1 Terms

1. $\langle Q \rangle_{\sqrt{g}} = \frac{\oiint \mathrm{d}\theta \mathrm{d}\zeta \sqrt{g}\, Q}{\oiint \mathrm{d}\theta \mathrm{d}\zeta \sqrt{g}}$ is the flux surface average of Q.
2. $\langle Q \rangle = \frac{\oiint \mathrm{d}\theta \mathrm{d}\zeta\, Q}{\oiint \mathrm{d}\theta \mathrm{d}\zeta}$ is the flux surface average without $\sqrt{g}$ of Q
3. $I = \langle B_\theta \rangle$ is the net toroidal current in tesla-meters
4. `data["I"]` $= \frac{\mu_0}{2\pi}$ `data["current"]`
5. ψ is the toroidal flux
6. $\frac{2\pi}{\mathrm{d}\psi / \mathrm{d}\rho}$ is `1 / data["psi_r"]`
7. λ is the stream function of θ and ζ
8. ι is the rotational transform

2 Rotational transform

Any vector can be written as a contravariant coordinate vector dot product with covariant basis vectors.

$$\boldsymbol{B} = B^\rho \boldsymbol{e}_\rho + B^\theta \boldsymbol{e}_\theta + B^\zeta \boldsymbol{e}_\zeta \tag{1}$$

Assume nested flux surfaces.

$$\boldsymbol{B} = B^\theta \boldsymbol{e}_\theta + B^\zeta \boldsymbol{e}_\zeta \tag{2}$$

The components of the magnetic field are defined as

$$B^\theta = \frac{\mathrm{d}\psi / \mathrm{d}\rho}{2\pi \sqrt{g}} \left(\iota - \frac{\partial \lambda}{\partial \zeta} \right) \tag{3}$$

$$B^\zeta = \frac{\mathrm{d}\psi / \mathrm{d}\rho}{2\pi \sqrt{g}} \left(1 + \frac{\partial \lambda}{\partial \theta} \right) \tag{4}$$

Rearranging gives an equation for the rotational transform.

$$\iota = \frac{2\pi \sqrt{g}}{\mathrm{d}\psi / \mathrm{d}\rho} B^\theta + \frac{\partial \lambda}{\partial \zeta} \tag{5}$$

This suggests you can chain a relation between ι and I through the common terms B^θ and B_θ in the equations for ι and I, respectively.

$$I = \langle B_\theta \rangle \tag{6}$$

$$= \langle \boldsymbol{B} \cdot \boldsymbol{e}_\theta \rangle \tag{7}$$

$$= \left\langle \left(B^\theta \boldsymbol{e}_\theta + B^\zeta \boldsymbol{e}_\zeta \right) \cdot \boldsymbol{e}_\theta \right\rangle \tag{8}$$

Plugging in definitions for the coefficients of the metric tensor g,

$$I = \left\langle B^\theta g_{\theta\theta} + B^\zeta g_{\theta\zeta} \right\rangle \tag{9}$$

Surface average operations are additive homomorphisms.

$$I = \left\langle B^\theta g_{\theta\theta} \right\rangle + \left\langle B^\zeta g_{\theta\zeta} \right\rangle \tag{10}$$

Some rearranging eventually yields ι as a function of I.

$$\left\langle B^\theta g_{\theta\theta} \right\rangle = I - \left\langle B^\zeta g_{\theta\zeta} \right\rangle \tag{11}$$

$$\left\langle \frac{\mathrm{d}\psi/\mathrm{d}\rho}{2\pi\sqrt{g}} \left(\iota - \frac{\partial\lambda}{\partial\zeta} \right) g_{\theta\theta} \right\rangle = \tag{12}$$

$$\left\langle \frac{\mathrm{d}\psi/\mathrm{d}\rho}{2\pi\sqrt{g}} \iota g_{\theta\theta} \right\rangle - \left\langle \frac{\mathrm{d}\psi/\mathrm{d}\rho}{2\pi\sqrt{g}} \frac{\partial\lambda}{\partial\zeta} g_{\theta\theta} \right\rangle = \tag{13}$$

For flux surface functions, surface average operations are the identity operation. So you may pull them out of the surface average.

$$\frac{\mathrm{d}\psi/\mathrm{d}\rho}{2\pi} \left(\iota \left\langle \frac{g_{\theta\theta}}{\sqrt{g}} \right\rangle - \left\langle \frac{\partial\lambda}{\partial\zeta} \frac{g_{\theta\theta}}{\sqrt{g}} \right\rangle \right) = I - \left\langle B^\zeta g_{\theta\zeta} \right\rangle \tag{14}$$

$$\iota = \left(\frac{2\pi}{\mathrm{d}\psi/\mathrm{d}\rho} \left(I - \left\langle B^\zeta g_{\theta\zeta} \right\rangle \right) + \left\langle \frac{\partial\lambda}{\partial\zeta} \frac{g_{\theta\theta}}{\sqrt{g}} \right\rangle \right) \Big/ \left\langle \frac{g_{\theta\theta}}{\sqrt{g}} \right\rangle \tag{15}$$

3 Equivalent arrangements

The following are mathematically equivalent rearrangements of (15). I am unsure which are more robust numerically. Small magnitudes in the denominator will be bad. Subtracting similar magnitude terms also results in floating point errors.

The rearrangement below merges the surface average in the numerator. It is good to minimize the number of surface average operations to reduce compute time.

$$\iota = \left(\frac{2\pi}{\mathrm{d}\psi/\mathrm{d}\rho} I + \left\langle \frac{\partial\lambda}{\partial\zeta} \frac{g_{\theta\theta}}{\sqrt{g}} - \frac{2\pi}{\mathrm{d}\psi/\mathrm{d}\rho} B^\zeta g_{\theta\zeta} \right\rangle \right) \Big/ \left\langle \frac{g_{\theta\theta}}{\sqrt{g}} \right\rangle \tag{16}$$

Replacing B^ζ gives

$$\boxed{\iota = \left(\frac{2\pi}{\mathrm{d}\psi/\mathrm{d}\rho} I + \left\langle \frac{\frac{\partial\lambda}{\partial\zeta} g_{\theta\theta} - \left(1 + \frac{\partial\lambda}{\partial\theta}\right) g_{\theta\zeta}}{\sqrt{g}} \right\rangle \right) \Big/ \left\langle \frac{g_{\theta\theta}}{\sqrt{g}} \right\rangle} \tag{17}$$

Integral form

$$\iota = \left(\frac{8\pi^3}{\mathrm{d}\psi/\mathrm{d}\rho} I + \oiint \mathrm{d}\theta \mathrm{d}\zeta \frac{\frac{\partial\lambda}{\partial\zeta} g_{\theta\theta} - \left(1 + \frac{\partial\lambda}{\partial\theta}\right) g_{\theta\zeta}}{\sqrt{g}} \right) \Big/ \oiint \mathrm{d}\theta \mathrm{d}\zeta \frac{g_{\theta\theta}}{\sqrt{g}} \tag{18}$$

You can also push $\mathrm{d}\psi/\mathrm{d}\rho$ into the surface averages or integrals.

4 Limit of rotational transform at axis

As defined above, $\lim_{\rho\to 0} \iota$ is of the indeterminate form $0/0$.[1] This follows from $\psi \sim \rho^2$ and the behavior of $\boldsymbol{e}_\theta$ as $(\rho) \to 0$.

$$\lim_{\rho\to 0} \|\boldsymbol{e}_\theta\| = 0 \implies \begin{cases} \lim_{\rho\to 0} g_{\theta\theta} & = 0 \\ \lim_{\rho\to 0} \frac{\partial g_{\theta\theta}}{\partial \rho} & = 0 \\ \lim_{\rho\to 0} g_{\theta\zeta} & = 0 \\ \lim_{\rho\to 0} \sqrt{g} & = 0 \\ \lim_{\rho\to 0} I & = 0 \end{cases} \tag{19}$$

$$\left.\begin{aligned} \lim_{\rho\to 0} \frac{\partial \|\boldsymbol{e}_\theta\|}{\partial \rho} & \neq 0 \\ \lim_{\rho\to 0} \|\boldsymbol{e}_\zeta\| & \neq 0 \\ \lim_{\rho\to 0} \|\boldsymbol{e}_\rho\| & \neq 0 \end{aligned}\right\} \implies \begin{cases} \lim_{\rho\to 0} \frac{\partial^2 g_{\theta\theta}}{\partial \rho^2} & \neq 0 \\ \lim_{\rho\to 0} \frac{\partial \sqrt{g}}{\partial \rho} & \neq 0 \end{cases} \tag{20}$$

The latter set of limits suggests ι is finite at the axis, and that we can find that limit analytically, without evaluating integrals. With that goal, we define

$$\alpha \stackrel{\text{def}}{=} \frac{8\pi^3}{\mathrm{d}\psi/\mathrm{d}\rho} I \tag{21}$$

$$\beta \stackrel{\text{def}}{=} \oiint \mathrm{d}\theta \mathrm{d}\zeta \frac{\frac{\partial \lambda}{\partial \zeta} g_{\theta\theta} - \left(1 + \frac{\partial \lambda}{\partial \theta}\right) g_{\theta\zeta}}{\sqrt{g}} \tag{22}$$

$$\gamma \stackrel{\text{def}}{=} \oiint \mathrm{d}\theta \mathrm{d}\zeta \frac{g_{\theta\theta}}{\sqrt{g}} \tag{23}$$

Applying l'Hôpital's rule

$$\lim_{\rho\to 0} \iota = \lim_{\rho\to 0} \frac{\frac{\mathrm{d}(\alpha+\beta)}{\mathrm{d}\rho}}{\frac{\mathrm{d}\gamma}{\mathrm{d}\rho}} \tag{24}$$

---

[1] I assume interchanging the order of the limit and integral operations is valid.

A sufficient condition for interchanging is uniform convergence of the integrand. Other ways to prove validity of the interchange involve finding some region where the integrand is monotonic. I have failed to prove such properties. I think knowledge of relations that depend on the stellarator is required to deduce such properties. For example, uniform convergence of the simplest integrand in the denominator of (18) requires finding a region, dependent on ϵ, where $\|\boldsymbol{e}_\theta\| < \epsilon \boldsymbol{e}_\theta / \|\boldsymbol{e}_\theta\| \cdot (\boldsymbol{e}_\zeta \times \boldsymbol{e}_\rho)$ for all $\epsilon > 0$. This task reduces to showing the covariant basis vectors are never orthogonal, with supremum of angle separation less than $\pi/2$. The alternative requires deducing if $g_{\theta\theta}/\sqrt{g}$ is monotonic near $\rho = 0$. Also, it seems futile to evade these requirements by returning to the Fourier-Zernike basis functions because their spectral coefficients depend on the stellarator.

In any case, my justification for the interchange follows from our use of finite sums to compute the integrals. When the integrals are replaced with finite sums, the interchange is valid because the limit of each term to be summed exists. Perhaps more importantly, computation shows that the limit we derive here appears to be the continuous extension of (18) to the $\rho = 0$ axis. Since ι is a physical quantity, we should expect ι to be continuous over physical domains. This requires the limit to be a continuous extension. Since the extension is to a single point, it must be unique.

The bounds of the integrals do not depend on ρ, so we may differentiate under both integrals using the Leibniz integral rule.

$$\frac{\mathrm{d}\alpha}{\mathrm{d}\rho} = 8\pi^3 \frac{\frac{\mathrm{d}I}{\mathrm{d}\rho}\frac{\mathrm{d}\psi}{\mathrm{d}\rho} - I\frac{\mathrm{d}^2\psi}{\mathrm{d}\rho^2}}{\left(\frac{\mathrm{d}\psi}{\mathrm{d}\rho}\right)^2} \tag{25}$$

$$\frac{\mathrm{d}\beta}{\mathrm{d}\rho} = \oiint \mathrm{d}\theta\mathrm{d}\zeta \frac{\partial}{\partial\rho}\left(\frac{\frac{\partial\lambda}{\partial\zeta}g_{\theta\theta} - \left(1+\frac{\partial\lambda}{\partial\theta}\right)g_{\theta\zeta}}{\sqrt{g}}\right) \tag{26}$$

$$\begin{aligned} &= \oiint \mathrm{d}\theta\mathrm{d}\zeta \left(\frac{g_{\theta\theta}\left(\frac{\partial^2\lambda}{\partial\rho\partial\zeta}\sqrt{g} - \frac{\partial\lambda}{\partial\zeta}\frac{\partial\sqrt{g}}{\partial\rho}\right)}{\sqrt{g}^2}\right. \\ &\left. + \frac{g_{\theta\zeta}\left(\frac{\partial\sqrt{g}}{\partial\rho}\left(1+\frac{\partial\lambda}{\partial\theta}\right) - \sqrt{g}\frac{\partial^2\lambda}{\partial\rho\partial\theta}\right)}{\sqrt{g}^2} + \frac{\frac{\partial\lambda}{\partial\zeta}\frac{\partial g_{\theta\theta}}{\partial\rho} - \left(1+\frac{\partial\lambda}{\partial\theta}\right)\frac{\partial g_{\theta\zeta}}{\partial\rho}}{\sqrt{g}}\right) \end{aligned} \tag{27}$$

$$\frac{\mathrm{d}\gamma}{\mathrm{d}\rho} = \oiint \mathrm{d}\theta\mathrm{d}\zeta \frac{\partial}{\partial\rho}\left(\frac{g_{\theta\theta}}{\sqrt{g}}\right) \tag{28}$$

$$= \oiint \mathrm{d}\theta\mathrm{d}\zeta \frac{\frac{\partial g_{\theta\theta}}{\partial\rho}\sqrt{g} - g_{\theta\theta}\frac{\partial\sqrt{g}}{\partial\rho}}{\sqrt{g}^2} \tag{29}$$

Now, if each of the following limits exist independently, and the limit in the denominator is nonzero,[2] then the following relation holds by the algebraic limit theorem.

$$\lim_{\rho\to 0}\frac{\frac{\mathrm{d}(\alpha+\beta)}{\mathrm{d}\rho}}{\frac{\mathrm{d}\gamma}{\mathrm{d}\rho}} = \frac{\lim_{\rho\to 0}\frac{\mathrm{d}\alpha}{\mathrm{d}\rho} + \lim_{\rho\to 0}\frac{\mathrm{d}\beta}{\mathrm{d}\rho}}{\lim_{\rho\to 0}\frac{\mathrm{d}\gamma}{\mathrm{d}\rho}} \tag{30}$$

Applying l'Hôpital's rule once to the toroidal current term yields

$$\lim_{\rho\to 0}\frac{\mathrm{d}\alpha}{\mathrm{d}\rho} = \frac{4\pi^3}{\mathrm{d}^2\psi/\mathrm{d}\rho^2}\frac{\mathrm{d}^2 I}{\mathrm{d}\rho^2} \tag{31}$$

Assume the order of limit and integral operations can be interchanged.

$$\lim_{\rho\to 0}\frac{\mathrm{d}\beta}{\mathrm{d}\rho} = \oiint \mathrm{d}\theta\mathrm{d}\zeta \lim_{\rho\to 0}\frac{\mu}{\xi} \tag{32}$$

$$\lim_{\rho\to 0}\frac{\mathrm{d}\gamma}{\mathrm{d}\rho} = \oiint \mathrm{d}\theta\mathrm{d}\zeta \lim_{\rho\to 0}\frac{\nu}{\xi} \tag{33}$$

Both limits are of indeterminate form 0/0. Applying l'Hôpital's rule…

$$\lim_{\rho\to 0}\frac{\mu}{\xi} = \lim_{\rho\to 0}\frac{\frac{\partial\mu}{\partial\rho}}{\frac{\partial\xi}{\partial\rho}} = \lim_{\rho\to 0}\frac{\frac{\partial^2\mu}{\partial\rho^2}}{\frac{\partial^2\xi}{\partial\rho^2}} \tag{34}$$

$$\lim_{\rho\to 0}\frac{\nu}{\xi} = \lim_{\rho\to 0}\frac{\frac{\partial\nu}{\partial\rho}}{\frac{\partial\xi}{\partial\rho}} = \lim_{\rho\to 0}\frac{\frac{\partial^2\nu}{\partial\rho^2}}{\frac{\partial^2\xi}{\partial\rho^2}} \tag{35}$$

[2]We will later discover this precondition is true.

The derivatives are given below.

$$
\begin{aligned}
\frac{\partial^2 \mu}{\partial \rho^2} = &-\frac{\partial^3 \sqrt{g}}{\partial \rho^3}\frac{\partial \lambda}{\partial \zeta} g_{\theta\theta} + \frac{\partial^3 \sqrt{g}}{\partial \rho^3} g_{\theta\zeta}\frac{\partial \lambda}{\partial \theta} + \frac{\partial^3 \sqrt{g}}{\partial \rho^3} g_{\theta\zeta} \\
&- \frac{\partial \lambda}{\partial \zeta}\frac{\partial^2 \sqrt{g}}{\partial \rho^2}\frac{\partial g_{\theta\theta}}{\partial \rho} - g_{\theta\theta}\frac{\partial^2 \sqrt{g}}{\partial \rho^2}\frac{\partial^2 \lambda}{\partial \rho \partial \zeta} + g_{\theta\zeta}\frac{\partial^2 \sqrt{g}}{\partial \rho^2}\frac{\partial^2 \lambda}{\partial \rho \partial \theta} \\
&+ \frac{\partial g_{\theta\zeta}}{\partial \rho}\left(\frac{\partial^2 \sqrt{g}}{\partial \rho^2}\left(1 + \frac{\partial \lambda}{\partial \theta}\right) - 3\sqrt{g}\frac{\partial^3 \lambda}{\partial \rho^2 \partial \theta}\right) \\
&+ \frac{\partial \sqrt{g}}{\partial \rho}\left(2\frac{\partial^2 \lambda}{\partial \rho \partial \zeta}\frac{\partial g_{\theta\theta}}{\partial \rho} + \frac{\partial \lambda}{\partial \zeta}\frac{\partial^2 g_{\theta\theta}}{\partial \rho^2} + g_{\theta\theta}\frac{\partial^3 \lambda}{\partial \rho^2 \partial \zeta}\right. \\
&\left. -2\frac{\partial g_{\theta\zeta}}{\partial \rho}\frac{\partial^2 \lambda}{\partial \rho \partial \theta} - \left(1 + \frac{\partial \lambda}{\partial \theta}\right)\frac{\partial^2 g_{\theta\zeta}}{\partial \rho^2} - g_{\theta\zeta}\frac{\partial^3 \lambda}{\partial \rho^2 \partial \theta}\right) \\
&+ 3\sqrt{g}\frac{\partial^2 \lambda}{\partial \rho \partial \zeta}\frac{\partial^2 g_{\theta\theta}}{\partial \rho^2} + 3\sqrt{g}\frac{\partial^3 \lambda}{\partial \rho^2 \partial \zeta}\frac{\partial g_{\theta\theta}}{\partial \rho} + \sqrt{g}\frac{\partial \lambda}{\partial \zeta}\frac{\partial^3 g_{\theta\theta}}{\partial \rho^3} \\
&+ \sqrt{g}\frac{\partial^4 \lambda}{\partial \rho^3 \partial \zeta} g_{\theta\theta} - 3\sqrt{g}\frac{\partial^2 g_{\theta\zeta}}{\partial \rho^2}\frac{\partial^2 \lambda}{\partial \rho \partial \theta} - \sqrt{g}\frac{\partial^3 g_{\theta\zeta}}{\partial \rho^3}\frac{\partial \lambda}{\partial \theta} \\
&- \sqrt{g} g_{\theta\zeta}\frac{\partial^4 \lambda}{\partial \rho^3 \partial \theta} - \sqrt{g}\frac{\partial^3 g_{\theta\zeta}}{\partial \rho^3}
\end{aligned} \tag{36}
$$

$$
\frac{\partial^2 \nu}{\partial \rho^2} = \sqrt{g}\frac{\partial^3 g_{\theta\theta}}{\partial \rho^3} + \frac{\partial \sqrt{g}}{\partial \rho}\frac{\partial^2 g_{\theta\theta}}{\partial \rho^2} - \frac{\partial^2 \sqrt{g}}{\partial \rho^2}\frac{\partial g_{\theta\theta}}{\partial \rho} - \frac{\partial^3 \sqrt{g}}{\partial \rho^3} g_{\theta\theta} \tag{37}
$$

$$
\frac{\partial^2 \xi}{\partial \rho^2} = 2\left(\left(\frac{\partial \sqrt{g}}{\partial \rho}\right)^2 + \sqrt{g}\frac{\partial^2 \sqrt{g}}{\partial \rho^2}\right) \tag{38}
$$

It follows that

$$
\lim_{\rho \to 0} \frac{\frac{\partial^2 \mu}{\partial \rho^2}}{\frac{\partial^2 \xi}{\partial \rho^2}} = \frac{\frac{\partial g_{\theta\zeta}}{\partial \rho}\frac{\partial^2 \sqrt{g}}{\partial \rho^2}\left(1 + \frac{\partial \lambda}{\partial \theta}\right)}{2\left(\frac{\partial \sqrt{g}}{\partial \rho}\right)^2} + \frac{\frac{\partial \lambda}{\partial \zeta}\frac{\partial^2 g_{\theta\theta}}{\partial \rho^2} - 2\frac{\partial g_{\theta\zeta}}{\partial \rho}\frac{\partial^2 \lambda}{\partial \rho \partial \theta} - \left(1 + \frac{\partial \lambda}{\partial \theta}\right)\frac{\partial^2 g_{\theta\zeta}}{\partial \rho^2}}{2\frac{\partial \sqrt{g}}{\partial \rho}} \tag{39}
$$

Observe that $\lim_{\rho \to 0}(\nu/\xi) \neq 0$ as required by (30).

$$
\lim_{\rho \to 0} \frac{\frac{\partial^2 \nu}{\partial \rho^2}}{\frac{\partial^2 \xi}{\partial \rho^2}} = \frac{\frac{\partial^2 g_{\theta\theta}}{\partial \rho^2}}{2\frac{\partial \sqrt{g}}{\partial \rho}} \neq 0 \tag{40}
$$

If we had attempted to apply the algebraic limit theorem prior to the first application of l'Hôpital's rule in (24), then the requirement that the limit in the denominator be nonzero would not be satisfied.

Collecting our results, we conclude:

$$\lim_{\rho\to 0}\iota = \left(\frac{8\pi^3}{\mathrm{d}^2\psi/\mathrm{d}\rho^2}\frac{\mathrm{d}^2 I}{\mathrm{d}\rho^2} + \oiint \mathrm{d}\theta\mathrm{d}\zeta\left(\frac{\frac{\partial g_{\theta\zeta}}{\partial\rho}\frac{\partial^2\sqrt{g}}{\partial\rho^2}\left(1+\frac{\partial\lambda}{\partial\theta}\right)}{\left(\frac{\partial\sqrt{g}}{\partial\rho}\right)^2} + \frac{\frac{\partial\lambda}{\partial\zeta}\frac{\partial^2 g_{\theta\theta}}{\partial\rho^2} - 2\frac{\partial g_{\theta\zeta}}{\partial\rho}\frac{\partial^2\lambda}{\partial\rho\partial\theta} - \left(1+\frac{\partial\lambda}{\partial\theta}\right)\frac{\partial^2 g_{\theta\zeta}}{\partial\rho^2}}{\frac{\partial\sqrt{g}}{\partial\rho}}\right)\right)\Bigg/\oiint \mathrm{d}\theta\mathrm{d}\zeta\frac{\frac{\partial^2 g_{\theta\theta}}{\partial\rho^2}}{\frac{\partial\sqrt{g}}{\partial\rho}} \tag{41}$$

In surface average form

$$\boxed{\lim_{\rho\to 0}\iota = \left(\frac{2\pi}{\mathrm{d}^2\psi/\mathrm{d}\rho^2}\frac{\mathrm{d}^2 I}{\mathrm{d}\rho^2} + \left\langle\frac{\frac{\partial g_{\theta\zeta}}{\partial\rho}\frac{\partial^2\sqrt{g}}{\partial\rho^2}\left(1+\frac{\partial\lambda}{\partial\theta}\right)}{\left(\frac{\partial\sqrt{g}}{\partial\rho}\right)^2} + \frac{\frac{\partial\lambda}{\partial\zeta}\frac{\partial^2 g_{\theta\theta}}{\partial\rho^2} - 2\frac{\partial g_{\theta\zeta}}{\partial\rho}\frac{\partial^2\lambda}{\partial\rho\partial\theta} - \left(1+\frac{\partial\lambda}{\partial\theta}\right)\frac{\partial^2 g_{\theta\zeta}}{\partial\rho^2}}{\frac{\partial\sqrt{g}}{\partial\rho}}\right\rangle\right)\Bigg/\left\langle\frac{\frac{\partial^2 g_{\theta\theta}}{\partial\rho^2}}{\frac{\partial\sqrt{g}}{\partial\rho}}\right\rangle} \tag{42}$$

5 toroidal angle

$$\boxed{\iota = \left(\frac{2\pi}{\mathrm{d}\psi/\mathrm{d}\rho} I + \left\langle \frac{\frac{\partial\lambda}{\partial\zeta} g_{\theta\theta} - \left(1 + \frac{\partial\lambda}{\partial\theta}\right) g_{\theta\zeta}}{\sqrt{g}} \right\rangle\right) \Big/ \left\langle \frac{(1+\omega_\zeta) g_{\theta\theta} - \omega_\theta g_{\theta\zeta}}{\sqrt{g}} \right\rangle}$$