MTH 151 (47 points) Fall 2022, Final Exam

Name: ______________________________ #(     ) December 13, 2022

Read all directions carefully and write your answers in the space provided. **To receive full credit, you must show all of your work.**

1. (3 points) Evaluate the following integral using u-substitution.

$$\int \frac{1}{x(\ln x)^2}\,dx$$

let $u = \ln x \longrightarrow du = \frac{1}{x}\,dx$

$$\int \frac{1}{x(\ln x)^2}\,dx = \int \frac{1}{u^2}\,du = \frac{-1}{u} + C$$

$$= \frac{-1}{\ln x} + C$$

2. (3 points) Evaluate the following integral using integration by parts.

$$\int_0^1 xe^{-2x}\,dx$$

| $u$ | $dv$ |
|---|---|
| $x$ | $e^{-2x}$ |
| $1$ | $\frac{-1}{2}e^{-2x}$ |
| $0$ | $\frac{+1}{4}e^{-2x}$ |

$+$ $-$

$$\int_0^1 xe^{-2x}\,dx = \frac{-1}{2}xe^{-2x} - \frac{1}{4}e^{-2x}\Big|_0^1$$

$$= \left[\frac{-1}{2}(1)e^{-2(1)} - \frac{1}{4}e^{-2(1)}\right] - \left[\cancel{\frac{-1}{2}(0)e^{-2(0)}}^{\,0} - \frac{1}{4}e^{\cancel{-2(0)}}{}^{\,1}\right]$$

$$= \frac{-1}{2}e^{-2} - \frac{1}{4}e^{-2} + \frac{1}{4} = -\frac{3}{4}e^{-2} + \frac{1}{4} = \frac{1}{4}\left(1 - 3e^{-2}\right)$$

3. (3 points) Estimate the maximum error (i.e. the error bound) involved in approximating $\int_0^{\pi/2} \cos(x)\,dx$ with $n = 6$ subintervals using Midpoint Rule.

$a = 0,\ b = \pi/2,\ n = 6,\ K??$

$f(x) = \cos x$

$f'(x) = -\sin x$

$f''(x) = -\cos x$

Now, $|f''(x)| = |-\cos x| \leq 1 = K$ since $-1 \leq \cos x \leq 1$

So, $|E_M| \leq \frac{K(b-a)^3}{24n^2} = \frac{1\cdot(\pi/2 - 0)^3}{24(6)^2} = \frac{\pi^3}{8*6*24}$ ##

4. (4 points) Find the indefinite integral

$$\int 3\sin^3(x)\cos^2(x)\,dx$$

$\int 3\sin^3 x \cos^2 x\,dx = \int 3\sin x \sin^2 x \cos^2 x\,dx$

$= \int 3\sin x(1-\cos^2 x)\cos^2 x\,dx$ $\quad u = \cos x \to du = -\sin x\,dx$

$= -3\int (1-u^2)u^2\,du = -3\int u^2 - u^4\,du$

$= -3\left[\frac{u^3}{3} - \frac{u^5}{5}\right] + C$

$= \frac{3}{5}u^5 - u^3 + C$

$= \frac{3}{5}(\cos x)^5 - (\cos x)^3 + C$ ##

5. (5 points) Compute the arc length of the graph of the given function on the interval given.

$$f(x) = 2(x-1)^{3/2} \quad \text{on } [1, 5]$$

$f'(x) = 3(x-1)^{1/2}$

$[f'(x)]^2 = 9(x-1)$

$1 + [f'(x)]^2 = 1 + 9x - 9 = 9x - 8$

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx = \int_{x=1}^{x=5} \sqrt{9x-8}\, dx$$

let $u = 9x - 8 \longrightarrow du = 9\, dx \longrightarrow dx = \frac{du}{9}$

If $x = 1 \longrightarrow u = 9 - 8 = 1$

$x = 5 \longrightarrow u = 9 \times 5 - 8 = 37$

$$= \int_{u=1}^{u=37} \sqrt{u} \cdot \frac{du}{9} = \frac{1}{9} \int_{u=1}^{u=37} u^{1/2}\, du$$

$$= \frac{1}{9} \cdot \frac{2}{3} u^{3/2} \Big|_{u=1}^{u=37}$$

$$= \frac{2}{27} \left( 37^{3/2} - 1^{3/2} \right)$$

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6. Use an appropriate test for sequence convergence to determine whether each of the following sequences converges or diverges.

(a) (2 points) $\frac{n}{2n+1}$

$\lim_{n\to\infty} \frac{n}{2n+1} = \lim_{n\to\infty} \frac{n}{2n} = \frac{1}{2}$ conv. #

(b) (2 points) $\frac{(-1)^n n}{2n+1}$

$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{n}{2n+1} \overset{\text{by (a)}}{=} \frac{1}{2} \neq 0$ div. #

by Thm 3 page 9

(c) (2 points) $\frac{1-2n^2}{3-5n^2}$

$\lim_{n\to\infty} \frac{1-2n^2}{3-5n^2} = \lim_{n\to\infty} \frac{-2n^2}{-5n^2} = \frac{-2}{-5}$ conv.

$= 2/5$ #

7. (4 points) Find the sum of the following geometric series $\sum_{n=k}^{\infty} \left(\frac{1}{3}\right)^{n-k+1}$.

two ways

1 $\sum_{n=k}^{\infty} \left(\frac{1}{3}\right)^{n-k+1} = \left(\frac{1}{3}\right)^{\cancel{k}-\cancel{k}+1} + \left(\frac{1}{3}\right)^{(\cancel{k}+1)-\cancel{k}+1)}$

$= \underset{a}{\frac{1}{3}} + \underset{ar}{\left(\frac{1}{3}\right)^2}$

$a = 1/3$, $r = \frac{ar}{a} = 1/3$

OR

2 $\sum_{n=k}^{\infty} \left(\frac{1}{3}\right)^{n-k+1} = \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n = \sum_{n=1}^{\infty} \underset{a}{\left(\frac{1}{3}\right)} \underset{r}{\left(1/3\right)}^{n-1}$

since $|\frac{1}{3}| < 1$ conv so, sum $= \frac{a}{1-r} = \frac{1}{3}/1-1/3 = \frac{\frac{1}{3}}{\frac{2}{3}} = 1/2$. #

8. (7 points) Determine if the following series converges:

$$\sum_{n=1}^{\infty} \frac{n}{2n+1}$$

.

by Q6 part (a)

DT:- $\lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2} \neq 0$

div.

9. (7 points) Determine if the following series converges and if so, find its sum:

$$\sum_{n=3}^{\infty} \frac{1}{n^2+n-6}$$

.

$$\frac{1}{n^2+n-6} = \frac{1}{(n+3)(n-2)} = \frac{A}{n+3} + \frac{B}{n-2}$$

$n=-3 \Rightarrow A = \frac{1}{-3-2} = -1/5$, $n=2 \Rightarrow B = \frac{1}{2+3} = 1/5$

So, $\sum_{n=3}^{\infty} \frac{-1/5}{(n+3)} + \frac{1/5}{n-2} = \frac{1}{5}\sum_{n=3} \frac{-1}{n+3} + \frac{1}{n-2}$

$S_n = \frac{1}{5}\sum_{k=3}^{n} \frac{1}{k-2} - \frac{1}{k+3} =$

$\frac{1}{5}\Big[\underset{k=3}{\left(\frac{1}{1} - \cancel{\frac{1}{6}}\right)} + \underset{k=4}{\left(\frac{1}{2} - \cancel{\frac{1}{7}}\right)} + \underset{k=5}{\left(\frac{1}{3} - \cancel{\frac{1}{8}}\right)} + \underset{k=6}{\left(\frac{1}{4} - \cancel{\frac{1}{9}}\right)}$
$+ \underset{k=7}{\left(\frac{1}{5} - \frac{1}{10}\right)} + \underset{k=8}{\left(\cancel{\frac{1}{6}} - \frac{1}{11}\right)} + \underset{k=9}{\left(\cancel{\frac{1}{7}} - \frac{1}{12}\right)}$
$+ \underset{k=10}{\left(\cancel{\frac{1}{8}} - \frac{1}{13}\right)} + \underset{k=11}{\left(\cancel{\frac{1}{9}} - \frac{1}{14}\right)} + \cdots + \underset{k=n}{\left(\frac{1}{n-2} - \frac{1}{n+3}\right)}$

$S_n = \frac{1}{5}\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \cancel{\frac{1}{n+3}}\right) \to 0$

$\lim_{n\to\infty} S_n = \boxed{\frac{1}{5}\left(1 + \frac{1}{2} + 1/3 + \frac{1}{4}\right)}$

This question is long. try to do

$$\sum_{n=1}^{\infty} Ln\left(\frac{n}{n+2}\right)$$

See the last page #

10. (5 points) Determine whether the following series converges or diverges.

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$$

**Hint**: Use the **integral test** and question 1. Assume that $f(x) = \dfrac{1}{x(\ln x)^2}$ is a decreasing and positive function.

by Integral test

consider $\displaystyle\int_2^{\infty} \frac{1}{x(\ln x)^2}\, dx$

by Q1: $\displaystyle\int \frac{1}{x(\ln x)^2}\, dx = \frac{-1}{\ln x} + C$

Now, $\displaystyle\int_2^{\infty} \frac{1}{x(\ln x)^2}\, dx = \lim_{t\to\infty} \left.\frac{-1}{\ln x}\right|_2^t$

$$= -\left(\cancel{\lim_{t\to\infty} \frac{1}{\ln t}}^{\,0} - \frac{1}{\ln 2}\right)$$

$$= \frac{1}{\ln 2} \qquad \text{conv.}$$

Hence, $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ conv. by Integral test

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$$\sum_{k=1}^{n} \ln\left(\frac{k}{k+2}\right) = \sum_{k=1}^{n} \ln(k) - \ln(k+2)$$

$$= (\overset{0}{\ln 1} - \cancel{\ln 3}) + (\ln 2 - \cancel{\ln 4}) + (\cancel{\ln 3} - \cancel{\ln 5})$$

$$+ \cdots (\cancel{\ln(n-1)} - \ln(n+1)) + (\cancel{\ln(n)} - \ln(n+2))$$

$S_n = \ln 2 - \ln(n+2)$

$\lim S_n \to \infty$ div.