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Homework 1 - Problem 4

In this question, we explore the relationship between Information Gain, KL Divergence and Entropy. One way to understand KL divergence is to view it as a measure to estimate distance of a probability distribution, p(x), from another probability distribution, q(x). It is defined as:

$$KL(p||q) = -\sum p(x) log_2 \frac{q(x)}{p(x)}$$

It is possible to define Information Gain (IG) as the KL-divergence from the product of the observed marginals of X and Y to their observed joint distribution.

$$IG(x, y) = KL(p(x, y)||p(x)p(y)) = -\sum_x \sum_y p(x, y) log_2 \frac{p(x)p(y)}{p(x, y)}$$

We, however, learned a different definition of IG in class: IG(x, y) = H[x] − H[x|y] = H[y] − H[y|x]. Show that this definition of IG is same as its definition in terms of KL-divergence.

We know that $H(x) = -\sum_x p(x)\ log_2\big(p(x)\big) = -\sum_x \sum_y p(x, y)\ log_2\big(p(x)\big)$

So $H(y) = -\sum_{x,y} p(x, y)\ log_2\big(p(y)\big)$ and $H(x, y) = -\sum_{x,y} P(x, y)\ log_2\big(p(x, y)\big)$

We also know the property $H(X|Y) = H(X, Y) - H(Y)$

So $IG(x, y) = H(x) - H(x|y) = H(x) - \big[H(x, y) - H(y)\big] = -H(x, y) + H(x) + H(y)$

With this:

$$IG(x, y) = \sum_{x,y} p(x, y)\ log_2\big(p(x, y)\big) - \sum_{x,y} p(x, y)\ log_2\big(p(x)\big) - \sum_{x,y} p(x, y)\ log_2\big(p(y)\big) =$$

$$= \sum_{x,y} p(x, y)\Big[\ log_2\big(p(x, y)\big) -\ log_2\big(p(x)\big) -\ log_2\big(p(y)\big)\Big] = \sum_{x,y} p(x, y)\ log_2\left[\frac{p(x, y)}{p(x)p(y)}\right]$$

$$IG(x, y) = -\sum_x \sum_y p(x, y)\ log_2\left[\frac{p(x)p(y)}{p(x, y)}\right] = KL(p(x, y)||p(x)p(y))$$