Review: Calculus I

Section 4.9: Antiderivatives

Objective: In this lesson, you learn

☐ how to find antiderivatives of functions.

I. Antiderivatives

Definition

A function F is called an **antiderivative** of f on an interval I if

$$F'(x) = f(x)$$

for all x in I.

$F(x) = x^2$
$f(x) = 2x$
$F'(x) = 2x = f(x)$

$G(x) = x^2 + 4$
$G'(x) = f(x)$
$= 2x$

Remark: If two functions have identical derivatives on an interval, then by the Mean Value Theorem, they must differ by a constant.

Thus, if F and G are any two antiderivatives of f, then $F'(x) = f(x) = G'(x)$. So

$$G(x) = F(x) + C.$$

Example 1: Find an antiderivative of f of the following functions

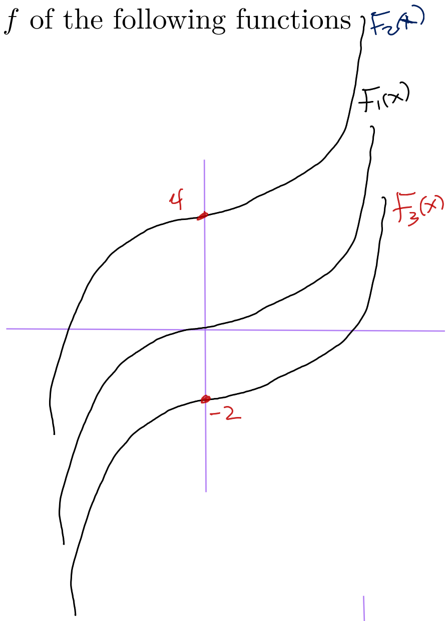
a. $f(x) = x^2$

$F_1(x) = \frac{1}{3}x^3$

$F_2(x) = \frac{1}{3}x^3 + 4$

$F_3(x) = \frac{1}{3}x^3 - 2$

⋮



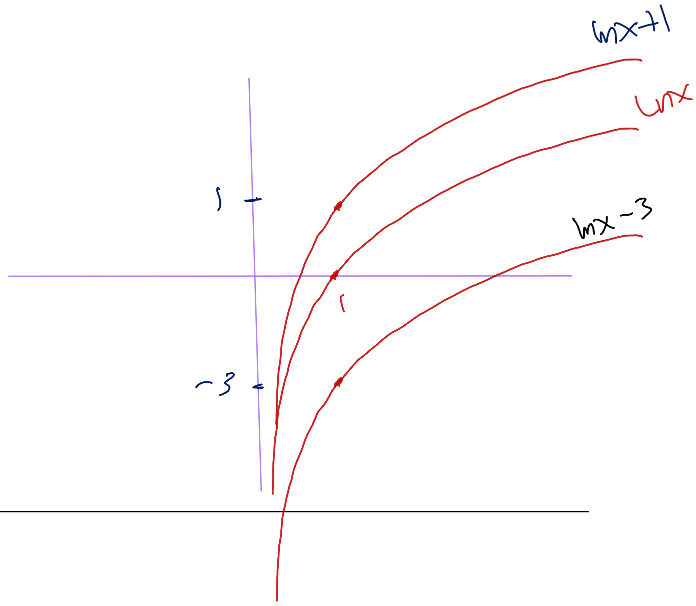
b. $f(x) = \frac{1}{x}$

$F_1(x) = \ln x$

$F_2(x) = \ln x + 1$

$F_3(x) = \ln x - 3$

⋮



Theorem

If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is

$$F(x) + C$$

where C is an arbitrary constant.

II. Antidifferentiation

The following table lists some of Antiderivative Formulas. (Assume $F' = f$ and $G' = g$.)

① $f(x)=3$
$f'(x)=0$
$F(x)=3x+C$

② $f(x)=x^4$
$f'(x)=4x^3$
$F(x)=\frac{x^5}{5}+C$

Some of Antiderivative/derivative Formulas

Antiderivative	**Function**	**Derivative**
$F(x) + C$	$f(x)$	$f'(x)$
$F(x) \pm G(x) + C$	$f(x) \pm g(x)$	$f'(x) \pm g'(x)$
① $kx + C$	k, k–constant	0
② $\frac{x^{n+1}}{n+1} + C$	$\cancel{x^n}$ $\boxed{(n \neq -1)}$	nx^{n-1}
③ $\frac{2x^{3/2}}{3} + C$ $\quad \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}$	$\sqrt{x} = x^{\frac{1}{2}}$	$\frac{1}{2\sqrt{x}}$ $\quad \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-1/2} = \frac{1}{2}\frac{1}{x^{1/2}} = \frac{1}{2\sqrt{x}}$
④ $\frac{b^x}{\ln b} + C$	b^x	$b^x \ln b$
$e^x + C$	e^x	e^x
$\ln \lvert x \rvert + C$, $x \neq 0$	$\frac{1}{x} = x^{-1}$, $x \neq 0$	$\frac{-1}{x^2}$ $\quad f(x)=x^{-1}$, $f'(x)=-1x^{-2} = \frac{-1}{x^2}$
$-\cos x + C$	$\sin x$	$\cos x$
$\sin x + C$	$\cos x$	$-\sin x$
$\tan x + C$	$\sec^2 x$	$2\sec^2 x \tan x$
$-\cot x + C$	$\csc^2 x$	$-2\csc^2 x \cot x$
$\sec x + C$	$\sec x \tan x$	$\sec x (\sec^2 x + \tan^2 x)$
$\csc x + C$	$-\csc x \cot x$	$\csc x (\csc^2 x + \cot^2 x)$
$\tan^{-1} x + C$	$\frac{1}{1+x^2}$	$\frac{-2x}{(1+x^2)^2}$
$\sin^{-1} x + C$	$\frac{1}{\sqrt{1-x^2}}$	$\frac{x}{(1-x^2)^{(5/2)}}$

$(h \cdot g)' = h \cdot g' + g \cdot h'$

$f(x) = \sec x \tan x$
$f'(x) = \sec x \sec^2 x + \tan x \sec x \tan x = \sec x(\sec^2 x + \tan^2 x)$

③ $\sqrt{2x+1}$
$f'(x) = \frac{1}{2\sqrt{2x+1}} * 2$
Chain Rule $= \frac{1}{\sqrt{2x+1}}$

$f(x) = \sqrt{g(x)}$
$f'(x) = \frac{g'(x)}{2\sqrt{g(x)}}$

④ $f(x) = e^{g(x)}$
$f'(x) = g'(x) e^{g(x)}$
$f(x) = e^{x^2}$
$f'(x) = 2x e^{x^2}$

⑤ $f(x) = \ln(g(x))$
$f'(x) = \frac{g'(x)}{g(x)}$
$f(x) = \ln(-x)$, $x<0$
$f'(x) = \frac{-1}{-x} = \frac{1}{x}$

⑥ $f(x) = \frac{1}{g(x)}$
$f'(x) = \frac{-g'(x)}{(g(x))^2}$

$f(x) = \sec^2 x = (\sec x)^2$
$f'(x) = 2(\sec x) * \sec x \tan x = 2\sec^2 x \tan x$

$\frac{x^{3/2}}{\frac{2}{3}} = x^{3/2} \div \frac{2}{3} = x^{3/2} * \frac{3}{2} = \frac{3x^{3/2}}{2}$

$\lvert x \rvert = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$

Example 2: Find all anti-derivative of

$$f(x) = x^3 + 3\sqrt{x} + \frac{4}{x} + 2$$

$$F(x) = \frac{x^{3+1}}{3+1} + \cancel{3} \cdot \frac{2x^{3/2}}{\cancel{3}} + 4\ln|x| + 2x + C$$

$$= \frac{1}{4}x^4 + 2x^{3/2} + 4\ln|x| + 2x + C.$$

Example 3:

a. Find f if $f'(t) = 2\cos(t) + 3e^t$

$$f(t) = 2\sin t + 3e^t + C$$

b. Which of the functions in part (a) satisfies $f(0) = 0$?

$$0 = f(0) = 2\sin^{0}(0) + 3e^{0} + C$$

$$0 = 3 + C$$

$$\boxed{C = -3}$$

$$f(x) = 2\sin t + 3e^t - 3$$