Section 7.3: Trigonometric substitution

Objective: In this lesson, you learn

☐ How to evaluate integrals of certain forms using trigonometric substitutions.

I. Trigonometric Substitution.

Problem: Find the area of a circle or an ellipse. i.e.

$$\int \sqrt{a^2 - x^2}\, dx \text{ where } a > 0?$$

Note that the difference between the substitution $u = a^2 - x^2$ and $x = a\sin\theta$ is

- $u = a^2 - x^2$, u (new) is a function of (old) x.
- $x = a\sin\theta$, x (old) is a function of (new) θ.

In general, to integrate $\int f(x)dx$, we can make a substitution of the form $x = g(t)$ by using the Substitution Rule in reverse, called an **inverse substitution.** To make the calculations simpler, assume that g has an **inverse function**, that is, $x = g(t)$ is a one-to-one function. So if $x = g(t)$, then

$$\int f(x)\ dx = \int f(g(t))\, g'(t)\, dt.$$

Example 1: Integrate $\int \sqrt{a^2 - x^2}\, dx$

Trigonometric substitution:

Here are trigonometric substitutions:

a. For the expression $\sqrt{a^2 - x^2}$, make a substitution $x = a\sin\theta$ defined on $[-\pi/2,\ \pi/2]$, and use the identity $1 - \sin^2\theta = \cos^2\theta$.

b. For the expression $\sqrt{a^2 + x^2}$, make a substitution $x = a\tan\theta$ defined on $(-\pi/2,\ \pi/2)$, and use the identity $1 + \tan^2\theta = \sec^2\theta$.

c. For the expression $\sqrt{x^2 - a^2}$, make a substitution $x = a\sec\theta$ defined on $[0,\ \pi/2)$ or $[\pi,\ 3\pi/2)$, and use the identity $\sec^2\theta - 1 = \tan^2\theta$.

Example 2: Evaluate $\int_0^1 x^3\sqrt{1 - x^2}dx$.

Example 3: Evaluate $\int \frac{1}{1+x^2}dx$.

Example 4: Evaluate $\displaystyle\int \frac{1}{x^2\sqrt{x^2+4}}$

Example 5: Evaluate $\int \frac{x^2+1}{(x^2-2x+2)^2}\,dx$.

Example 6: Evaluate $\int \frac{1}{x^2\left(x^2-9\right)^{1/2}} dx$.