Chapter 3

Partial Differential Equations

A partial differential equation (PDE) is a differential equation that involves partial derivatives of one or more dependent variables with respect to one or more independent variables. In general, a partial differential equation of variables x, y can be written as

$$f(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}, ...) = 0$$

where $u = u(x, y)$ is the solution.

Example 3.1. *Consider* $\dfrac{\partial u}{\partial x} = x^2 + y^2$. *The solution is*

$$u = \int (x^2 + y^2)\ dx + g(y)$$

.

Remark 3.2. ***Order*** *of a partial differential equation is the same as that of the order of the highest differential co-efficient in it.*

Remark 3.3. *If the dependent variable and all its partial derivatives occur linearly in any PDE, then such an equation is called* ***linear PDE*** *otherwise a non-linear PDE.*

Remark 3.4. *A PDE is called a* ***quasi–linear PDE*** *if all the terms with highest order derivatives of dependent variables occur linearly. That is, the coefficients of such terms are functions of only lower-order derivatives of the dependent variables.*

Remark 3.5. *If all the terms of a PDE contain the dependent variable or its partial derivatives, then such a PDE is called a* ***homogeneous PDE*** *or non-homogeneous otherwise.*

Example 3.6. *The following are some partial differential equations involving the independent variables x, y.*

a) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0,$ *(linear, homogeneous)*

b) $u_{xx} + u_{yy} = 0$ *(linear, homogeneous)*

c) $ux\dfrac{\partial^2 u}{\partial x^2} + u^2 xy\dfrac{\partial^2 u}{\partial x \partial y} + uy\dfrac{\partial^2 u}{\partial y^2} + \left(\dfrac{\partial u}{\partial x}\right)^2 + \left(\dfrac{\partial u}{\partial y}\right)^2 + u^2 = 0$ *(non-linear but quasi-linear, homogeneous)*

d) $u_{xx} + u_{yy} = [(u_x)^2 + (u_y)^2]\, u$ *(quasi-linear)*

e) $(x^2 + y^2)\dfrac{\partial u}{\partial x} + \dfrac{\partial^2 u}{\partial x \partial y} - 3u = 0$ *(linear, 2nd order PDE)*

3.1 Classification of PDE

The classification of PDEs is an important concept because the general theory and solving methods usually apply only to a given class of equations.
The most general **linear partial differential equation of second order with two independent variables** has the form

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial y \partial x} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu + G = 0 \tag{3.1}$$

where A, B, C, D, E, F and G are functions of x, y and constant terms. The equation (3.1) may be written in the form

$$Au_{xx} + Bu_{xy} + Cu_{yy} + f(x, y, u_x, u_y, u) = 0 \tag{3.2}$$

Assume that $A, B,$ and C are continuous functions of x and y processing continuous partial derivatives of high orders as necessary.

The classification of PDE is motivated by the classification of second-order algebraic equations in two variables. In other words, the nature of equation (3.1) is determined by the principal part containing the highest partial derivatives. i.e.

$$L_u = Au_{xx} + Bu_{xy} + Cu_{yy} \tag{3.3}$$

The classification of equation (3.1) is done as follows.

1. If $B^2 - 4AC < 0$, the equation is said to be **elliptic**.

2. If $B^2 - 4AC > 0$, the equation is said to be **hyperbolic**.

3. If $B^2 - 4AC = 0$, the equation is said to be **parabolic**.

The above classification of (3.1) is still valid if the coefficients A, B, C, D, E, and F depend on x, y. In this case,

1. If $B^2(x,y) - 4A(x,y)C(x,y) < 0$, the equation is said to be **elliptic** at (x,y).
2. If $B^2(x,y) - 4A(x,y)C(x,y) > 0$, the equation is said to be **hyperbolic** at (x,y).
3. If $B^2(x,y) - 4A(x,y)C(x,y) = 0$, the equation is said to be **parabolic** at (x,y).

Example 3.7. *Consider the following PD equations.*

1. $u_{xx} + 2u_{yy} = 1$ *is elliptic.*
2. $u_{xx} - u_{yy} = 1$ *is hyperbolic.*
3. $u_{xx} + 3u_{yy} - 2u_x + 24U_y + 5u = 0$ *is elliptic.*
4. $u_{xx} + u_{yy} = 0$ *(Laplace equation) is an elliptic.*
5. $u_t = u_{xx}$ *(Heat equation) is of parabolic type.*
6. $u_{tt} - u_{xx} = 0$ *(Wave equation) is of hyperbolic type.*
7. $u_{xx} + xu_{yy} = 0$, $x \neq 0$ *(Tricomi equation) is hyperbolic for* $x < 0$ *and elliptic for* $x > 0$*. This example shows that equations with variable coefficients can change form in the different regions of the domain.*

In general, a partial differential equation of order n has a solution that contains at most n arbitrary functions. Therefore the general solution can be written as the linear combination of n arbitrary functions. This general solution can be particularized to a unique solution if appropriate extra conditions are provided. These are classified as **boundary conditions**. The kind of boundary conditions we need to specify depends on the nature of the problem.

3.2 Techniques for Solving PDEs

Different types of equations usually require different solving techniques. However, there are some methods that work for most linear partial differential equations with appropriate boundary conditions on a regular domain. These methods include the separation of variables, series expansions, similarity solutions, hybrid methods, and integral transform methods.

3.2.1 Solution by direct integration

The simplest form of a partial differential equation is one for which a solution can be determined through direct partial integration.

Example 3.8. *Solve the differential equation* $u_{xx} = 12x^2(t+1)$ *given that at* $x = 0, u = cos(2t)$ *and* $u_x = sint$.

Example 3.9. *Solve the differential equation* $u_{xy} = sin(x+y)$ *given that at* $y = 0, u_x = 1$ *and at* $x = 0, u = (y-1)^2$.

Initial conditions and boundary conditions: As with any differential equation, the arbitrary constants or arbitrary functions in a particular case are determined based on the additional information provided about the variables within the equation. These additional facts are referred to as the initial conditions or, more generally, as boundary conditions, as they may not always pertain to zero values of the independent variables.

3.2.2 Separation of Variables

This is a fundamental method that holds significant power in obtaining solutions to specific problems involving PDEs. While the method is applied to a relatively small class of problems, it encompasses numerous problems of significant physical interest.

The separation of variables attempts a solution of the form $u = X(x)Y(y)$ where $X(x)$ and $Y(y)$ are functions of x, y respectively. In order to determine these functions, they must satisfy the partial differential equation and the required boundary conditions. Consequently, the partial differential equation is usually transformed into two ordinary differential equations (ODEs). The final solution is then obtained by solving these ODEs. A solution that takes this form is referred to as separable in terms of both x and y, and the process of seeking solutions in this form is known as the method of separation of variables. The following examples illustrate this method by examining the wave equation, the heat equation, and the Laplace equation.

Two Dimensional Heat Flow

Suppose we want to find the temperature distribution in a rectangular metal plate under certain conditions. The plate is covered on its top and bottom faces by layers of thermal insulating material so that heat is constrained to flow mainly in the X and Y directions, as shown in the diagram below. Along the edges of the plate, various conditions are applied. These are known as boundary conditions.

The following assumptions are made when formulating a mathematical model.

- The metal is uniform in the sense that its thermal conductivity is the same at all points of the plate.
- The plate is sufficiently thin so that we neglect any heat flow in the directions perpendicular to its face.
- The temperature distribution is in a steady state. i.e., the temperature at any point in the plate does not depend on the time.

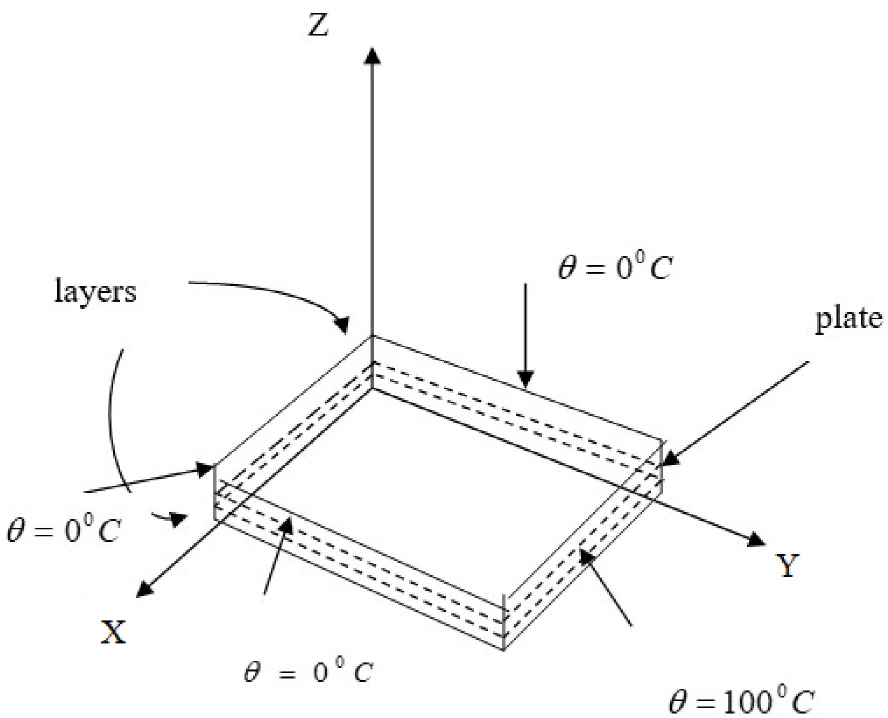

Figure 3.1

Let us consider the temperature function $\theta(x, y)$ depends on x and y. It can be shown that $\theta(x, y)$ satisfies the Laplace equation given by

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0. \tag{3.4}$$

Laplace equation possesses infinitely many solutions. For a unique solution, the following boundary conditions are used.

Boundary conditions:

$$\theta(0, y) = 0, \quad \text{for all } 0 \leq y < b, \tag{3.5a}$$

$$\theta(a, y) = 0, \quad \text{for all } 0 \leq y < b, \tag{3.5b}$$

$$\theta(x, 0) = 0, \quad \text{for all } 0 \leq x \leq a, \tag{3.5c}$$

$$\theta(x, b) = 100°C, \quad \text{for all } 0 \leq x \leq a. \tag{3.5d}$$

The method of separation of variables involves attempting to find solutions that can be expressed as sums or products of functions of a single variable. When dealing with the Laplace equation, we seek solutions in the following form:

$$\theta(x, y) = X(x)Y(y). \tag{3.6}$$

We now differentiate equation (3.6) and substitute into the equation (3.4).

$$Y(y)X''(x) + X(x)Y''(y) = 0. \tag{3.7}$$

From this, we obtain,

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)}. \tag{3.8}$$

Since X is a function of x only, the left member of (3.8) is also a function of x only. Similarly, the right member of (3.8) is a function of y only. Therefore, both of them must be equal to a constant (say $-k^2$). That yields two ordinary differential equations.

$$X''(x) + k^2 X(x) = 0, \tag{3.9a}$$

$$Y''(y) - k^2 Y(y) = 0. \tag{3.9b}$$

Solving the two ordinary differential equations, we can obtain the general solution,

$$\theta(x, y) = (A \sin kx + B \cos kx)\left(Ce^{ky} + De^{-ky}\right). \tag{3.10}$$

Applying the boundary conditions to the general solution, finally, we obtain

$$\theta(x, y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi y}{a}\right). \tag{3.11}$$

This is the Fourier sine series for $\theta = 100$ with $0 < x < L$. Solving this, we obtain,

$$\theta(x, y) = \sum_{n=1}^{\infty} \frac{400}{(2n-1)\pi} \frac{1}{\sinh\left(\frac{(2n-1)\pi b}{a}\right)} \sin\left(\frac{(2n-1)\pi x}{a}\right) \sinh\left(\frac{(2n-1)\pi y}{a}\right). \tag{3.12}$$

Exercise 3.10. *A rectangular plate with insulated surfaces is* $8cm$ *wide and so long compared to its width that it may be considered infinite in length. If the temperature along one short edge* $y = 0$ *is given by,* $u(x, 0) = 100 \sin \frac{\pi x}{8}$, $0 < x < 8$ *while the two long edges* $x = 0$ *and* $x = 8$, *as well as the other edge is kept at* $0°C$, *find steady state temperature* $u(x, y)$.

Laplace Equation in Polar Coordinates

The Laplace equation in polar coordinates is

$$r^2 \frac{\partial^2 T}{\partial r^2} + r\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial \theta^2} = 0. \tag{3.13}$$

where r is the radius and θ is the angle. Let

$$T(r, \theta) = R(r)\Theta(\theta). \tag{3.14}$$

We now differentiate equation (3.14) and substitute into the equation (3.13).

$$r^2\Theta(\theta)R''(r) + r\Theta(\theta)R'(r) + R(r)\Theta''(\theta) = 0. \tag{3.15}$$

From this we obtain,

$$\frac{r^2R''(r) + r\Theta(\theta)R'(r)}{R} = \frac{-\Theta''(\theta)}{\Theta(\theta)} = h, \quad \text{where } h \text{ is a constant.} \tag{3.16}$$

Considering all three possibilities for h $(h = 0, h > 0, h < 0)$ it can be shown that the general solution is,

$$T(r, \theta) = (Ar^k + Br^{-k})(C \cos k\theta + D \sin k\theta) \tag{3.17}$$

The boundary conditions are,

- $T(r, 0) = 0$, $0 \leq r \leq a$.
- $T(r, \pi) = 0$, $0 < r \leq a$.
- $T(a, \theta) = T_0$, $0 < \theta < \pi$.

Then, it can be shown that (exercise 3.11)

$$T(r, \theta) = \sum_{n=1}^{\infty} A_n a^n \sin n\theta \tag{3.18}$$

where $0 < \theta < \pi$. This is the Fourier half-range sine series of T at all the points. Therefore,

$$A_n a^n = \frac{2}{\pi}\int_0^{\pi} T_0 \sin n\theta \; d\theta = \frac{2T_0}{n\pi}\left(1 - (-1)^n\right). \tag{3.19}$$

Exercise 3.11. *The diameter of a semi-circular plate of radius a is kept at* $0°C$*, and the temperature at the semi-circular boundary is* $T_0^\circ C$*. Find the steady-state temperature in the plate.*

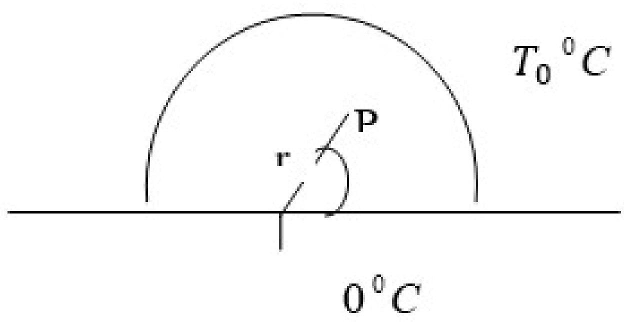

Figure 3.2

Exercise 3.12. *An infinitely long plane uniform plate is bounded by two parallel edges and an end at right angles to them. The breadth is* π*. This end is maintained at a temperature* $100°C$ *at all points, and other ends are at a zero temperature. Determine the temperature at any point of the plate.*

Heat Flow in One Dimension

Suppose that we have a long thin bar of length l, which is aligned along the x axis. We wish to determine the bar's temperature distribution $\theta(x,t)$. Assume the bar is insulated along its sides, and the heat flows in the x direction only.

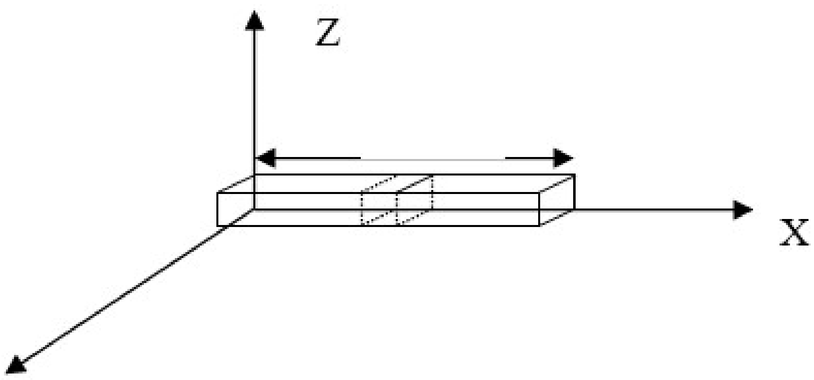

Figure 3.3

The following laws of heat flow are used.

- The amount of heat in a body is proportional to its mass and temperature.
- The heat flows from a point at a higher temperature to a lower temperature.
- The rate of heat flow through a plane surface is proportional to the area of the surface and the rate of change of temperature with respect to the distance in a direction perpendicular to the plane.

It can be shown that the temperature distribution $\theta(x,t)$ satisfies the following

$$\frac{\partial^2\theta}{\partial x^2} = \frac{1}{k}\frac{\partial\theta}{\partial t}. \tag{3.20}$$

where k is a positive constant.

Example 3.13. *Obtain the solution of the equation 3.20, satisfying the following boundary conditions,*

- $\theta(0,t) = 0,\ t \geq 0$
- $\theta(l,t) = 0,\ t \geq 0$
- $\theta(x,0) = f(x),\ 0 < x < l$

where $f(x)$ is a given function and l is the length of the bar (a constant).
After assuming $\theta(x,t) = X(x)T(t)$ and applying separation of variables, the general solution can be obtained. It is given by

$$\theta(x,t) = \sum_{r=1}^{\infty} B_r \sin\left(\frac{r\pi x}{l}\right) e^{-\frac{r^2\pi^2 kt}{l^2}}. \tag{3.21}$$

From Fourier half range sine series of $f(x)$,

$$B_r = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{r\pi x}{l}\right)dx. \tag{3.22}$$

Exercise 3.14. *A rod of length L has its ends A and B maintained at 20^oC and 40^oC respectively until steady state conditions prevail. The temperature at A is suddenly raised to 50^oC while that at B is lowered to 10^oC and maintained thereafter. Find the subsequent temperature distribution of the rod.*

Equation of a Vibrating String (One Dimensional Wave Equation)

Consider a string that is flexible and homogeneous, tightly stretched between two points labeled as O and A. We assume the tension in the string to be so large that gravity may be neglected in comparison with the tension. The differential equation governing the motion when the string is set vibrating in the vertical plane can be shown as,

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}. \tag{3.23}$$

where c is a constant. This is known as the one-dimensional wave equation.

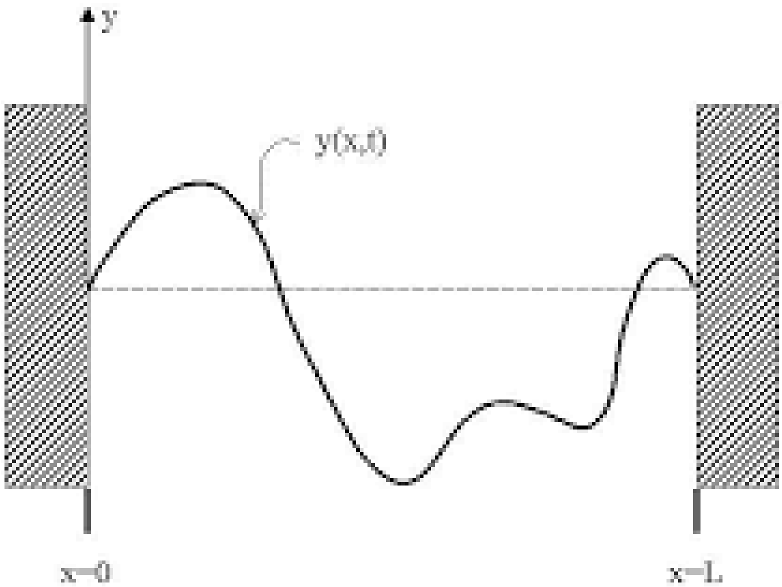

Figure 3.4: Example 3.4

Example 3.15. *A string is stretched and fastened to two points at a distance L apart. Motion is started by displacing the string in the form $y = f(x)$ from which it is released at a time $t = 0$. Find the displacement of any point at distance x from one end at time t. The boundary conditions are*

- $y(0, t) = y(L, t) = 0$ *for all* t,
- $y(x, 0) = f(x)$,
- $\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0.$

Step 1: Let $y(x, t) = X(x)T(t)$.

Step 2: Follow the steps of separation of variables.

The general solution is given by

$$y(x,t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L}.$$

Since $y(x,0) = f(x)$, *from Fourier half range sine series of* $f(x)$

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\ dx.$$

Exercise 3.16. *A perfectly elastic string is stretched between two points* $10cm$ *apart. Its center point is displaced* $2cm$ *from its position of rest at right angles to the original direction of the string and then released with zero velocity. Applying the equation* $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{1}{c^2}\dfrac{\partial u}{\partial t}$ *with* $c^2 = 1$ *to determine the subsequent motion* $u(x,t)$.

Partial Differential Equations: Exercises

1. Consider the following equations. In each case, determine whether the equation is hyperbolic, parabolic, or elliptic.

   (a) $u_{xx} - 7u_{xy} + 6u_{yy} = 0$

   (b) $u_{xx} + u_{xy} - 6u_{yy} = 0$

   (c) $4u_{xx} - 4u_{xy} + u_{yy} = 0$

   (d) $2u_{xx} + 3u_{yy} = 0$

   (e) $2u_{xy} + 3u_{yy} = 0$

2. Consider the equation $(x^2 - 1)\dfrac{\partial^2 u}{\partial x^2} + 2y\dfrac{\partial^2 u}{\partial x \partial y} - \dfrac{\partial^2 u}{\partial y^2} = 0$. Find the set points at which the equation is

   (a) hyperbolic,

   (b) parabolic, and

   (c) elliptic.

3. Solve the following equation.

$$\frac{\partial^2 u}{\partial x^2} = 24x^2(t-2)$$

   It is given that at $x = 0$, $u = e^{2t}$ and $\dfrac{\partial u}{\partial x} = 4t$.

4. A rectangular plate with insulated surfaces is 50 cm wide and so long compared to its width that it may be considered infinite in length. If the temperature along one short edge $y = 0$ is given by $u(x, 0) = 80$, $0 < x < 50$ while the two long edges $x = 0$ and $x = 50$, as well as the other edge is kept at $0°C$, find steady state temperature $u(x, y)$.

5. A rectangular plate $OPQR$ is bounded by the lines $x = 0, y = 0, x = 4$ and $y = 2$. Determine the potential distribution $u(x, y)$ over the rectangle using the Laplace equation, $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$, subject to the following boundary conditions.

   - $u(0, y) = 0$ and $u(4, y) = 0$ for all $0 \le y \le 2$,
   - $u(x, 2) = 0$ for all $0 \le x \le 4$,
   - $u(x, 0) = x(4 - x)$ for all $0 \le x \le 4$.

6. Solve Laplace's equation in plane polar coordinates

$$r^2\frac{\partial^2 T}{\partial r^2} + r\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial \theta^2} = 0$$

in the circular region $x^2 + y^2 = 1$ of the plane where

- $T(r, \theta)$ is finite for $0 \leq r \leq 1$ and for all θ,
- $T(1, \theta) = \sin 2\theta - 4\cos\theta$ for all θ, and
- $T(r, \theta + 2\pi) = T(r, \theta)$ for $0 \leq r \leq 1$ and for all θ.

7. An insulated uniform metal bar, 10 units long, has the temperature of its ends maintained at $0°C$, and at $t = 0$ the temperature distribution along the bar is defined by $f(x) = x(10 - x)$. Solve the heat conduction equation $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{1}{c^2}\dfrac{\partial u}{\partial t}$ with $c^2 = 4$ to determine the temperature u of any point in the bar at time t.