2] Considere que:

- $r = xi + yj + zk = x^i e_i$)
- $a = a(r) = a(x,y,z) = a^i(x,y,z)e_i$ y $b = b(r) = b(x,y,z) = b^i(x,y,z)i_i$,
- $\phi = \phi(r) = \phi(x,y,z)$ y $\psi = \psi(r) = \psi(x,y,z)$

Utilizando la notación de índices e inspirándose en las secciones: 1.4.5, 1.5.5 y 1.5.7 demuestre las siguientes identidades vectoriales:

a) $\nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi$.

$$\nabla(\phi\psi) = \partial^i(\phi\psi)e_i = \underbrace{\phi\partial^i(\psi)e_i}_{\phi\nabla\psi} + \underbrace{\psi\partial^i(\phi)e_i}_{\psi\Delta\phi}$$

$$\nabla(\phi\psi) = \phi\nabla\psi + \psi\Delta\phi$$ DEMOSTRADO

d) $\nabla\cdot(\nabla\times a)$ ¿Qué puede decir de $\nabla\times(\nabla\cdot a)$?

Podemos afirmar que ésta operación no es permitida, ya que la operación rotacional es vectorial, es decir que solo se puede aplicar a vectores, ya que el producto punto es escalar y el producto cruz es aplicado vectorialmente, entonces podemos afirmar que $\nabla\times(\nabla\cdot a)$ no se puede realizar.

f) $\nabla\times(\nabla\times a) = \nabla(\nabla\cdot a) - \nabla^2 a$

$$\varepsilon^{ijk}\partial_j\varepsilon_{kmn}\partial^m a^n = \varepsilon^{ijk}\varepsilon_{kmn}\partial_j\partial^m a^n = \varepsilon^{ijk}\varepsilon_{mnk}\partial_j\partial^m a^k$$

$$= (\delta^i_m\delta^j_n - \delta^i_n\delta^j_m)\partial_j\partial^m a^n$$

$$= \delta^i_m\delta^j_n\partial_j\partial^m a^n - \delta^i_n\delta^j_m\partial_j\partial^m a^n$$

$$= \delta^i_M\partial^m\delta^j_n\partial_j a^n - \delta^i_n a^n\delta^j_m\partial_j\partial^m$$

$$= \partial^m\underbrace{\partial_j a^j}_{(\nabla\cdot a)} - a^n\underbrace{\partial_m\partial^m}_{(\nabla\cdot\nabla)}$$

$$= \nabla(\nabla\cdot a) - a(\nabla\cdot\nabla) = \nabla(\nabla\cdot a) - a|\nabla|^2$$

a) Demuestre

$$\cos(3\alpha) = \cos^3(\alpha) - 3\cos(\alpha)\,sen^2(\alpha)$$

• Sabemos...

$$sen(2\alpha) = 2\,sen(\alpha).\cos(\alpha)$$
$$\cos(2\alpha) = 2\cos^2(\alpha) - 1$$
$$sen^2(\alpha) + \cos^2(\alpha) = 1$$

$$\cos(2\alpha+\alpha) = \cos^3(\alpha) - 3\cos(\alpha)\left[1-\cos^2(\alpha)\right]$$

$$\cos(2\alpha).\cos\alpha - sen(2\alpha).sen(\alpha) = \cos^3(\alpha) - 3\cos(\alpha)\left[1-\cos^2(\alpha)\right]$$

$$\left[2\cos^2(\alpha)-1\right].\cos\alpha - \left[2\,sen(\alpha).\cos(\alpha)\right].sen(\alpha) = \cos^3(\alpha) - 3\cos(\alpha)\left[1-\cos^2(\alpha)\right]$$

$$2\cos^3(\alpha) - \cos(\alpha) - 2\,sen^2(\alpha).\cos(\alpha) = \cos^3(\alpha) - 3\cos(\alpha) + 3\cos^3(\alpha)$$

$$2\cos^3(\alpha) - \cos(\alpha) - 2\left[1-\cos^2(\alpha)\right].\cos(\alpha) = \cos^3(\alpha) - 3\cos(\alpha) + 3\cos^3(\alpha)$$

$$2\cos^3(\alpha) - \cos(\alpha) - 2\cos(\alpha) + 2\cos^3(\alpha) = \cos^3(\alpha) - 3\cos(\alpha) + 3\cos^3(\alpha)$$

$$\boxed{4\cos^3(\alpha) - 3\cos(\alpha) = 4\cos^3(\alpha) - 3\cos(\alpha)}$$

Demostrado

b) $sen(3\alpha) = 3\cos^2(\alpha)\,sen(\alpha) - sen^3(\alpha)$

• Sabemos que.

$$sen(2\alpha) = 2\,sen(\alpha).\cos(\alpha)$$
$$\cos(2\alpha) = 2\cos^2(\alpha) - 1$$
$$sen^2(\alpha) + \cos^2(\alpha) = 1$$

$$sen(2\alpha+\alpha) = 3\cos^2(\alpha)\,sen(\alpha) - sen^3(\alpha)$$

$$sen(2\alpha)\cos(\alpha) + \cos(2\alpha)\,sen(\alpha) = 3\cos^2(\alpha)\,sen(\alpha) - sen^3(\alpha)$$

$$\left[2\,sen(\alpha).\cos(\alpha)\right].\cos(\alpha) + \cos(2\alpha)\,sen(\alpha) = 3\,sen(\alpha).\cos^2(\alpha) - sen^3(\alpha)$$

$$2\,sen(\alpha).\cos^2(\alpha) + \left[2\cos^2(\alpha)-1\right]sen(\alpha) = 3\,sen(\alpha).\left[1-sen^2(\alpha)\right] - sen^3(\alpha)$$

$$2\,sen(\alpha).\cos^2(\alpha) + 2\cos^2(\alpha)\,sen(\alpha) - sen(\alpha) = 3\,sen(\alpha) - 3\,sen^3(\alpha) - sen^3(\alpha)$$

$$\cos^2(\alpha)\left[2\,sen(\alpha) + 2\,sen(\alpha)\right] - sen(\alpha) = 3\,sen(\alpha) - 4\,sen^3(\alpha)$$

$$\left[1-sen^2(\alpha)\right]\left[4\,sen(\alpha)\right] - sen(\alpha) = 3\,sen(\alpha) - 4\,sen^3(\alpha)$$

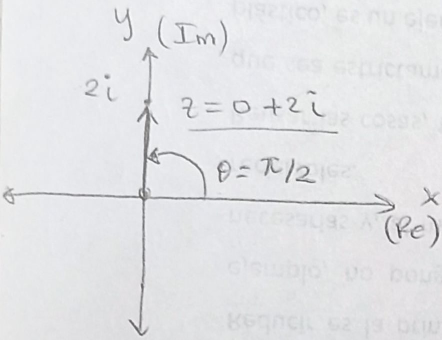
$$4\,sen(\alpha) - 4\,sen^3(\alpha) - sen(\alpha) = 3\,sen(\alpha) - 4\,sen^3(\alpha)$$

$$\boxed{3\,sen(\alpha) - 4\,sen^3(\alpha) = 3\,sen(\alpha) - 4\,sen^3(\alpha)}$$

Demostrado

⑤ Encuentre las raíces de:

a) $(2i)^{1/2} = (0+2i)^{1/2}$



$r = 2$

$\theta = \pi/2$

entonces...

$$\boxed{z_k = r^{1/n}\left[\cos\left(\frac{\theta+2\pi k}{n}\right) + i\,\text{sen}\left(\frac{\theta+2\pi k}{n}\right)\right]}$$

FORMULA...

con
$n = 2$
$k = 0, 1$
$\theta = \frac{\pi}{2}$
$r = 2$

entonces $z_k = (2)^{1/2}\left[\cos\left(\frac{\frac{\pi}{2}+2\pi k}{2}\right) + i\,\text{sen}\left(\frac{\frac{\pi}{2}+2\pi k}{2}\right)\right]$

con $k=0$

$$z_0 = 2^{1/2}\left[\cos\left(\frac{\frac{\pi}{2}+\cancel{2\pi(0)}^{0}}{2}\right) + i\,\text{sen}\left(\frac{\frac{\pi}{2}+\cancel{2\pi(0)}^{0}}{2}\right)\right]$$

$$z_0 = 2^{1/2}\left[\cos\left(\frac{\pi}{4}\right) + i\,\text{sen}\left(\frac{\pi}{4}\right)\right]$$

$$z_0 = 2^{1/2}\left[\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right]$$

$$z_0 = 2^{1/2}\left[\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right]$$

$$z_0 = \frac{\sqrt{2}\cdot\sqrt{2}}{2} + \frac{\sqrt{2}\cdot\sqrt{2}}{2}i$$

$$z_0 = \frac{\cancel{\sqrt{2}}^{2}}{2} + \frac{\cancel{\sqrt{2}}^{2}}{2}i$$

$$z_0 = \frac{2}{2} + \frac{2}{2}i$$

$$\boxed{z_0 = 1 + i}$$

con $k=1$

$$z_1 = (2)^{1/2}\left[\cos\left(\frac{\frac{\pi}{2}+2\pi(1)}{2}\right) + i\,\text{sen}\left(\frac{\frac{\pi}{2}+2\pi(1)}{2}\right)\right]$$

$$z_1 = \sqrt{2}\left[\cos\left(\frac{5\pi}{4}\right) + i\,\text{sen}\left(\frac{5\pi}{4}\right)\right]$$

$$z_1 = \sqrt{2}\left[-\frac{\sqrt{2}}{2} + \left(-\frac{\sqrt{2}}{2}i\right)\right]$$

$$= \sqrt{2}\left[-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right]$$

$$= -\frac{(\cancel{\sqrt{2}})^{2}}{2} - \frac{(\cancel{\sqrt{2}})^{2}}{2}i$$

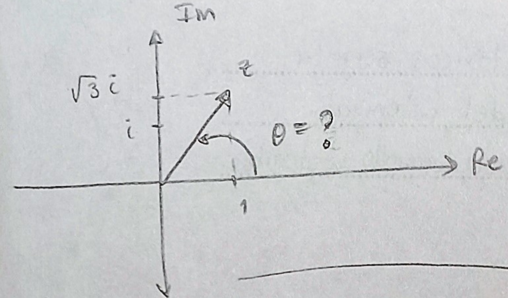
$$\boxed{z_1 = -1 - i}$$

entonces

$$\boxed{a)\ (2i)^{1/2} = \begin{cases} z_0 = 1 + i \\ z_1 = -1 - i \end{cases}}$$

← Respuesta

b) $(1-\sqrt{3}i)^{1/2} \Rightarrow z = 1-\sqrt{3}i$



entonces

$\theta = \arc\tan\left(\frac{\sqrt{3}}{1}\right) \Rightarrow \theta = \arc\tan(\sqrt{3})$

$\Rightarrow \theta = \pi/3$

$n = 2$

$k = 0,1$

$r = \sqrt{1^2+(\sqrt{3})^2} = \sqrt{1+3} = \sqrt{4} = 2$

$r = 2$

Entonces $z_k = r^{1/n}\left[\cos\left(\frac{\theta+2\pi k}{n}\right) + i\,sen\left(\frac{\theta+2\pi k}{n}\right)\right]$

con $k=0$

$z_0 = (2)^{1/2}\left[\cos\left(\frac{\pi}{6}\right) + i\,sen\left(\frac{\pi}{6}\right)\right]$

$z_0 = \sqrt{2}\left[\frac{\sqrt{3}}{2} + \frac{1}{2}i\right]$

$z_0 = \frac{\sqrt{2}.\sqrt{3}}{2} + \frac{\sqrt{2}}{2}i \Rightarrow \boxed{z_0 = \frac{\sqrt{6}}{2} + \frac{\sqrt{2}}{2}i}$

con $k=1$

$z_1 = (2)^{1/2}\left[\cos\left(\frac{\frac{\pi}{3}+2\pi}{2}\right) + i\,sen\left(\frac{\frac{\pi}{3}+2\pi}{2}\right)\right]$

$z_1 = \sqrt{2}\left[\cos\left(\frac{7\pi}{6}\right) + i\,sen\left(\frac{7\pi}{6}\right)\right]$

$z_1 = \sqrt{2}\left[-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right]$

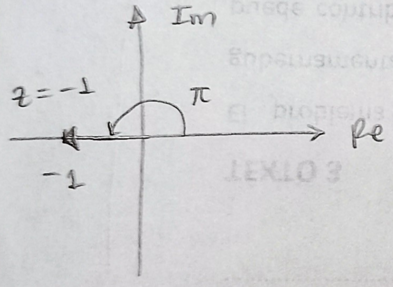
$z_1 = -\frac{\sqrt{2}\sqrt{3}}{2} - \frac{\sqrt{2}}{2}i \Rightarrow \boxed{z_1 = -\frac{\sqrt{6}}{2} - \frac{\sqrt{2}}{2}i}$

entonces las Raices de

$$(1-\sqrt{3}i)^{1/2} = \begin{cases} z_0 = \frac{\sqrt{6}}{2} + \frac{\sqrt{2}}{2}i \\ z_1 = -\frac{\sqrt{6}}{2} - \frac{\sqrt{2}}{2}i \end{cases}$$

Respuesta

c) $(-1)^{1/3} = (-1+0i)^{1/3}$



entonces

$n = 3$

$k = 0, 1, 2$

$\theta = \pi$

$r = 1$

Entonces $z_k = r^{1/n}\left[\cos\left(\dfrac{\theta + 2\pi k}{n}\right) + i\,\text{sen}\left(\dfrac{\theta + 2\pi k}{n}\right)\right]$.

con $k=0$

$$z_0 = (1)^{1/3}\left[\cos\left(\frac{\pi + 2\pi k}{3}\right) + i\,\text{sen}\left(\frac{\pi + 2\pi k}{3}\right)\right]$$

$$z_0 = \cos\left(\frac{\pi}{3}\right) + i\,\text{sen}\left(\frac{\pi}{3}\right) \Rightarrow \boxed{z_0 = \frac{1}{2} + \frac{\sqrt{3}}{2}i}$$

con $k=1$

$$z_1 = \cos(\pi) + i\,\text{sen}(\pi) \Rightarrow \boxed{z_1 = -1 + 0i}$$

con $k=2$

$$z_2 = \cos\left(\frac{5\pi}{3}\right) + i\,\text{sen}\left(\frac{5\pi}{3}\right) \Rightarrow \boxed{z_2 = \frac{1}{2} - \frac{\sqrt{3}}{2}i}$$

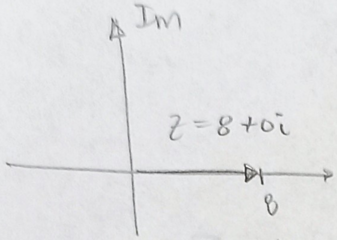
entonces

$$c)\ (-1)^{1/3} = \begin{cases} z_0 = \frac{1}{2} + \frac{\sqrt{3}}{2}i \\ z_1 = -1 + 0i \\ z_2 = \frac{1}{2} - \frac{\sqrt{3}}{2}i \end{cases}$$

⇐ Respuesta

d) $8^{1/6} = (8+0i)^{1/6}$

$z_k = r^{1/n}\left[\cos\left(\frac{\theta+2\pi k}{n}\right) + i\,\text{sen}\left(\frac{\theta+2\pi k}{n}\right)\right]$



$n = 6$

$k = 0,1,2,3,4,5$

$\theta = \arctan\left(\frac{Im}{Re}\right)$

$= \arctan\left(\frac{0}{8}\right)$

$= \arctan(0)$

$\theta = 0$

$r = 8$

con $k=0$

$z_0 = 8^{1/6}\left[\cancel{\cos(0)}^{1} + \cancel{i\,\text{sen}(0)}^{0}\right] \Rightarrow \boxed{z_0 = 8^{1/6}}$

con $k=1$

$z_1 = 8^{1/6}\left[\cos\left(\frac{\pi}{3}\right) + i\,\text{sen}\left(\frac{\pi}{3}\right)\right] \Rightarrow \boxed{z_1 = 8^{1/6}\left[\frac{1}{2} + \frac{\sqrt{3}}{2}i\right]} \Rightarrow \boxed{z_1 = \frac{8^{1/6}}{2} + \frac{8^{1/6}\sqrt{3}}{2}i}$

con $k=2$

$z_2 = 8^{1/6}\left[\cos\left(\frac{2\pi}{3}\right) + i\,\text{sen}\left(\frac{2\pi}{3}\right)\right] \Rightarrow \boxed{z_2 = 8^{1/6}\left[-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right]}$

con $k=3$

$z_3 = 8^{1/6}\left[\cos(\pi) + i\,\text{sen}(\pi)\right] \Rightarrow \boxed{z_3 = 8^{1/6}\left[-1+0i\right]}$

con $k=4$

$z_4 = 8^{1/6}\left[\cos\left(\frac{4\pi}{3}\right) + i\,\text{sen}\left(\frac{4\pi}{3}\right)\right] \Rightarrow \boxed{z_4 = 8^{1/6}\left[-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right]}$

con $k=5$

$z_5 = 8^{1/6}\left[\cos\left(\frac{5\pi}{3}\right) + i\,\text{sen}\left(\frac{5\pi}{3}\right)\right] \Rightarrow \boxed{z_5 = 8^{1/6}\left[\frac{1}{2} - \frac{\sqrt{3}}{2}i\right]}$

con $k=6$

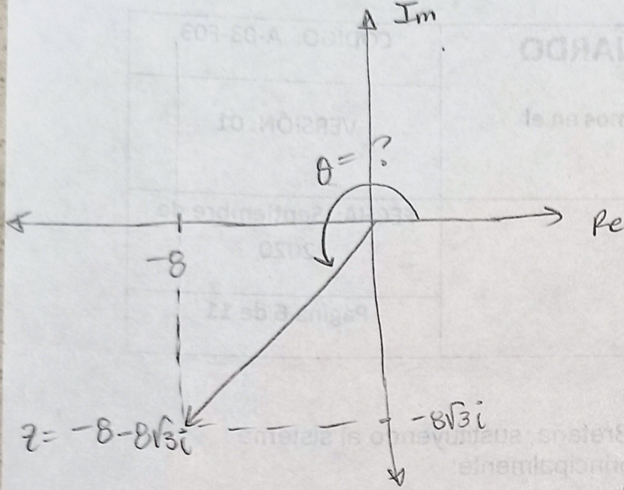
$z_6 = 8^{1/6}\left[\cos(6\pi) + i\,\text{sen}(6\pi)\right] \Rightarrow \boxed{z_6 = 8^{1/6}\left[1+0i\right]}$

entonces

$$8^{1/6} = \begin{cases} z_0 = 8^{1/6} \\ z_1 = 8^{1/6}\left[\frac{1}{2} + \frac{\sqrt{3}}{2}i\right] \\ z_2 = 8^{1/6}\left[-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right] \\ z_3 = 8^{1/6}\left[-1+0i\right] \\ z_4 = 8^{1/6}\left[-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right] \\ z_5 = 8^{1/6}\left[\frac{1}{2} - \frac{\sqrt{3}}{2}i\right] \\ z_6 = 8^{1/6}\left[1+0i\right] \end{cases}$$

← Respuesta

e) $(-8-8\sqrt{3}i)^{1/4} \Rightarrow \boxed{z = -8-8\sqrt{3}i}$ entonces $z_k = r^{1/n}\left[\cos\left(\frac{\theta+2\pi k}{n}\right)+i\,\text{sen}\left(\frac{\theta+2\pi k}{n}\right)\right]$



$n = 4$

$k = 0, 1, 2, 3$

$\theta = \arctan\left(\frac{Im}{Re}\right)$

$= \arctan\left(\frac{-\cancel{8}\sqrt{3}}{-\cancel{8}}\right)$

$= \arctan(\sqrt{3})$

$\theta = \frac{\pi}{3}$

$r = \sqrt{(-8)^2+(-8\sqrt{3})^2}$

$r = \sqrt{64+(64\cdot 3)}$

$r = \sqrt{64+192}$

$r = \sqrt{256} = 16$

$r = 16$

con $k=0$

$z_0 = 16^{1/4}\left[\cos\left(\frac{\pi}{12}\right)+i\,\text{sen}\left(\frac{\pi}{12}\right)\right] \Rightarrow \boxed{z_0 = 2\left[\frac{\sqrt{6}+\sqrt{2}}{4}+\frac{\sqrt{6}-\sqrt{2}}{4}i\right]}$

con $k=1$

$z_1 = 16^{1/4}\left[\cos\left(\frac{7\pi}{12}\right)+i\,\text{sen}\left(\frac{7\pi}{12}\right)\right] \Rightarrow \boxed{z_1 = 2\left[\frac{-\sqrt{6}+\sqrt{2}}{4}+\frac{\sqrt{6}+\sqrt{2}}{4}i\right]}$

con $k=2$

$z_2 = 2\left[\cos\left(\frac{13}{12}\pi\right)+i\,\text{sen}\left(\frac{13}{12}\pi\right)\right] \Rightarrow \boxed{z_2 = 2\left[\frac{-\sqrt{6}-\sqrt{2}}{4}-\frac{\sqrt{6}+\sqrt{2}}{4}i\right]}$

con $k=3$

$z_3 = 2\left[\cos\left(\frac{19}{12}\pi\right)+i\,\text{sen}\left(\frac{19}{12}\pi\right)\right] \Rightarrow \boxed{z_3 = 2\left[\frac{\sqrt{6}-\sqrt{2}}{4}-\frac{\sqrt{6}-\sqrt{2}}{4}\right]}$

entonces

$$(-8-8\sqrt{3}i)^{1/4} = \begin{cases} z_0 = \frac{\sqrt{6}+\sqrt{2}}{2}+\frac{\sqrt{6}-\sqrt{2}}{2}i \\ z_1 = \frac{-\sqrt{6}+\sqrt{2}}{2}+\frac{\sqrt{6}-\sqrt{2}}{2}i \\ z_2 = \frac{-\sqrt{6}-\sqrt{2}}{2}-\frac{\sqrt{6}+\sqrt{2}}{2}i \\ z_3 = \frac{\sqrt{6}-\sqrt{2}}{2}-\frac{\sqrt{6}-\sqrt{2}}{2}i \end{cases} \Rightarrow \text{Respuesta}$$