## Section 7.4: Integration of Rational Functions by Partial Fractions

**Objective:** In this lesson, you learn

- ☐ How to integrate any rational function by expressing it as a sum of partial fractions.
- ☐ How to convert a nonrational function to a rational function by an appropriate substitution.

### I. Integration of Rational Functions by Partial Fractions.

To integrate any rational function (a ratio of polynomials), express it as a sum of simpler fractions, called **partial fractions**, which we already know how to integrate.

**Problem**: Evaluate

$$\int \frac{x-1}{x^2-5x+6}\,dx$$

**General Problem:** Evaluate

$$\int f(x)\,dx = \int \frac{P(x)}{Q(x)}\,dx,$$

where $P(x)$ (dividend) and $Q(x)$ (divisor) are polynomials.

If the $\deg(P(x)) > \deg(Q(x))$ then (by the long division) there are two polynomials $q(x)$ and $r(x)$ such that

$$f(x) = \frac{P(x)}{Q(x)} = q(x) + \frac{r(x)}{Q(x)}.$$

where $r(x) = 0$ or $\deg(r(x)) < \deg(Q(x))$. The polynomial $q(x)$ is the quotient and $r(x)$ is the remainder produced by the long division process.

If $r(x) = 0$, then $\dfrac{P(x)}{Q(x)}$ is really just a polynomial, so we can ignore that case here.

Now,

$$\int f(x)\,dx = \int \frac{P(x)}{Q(x)} = \int q(x)\;dx + \int \frac{r(x)}{Q(x)}\,dx.$$

We can easily integrate the polynomial $q(x)$, so the general problem reduces to the problem of integrating a rational function $\dfrac{r(x)}{Q(x)}$ with $\deg r(x) < \deg Q(x)$.

So, for the purposes of investigating how to intergrate a rational function we can suppose

$$f(x) = \frac{P(x)}{Q(x)}$$

with $\deg P(x) < \deg Q(x)$.

**Fact about every $Q(x)$:**

$Q(x)$ can be factored as a product of linear factors (i.e. of the form $ax+b$) and/or irreducible quadratic forms (i.e. of the form $ax^2+bx+c$, where $b^2-4ac<0$).

Our strategy to integrate the rational function $f(x)$ is as follows:

- Factor $Q(x)$ into linear and irreducible quadratic factors.
- write $f(x)$ as a sum of partial fractions, where each fraction is of the form

$$\frac{K}{(ax+b)^s} \qquad \text{or} \qquad \frac{Lx+M}{(ax^2+bx+c)^t}$$

- integrate each partial fraction in the sum.

**Question:** How do we find $K, L$, and $M$?

**Partial-Fraction Decomposition: General Techniques**

**Case 1:** The denominator $Q(x)$ is a product of distinct linear factors, that is, $Q(x) = (a_1x+b_1)(a_2x+b_2)\cdots(a_kx+b_k)$, where no factor is repeated and no factor is a constant multiple of another. In this case, there exists constants $A_1, A_2, \ldots A_k$ such that

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \cdots \frac{A_k}{a_kx+b_k}.$$

**Example 1:** Evaluate $\displaystyle\int \frac{2x+1}{x^2-1}\,dx$

**Example 2:** Evaluate $\int \frac{4x^2 - 3x - 4}{x^3 + x^2 - 2x}\, dx$

**Case 2:** The denominator $Q(x)$ is a product of linear factors, some of which are repeated. Suppose the first linear factor $(a_1x + b_1)$ is repeated $r$ times so that $(a_1x + b_1)^r$ occurs in the factorization of $Q(x)$. Then instead of the single term $A_1/(a_1x + b_1)$ (in **case 1**) we would use

$$\frac{A_1}{a_1x + b_1} + \frac{A_2}{(a_1x + b_1)^2} + \cdots \frac{A_r}{(a_1x + b_1)^r}.$$

**Example 3:** Evaluate $\int \frac{2x-1}{(x-5)^2}\,dx$

**Example 4:** Evaluate $\int \frac{x^3 - 4x - 1}{x(x-1)^3} \, dx$

**Case 3:** The denominator $Q(x)$ contains irreducible quadratic factors, none of which is repeated. If $Q(x)$ has $ax^2 + bx + c$, where $b^2 - 4ac < 0$, then, in the sum of its partial fractions, the expression $\frac{P(x)}{Q(x)}$ will have terms of the form

$$\frac{Ax + B}{ax^2 + bx + c},$$

where $A$ and $B$ are constants to be determined.

**Example 5:** Evaluate $\displaystyle\int \frac{5x^3 - 3x^2 + 2x - 1}{x^4 + x^2}\, dx$

**Example 6:** Evaluate $\int \frac{x^4}{x^4 - 1}\, dx$

**Case 4:** The denominator $Q(x)$ contains a repeated irreducible quadratic factor. If $Q(x)$ has the $(ax^2+bx+c)^r$, where $b^2-4ac<0$, then, in the sum of its partial fractions, the expression $\frac{P(x)}{Q(x)}$ will have a sum

$$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots\frac{A_rx+B_r}{(ax^2+bx+c)^r}.$$

**Example 7:** Evaluate $\displaystyle\int\frac{dx}{x(x^2+1)^2}$

## II. Rationalizing Substitutions.

Some nonrational functions can be changed into rational functions by means of appropriate substitutions. For example, if an expression of the form $\sqrt[n]{g(x)}$ is in an integrand, then the substitution

$$u = \sqrt[n]{g(x)}$$

may be effective.

**Example 8:** Evaluate $\displaystyle\int \frac{\sqrt{x+4}}{x}\,dx$

**Example 9:** Evaluate $\int \frac{\sqrt{x}}{x^2 + x}\, dx$