2.5 The Derivatives of Composite Functions "The Chain Rule"

Given: $y = (4 - 3x)^3$ let $u = 4 - 3x$ then $y = u^3$

y is a function of u and u is a function of x is a *composite function*.

$\boxed{\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}}$ provided $\dfrac{dy}{du}$ and $\dfrac{du}{dx}$ exist.

Therefore if $h(x) = (f \circ g)(x)$, then $h'(x) = f'(g(x)) \cdot g'(x)$

$$\frac{d[h(x)]}{dx} = \frac{d[f(g(x))]}{d[g(x)]} \cdot \frac{d[g(x)]}{dx}$$

Example 1: Determine $\dfrac{dy}{dx}$ if $y = (4x + 6)^3$

$$\frac{dy}{dx} = \frac{d[(4x+6)^3]}{dx}$$

$$\frac{dy}{dx} = \frac{d[(4x+6)^3]}{d[(4x+6)]} \cdot \frac{d[(4x+6)]}{dx}$$

$$= 3(4x + 6)^2(4)$$

$$= 12(4x + 6)^2$$

If we had expanded $y = (4x + 6)^3$ and found the derivative of the product, the result would have been the same.

The Chain Rule with the Power Rule

$$\frac{d[g(x)^n]}{dx} = \frac{d[g(x)^n]}{d[g(x)]} \cdot \frac{d[g(x)]}{dx}$$

$= n[g(x)]^{n-1} \cdot g'(x)$, where n is a constant

Take the derivative of the '*outer*' function multiplied by the derivative of the '*inner*' function.

Example 2: Determine $f'(s)$ if $f(s) = (2s^3 - 5)^4$

$$f'(s) = \underline{4(2s^3 - 5)^3}\,\underline{(6s^2)}$$

derivative of outer · derivative of inner

$$f'(s) = 24s^2(2s^3 - 5)^3$$

Example 3: Determine $\dfrac{dy}{dx}$ if $y=\sqrt{(x+3)^2+1}$

$$y=\sqrt{x^2+6x+10}$$

$$y=(x^2+6x+10)^{\frac{1}{2}}$$

$$\frac{dy}{dx}=\frac{d[(x^2+6x+10)^{\frac{1}{2}}]}{dx}$$

$$\frac{dy}{dx}=\frac{d[(x^2+6x+10)^{\frac{1}{2}}]}{d[(x^2+6x+10)]}\cdot\frac{d[(x^2+6x+10)]}{dx}$$

$$=\frac{1}{2}(x^2+6x+10)^{-\frac{1}{2}}(2x+6)$$

$$=\frac{x+3}{\sqrt{x^2+6x+10}}$$

Differentiating a quotient with constant numerator.

Example 4: Determine $f'(t)$ if $f(t) = \dfrac{5}{(4t+3)^3}$

$$f(t) = 5(4t + 3)^{-3}$$

$$f'(t) = \underbrace{(-3)(5)(4t + 3)^{-3-1}}_{\text{derivative of outer}} \cdot \underbrace{(4)}_{\text{derivative of inner}}$$

$$f'(t) = (-60)(4t + 3)^{-4}$$

$$= \frac{-60}{(4t+3)^4}$$