Appendix A. Bounce-averaged drifts

For the analytical, shifted circle model, the important geometric coefficients are defined below

$$\text{gradpar} = \boldsymbol{b}\cdot\boldsymbol{\nabla\theta} = G_0(1-\epsilon\cos(\theta)) \tag{A 1}$$

where $\epsilon \ll 1$ is the aspect ratio of the flux surface of interest. In this analytial model, we can also write the integrated local shear

$$\text{gds21} = \frac{\boldsymbol{\nabla}\psi\cdot\boldsymbol{\nabla}\alpha}{|\boldsymbol{\nabla}\psi|^2} = -\hat{s}\left(\hat{s}\theta - \frac{\alpha_{\text{MHD}}}{B^4}\sin(\theta)\right), \quad \alpha_{\text{MHD}} = -\frac{0.5}{\iota^2}\frac{dP}{d\rho} \tag{A 2}$$

and the binormal component of the grad-B drift

$$\begin{aligned}\text{gbdrift} &= \frac{1}{B^2}(\boldsymbol{b}\times\boldsymbol{\nabla}B)\cdot\boldsymbol{\nabla}\alpha &\text{(A 3)}\\ &= f_2\left[-\hat{s} + \left(\cos(\theta) - \frac{\text{gds21}}{\hat{\text{s}}}\sin(\theta)\right)\right] \\ &= f_2\left[-\hat{s} + \left(\cos(\theta) + \hat{s}\theta\sin(\theta) - \frac{\alpha_{\text{MHD}}}{B^4}\sin(\theta)^2\right)\right] &\text{(A 4)}\end{aligned}$$

where we have used (A 2) to obtain the final expression for gbdrift. The geometric factor corresponding to the binormal component of the curvature drift

$$\begin{aligned}\text{cvdrift} &= \frac{1}{B^2}\left[\boldsymbol{b}\times\boldsymbol{\nabla}\left(p+\frac{B^2}{2}\right)\right]\cdot\boldsymbol{\nabla}\alpha &\text{(A 5)}\\ &= \text{gbdrift} + f_3\frac{1}{B^2}\frac{dP}{d\rho} &\text{(A 6)}\\ &= f_2\left[-\hat{s} + \left(\cos(\theta) + \hat{s}\theta\sin(\theta) - \frac{\alpha_{\text{MHD}}}{B^4}\sin(\theta)^2\right)\right] + f_3\frac{\alpha_{\text{MHD}}}{B^2} &\text{(A 7)}\end{aligned}$$

Note that the quantities f_2 and f_3 are scalar factors used to match analytical expressions with their numerical values. Using all these quantities, we can calculate the bounce-averaged drift

$$\langle\omega_{\text{Ds}}\rangle = \int_{-\theta_{b1}}^{\theta_{b2}} \frac{d\theta}{\boldsymbol{b}\cdot\boldsymbol{\nabla}\theta}\frac{1}{w_\parallel}\left[w_\parallel^2\text{cvdrift} + \frac{w_\perp^2}{2}\text{gbdrift}\right] \tag{A 8}$$

where θ_{b1} and θ_{b2} are bounce angles. As used in Connor *et al.* and shown by Hegna, in the limit of a large aspect ratio shifted circle model, the parallel speed of a particle with a fixed energy E is

$$w_\parallel = \sqrt{\frac{2E}{m}}\sqrt{1-\lambda B} = \sqrt{\frac{2E}{m}}\sqrt{1-\lambda B_0(1-\epsilon\cos(\theta))} = 2\sqrt{\frac{E}{m}}\sqrt{\epsilon\lambda B_0}\sqrt{k^2-\sin(\theta/2)^2} \tag{A 9}$$

where $\lambda = \mu/E$ is the pitch angle, $\mu = mw_\perp^2/(2B)$ is the magnetic moment, $E = mw^2/2$ the particle energy, and the parameter

$$k^2 = \frac{1}{2}\left(\frac{1-\lambda B_0}{\epsilon\lambda B_0} + 1\right), \tag{A 10}$$

is a reparametrization of the pitch angle λ. Using these geometric simplifications, and $w_\perp^2/2 = E - w_\parallel^2/2$, we can write the bounce-averaged drift as

$$\langle \omega_D \rangle = \int_{-2\sin^{-1}(k)}^{2\sin^{-1}(k)} \frac{d\theta}{\boldsymbol{b}\cdot\boldsymbol{\nabla}\theta} \Big[\sqrt{\epsilon\lambda B_0}\sqrt{(k^2 - (\sin(\theta/2)^2)}\,\mathrm{cvdrift} \\ - \sqrt{\epsilon\lambda B_0}\frac{\sqrt{(k^2 - (\sin(\theta/2)^2)}}{2}\mathrm{gbdrift} + \frac{1}{\sqrt{\epsilon\lambda B_0}}\frac{1}{\sqrt{k^2 - (\sin(\theta/2))^2}}\frac{\mathrm{gbdrift}}{2}\Big] \tag{A 11}$$

Using the following identities, we can further simplify (A 11)

$$\mathsf{I}_0 = \int_{-2\sin^{-1}(k)}^{2\sin^{-1}(k)} \frac{d\theta}{\sqrt{k^2 - \sin(\theta/2)^2}} = \frac{4}{k}K\left(\sin^{-1}(k), \frac{1}{k^2}\right) \tag{A 12}$$

$$\mathsf{I}_1 = \int_{-2\sin^{-1}(k)}^{2\sin^{-1}(k)} d\theta\sqrt{k^2 - \sin(\theta/2)^2} = 4kE\left(\sin^{-1}(k), \frac{1}{k^2}\right) \tag{A 13}$$

$$\mathsf{I}_2 = \int_{-2\sin^{-1}(k)}^{2\sin^{-1}(k)} \frac{d\theta}{\sqrt{k^2 - \sin(\theta/2)^2}}\theta\sin(\theta) = 16k\,E \tag{A 14}$$

$$\mathsf{I}_3 = \int_{-2\sin^{-1}(k)}^{2\sin^{-1}(k)} d\theta\sqrt{k^2 - \sin(\theta/2)^2}\,\theta\sin(\theta) \\ = \frac{16k}{9}\left[2(-1+2k^2)\,E - (-1+k^2)\,K\right] \tag{A 15}$$

$$\mathsf{I}_4 = \int_{-2\sin^{-1}(k)}^{2\sin^{-1}(k)} \frac{d\theta}{\sqrt{k^2 - \sin(\theta/2)^2}}\,(\sin(\theta))^2 \\ = \frac{16k}{3}\left[(-1+2k^2)\,E - 2(-1+k^2)\,K\right] \tag{A 16}$$

$$\mathsf{I}_5 = \int_{-2\sin^{-1}(k)}^{2\sin^{-1}(k)} d\theta\sqrt{k^2 - \sin(\theta/2)^2}\,(\sin(\theta))^2 \\ = \frac{32k}{30}\left[2(1-k^2+k^4)\,E - (1-3k^2+2k^4)\,K\right] \tag{A 17}$$

$$\mathsf{I}_6 = \int_{-2\sin^{-1}(k)}^{2\sin^{-1}(k)} \frac{d\theta}{\sqrt{k^2 - \sin(\theta/2)^2}}\,\cos(\theta) \\ = \frac{4}{k}\left[2k^2\,E + (1-2k^2)\,K\right] \tag{A 18}$$

$$\mathsf{I}_7 = \int_{-2\sin^{-1}(k)}^{2\sin^{-1}(k)} d\theta\sqrt{k^2 - \sin(\theta/2)^2}\cos(\theta) \\ = \frac{2\,|k|}{3}\left[(-2+4k^2)E - 4(-1+k^2)K\right]$$

where K and E are incomplete elliptic integrals of the first and second kind, respectively. Using these formulae, to lowest order, we can write all the three terms in the above equation, and the evaluated analytical bounce-averaged drifts give us

$$\langle \omega_D \rangle = \frac{1}{G_0}\Bigg\{ \left(f_3 \frac{\alpha_{\text{MHD}}}{B_0^2} - f_2 \frac{\hat{s}}{2} \right) \mathsf{I}_1 + \frac{f_2}{2} \left(\hat{s}\, \mathsf{I}_3 - \frac{\alpha_{\text{MHD}}}{B_0^4} \mathsf{I}_5 + \mathsf{I}_7 \right) \\ + \frac{f_2}{2} \left[-\hat{s}\left(\mathsf{I}_0 - \mathsf{I}_2\right) - \frac{\alpha_{\text{MHD}}}{B_0^4} \mathsf{I}_4 + \mathsf{I}_6 \right] \Bigg\} \tag{A 19}$$

After rearranging the terms, we can write

$$\langle \omega_D \rangle = \frac{1}{G_0}\Bigg\{ f_3 \frac{\alpha_{\text{MHD}}}{B_0^2} \mathsf{I}_1 - \frac{f_2}{2} \left[\hat{s}\left(\mathsf{I}_0 + \mathsf{I}_1 - \mathsf{I}_2 - \mathsf{I}_3\right) + \frac{\alpha_{\text{MHD}}}{B_0^4} \left(\mathsf{I}_4 + \mathsf{I}_5\right) - \left(\mathsf{I}_6 + \mathsf{I}_7\right) \right] \Bigg\} \tag{A 20}$$

To be removed later To resolve the smaller discrepancies between the analytical and numerical values, it is important to note that we have dropped all terms of the order ϵ from the analytical integrals.