Solving 3D MHD Equation Using Poincare Boundary Condition in DESC

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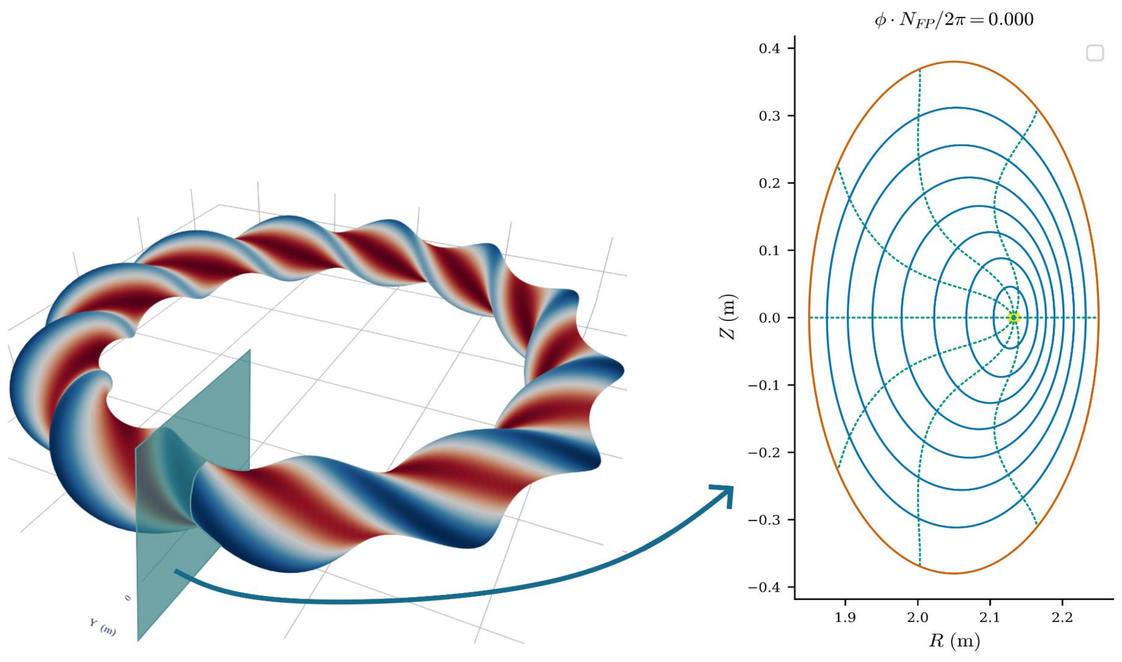
Motivation

3D MHD equilibrium has key importance on design and optimization of stellarators that propose better stability than tokamaks due to their advanced magnetic fields. 3D MHD equations, comprising a set of three non-linear partial differential equations (PDEs), pose challenges in determining well-defined necessary boundary conditions. Although the common practice involves prescribing the last closed flux surface (LCFS) as a boundary condition, this approach limits the optimization space and prevents researchers from exploring the totality of the design options. **In response, we propose the implementation of the Poincaré section—a representation of magnetic field lines at a toroidal cross-section—as a novel boundary condition for ideal MHD equations within the DESC stellarator optimization code.**



NAME	L	M	N	# LCFS	# Poincare
HELIOTRON	24	12	3	**175**	**319**
ATF	24	12	4	**225**	**319**
W7-X	24	12	12	**625**	**319**
ARIES-CS	24	12	8	**425**	**319**
ESTELL	12	12	6	**325**	**133**
precise_QA	8	8	8	**289**	**65**
DSHAPE	26	13	0	**27**	**371**

The resolutions and corresponding number of parameters to define a last close flux surface and Poincare cross section

Ideal MHD Equilibrium

An equilibrium can be described either using minimum energy principle or zero force balance. The former is used in legacy stellarator equilibrium code VMEC. In DESC, both methods are implemented and it is seen that force balance equation results in better convergence. The force balance equation along with magnetic Gauss Law can be written as,

$$(\nabla \times \mathbf{B}) \times \mathbf{B} = \mu_0 \nabla p \qquad \nabla \cdot \mathbf{B} = 0$$

The partial differential equation can be solved by first transforming it to an optimization problem on spectral coefficients and finding the coefficients that minimizes the force error on number of collocation points.

$$\min_x ||(\nabla \times \mathbf{B}) \times \mathbf{B} - \mu_0 \nabla p||^2 \qquad \text{where } \mathbf{C}x = \mathrm{b}$$

Here the state vector **x** is comprised of the spectral coefficients R_{lmn}, Z_{lmn} and λ_{lmn}, and **C** is the linear constraint matrix. In DESC, Fourier-Zernike basis is chosen for the spectral method. Cylindrical radial position can be shown in this basis as.

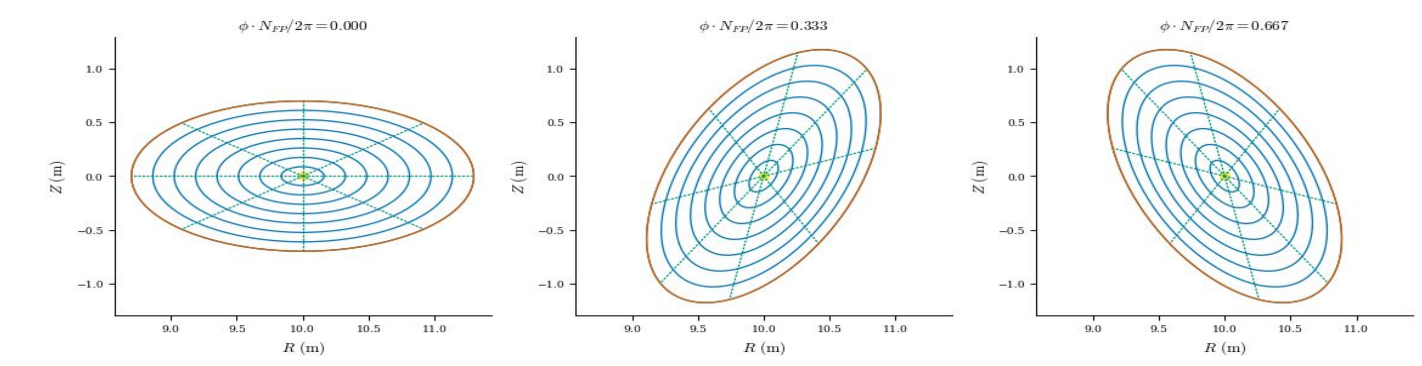
$$R(\rho,\theta,\zeta) = \sum_{m=-M,n=-N,l=0}^{M,N,L} R_{lmn} \mathcal{Z}_l^m(\rho,\theta)\mathcal{F}^n(\zeta)$$

Conventional Approach

- Give LCFS in terms of $\boldsymbol{R}^{\boldsymbol{b}}_{mn}$ and $\boldsymbol{Z}^{\boldsymbol{b}}_{mn}$
- Using the property of Zernike polynomials, the linear constraint for boundary surface is,

$$Rb_{m'n'} = \sum_{l=0}^{L} R_{lm'n'}$$

- Scaled down LCFS to get the nested concentric surfaces and use them as the initial guess of the optimization.



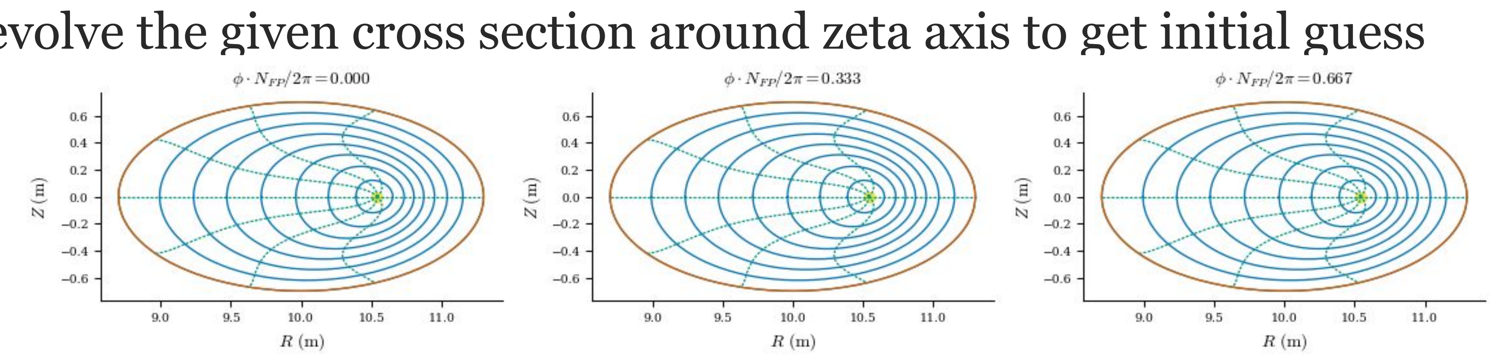
Proposed Boundary Condition

- Give Poincare cross section (ζ=0) in terms of $\boldsymbol{R}^{\boldsymbol{p}}_{lm}$, $\boldsymbol{Z}^{\boldsymbol{p}}_{lm}$ and $\boldsymbol{\lambda}^{\boldsymbol{p}}_{lm}$
- Linear constraints are then,

$$Rp_{l'm'} = \sum_{n=0}^{N} R_{l'm'n}$$

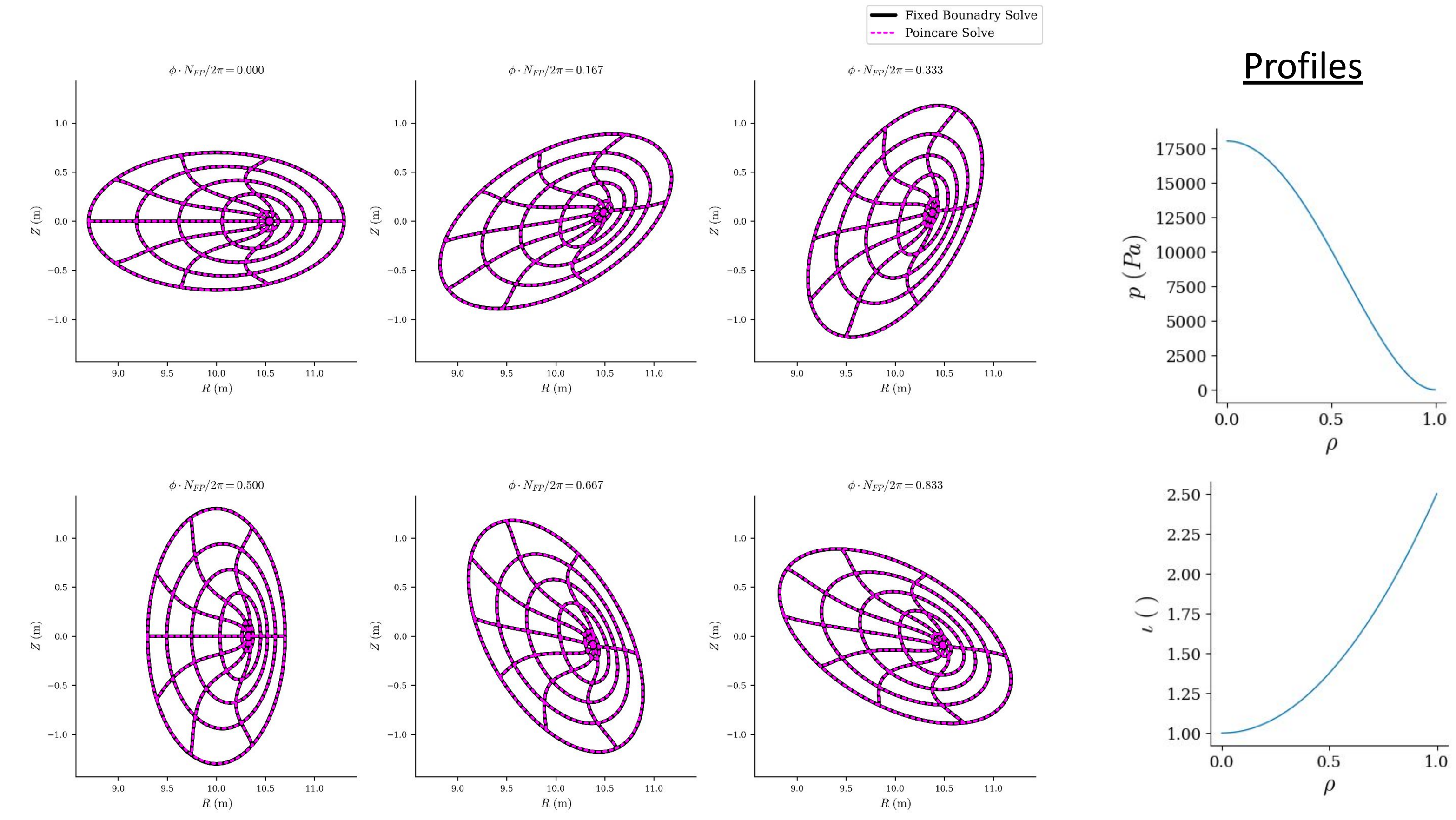
It can be seen that this results in a self consistency constraint on half of the modes (n>0), and the rest is free.

- Even if λp_{lm} (which is not supplied in LCFS) is given, the total number of free variables is higher with this new boundary condition for most geometries which makes the optimization space of Poincare boundary condition is larger
- Revolve the given cross section around zeta axis to get initial guess

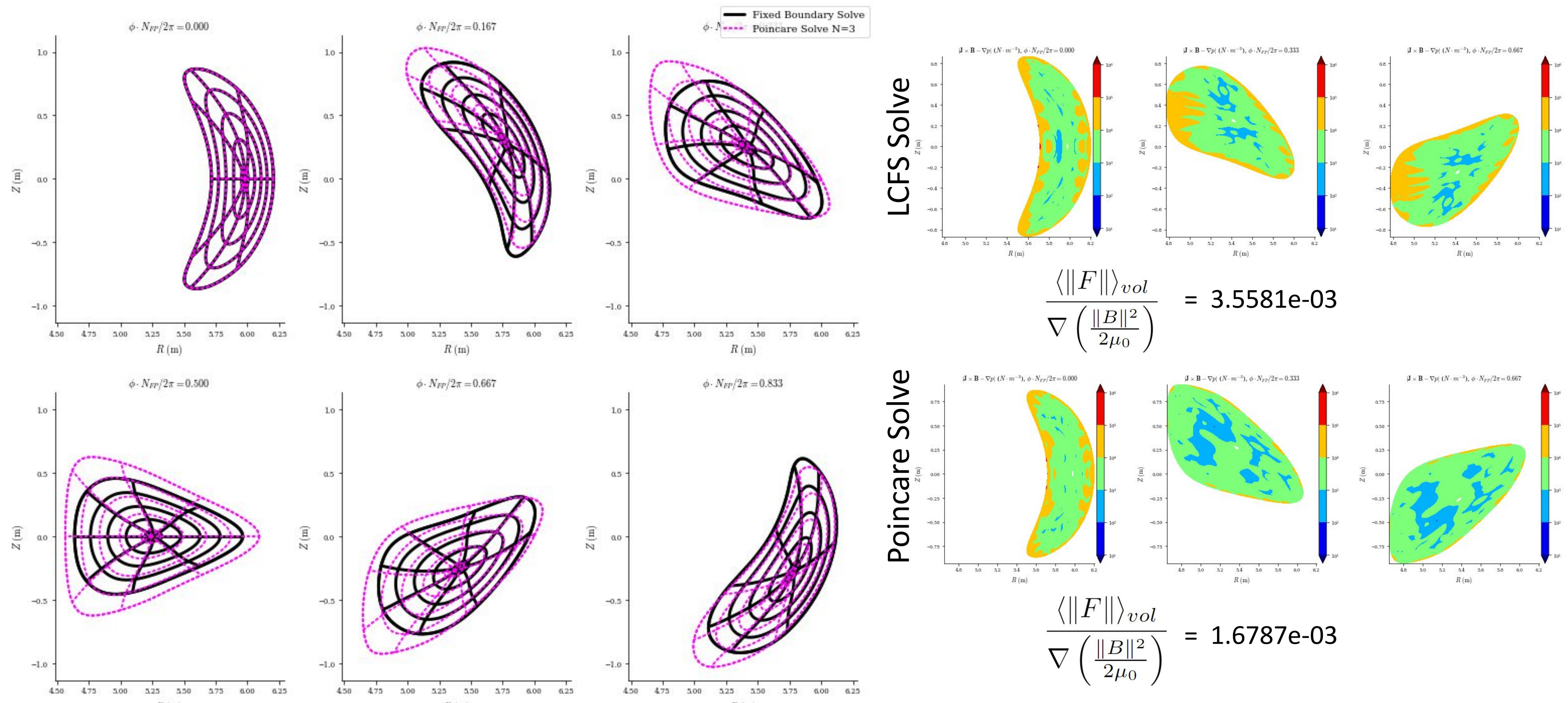


Results

- Results of Poincare BC on a HELIOTRON device with NFP=19



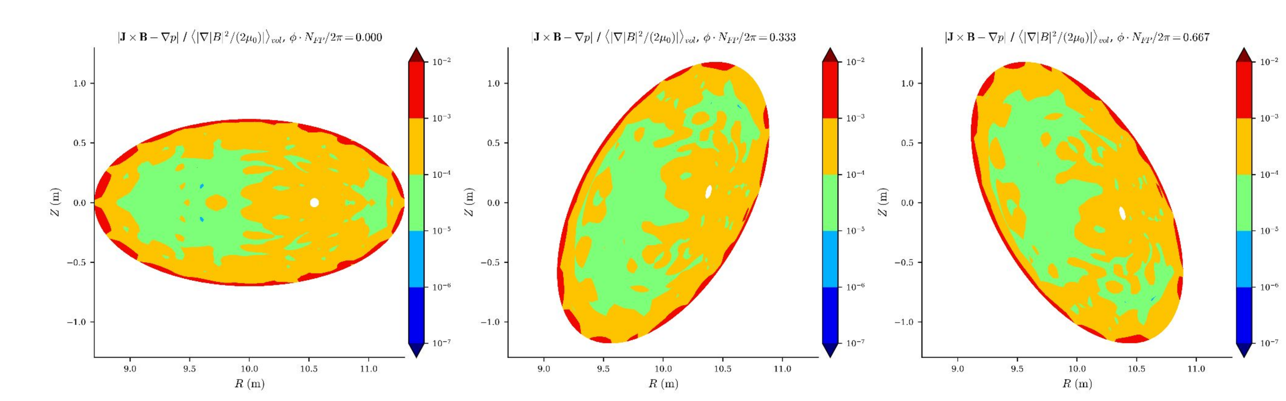
- However, **the new BC doesn't produce the same equilibrium if we start from an axisymmetric initial guess**. Especially for low NFP cases, the optimization finds a good force balance which is different than fixed boundary.
- Results for W7-X with continuation on max toroidal resolution upto N=3 (N=12 is the original equilibrium). Increasing N for Poincare BC improves the force error but takes the flux surfaces further away from LCFS ones.



LCFS Solve: $\frac{\langle \|F\| \rangle_{vol}}{\nabla \left(\frac{\|B\|^2}{2\mu_0}\right)}$ = 3.5581e-03

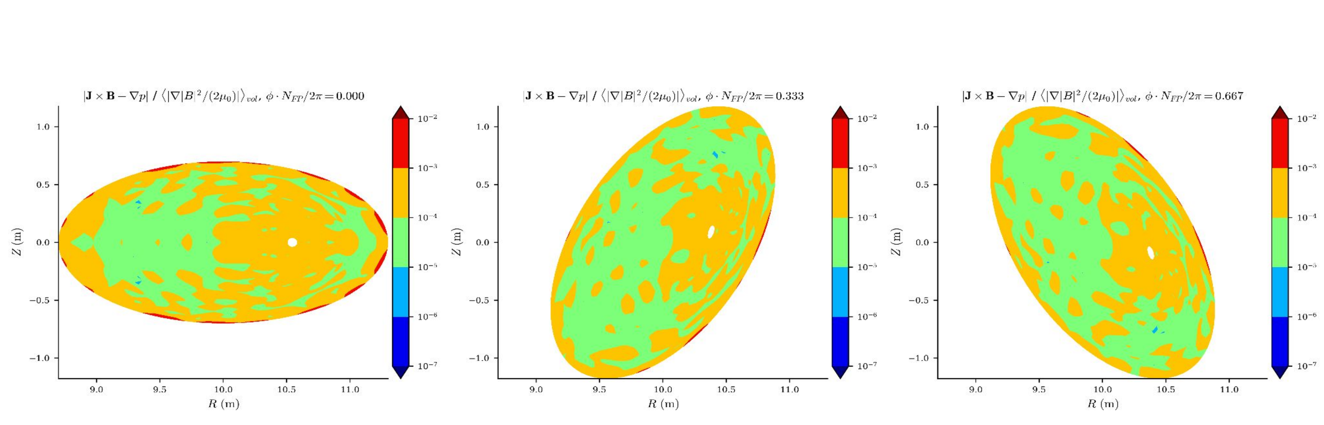
Poincare Solve: $\frac{\langle \|F\| \rangle_{vol}}{\nabla \left(\frac{\|B\|^2}{2\mu_0}\right)}$ = 1.6787e-03

- This behaviour is not observed if the LCFS optimized equilibrium fed back to Poincare BC (or vice versa) and resolved. **The effect of resolving is to change the LCFS slightly to achieve better force balance.**
- We can use this strategy to reduce the force balance error around the LCFS.



LCFS Solve

$\frac{\langle \|F\| \rangle_{vol}}{\nabla \left(\frac{\|B\|^2}{2\mu_0}\right)}$ = 2.7305e-04



Poincare Solve

$\frac{\langle \|F\| \rangle_{vol}}{\nabla \left(\frac{\|B\|^2}{2\mu_0}\right)}$ = 1.3002e-04

Conclusion & Future Work

- Poincare boundary condition is effective in achieving a **good force balance error**
- Same equilibrium as the fixed boundary condition is not guaranteed
- When the fixed boundary solution is used as an initial guess for the Poincare boundary condition, both methods converge to the same equilibrium, allowing the Poincare boundary condition to improve the force balance.
- New method of providing a boundary condition can **open new areas of optimization space**
- We will implement the new BC during optimization when force balance is a constraint

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