Section 11.2: Series

Objective: In this lesson, you learn how to

- ☐ define a series and determine its convergence or divergence using partial sums and analyze geometric series, as well as harmonic series.

I. Series

Definition: Infinite series or series

An **infinite series or series** is the sum of an infinite sequence $a_1 + a_2 + a_3 + \cdots$ and is denoted by

$$\sum_{n=1}^{\infty} a_n \textbf{ or } \sum a_n.$$

Definition: Partial sums

If $\sum_{n=1}^{\infty} a_n$ is a series, then

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$$

is called its n^{th} **Partial sum**.

Remark: s_n is the partial sum of terms in the sequence $\{a_n\}$ from 1 to n, therefore,

$s_1 = a_1,$

$s_2 = a_1 + a_2 = s_1 + a_2,$

$s_3 = a_1 + a_2 + a_3 = s_2 + a_3,$

$s_4 = a_1 + a_2 + a_3 + a_4 = s_3 + a_4.$ etc....

Example 1: Find the first five partial sum terms of $\sum_{i=1}^{n} n$.

Convergent and divergent

Given a series $\sum_{i=1}^{\infty} a_i = a_1 + a_2 + a_3 + \cdots$, let s_n denote its n^{th} partial sum $s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$.

- The series $\sum a_n$ converges if the sequence of partial sums $\{s_n\}$ is convergent and we have

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{i=1}^{n} a_i = \sum_{i=1}^{\infty} a_i = s,$$

  the number s is called **the sum of the series**.

- The series $\sum a_n$ diverges if the sequence of partial sums $\{s_n\}$ is divergent (i.e. $\lim_{n\to\infty} s_n =$ DNE).

Example 2: Is the series $\sum_{i=1}^{\infty} n$ convergent or divergent?

How To Shift a Series:

Example 3: Adjust the series

$$\sum_{n=4}^{\infty} \frac{(-1)^{n+1}}{n-3},$$

so that the index will now start at $n = 1$.

Definition: Geometric series

The geometric series

$$\sum_{n=1}^{\infty} ar^{n-1},\ a \neq 0,$$

is convergent if $|r| < 1$ and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}, \quad |r| < 1$$

,

if $|r| \geq 1$, the geometric series is divergent

Proof :

Example 4: Present the number $6.2828282\cdots = 6.\overline{28}$ as a ratio of integers.

Example 5: Is the following series $\sum_{n=1}^{\infty} 2^{2n}3^{1-n}$ convergent or divergent?

Example 6: Find the sum of the following series

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$

Example 7: Find the value(s) of c so that

$$\sum_{n=1}^{\infty} (2c-5)^n$$

will converge. Find the sum for those values of c.

Definition: Telescoping series

A **telescoping series** is a series where each term b_n can be written as

$$b_n = a_n - a_{n+1},$$

for some series b_n.

Notice: To find the sum, we have

$$S_n = b_1 + b_2 + b_3 + \ldots + b_n = (a_1 - a_2) + (a_2 - a_3) + (a_3 - a_4) + \ldots + (a_n - a_{n+1}) = a_1 - a_{n+1}.$$

Therefore,

$$S_n = a_1 - a_{n+1}.$$

Thus, the telescoping series is convergent if $a_{n+1} \longrightarrow$ **finite number**, and the sum is

$$S = \lim_{n\to\infty} S_n = a_1 - \lim_{n\to\infty} a_{n+1}$$

Example 8: Show that the series

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)}$$

is convergent and find its sum.

Example 9: Show that the **harmonic**

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

is divergent.

Theorem 1

If the series $\sum_{n=1}^{\infty} a_n$ converges, then $\lim\limits_{n\to\infty} a_n = 0$.

Note: The converse is not true ingeneral.

$$\text{if } \lim_{n\to\infty}, \textbf{ we cannnot conclude that } \sum a_n \text{ is convergent}$$

Test for Divergence.

If $\lim\limits_{n\to\infty} a_n$ does not exist or if $\lim\limits_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

Remark:

- The statement follows from the theorem immediately preceding it, since if the series is not divergent, then it is convergent, and so $\lim\limits_{n\to\infty} a_n$ =0.
- Note that if $\lim\limits_{n\to\infty} a_n = 0$, the series $\sum_{n=1}^{\infty} a_n$ **may converge or it may diverge.**

Example 12: Determine whether the series is convergent or divergent. If it is convergent, find it sum

a. $\sum_{n=1}^{\infty} \dfrac{n-1}{3n-1}$

b. $\sum_{n=1}^{\infty} n \sin\left(\dfrac{1}{n}\right)$

c. $\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$

Theorem.

If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent series, then so are the series $\sum_{n=1}^{\infty} ca_n$ (where c is a constant), $\sum_{n=1}^{\infty} (a_n + b_n)$, and $\sum_{n=1}^{\infty} (a_n - b_n)$, and

a. $\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$.

b. $\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$.

c. $\sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$.

Example 13: Find the sum of the series $\displaystyle\sum_{n=1}^{\infty} \left(\frac{5}{2^n} - \frac{26}{(n+1)(n+2)} \right)$?