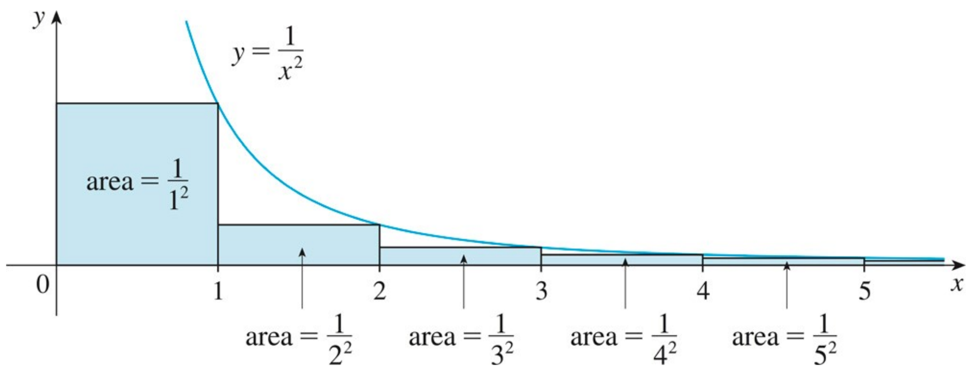
Section 11.3: The Integral Test and Estimates of Sums

Objective: In this lesson, you learn

- ☐ how to develop the Integral Test to determine whether or not a series is convergent or divergent without explicitly finding its sum.

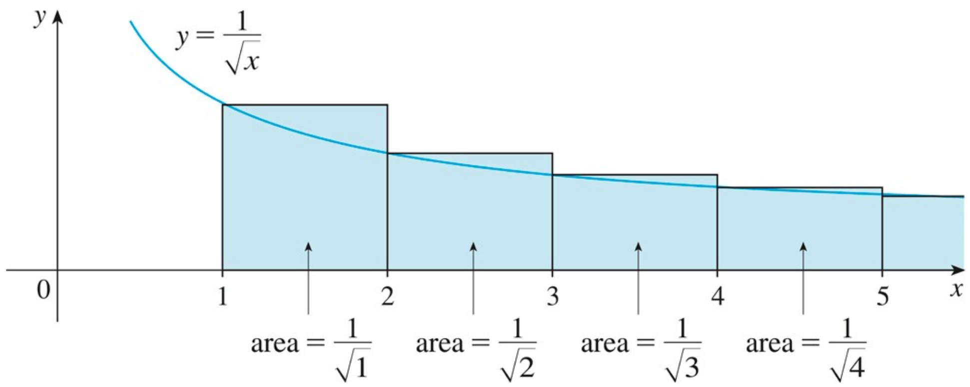
Problem: Compare

$$\int_1^{\infty} \frac{1}{x^2}\, dx \qquad \text{and} \qquad \sum_{n=1}^{\infty} \frac{1}{n^2}$$



Problem: Compare

$$\int_1^\infty \frac{1}{\sqrt{x}}\,dx \qquad \text{and} \qquad \sum_{n=1}^\infty \frac{1}{\sqrt{n}}$$



I. The Integral Test

Except for **geometric** series and the **telescoping** series, it is difficult to find the exact sum of a series. So we try to determine the convergence of a series without explicitly finding the sum.

The Integral Test

Suppose f is a continuous, positive, decreasing function on $[a, \infty)$ and let $a_n = f(n)$. Then the series

$$\sum_{n=a}^{\infty} a_n$$

is **convergent** if and only if the **improper integral**

$$\int_a^{\infty} f(x)\, dx$$

is covergent. That is,

a. If $\int_a^{\infty} f(x)\, dx$ is convergent, then $\sum_{n=a}^{\infty} a_n$ is convergent.

b. If $\int_a^{\infty} f(x)\, dx$ is divergent, then $\sum_{n=a}^{\infty} a_n$ is divergent.

Note that: In general, $\sum_{n=1}^{\infty} a_n \neq \displaystyle\int_1^{\infty} f(x)\, dx$.

Example 1: For what values of p is the $p-$**series** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ convergent?

Example 2: Does the series converge or diverge?

$$\sum_{n=1}^{\infty} \frac{n^2}{1+n^2}$$

Example 3: Does the series converge or diverge?

$$\sum_{n=1}^{\infty} \frac{1}{1+n^2}$$

Example 4: Test the series

$$\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n}$$

for convergence or divergence.

II. Estimating the Sum of a Series

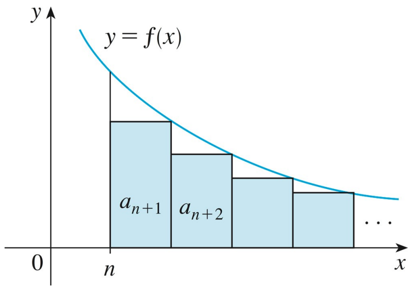
Remainder Estimate for the Integral Test

Once convergence has been established, we want to determine how large the **error** (the **remainder**)

$$R_n = s - s_n$$

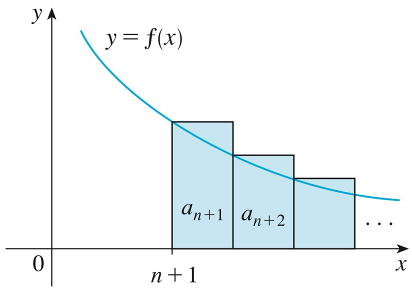
is when s_n, the sum of the first n terms,

is used as an approximation to the total sum.



$R_n = s - s_n = a_{n+1} + a_{n+2} + \cdots \leq \int_n^\infty f(x)\, dx$

and



$R_n = a_{n+1} + a_{n+2} + \cdots \geq \int_{n+1}^\infty f(x)\, dx.$

So we have established the following error estimate.

Suppose $f(k) = a_k$, where f is a continuous, positive, decreasing function for $x \geq n$ and $\sum_{n=1}^\infty a_n$ is convergent. If $R_n = s - s_n$, then

$$\int_{n+1}^\infty f(x)\, dx \leq R_n \leq \int_n^\infty f(x)\, dx.$$

Note: Add s_n to each side of the inequality to obtain

$$s_n + \int_{n+1}^\infty f(x)\, dx \leq s \leq s_n + \int_n^\infty f(x)\, dx.$$

Example 5: Consider the following series $\sum_{n=1}^{\infty} 30/n^4$.

a. Find the bounds on R_{10}

b. What bounds are on the sum for this series using $n = 10$?

c. If you wanted the error to be less than 0.005, what is the smallest value of n should you use?

Example 6: Consider the following series

$$\sum_{n=1}^{\infty} e^{-n}.$$

a. Approximate the sum of the series by using the sum of the first 5 terms. Estimate the error involved in this approximation.

b. How many terms are required to ensure that the sum is accurate to within 0.001?

c. Estimate $s = \sum_{n=1}^{\infty} e^{-n}$, with $n = 5$ terms.