{OB}\right|}\right) = \cos^{-1}\left(\frac{13 + 10 - 17}{2 \cdot \sqrt{13} \cdot \sqrt{10}}\right) = 74.74°$$

Use the sine law to determine the next:

$$\frac{\left|\overrightarrow{OA} - \overrightarrow{OB}\right|}{\sin 74.74°} = \frac{\left|\overrightarrow{OA}\right|}{\sin\alpha} \Rightarrow \alpha = \sin^{-1}\left(\frac{\left|\overrightarrow{OA}\right|\sin 74.74°}{\left|\overrightarrow{OA} - \overrightarrow{OB}\right|}\right) = 57.53°$$

Use the fact that the sum of a triangles angles is 180° to determine the last:

180 – 74.74 – 57.53 = 47.73°

PTS: 1 REF: Thinking OBJ: 6.6 - Operations with Algebraic Vectors in R^2

9. ANS:

a. $\left|\overrightarrow{OA}\right| = \sqrt{1^2 + 2^2} = \sqrt{5}$

$\left|\overrightarrow{OB}\right| = \sqrt{(-6)^2 + 3^2} = 3\sqrt{5}$

So the perimeter is $2\left|\overrightarrow{OA}\right| + 2\left|\overrightarrow{OB}\right| = 2\sqrt{5} + 6\sqrt{5} = 8\sqrt{5}$

b. Area is $\left|\overrightarrow{OA}\right| \cdot \left|\overrightarrow{OB}\right| = \sqrt{5} \cdot 3\sqrt{5} = 15$

c. There are two ways to calculate the length of the diagonals:

1. To determine the vector that could represent the diagonal and determine its magnitude

or

2. To use the Pythagorean theorem using the sides of the rectangle.

Since we have already found the sides, we will use the Pythagorean theorem. Regardless, the answer will be the same. Thus, the length of the diagonal is:

$$\sqrt{\left|\overrightarrow{OA}\right|^2 + \left|\overrightarrow{OB}\right|^2} = \sqrt{5 + 45} = 5\sqrt{2}$$

PTS: 1 REF: Thinking OBJ: 6.6 - Operations with Algebraic Vectors in R^2

10. ANS:

The vector representing its length could either be (10, 0, 0) or (–10, 0, 0). They both represent a length of 10 units along the *x*-axis or direction.
The vector representing its height could either be (0, 0, 3) or (0, 0, –3). They both represent a height of 3 units along the *z*-axis or direction.
The vector representing its width, would have to have a magnitude of 2, since volume is length times width times height. In addition to the previous fact, it would have to be in the *y*-direction since it's a rectangular prism. So, the vector could either be (0, 2, 0) or (0, –2, 0).
Any combination of each of the 3 vectors found would result in a rectangular prism with the above properties.

PTS: 1 REF: Thinking OBJ: 6.7 - Operations with Vectors in R^3

11. ANS:

Looking at the *x*-component and solving for *b*:
$b - c = -2$
$b = -2 + c$
Looking at the *y*-component and substituting in *b*:
$-a + b - 2c = 3$
$-a - 2 + c - 2c = 3$
Solving for *a*:
$a = -5 - c$
Looking at the *z*-component:
$3a + b - 5c = -8$
Substituting in *a* and *b*:
$3(-5 - c) - 2 + c - 5c = -8$
Solving for *c*:
$-17 - 7c = -8$
$c = -\frac{9}{7}$
Substituting *c* into the *b* equation:
$b = -2 - \frac{9}{7} = -\frac{23}{7}$
Substituting *c* into the *a* equation:
$a = -5 + \frac{9}{7} = -\frac{26}{7}$
Checking the results:
$(0 - \frac{23}{7} + \frac{9}{7}, \frac{26}{7} - \frac{23}{7} + \frac{18}{7}, -\frac{78}{7} - \frac{23}{7} + \frac{45}{7}) = (-2, 3, -8)$

PTS: 1 REF: Thinking OBJ: 6.8 - Linear Combinations and Spanning Sets