

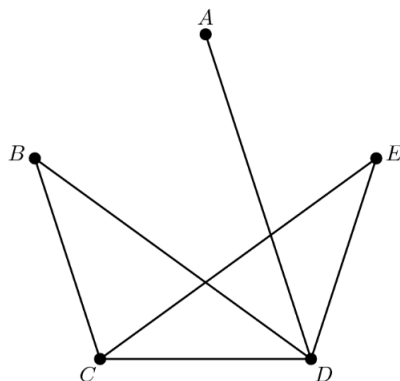
Combinatorics & Computer Science

1. The intramural squash league has 5 players, namely Albert, Bassim, Clara, Daniel, and Eugene. Albert has played one game, Bassim has played two games, Clara has played 3 games, and Daniel has played 4 games. Assuming no two players in the league play each other more than one time, how many games has Eugene played?

Proposed by Daniel Li

Answer: 2

Solution. Daniel must play each of Albert, Bassim, Clara, and Eugene; this uses up the only game Albert plays, and so Clara must also play Bassim and Eugene. But now Bassim has played two games, and so all conditions are satisfied. Thus Eugene has played 2 games.



2. David is taking a true/false exam with 9 questions. Unfortunately, he doesn't know the answer to any of the questions, but he does know that exactly 5 of the answers are True. In accordance with this, David guesses the answers to all 9 questions, making sure that exactly 5 of his answers are True. What is the probability he answers at least 5 questions correctly?

Proposed by David Altizio

Answer: $\frac{9}{14}$

Solution. Without loss of generality, assume the first five questions have answer **True** and the last four questions have answer **False**; this has no bearing on the answer to the question.

Suppose David answers k of the **True** questions with answer **True**. Then David has used $5 - k$ of his allotted **False** answers, so $4 - (5 - k) = k - 1$ of the last four questions are answered **False**. Thus, David answered $2k - 1$ of the questions correctly; in turn, he answers at least 5 questions correctly if and only if $k \geq 3$.

Finally, the number of ways for David to answer k of the **True** questions with **True** is $\binom{5}{k} \binom{4}{k-1}$, so the desired probability is

$$\frac{\binom{5}{3} \binom{4}{2} + \binom{5}{4} \binom{4}{1} + \binom{5}{5} \binom{4}{0}}{\binom{9}{5}} = \frac{81}{126} = \frac{9}{14}.$$

3. Consider a 1-indexed array that initially contains the integers 1 to 10 in increasing order.

The following action is performed repeatedly (any number of times):

`def action():`

`Choose an integer n between 1 and 10 inclusive`

`Reverse the array between indices 1 and n inclusive`

`Reverse the array between indices n+1 and 10 inclusive (If n = 10, we do nothing)`

How many possible orders can the array have after we are done with this process?

Proposed by Dilhan Salgado

Answer: 20

Solution. The move is equivalent to reversing the array and cycling the elements. For the final position, there are 2 options for the direction of the array and 10 options for the first element (which together uniquely determine the entire array). Thus there are $2 \cdot 10 = 20$ total orderings.

4. The continent of Trianglandia is an equilateral triangle of side length 9, divided into 81 triangular countries of side length 1. Each country has the resources to choose at most 1 of its 3 sides and build a “wall” covering that entire side. However, since all the countries are at war, no two countries are willing to have their walls touch, even at a corner. What is the maximum number of walls that can be built in Trianglandia?

Proposed by Adam Bertelli

Answer: 27

Solution. We claim the answer is $\boxed{27}$.

Note that there are 55 total “corners” of individual triangular countries, and each wall takes up exactly two corners. In addition, no two walls can share a corner; thus there are at most $\lfloor \frac{55}{2} \rfloor = 27$ walls. To construct this, we can simply start at the topmost point, and follow a path that “snakes” through each row from top to bottom, alternating left and right on each row. If we then turn every other segment in this path into a wall, since this path covers every point, we will end up with 27 walls as desired.

5. Seven cards numbered 1 through 7 lay stacked in a pile in ascending order from top to bottom (1 on top, 7 on bottom). A shuffle involves picking a random card *of the six not currently on top* and putting it on top. The relative order of all the other cards remains unchanged. Find the probability that, after 10 shuffles, 6 is higher in the pile than 3.

Proposed by Sam Delatore

Answer: $\frac{3^{10}-2^{10}}{2 \cdot 3^{10}}$ OR $\frac{58025}{118098}$

Solution. If neither a three or six is ever picked, the three will be on top of the six so the condition will never happen. This occurs with probability $(\frac{4}{6})^{10} = \frac{2^{10}}{3^{10}}$. Over all cases where at least one is picked, the probability is $\frac{1}{2}$ by symmetry. Thus the answer is

$$\frac{1}{2} \left(1 - \frac{2^{10}}{3^{10}} \right) = \boxed{\frac{3^{10} - 2^{10}}{2 \cdot 3^{10}}}.$$

6. The nation of CMIMCland consists of 8 islands, none of which are connected. Each citizen wants to visit the other islands, so the government will build bridges between the islands. However, each island has a volcano that could erupt at any time, destroying that island and any bridges connected to it. The government wants to guarantee that after any eruption, a citizen from any of the remaining 7 islands can go on a tour, visiting each of the remaining islands exactly once and returning to their home island (only at the end of the tour). What is the minimum number of bridges needed?

Proposed by Daniel Li

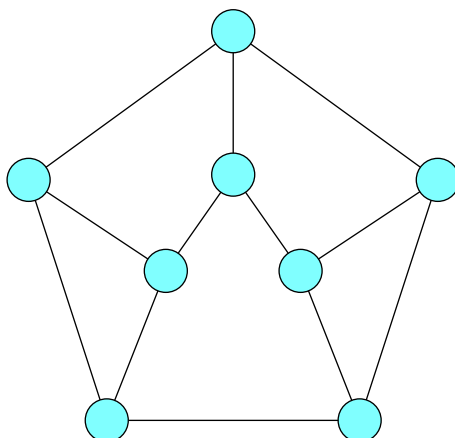
Answer: 12

Solution. The critical claim is that every island must have at least 3 bridges. First note that for each island, any tour through it requires the use of at least 2 distinct bridges. Assume for the sake of contradiction that some island I has at most 2 bridges. Then consider the case

where one of the islands adjacent to I is destroyed. Then there is only one remaining bridge connected to I , so we cannot form a complete tour.

Now as each island has at least 3 bridges, the total number of “bridge ends” must be at least $3 \cdot 8 = 24$, and as each bridge provides exactly 2 “bridge ends” there must be at least $\frac{24}{2} = 12$ bridges.

The following construction shows that 12 is achievable.



7. Consider a complete graph of 2020 vertices. What is the least number of edges that need to be marked such that each triangle (3-vertex subgraph) has an odd number of marked edges?

Proposed by Joshua Abrams

Answer: 1019090

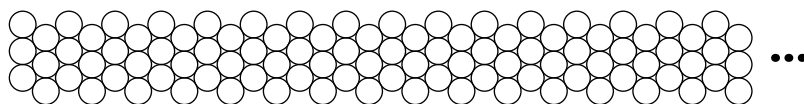
Solution. Suppose we have at least 2 connected components of marked edges. If we choose two points in a connected component of marked edges, and a third point outside of this connected component, then no edge containing the third point can be marked, thus the edge between the first two must be marked, i.e. any connected component must be a complete graph. In addition, we cannot have more than 3 connected component, otherwise we could pick one point from each and get a triangle with no marked edges. Thus in this case, we have $\binom{n}{2} + \binom{2020-n}{2}$ edges, which is minimized at $n = 1010$ by convexity of the function $\frac{n(n-1)}{2}$.

Otherwise, if there is only one connected component, suppose some two points, A and B , have an unmarked edge between them. Then, every other point must share a marked edge with exactly one of these two (because the triangle among all 3 must have one marked edge). In addition, if some point P connects to A , and some point Q connects to B , P and Q cannot share a marked edge by triangle APQ . Thus we actually end up with (at least) two disconnected components, those adjacent to A and those adjacent to B , contradiction. Therefore we could not have started with a missing edge, so our graph was actually the complete graph, giving us $\binom{2020}{2}$ marked edges.

The first case is clearly smaller, so our answer is

$$2 \binom{1010}{2} = 1010 \cdot 1009 = \boxed{1019090}.$$

8. Catherine has a plate containing 300 circular crumbling mooncakes, arranged as follows:



(This continues for 100 total columns). She wants to pick some of the mooncakes to eat, however whenever she takes a mooncake all adjacent mooncakes will be destroyed and cannot be eaten. Let M be the maximal number of mooncakes she can eat, and let n be the number of ways she can pick M mooncakes to eat (Note: the order in which she picks mooncakes does not matter). Compute the ordered pair (M, n) .

Proposed by Dilhan Salgado

Answer: $(100, F_{104} + 1223)$

Solution. First, note that $M = 100$. This is certainly achievable: take the first and third mooncake in the odd numbered columns. In any two adjacent columns $(i, i+1)$, she can take at most 2 mooncakes so this gives the desired bound.

In fact, every configuration of M mooncakes satisfies that for all $(i, i+1)$ with i odd Catherine takes exactly 2 mooncakes (this is both necessary and sufficient).

Let $f_{a,b,n}$ denote the number of ways to eat all the mooncakes when in columns $(1, 2)$ we eat cakes a and b (where the cakes are numbered $1, 2, \dots, 6$ in order left to right, top to bottom), and there are n pairs of columns. Notice that $f_{a,b,1} = 1$ for all a, b . Then the final answer is $f_{50} = f_{1,4,50} + f_{1,5,50} + f_{1,6,50} + f_{2,5,50} + f_{2,6,50} + f_{3,6,50}$.

Note that the following are true:

$$\begin{aligned} f_{1,4,n} &= f_{1,4,n-1} + f_{1,6,n-1} + f_{2,6,n-1} \\ f_{1,5,n} &= f_{n-1} \\ f_{1,6,n} &= f_{1,4,n-1} + f_{1,6,n-1} + f_{2,6,n-1} + f_{3,6,n-1} \\ f_{2,5,n} &= f_{2,5,n-1} + f_{2,6,n-1} \\ f_{2,6,n} &= f_{2,6,n-1} \\ f_{3,6,n} &= f_{1,4,n-1} + f_{1,6,n-1} + f_{2,6,n-1} + f_{3,6,n-1} \end{aligned}$$

Since $f_{2,6,1} = 1$ this implies that $f_{2,6,n} = 1$. From $f_{2,5,n} = f_{2,5,n-1} + 1$ we then have $f_{2,5,n} = n$. Finally, $f_{1,6,n} = f_{3,6,n}$ and $f_{1,4,n} = f_{1,6,n} - f_{1,6,n-1}$.

This implies that

$$f_{1,6,n} = 3f_{1,6,n-1} - f_{1,6,n-2} + 1$$

Inductively, this gives $f_{1,6,n} = F_{2n+1} - 1$ where F_n are the Fibonacci numbers (one way to see this is that the roots of the characteristic polynomial are the squares of those for the Fibonacci recurrence). Hence, $f_{1,4,n} = F_{2n}$.

Finally,

$$f_n = f_{n-1} + F_{2n} + 2F_{2n+1} + n - 1 = f_{n-1} + F_{2n+3} + n - 1 \implies f_n = F_{2n+4} + \binom{n}{2} - 2$$

where the last implication follows from the base case $f_1 = 6$. This gives $f_{50} = F_{104} + 1223$, and so the desired answer is $\boxed{(100, F_{104} + 1223)}$.

9. Let $\Gamma = \{\varepsilon, 0, 00, \dots\}$ be the set of all finite strings consisting of only zeroes. We consider *six-state unary DFAs* $D = (F, q_0, \delta)$ where F is a subset of $Q = \{1, 2, 3, 4, 5, 6\}$, not necessarily strict and possibly empty; $q_0 \in Q$ is some *start state*; and $\delta : Q \rightarrow Q$ is the *transition function*. For each such DFA D , we associate a set $F_D \subseteq \Gamma$ as the set of all strings $w \in \Gamma$ such that

$$\underbrace{\delta(\dots(\delta(q_0))\dots)}_{|w| \text{ applications}} \in F,$$

We say a set \mathcal{D} of DFAs is *diverse* if for all $D_1, D_2 \in \mathcal{D}$ we have $F_{D_1} \neq F_{D_2}$. What is the maximum size of a diverse set?

Proposed by Misha Ivkov

Answer: 306

Solution. Define the minimal DFA D' associated with a language L as the unique DFA on a minimal number of states with $L(D') = L$. We are guaranteed such a DFA by Myhill-Nerode, whose proof we defer to the first ever CMIMC Power Round.

Note that for any DFA D on $< n$ states, we can just add some unreachable states so that it has size exactly n . Hence we just need to count minimal DFAs of size at most n .

First, let us characterize this set. Since every node has exactly 1 out-transition, we can view a unary DFA as being a linked list where it is possible for there to be a cycle. Consider the loop, and associate a binary word v with it where $v_i = 1$ if the i th node of the loop is in F and 0 otherwise. Then to be a minimal DFA, we must have that this loop is minimal in the sense that there does not exist x such that $v = x^k$ for $k \geq 2$. Certainly, we also cannot have any unreachable nodes in our linked list. There is only one other constraint: the end of the loop cannot have the same output as the node directly before the loop. Otherwise, we could simply redirect the second to last node in the loop to the one before the loop.

These three conditions characterize minimal DFAs, so all we have left to do is count. Let $f(n)$ denote the number of “minimal” (or “primitive”, as often referred to in the literature) words v of length n . This is derived in CMIMC 2019 Combo/CS #7, but we will not use its exact value until the end.

Note that the number of minimal DFAs on n states is

$$m(n) = f(n) + \sum_{i=1}^{n-1} f(i)2^{n-i-1}$$

by utilizing the last constraint and reindexing the second sum. Hence the number of minimal DFAs on at most n states is

$$M(n) = \sum_{k=1}^n \left(f(k) + \sum_{i=1}^{k-1} f(i)2^{k-i-1} \right) = \sum_{k=1}^n f(k)2^{n-k}.$$

Now recalling that

$$f(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) 2^d$$

and computing gives $M(6) = \boxed{306}$.

10. Define a string to be doubly palindromic if it can be split into two (non-empty) parts that are read the same both backwards and forwards. For example hannahhuh is doubly palindromic as it can be split into hannah and huh. How many doubly palindromic strings of length 9 using only the letters $\{a, b, c, d\}$ are there?

Proposed by Dilhan Salgado

Answer: 8104

Solution. First consider all possible splitting points. We can split after the first i characters for $1 \leq i \leq 8$. The number of palindromes of length l is $4^{\lceil \frac{l}{2} \rceil}$. Thus the total number of possible double palindromes for each splitting point will be $4^{\lceil \frac{i}{2} \rceil + \lceil \frac{9-i}{2} \rceil} = 4^5$. (Note exactly one of the divisions will round up). Thus the initial answer is $8 \cdot 4^5 = 8192$.

However, we have over-counted strings such as ‘aaaaaaaa’ which can be split in multiple places. Assume that a string s can be split in two places. Thus $s = xyz$ where x, yz are palindromes, and xy, z are palindromes. Let v' denote the reverse of v for arbitrary v . We thus know: $x = x', yz = z'y', xy = y'x', z = z'$. We can then see $zxy = zy'x' = z'y'x = yzx$, so $zxy = yzx$ which is a non-trivial cyclic shift. Thus we know that s is a nontrivial cyclic-shift of itself. As the length of s is 9, this implies $s = ttt$ for some t of length 3. We now case on t .

- $t = aaa$ (and symmetric - 4 total ways) \rightarrow 8 splitting points, overcount of 7.
- $t = aab$ (and symmetric - 12 total ways) \rightarrow 3 splitting points, overcount of 2.
- $t = aba$ (and symmetric - 12 total ways) \rightarrow 2 splitting points, overcount of 1.
- $t = abb$ (and symmetric - 12 total ways) \rightarrow 3 splitting points, overcount of 2.
- $t = abc$ (and symmetric - 12 total ways) \rightarrow 0 splitting points, no overcount.

So, the total overcount is $4 \cdot 7 + 12 \cdot 2 + 12 \cdot 1 + 12 \cdot 2 = 88$. Subtracting from 8192 gives the desired answer of $\boxed{8104}$.

11. (Estimation) Max flips 2020 fair coins. Let the probability that there are at most 505 heads be p . Estimate $-\log_2(p)$ to 5 decimal places, in the form $x.abcd e$ where x is a positive integer and a, b, c, d, e are decimal digits.

Proposed by Misha Ivkov

Answer: (5.3605584085, 117)