Section 6.2: Volumes

Objective: In this lesson, you learn

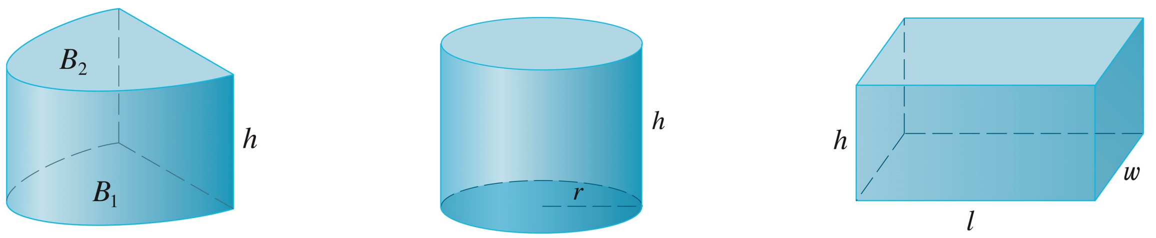
☐ How to establish the volume of a solid of revolution as the limit of a Riemann sum.

I. Volumes

The volume of a cylindrical solid is always defined to be its base area times its height:

$$\textbf{Volume} = \textbf{Area} \times \textbf{Height}$$

Some classic formulas: A solid S that is a right cylinder (a known base area A and height h).



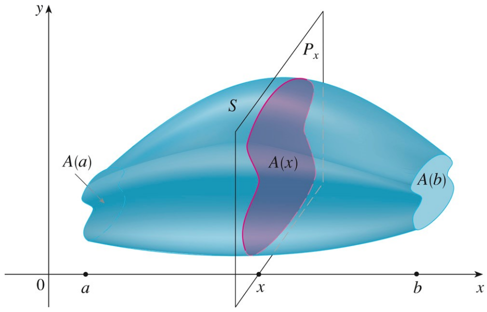
(a) Cylinder $V = Ah$ (b) Circular cylinder $V = \pi r^2 h$ (c) Rectangular box $V = lwh$

Problem: How do we define the volume of a solid S that is not a right cylinder?

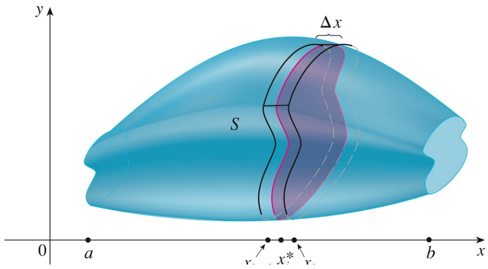
Volumes

For a solid S that is not a right cylinder, we "cut" into pieces and approximate each piece by a cylinder. Then the volume is estimated by adding the volumes of the cylinders.

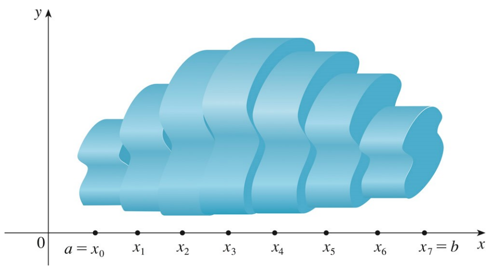
1. Start by intersection S with a plane region that is called a **cross-section** of S.
2. Let $A(x)$ be the area of the cross-section of S in a plane P_x cutting S perpendicularly to the $x-$axis and passing through the point x, where $a \leq x \leq b$.



3. Divide S into n "slabs" of equal width $\triangle x$ by using the planes $P_{x_1}, P_{x_2}, \cdots, P_{x_n}$ to slice the solid.
4. If we choose samples points x^* in $[x_{i-1}, x_i]$, we can approximate the i^{th} slab S_i by a cylinder with base area $A(x_i^*)$ and height $\triangle x$.
5. The volume of this cylinder is $A(x_i^*)\triangle x$.



6. Add the volumes of the slabs to approximate the total volume, V, which is the limit of
$$\sum_{i=1}^{n} A(x_i^*)\triangle x \ .$$
7. The approximation gets better as $n \to \infty$.

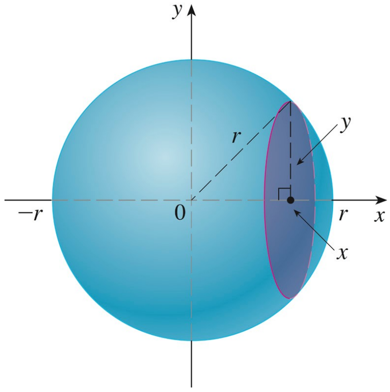


So we can define the volume of a solid as follows:

Let S be a solid that lies between $x = a$ and $x = b$. If the cross-sectional area of S in the plane P_x, through S and perpendicular to the x-axis, is $A(x)$, where A is a continuous function, then the volume of S is
$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\,\Delta x = \int_a^b A(x)\,dx.$$

Remark: Note that in the volume formula $V = \int_a^b A(x)dx$, $A(x)$ is the area of a moving cross-section obtained by slicing through x perpendicular to the x-axis.

Example 1: Show that the volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$



II. Solids of revolution

Solids of revolution are obtained by revolving a region about a line.

Compute the volume of a solid of revolution by using the basic formula $V = \int_a^b A\,(x)\,dx$ or $V = \int_c^d A\,(y)\,dy$, and find the cross-sectional area $A(x)$ or $A(y)$ in one of the following ways:

i. If the cross-section is a disk, find the radius of the disk (in terms of x or y) and use

$$A = \pi(\text{radius})^2.$$

ii. If the cross-section is a washer, find the inner radius and outer radius and use

$$A = \pi(\text{outer radius})^2 - \pi(\text{inner radius})^2.$$

Example 2: Calculate the volume of the solid, obtained by rotating the curve $y = x^3$ for $0 \leq x \leq 1$ around the $x-$axis?

Example 3: Rotate the same curve, in example 2, around the $y-$axis?

Example 4: Rotate the same curve, in example 2, around the $y = -1$?

Example 5: Calculate the volume of the solid, obtained by rotating the region R around the $x-$axis, where the region R is enclosed by the curves $y = x^2$ and $y = x$?

Example 6: Rotate the same curve, in example 5, around the $y-$axis?

Example 7: Rotate the same curve, in example 5, around the $y = 2$?

Homework: Rotate the same curve, in example 5, around the $x = -1$ or $x = 2$?