MTH 151 (32 points) Fall 2023, Exam 2

Name: ______________________ #(   ) Nov 15, 2023

No notes or other aids are allowed. Read all directions carefully and write your answers in the space provided. To receive full credit, you must show all of your work. The final answer without steps/work is only worth 0.25 of the points of the question

1. (6 points) Evaluate the following integral using a trigonometric substitution.

$$\int \frac{1}{x\sqrt{x^2+25}}\,dx$$

let $\boxed{x = 5\tan\theta} \to \boxed{dx = 5\sec^2\theta\,d\theta}$

$x^2+25 = 25\tan^2\theta + 25 = 25(\tan^2\theta+1) = 25\sec^2\theta$

$\boxed{\sqrt{x^2+25} = 5\sec\theta}$

$$\int \frac{1}{x\sqrt{x^2+25}}\,dx = \int \frac{1}{5\tan\theta\cdot\cancel{5\sec\theta}}\,\cancel{5}\sec^{\cancel{2}}\theta\,d\theta$$

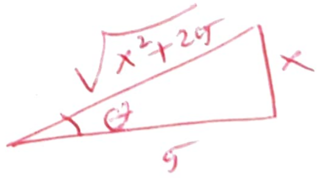
$$= \frac{1}{5}\int \frac{\sec\theta}{\tan\theta}\,d\theta$$

$$= \frac{1}{5}\int \frac{\frac{1}{\cancel{\cos\theta}}}{\frac{\sin\theta}{\cancel{\cos\theta}}}\,d\theta = \frac{1}{5}\int \csc\theta\,d\theta$$

$$= -\frac{1}{5}\ln|\csc\theta - \cot\theta| + C$$

$\tan\theta = \frac{x}{5}$

$$= -\frac{1}{5}\ln\left|\frac{\sqrt{x^2+25}}{x} - \frac{5}{x}\right| + C$$



2. (3 points) Write down the general form of the partial fraction decomposition of the following integral. You are **not required** to solve for the constants in the numerator. **Don't integrate.**

$$\int \frac{x^2+3}{x(x-1)(x^2+1)}\,dx$$

$$\frac{x^2+3}{x(x-1)(x^2+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{Cx+D}{x^2+1}$$

3. (6 points) Evaluate the following integral using partial fraction

$$\int \frac{1}{(x+1)(x^2+1)}\,dx$$

$$\frac{1}{(x+1)(x^2+1)} = \frac{A^{=1/2}}{x+1} + \frac{Bx+C^{=1/2}}{x^2+1}$$

$x=-1 \Rightarrow \quad A = \frac{1}{2}$

$x=0 \Rightarrow \quad \frac{1}{(1)(1)} = \frac{1}{2} + \frac{C}{1} \Rightarrow C = 1-\frac{1}{2} = 1/2$

$x=1 \Rightarrow \quad \frac{1}{(2)(2)} = \frac{1/2}{2} + \frac{B+1/2}{2}$

$\cancel{1} = \cancel{1} + 2B + 1 \Rightarrow \boxed{B = -1/2}$

$$\int \frac{1}{(x+1)(x^2+1)}\,dx = \int \frac{\frac{1}{2}}{x+1}\,dx + \int \frac{1/2\,dx}{x^2+1} - \frac{1}{2}\int \frac{x}{x^2+1}\,dx$$

$$= \frac{1}{2}\ln|x+1| + \frac{1}{2}\tan^{-1}(x) - \frac{1}{4}\ln|x^2+1| + C.$$

4. (a) (4 points) Approximate the value of

$$\int_1^7 \frac{dx}{x^2}$$

with $n = 6$ subintervals to 6 decimal places using Simpson's Rule.

$\Delta x = \frac{b-a}{n} = \frac{7-1}{6} = 1$

$x_0 \quad x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_6$

$1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7$

$S_6 = \frac{\Delta x}{3}\left(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + 4f(x_5) + f(x_6)\right)$

$= \frac{1}{3}\left(1 + 4\cdot\frac{1}{4} + 2\cdot\frac{1}{9} + 4\cdot\frac{1}{16} + 2\cdot\frac{1}{25} + 4\cdot\frac{1}{36} + \frac{1}{49}\right)$

(b) (3 points) How large should n be in order to guarantee that the Simpson's Rule estimate for $\int_1^7 \frac{dx}{x^2}$ is accurate to within $0.001 = 10^{-3}$?

$f(x) = x^{-2}$, $f'(x) = -2x^{-3}$, $f''(x) = 6x^{-4}$, $f'''(x) = -24x^{-5}$, $f^{(4)}(x) = 120x^{-6}$

$|f^{(4)}(x)| \leq 120$ on $[1, 7]$.

$|E_s| \leq 10^{-3}$

$\frac{120(7-1)^5}{180n^4} \leq 10^{-3} \Rightarrow \frac{120(6)^5}{180 * 10^{-3}} \leq n^4$

$n^4 \geq 5{,}184{,}000$

$n \geq \sqrt[4]{5{,}184{,}000} \approx 47.7$

$\boxed{n \geq 48}$

5. (5 points) Determine if the following integral converges of diverges. Explain why.

$$\int_0^\infty 4xe^{-x^2}\,dx$$

$$\int_0^\infty 4x e^{-x^2} dx = \lim_{t\to\infty} \int_0^t 4x e^{-x^2} dx$$

let $u = x^2 \rightarrow du = 2x\,dx \rightarrow \frac{du}{2x} = dx$

$=$ ~~lim~~

$$\int 4x e^{-x^2} dx = \int \cancel{4x} e^{-u} \frac{du}{\cancel{2x}}$$

$$= 2\int e^{-u} du = -2e^{-u} + C$$

$$\int_0^\infty 4x e^{-x^2} dx = \lim_{t\to\infty} \int_0^t 4x e^{-x^2} dx$$

$$= \lim_{t\to\infty} -2e^{-x^2} \Big|_0^t$$

$$= \lim_{t\to\infty} -2(\cancel{e^{-t^2}} \cancel{+} - e^{-0})$$

$$= -2(0-1) = 2 \quad \text{conv.}$$

6. (5 points) Compute the arc length of the curve $f(x) = \ln(\cos x)$ over the interval $[0, \pi/4]$.

$$L = \int_a^b \sqrt{1+[f'(x)]^2}\, dx$$

$$f(x) = \ln|(\cos x)| \rightarrow f'(x) = \frac{-\sin x}{\cos x} = -\tan x$$

$$1+[f'(x)]^2 = 1+\tan^2 x = \sec^2 x$$

$$\sqrt{1+[f'(x)]^2} = \sqrt{\sec^2 x} = \sec x$$

$$L = \int_0^{\pi/4} \sec x\, dx$$

$$= \ln|\sec x + \tan x| \Big|_0^{\pi/4}$$

$$= \ln|\sec \pi/4 + \tan \pi/4| \overset{0}{\cancel{- \ln|\overset{1}{\sec 0} + \overset{0}{\tan(0)}|}}$$

$$= \ln|\sqrt{2}+1|$$