Differential Equations

MA 2013/2014

B. Sc. Engineering Honours Degree
Semester 3 – 2021 Intake

Department of Mathematics
Faculty of Engineering
University of Moratuwa

Learning Outcomes

After completing this course, you should be able to

- Demonstrate the knowledge of Laplace Transform and its applications.
- Apply Fourier series to transform functions to trigonometric infinite series.
- Classify Partial Differential Equations (PDE) and reduce PDEs to canonical form.

Outline of the Syllabus

Laplace Transform and Application to DE

- Laplace transforms of elementary functions and some basic theorems on Laplace transform.
- Inverse Laplace transform, methods to find inverse transform, Convolution theorem.
- Application of Laplace transforms to find solutions to ODEs and systems of ODEs.
- Transfer functions, concepts of stability and controllability.
- Complex Inversion formula.

Fourier Series

- Fourier coefficients, Dirichlet's condition, odd and even functions. Half-range series.
- Parseval's Theorem
- Complex Fourier Series
- Fourier series as the norm minimizer

Partial Differential Equations

- Canonical Forms
- Classification of second-order partial differential equations: Hyperbolic, Parabolic, and Elliptic
- Linear and Nonlinear First Order Partial Differential Equations. Equations solvable by direct integration.

- Solutions by separation of variables.
- Fourier series application to boundary value problems.
- Solve partial differential equations by using Laplace transform.

Ordinary Linear Differential Equations with Variable Coefficients

- Solutions in series form, Frobenius method.
- Special functions: Introduction of Legendre Polynomials and Bessel's functions.

Evaluation:

- Final exam paper 70%
- Continuous assessment 30%

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Chapter 1

Laplace Transformation

In the first chapter, we introduce a concept called *Laplace transform*, which has many applications in engineering and science. Laplace transform can be used to solve differential equations with initial values. It takes a function of the real variable t (in the time domain) and transforms it into a function of the real variable s (in the frequency domain).

Since the transform is invertible, no information is lost, and it is reasonable to think of a function $f(t)$ and its Laplace transform $F(s)$ as two views of the same phenomenon. Each view has its own uses, and some features of the phenomenon are easier to understand in one view or the other.

Definition 1.1. *Let f be a real-valued function of the real variable t, defined for $t \geq 0$. Let s be a variable that assumed to be real. Then the Laplace transform of f can be defined by*

$$F(s) = \int_0^\infty f(t)e^{-st}\, dt, \tag{1.1}$$

for all values of s for which this integral is defined. We denote the Laplace transform F of f more precisely by $\mathcal{L}\{f(t)\}$.

1.1 Important Formulae

- $\mathcal{L}\{1\} = \dfrac{1}{s}$; $s > 0$
- $\mathcal{L}\{t\} = \dfrac{1}{s^2}$; $s > 0$
- $\mathcal{L}\{t^n\} = \dfrac{n!}{s^{n+1}}$; n is a positive integer, $s > 0$
- $\mathcal{L}\{e^{at}\} = \dfrac{1}{s-a}$; $s - a > 0$

- $\mathcal{L}\{\sin at\} = \dfrac{a}{s^2 + a^2}$; $s > 0$
- $\mathcal{L}\{\cos at\} = \dfrac{s}{s^2 + a^2}$; $s > 0$
- $\mathcal{L}\{\sinh at\} = \dfrac{a}{s^2 - a^2}$; $s > 0$ so that $s > |a|$
- $\mathcal{L}\{\cosh at\} = \dfrac{s}{s^2 - a^2}$; $s > 0$ so that $s > |a|$

Exercise 1.2. *Derive the above formulae.*

Exercise 1.3. *Find the Laplace transform of*

$$f(t) = \begin{cases} -1 & \textit{if} \quad 0 \leq t < 2\ , \\ 1 & \textit{if} \quad 2 \leq t. \end{cases}$$

1.2 Basic Properties of the Laplace Transforms

1.2.1 The Linear Property

Let f_1 and f_2 be functions whose Laplace transforms exist. Then for any constants c_1 and c_2,

$$\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} = c_1 \mathcal{L}\{f_1(t)\} + c_2 \mathcal{L}\{f_2(t)\}.$$

1.2.2 The First Shifting Theorem (s-Shifting)

[illegible]. *Suppose f is a function so that $\mathcal{L}\{f\}$ exists for $s > \alpha$. Then for any constant a,*

$$\mathcal{L}\{e^{at} f(t)\} = F(s - a)$$

for $s > \alpha + a$, where $F(s)$ denotes $\mathcal{L}\{f(t)\}$.

Exercise 1.5. *Show that*

- $\mathcal{L}\{e^{at} t^n\} = \dfrac{n!}{(s-a)^{n+1}}$
- $\mathcal{L}\{e^{at} \cosh bt\} = \dfrac{s-a}{(s-a)^2 - b^2}$
- $\mathcal{L}\{e^{at} \sinh bt\} = \dfrac{b}{(s-a)^2 - b^2}$
- $\mathcal{L}\{e^{at} \cos bt\} = \dfrac{s-a}{(s-a)^2 + b^2}$

Theorem 1.4 - Proof

If $L\{f(t)\} = F(s)$ then $L\{e^{at}f(t)\} = F(s-a)$

Proof

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

So that

$$F(s-a) = \int_0^\infty e^{-(s-a)t} f(t)\, dt$$

$$F(s-a) = \int_0^\infty e^{at} \cdot e^{-st} f(t)\, dt$$

$$F(s-a) = \int_0^\infty e^{-st} \left[e^{at} f(t) \right] dt$$

$$\therefore \; F(s-a) = L\{e^{at} f(t)\}$$

☆Exercise: Find the laplace transformation of,

1) $f(t) = t e^{t} + e^{t}\sin t + e^{t}\sinh 2t$

$$L\{f(t)\} = \frac{1}{(s-1)^2} + \frac{1}{(s-1)^2+1} + \frac{2}{(s-1)^2-2^2}$$

2) $f(t) = e^{at} t^3 + e^{at}\cos t + e^{2t}\sin 2t$

$$L\{f(t)\} = \frac{3!}{(s-a)^4} + \frac{(s-a)}{(s-a)^2+1} + \frac{2}{(s-2)^2+2^2}$$

3) $f(t) = e^{2t}\sin 2t + e^{3t}\underbrace{\cos^3 t}_{\frac{1}{4}(\cos 3\theta + 3\cos\theta)}$

$$L\{f(t)\} = \frac{2}{(s-2)^2+2^2} + \frac{1}{4}\left[\frac{(s-3)}{(s-3)^2+9} + \frac{3(s-3)}{(s-3)^2+1}\right]$$

- $\mathcal{L}\{e^{at} \sin bt\} = \dfrac{b}{(s-a)^2 + b^2}$

Exercise 1.6. *Evaluate the Laplace transform of the following functions.*

- $f(t) = t^2 + e^{5t} + \cos 3t$
- $g(t) = t^3 e^{2t} + e^{4t} \cos 3t + e^{2t} \sin 5t$

Exercise 1.7. *Find* $\mathcal{L}\{e^{at} \cos^2 t\}$.

2.

1.3 Differentiation and Integration of the Laplace Transform

Knowing the Laplace transform of the derivative of y is necessary to solve the differential equations using the method we will describe in this course.

Suppose that y is continuous on $[0, \infty)$ and the derivative of y is continuous on $(0, \infty)$. Then

$$\mathcal{L}\{y'(t)\} = s\mathcal{L}\{y(t)\} - y(0). \tag{1.2}$$

Use integrating by parts.

To treat differential equations, we will also need to know $\mathcal{L}\{y''\}$. The corresponding formula for y'' can be obtained by replacing y in Equation 1.2.

$$\mathcal{L}\{y''(t)\} = s\mathcal{L}\{y'(t)\} - y'(0)$$

$$\mathcal{L}\{y''(t)\} = s\left(s\mathcal{L}\{y(t)\} - y(0)\right) - y'(0)$$

$$\mathcal{L}\{y''(t)\} = s^2\mathcal{L}\{y(t)\} - sy(0) - y'(0) \tag{1.3}$$

In general, for any given positive integer n,

$$\mathcal{L}\{y^n(t)\} = s^n\mathcal{L}\{y(t)\} - s^{n-1}y(0) - s^{n-2}y''(0) - \cdots - sy^{n-2}(0) - y^{n-1}(0). \tag{1.4}$$

Not only can the Laplace transform be differentiated, but it can be integrated as well, and the result is another Laplace transform.

If $g(t) = \int_0^t f(x)\,dx$ then $g(0) = 0$ and $g'(t) = f(t)$. Observe that the Laplace transform of $g'(t)$ is $s\mathcal{L}\{g(t)\}$. Hence,

$$\mathcal{L}\left\{\int_0^t f(x)\,dx\right\} = \frac{1}{s}F(s) \tag{1.5}$$

provided that $s \neq 0$.

3.

1.4 Laplace Transform of a Function with t^n Terms

1.4.1 Multiplication by t

If $\mathcal{L}\{f(t)\} = F(s)$ then

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}[F(s)]$$

where n is a positive integer.

Activity 1.9. *Prove the result.*

Example 1.10. *Find the Laplace transform of the following functions.*

1. $f(t) = t \sinh at$

2. $f(t) = t^2 \cos at$

1.4.2 Division by t

If $\mathcal{L}\{f(t)\} = F(s)$ then

$$\mathcal{L}\left\{\frac{1}{t} f(t)\right\} = \int_s^\infty F(x)\ dx.$$

Activity 1.11. *Prove the result.*

Example 1.12. *Evaluate the Laplace transform of the following functions.*

1. $f(t) = \dfrac{\sin 3t}{t}$

2. $f(t) = e^{5t}\dfrac{\sin 2t}{t}$

4.

1.5 Unit Step Function

Let a be a fixed positive real number. The unit step function, which is denoted by $U_a(t)$, is defined as

$$U_a(t) = \begin{cases} 0 & \text{if} \quad 0 \le t < a \\ 1 & \text{if} \quad t \ge a. \end{cases}$$

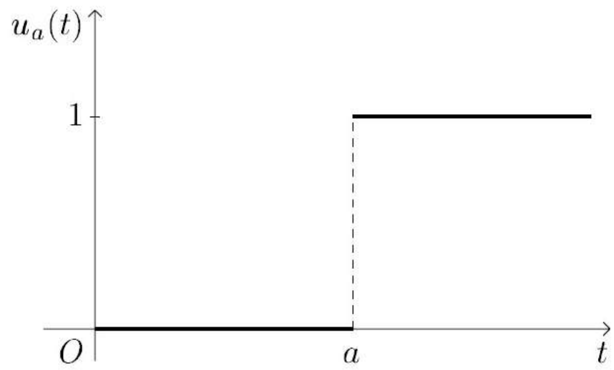


Figure 1.1: Unit step function $U_a(t)$

Consider a function $f(t)$ defined for $t \geq 0$. Suppose we need a new function $f_1(t)$ that results by translating f a distance of a units in the positive direction and then assigning 0 for $0 \leq t < a$. That is

$$f_1(t) = \begin{cases} 0 & \text{if} \quad 0 < t < a \ , \\ f(t-a) & \text{if} \quad a \leq t. \end{cases}$$

Notice that

$$f_1(t) = U_a(t) f(t-a).$$

Example 1.14. *Express the following functions in terms of the unit step functions and find their Laplace transform.*

$$f(t) = \begin{cases} 8 & \textit{if} \quad t < 2 \ , \\ 6 & \textit{if} \quad t \geq 2. \end{cases}$$

$$g(t) = \begin{cases} t-1 & \textit{if} \quad 1 \leq t < 2 \ , \\ 3-t & \textit{if} \quad 2 \leq t < 3. \end{cases}$$

1.5.1 The Second Shifting Theorem

Proposition 1.15. *Let $a \geq 0$ be given and $\mathcal{L}\{f(t)\} = F(s)$. Then*

$$\mathcal{L}\{U_a(t) f(t-a)\} = e^{-as} F(s).$$

Example 1.16. *Find $\mathcal{L}\{U_2(t) sin(t-2)\}$.*

Example 1.17. *Determine $\mathcal{L}\{U_2(t) t^2\}$.*

Example 1.18. *Determine $\mathcal{L}\{U_3(t) \sin \frac{\pi t}{2}\}$.*

If $L\{f(t)\} = F(s)$

then

$$L\{f(t-a)u(t-a)\} = e^{-as}F(s)$$

Proof

$$L\{f(t-a)u(t-a)\} = \int_0^\infty e^{-st} f(t-a)\, u(t-a)\, dt$$

$$\text{L.H.S} = \underbrace{\int_0^a e^{-st} f(t-a)(0)\, dt}_{0} + \int_a^\infty e^{-st} f(t-a)(1)\, dt$$

$$= \int_a^\infty e^{-st} f(t-a)\, dt$$

let $u = t-a$

$t \to a \quad t \to \infty$

$u \to 0 \quad u \to \infty$

$$= \int_0^\infty e^{-s(u+a)} \cdot f(u)\, du$$

$$= e^{-as} \int_0^\infty e^{-su} f(u)\, du$$

$$= e^{-as} F(s)$$

1) $L\{(t-1)u(t-1)\} = \dfrac{e^{-s}}{s^2}$

$L\{f(t)\} = F(s)$

$f(t) = t$

$F(s) = 1/s^2$

2) $L\{(t-2)^2 u(t-2)\} = e^{-2s} \cdot \dfrac{2!}{s^3}$

$f(t) = t^2$

$F(s) = \dfrac{2!}{s^3}$

3) $L\{\cos(t-1)u(t-1)\} = \dfrac{e^{-s} s}{s^2+1}$

$f(t) = \cos t$

$F(s) = \dfrac{s}{s^2+1}$

4) $L\{\sin(t-2)u(t-2)\} = e^{-2s} \cdot \dfrac{1}{s^2+1}$

5) $L\{e^{(t-3)} u(t-3)\} = e^{-3s} \cdot \dfrac{1}{s-1}$

Qs,

1) $$L^{-1}\left\{\frac{e^{-as}}{s^2+1}\right\} = u(t-a).\,\sin(t-a)$$

2) $$L^{-1}\left\{\frac{e^{-2s}}{s-3}\right\} = u(t-2).\,e^{-3(t-2)}$$

3) $$L^{-1}\left\{\frac{e^{-3s}}{s^4}\right\} = u(t-3).\,\frac{1}{3!}.\,(t-3)^3$$

1.5.2 Laplace Transform of Periodic Functions

If $f(t)$ is a periodic function with period $T > 0$. If the Laplace transform of $f(t)$ exists then

$$\mathcal{L}\{f(t)\} = \left\{ \frac{\int_0^T e^{-st} f(t)\, dt}{(1 - e^{-sT})} \right\}.$$

Exercise 1.20. *Find $\mathcal{L}\{|\sin t|\}$.*

5.

1.6 Inverse Laplace Transforms

To apply the Laplace transform to physical problems, it is necessary to invoke the inverse transform.

. *If $\mathcal{L}\{f(t)\} = F(s)$ then the inverse Laplace transform of $F(s)$ is*

$$\mathcal{L}^{-1}\{F(s)\} = f(t).$$

Remark 1.22. *If $\mathcal{L}\{f(t)\} = F(s)$ then*

$$\mathcal{L}^{-1}\{F(s+a)\} = e^{(-at)}\mathcal{L}^{-1}\{F(s)\}.$$

Remark 1.23. *If $\mathcal{L}\{f(t)\} = F(s)$ then*

$$\mathcal{L}^{-1}\{F(\alpha s)\} = \frac{1}{\alpha} f\left(\frac{t}{\alpha}\right)$$

provided that $\alpha > 0$.

The fact that **the inverse Laplace transform is linear** follows immediately from the linearity of the Laplace transform. When finding the Laplace transform of complicated equations, we need to simplify them via partial fractions and linearity to forms that have known inverse transforms.

Example 1.24. *Find the inverse Laplace transform of each function.*

1. $F(s) = \dfrac{1}{s^2 + a^2}$
2. $F(s) = \dfrac{1}{(s-a)^2 + b^2}$
3. $F(s) = \dfrac{s-a}{(s-a)^2 + b^2}$
4. $F(s) = \dfrac{1}{s(s+3)}$

ex:- Find the inverse Laplace transform of $F(s)$,

1)

$$F(s) = \frac{2s-11}{s^2-4s+8}$$

$$= \frac{2s-11}{(s-2)^2+2^2}$$

$$= \frac{2(s-2)}{(s-2)^2+2^2} - \frac{7}{(s-2)^2+2^2}$$

$$L^{-1}[F(s)] = 2e^{2t}\cos 2t - \frac{7}{2}e^{2t}\sin 2t$$

2) $$F(s) = \frac{3s+7}{(s^2+6s+9)}$$

$$= \frac{3s+7}{(s+3)^2}$$

$$= \frac{3(s+3)-2}{(s+3)^2}$$

$$= \frac{3}{(s+3)} - \frac{2}{(s+3)^2}$$

$$L^{-1}[F(s)] = 3e^{-3t} - 2te^{-3t}$$

5. $F(s) = \dfrac{3!}{(s-2)^4}$

6. $F(s) = \left(\dfrac{2(s-1)e^{-2s}}{(s-1)^2+1}\right) e^{-2s}$

Proposition 1.25. *Suppose $\mathcal{L}^{-1}\{F(s)\} = f(t)$. Then*

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = U_a(t)f(t-a)$$

where a is any real number.

Example 1.26. *Recall that $\mathcal{L}^{-1}\left(\dfrac{s}{s^2+4}\right) = \cos 2t$. Hence,*

$$\mathcal{L}^{-1}\left(\frac{s}{s^2+4}e^{-7s}\right) = \cos\left(2(t-7)\right)U_7(t).$$

The presence of e^{-7s} caused two changes to $\cos 2t$. The input t was replaced by $t-7$ and then $\cos\left(2(t-7)\right)$ was multiplied by $U_7(t)$.

Exercise 1.27. *Determine the inverse transform of each function.*

1. $\dfrac{1+e^{-\pi s}}{s^2+1}$

2. $\dfrac{e^{-7s}}{(s-5)^2+4}$

3. 1.7 Convolution

Definition 1.28. *Let $f(t)$ and $g(t)$ be piece-wise continuous functions. The convolution of $f(t)$ with $g(t)$ is defined by*

Here,

$$f(t) * g(t) = \int_0^t f(\tau)g(t-\tau)\, d\tau. \tag{1.6}$$

$f(t) = L^{-1}[F(s)]$

$g(t) = L^{-1}[G(s)]$

Remark 1.29. *The convolution is* ***commutative****. That is,*

$$(f * g)(t) = (g * f)(t).$$

Example 1.30. *Let $f(t) = t^2$ and $g(t) = 2t+3$. Find $(f * g)(t)$.*

Example 1.31. *Express the following integral as a convolution.*

$$\int_0^t \tau^3 cos(t-\tau)\, d\tau$$

Remark 1.32. *In general, $\mathcal{L}\{f(t)g(t)\} \neq \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}$.*

Convolution Theorem

If $f(t)$, $g(t)$ are Continuous functions $t>0$,

$$L^{-1}[F(s)] = f(t) \quad \text{and} \quad L^{-1}[G(s)] = g(t)$$

then,

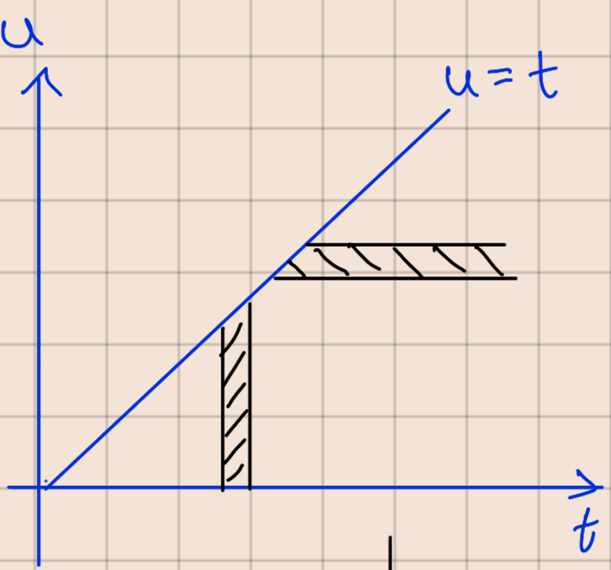
$$L^{-1}[F(s).G(s)] = \int_0^t f(u).g(t-u)\,du$$

Proof

Consider

$$L\left\{\int_0^t f(u)\,g(t-u)\,du\right\}$$

$$= \int_{t=0}^{\infty} e^{-st}\left[\int_{u=0}^{t} f(u)g(t-u)\,du\right]dt$$



$t: 0 \to \infty$ | $u: 0 \to \infty$

$u: 0 \to t$ | $t: u \to \infty$

by changing the order of integration,

$$= \int_{u=0}^{\infty} f(u)\left[\int_{t=u}^{\infty} g(t-u)\,e^{-st}\,dt\right]du$$

$\therefore$ Changing the order of integrals give the same result.

$$= \int_{u=0}^{\infty} f(u).\,e^{-su}\,du.\int_{t=u}^{\infty} g(t-u)\,e^{-s(t-u)}\,dt.$$

Let $v = t - u \qquad dv = dt$

$$= \int_{u=0}^{\infty} e^{-su} f(u)\, du \,.\, \int_{v=0}^{\infty} e^{-sv} g(v)\, dv$$

$$= F(s) \,.\, G(s)$$

$$\therefore \quad \mathcal{L}^{-1}\{F(s) \,.\, G(s)\} = \int_0^t f(u) g(t-u)\, du$$

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1) Find inverse $\mathcal{L}$ of $F(s) = \dfrac{s}{(s^2+a^2)^2}$ using Convolution theorem.

$$\frac{s}{(s^2+a^2)^2} = \underbrace{\frac{s}{s^2+a^2}}_{F(s)} \cdot \underbrace{\frac{1}{s^2+a^2}}_{G(s)}$$

$$\mathcal{L}^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \int_0^t \cos(au)\, \frac{\sin(a(t-u))}{a}\, du$$

$$= \frac{1}{2a} \int_0^t 2 \sin a(t-u) \cos(au)$$

$$= \frac{1}{2a} \int_0^t \sin(at) - \sin(at - 2au)\, du$$

$$= \frac{1}{2a} \,.\, \sin at \, [u]_0^t - \frac{1}{2a}\left[\cancel{\frac{\cos(at-2au)}{2a}}\right]_0^t \;\; 0$$

$$= \frac{1}{2a} t \sin at$$

2) $$\frac{s^2}{(s^2+a^2)(s^2+b^2)} = \frac{s}{(s^2+a^2)} \cdot \frac{s}{(s^2+b^2)}$$

$$L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right] = \int_0^t \cos(au)\cos(b(t-u))\,du$$

$$= \frac{1}{2}\int_0^t \cos[(a-b)u+bt] + \cos[bt-(a+b)u]$$

$$= \frac{1}{2}\left[\frac{\sin[(a-b)u+bt]}{(a-b)}\right]_0^t + \frac{1}{2}\left[\frac{\sin[(a+b)u-bt]}{(a+b)}\right]_0^t$$

$$= \frac{1}{2(a-b)}\left[\sin(at)-\sin(bt)\right] + \frac{1}{2(a+b)}\left[\sin(at)+\sin(bt)\right]$$

$$= \frac{(a+b)[\sin at - \sin bt] + (a-b)[\sin(at)+\sin(bt)]}{2(a^2-b^2)}$$

$$= \frac{a\sin at - b\sin bt}{a^2-b^2}$$

3) $F(s) = \dfrac{1}{s(s+a)}$

7) $F(s) = \dfrac{e^{at}}{s(s^2+a^2)}$

4) $F(s) = \dfrac{1}{s^2(s^2+a^2)}$

5) $F(s) = \dfrac{1}{(s^2+a^2)^2}$

6) $F(s) = \dfrac{se^{at}}{s^2+a^2}$

3) $$L^{-1}\left[\frac{1}{s(s+a)}\right] = L^{-1}\left[\frac{1}{s}\cdot\frac{1}{s+a}\right]$$

$$= \int_0^t 1 \cdot e^{-a(u-t)} \cdot 1 \, du .$$

$$\frac{1}{s} - \frac{1}{s+a}$$

$$\frac{s+a-s}{s(s+a)}$$

$$= e^{at}\int_0^t e^{-au}\, du$$

$$= e^{at} \cdot \left[\frac{e^{-au}}{-a}\right]_0^t = \frac{1-e^{at}}{a}$$

4) $$L^{-1}\left[\frac{1}{s^2}\cdot\frac{a}{a(s^2+a^2)}\right] = \int_0^t \frac{\sin(au)}{a}\cdot(u-t)\, du$$

$$= \int_0^t \frac{\sin(au)}{a}(-t)\, du + \int_0^t \frac{\sin(au)}{a} u \, du$$

$$= \frac{-t}{a}\left[\frac{-\cos(au)}{a}\right]_0^t + \left[\frac{-\cos(au)\, u}{a}\right]_0^t - \left[\frac{-\sin(au)}{a^2}\right]_0^t$$

$$= \frac{t\cos at}{a^2} - \frac{t}{a^2} - \frac{t\cos at}{a} + \frac{\sin at}{a^2}$$

One of the most significant properties possessed by the convolution in connection with the Laplace transform is that the product of the Laplace transforms of two functions is the Laplace transform of their convolution, which is called the convolution theorem.

Theorem 1.33 (Convolution Theorem). *If $f(t)$ and $g(t)$ are Laplace transformable functions then*

$$\mathcal{L}\{(f * g)(t)\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}.$$

Example 1.34. *Without evaluating the integral, find*

$$\mathcal{L}\left\{\int_0^t e^{2(t-\tau)}\tau^3 \, d\tau\right\}.$$

Exercise 1.35. *Use the convolution theorem to evaluate*

1. $\mathcal{L}\{\int_0^t e^{t+\theta} \sin(t-\theta) d\theta\}$
2. $\mathcal{L}^{-1}\left\{\dfrac{s}{(s^2+a^2)^2}\right\}$

1.8 Solutions for Ordinary Differential Equations

The Laplace transformation is useful in solving ordinary differential equations, simultaneous differential equations, special integral differential equations, linear intro-differential equations, etc. Using the differentiation and integration properties of Laplace transforms, each term in the equation is transformed. Initial conditions are automatically taken into account. We solve the resulting algebraic equation in the s-domain. We convert the solution back to the time domain by using inverse transform.

Example 1.36. *Solve the initial value problem $y' - 5y = e^{-2t}$ with $y(0) = 3$.*

Example 1.37. *Solve the IVP $y'' - y = e^{-t}$ with $y(0) = 1$ and $y'(0) = 0$.*

Example 1.38. *Solve the IVP $y'' + 5y' + 6y = 0$ with $y(0) = 2$ and $y'(0) = 3$.*

Example 1.39. *Solve the IVP $y'' + 7y = 10e^{2t}$ with $y(0) = 0$ and $y'(0) = 3$.*

Example 1.40. *Solve the IVP $y'(t) = 1 - \sin t - \int_0^t y(\tau)\, d\tau$ with $y(0) = 0$.*

A system of differential equations can also be readily handled by the Laplace transform method.

Example 1.41. *Consider,*

$$x' + 3x + y' = 1 \tag{1.7a}$$

$$x' - x + y' = e^t \tag{1.7b}$$

with $x(0) = 0$ *and* $y(0) = 0$. *By taking the Laplace transforms and solving the simultaneous equations, solutions for* x, y *can be found.*

Applications of Differential Equations.

- Consider 2^{nd} Order or higher order D:E of the form,

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = r(t)$$

$$y = y(t)$$

Solution procedure,

1) Step 1, Let $L[y(t)] = Y(s)$

2) Step 2, take L.T. on both sides,

$$aL\left\{\frac{d^2y}{dx^2}\right\} + bL\left\{\frac{dy}{dx}\right\} + cL\{y(t)\} = L\{r(t)\}$$

$$a\left[s^2Y(s) - sy(0) - y'(0)\right] + b\left[sY(s) - y(0)\right] + cY(s) = L\{r(t)\}$$

now from a function of

$$Y(s) = \frac{L\{r(t)\}}{g(s)}$$

taking inverse Laplace Transformation find $y(t)$ which is the solution.

Ex :- Solve D.E

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0 \quad ; \quad y(0) = 1, \; y'(0) = 1$$

Let $L[y(x)] = F(s)$

$$L\left[\frac{d^2y}{dx^2}\right] - 2L\left[\frac{dy}{dx}\right] + 2L[y(x)] = 0$$

$$\{s^2F(s) - s\,y(0) - y'(0)\} - 2\{sF(s) - y(0)\} + 2F(s) = 0$$

$$[s^2 - 2s + 2]F(s) - s - 1 + 2 = 0$$

$$F(s) = \frac{s-1}{s^2 - 2s + 2}$$

$$= \frac{(s-1)}{(s-1)^2 + 1}$$

$$= e^x \cos x$$

2) $\frac{d^2y}{dt^2} + \frac{dy}{dt} - 3y = 3\cos 3t - 11\sin 3t$; $y(0)=0$

$y'(0)=6$

$$L\left[\frac{d^2y}{dt^2} + \frac{dy}{dt} - 3y\right] = L\left[3\cos 3t - 11\sin 3t\right]$$

Let $L[y(t)] = Y(s)$

$$\left[s^2 Y(s) - s\underbrace{y(0)}_{0} - \overbrace{y'(0)}^{6}\right] + \left[sY(s) - \overbrace{y(0)}^{0}\right] - 3Y(s)$$

$$= \left(\frac{3s}{s^2+9} - \frac{11\times 3}{s^2+9}\right)$$

$$Y(s)\left[s^2+s-3\right] - 6 = \frac{3(s-11)}{s^2+9}$$

$$Y(s) = \frac{3s - 33 + 6s^2 + 54}{(s^2+9)\underbrace{(s^2+s-3)}_{(s+\frac{1}{2})^2 - \frac{13}{4}}}$$

$$= 3\left[2s^2 + s + 7\right]$$

Laplace Transform: Exercises

In each of the following exercises, use the Laplace transform to find the solution.

1. $\frac{dy}{dt} - y = e^{3t}, \quad y(0) = 2.$

2. $\frac{d^2y}{dt^2} + 7y = 10e^{2t}, \quad y(0) = 0, y'(0) = 3.$

3. $\frac{dy}{dt} = 1 - \sin t - \int_0^t y(\alpha)\ d\alpha, \quad y(0) = 0.$

4. $\frac{d^2y}{dt^2} + \frac{dy}{dt} - 12y = 0, \quad y(0) = 4, y'(0) = -1.$

5. $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = te^{-2t}, \quad y(0) = 1, y'(0) = 0.$

6. $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = 0, \quad y(0) = 2, y'(0) = 4.$

7. $\frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y = 18e^{-t}\sin 3t, \quad y(0) = 0, y'(0) = 3.$

8. $\frac{d^2y}{dt^2} + 7\frac{dy}{dt} + 10y = 4te^{-3t}, \quad y(0) = 0, y'(0) = -1.$

9. $\frac{dx}{dt} + y = 3e^{2t}, \quad \frac{dy}{dt} + x = 0, \quad x(0) = 2, y(0) = 0.$

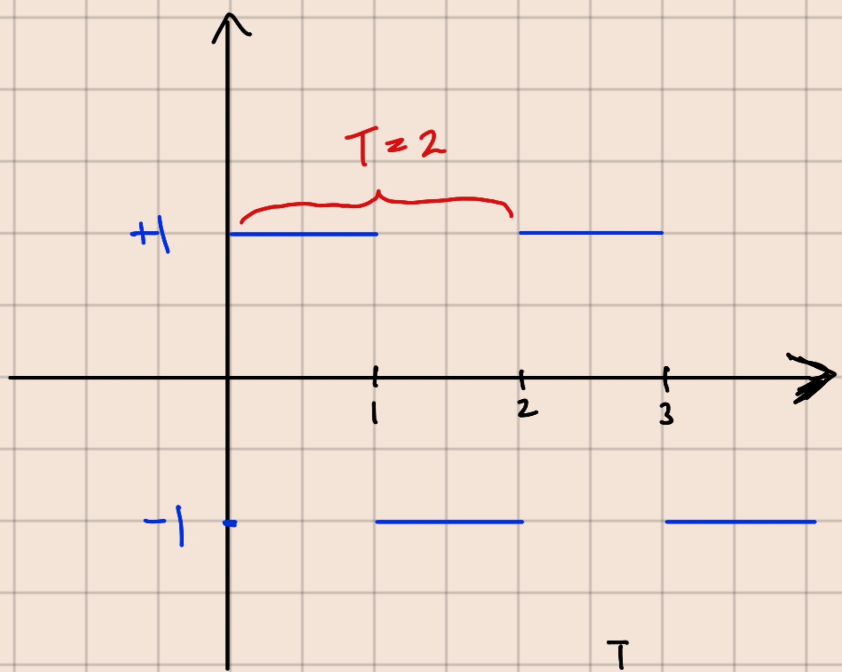
10. $\frac{dx}{dt} - 5x + 2y = 3e^{4t}, \quad \frac{dy}{dt} - 4x + y = 0, \quad x(0) = 3, y(0) = 0.$

11. $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = h(t), \quad y(0) = 0, y'(0) = 0, \quad \text{where } h(t) = \begin{cases} 1 & \text{if } 0 < t < \pi \\ 0 & \text{if } t > \pi. \end{cases}$

12. $\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = h(t), \quad y(0) = 0, y'(0) = 0, \quad \text{where } h(t) = \begin{cases} 2 & \text{if } 0 < t < 4 \\ 0 & \text{if } t > 4. \end{cases}$

Periodic

ex:- $f(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ -1 & 1 < x < 2 \end{cases}$



$$L\{f(t)\} = \frac{1}{1-e^{-sT}} \cdot \int_0^T e^{-sT} f(t)\,dt$$

$$= \frac{1}{1-e^{-2s}} \left\{ \int_0^1 e^{-st} \underbrace{f(t)}_{1}\,dt + \int_1^2 e^{-st} \underbrace{f(t)}_{-1}\,dt \right\}$$

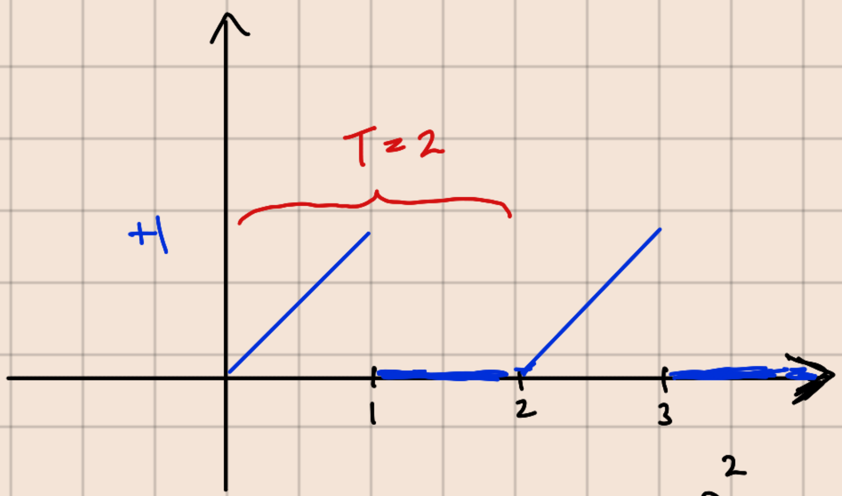
$$= \frac{1}{1-e^{-2s}} \left\{ \int_0^1 e^{-st}\,dt - \int_1^2 e^{-st}\,dt \right\}$$

$$= \frac{1}{1-e^{-2s}} \left\{ \left[\frac{e^{-st}}{-s}\right]_0^1 - \left[\frac{e^{-st}}{-s}\right]_1^2 \right\}$$

$$= \frac{1}{1-e^{-2s}} \left\{ \frac{-e^{-s}}{s} + \frac{1}{s} + \frac{e^{-2s}}{s} - \frac{e^{-s}}{s} \right.$$

$$= \frac{1}{s(1-e^{-2s})} \left\{ 1 - 2e^{-s} + e^{-2s} \right\}$$

2) $f(t) = \begin{cases} & 0 \leq t \leq 1 \\ 0 & 1 < t \leq 2 \end{cases}$



$$L[f(x)] = \frac{1}{1-e^{-2s}} \cdot \int_0^2 e^{-st} f(t)\, dt$$

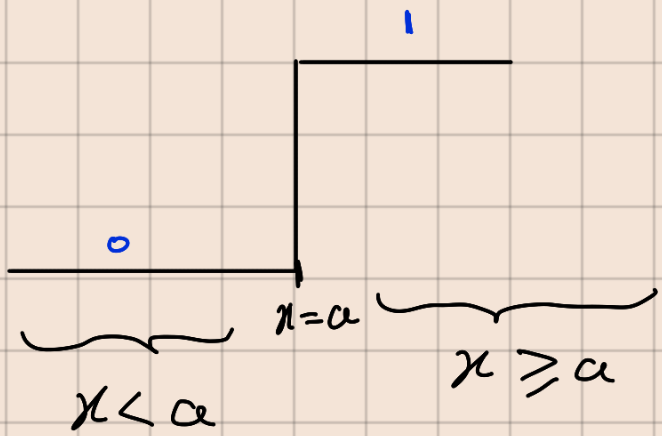
$$= \frac{1}{1-e^{-2s}} \cdot \left\{ \int_0^1 e^{-st} \cdot t\, dt + \int_1^2 e^{-st} \cdot 0\, dt \right\}$$

$$= \frac{1}{1-e^{-2s}} \left\{ \left[\frac{e^{-st}}{-s} \cdot t \right]_0^1 - \left[\frac{e^{-st}}{s^2} \right]_0^1 \right\}$$

$$= \frac{1}{1-e^{-2s}} \left\{ \left(\frac{e^{-s}}{-s} - 0 \right) - \left(\frac{e^{-s}}{s^2} - \frac{1}{s^2} \right) \right\}$$

$$= \frac{1}{(1-e^{-2s})s^2} \left\{ 1 - e^{-s} - s e^{-s} \right\}$$

Unit step function,



$$U(x-a) = \begin{cases} 1 & x \geq a \\ 0 & x < a \end{cases}$$

Theorem

$$L\{U(t-a)\} = \frac{e^{-as}}{s}$$

when $U(t-a)$ is a unit step function.

Proof

$$L\{U(t-a)\} = \int_0^\infty e^{-st}\, U(t-a)\, dt$$

$$= \int_0^a e^{-st} \underbrace{U(t-a)}_{0} dt + \int_a^\infty e^{-st} \underbrace{U(t-a)}_{1} dt$$

$$= \int_a^\infty e^{-st}\, dt$$

$$= \left[\frac{e^{-st}}{-s} \right]_a^\infty$$

$$= \frac{e^{-as}}{s}$$

Special Case

if $a = 0$, $U(t-0) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 1 \end{cases}$

$$\therefore L\{U(t)\} = \frac{1}{s}$$

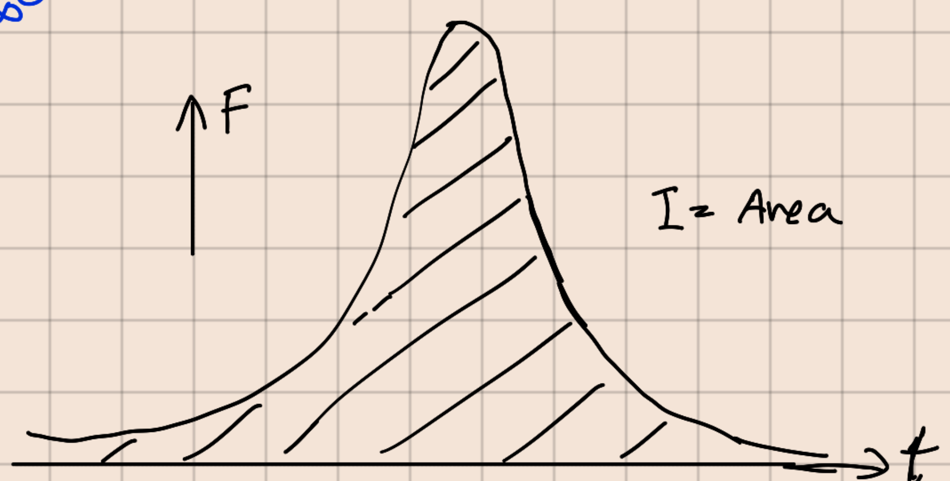
Impulse function

Impulse function is defined and denoted by,

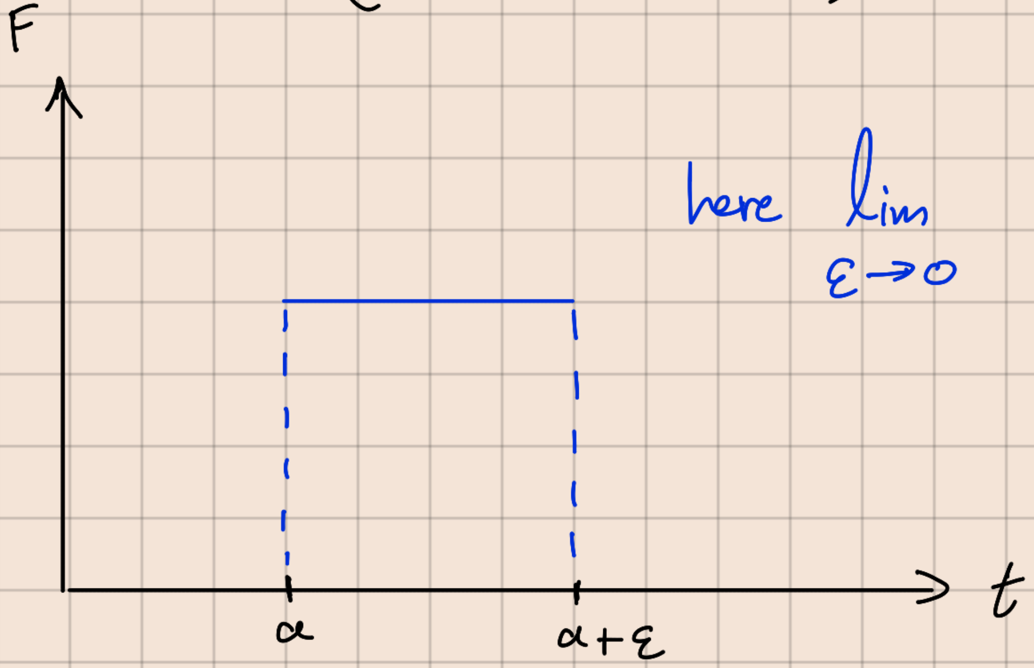
$$\delta(t-a) = \begin{cases} \infty & t = a \\ 0 & \text{otherwise} \end{cases}$$

Such that

$$\int_{-\infty}^{\infty} \delta(t-a)\, dt = I$$ * hypothetically.



$$\delta(t-a) = \begin{cases} \frac{1}{\varepsilon} & a < t < a+\varepsilon \\ 0 & \text{otherwise} \end{cases}$$



here $\lim_{\varepsilon \to 0}$

Theorem

If $f(t)$ be a continuous function at $t=a$

$$\int_0^{a} f(t)\,\delta(t-a)\,dt = f(a)$$

Proof (use mean value theorem of integrals)

$$\int_0^\infty f(t)\,\delta(t-a)\,dt = \int_0^a f(t)\,\cancel{\underbrace{\delta(t-a)}_{=0}}\,dt + \int_a^{a+\varepsilon} f(t)\,\underbrace{\delta(t-a)}_{1/\varepsilon}\,dt + \int_{a+\varepsilon}^\infty f(t)\,\cancel{\underbrace{\delta(t-a)}_{=0}}\,dt$$

$$L.H.S = \int_a^{a+\varepsilon} f(t) \cdot \frac{1}{\varepsilon}$$

by mean value theorem of integrals.

$$= f(\theta) \int_a^{a+\varepsilon} \frac{1}{\varepsilon}\, dt \qquad a < \theta < a+\varepsilon$$

$$= f(\theta) \cdot \frac{1}{\varepsilon} \left[t \right]_a^{a+\varepsilon}$$

$$= f(\theta) \cdot \frac{1}{\varepsilon} \cdot \varepsilon$$

$$= f(\theta)$$

when $\lim_{\varepsilon \to 0} \theta = a$

$\therefore \quad L.H.S = f(a)$

Laplace Transformation,

$$L[\delta(t-a)] = \int_0^{\infty} e^{-st} \delta(t-a)\, dt$$

$$= e^{-as}$$

Applications

- The deflection of a beam of length L clamped horizontally at both ends and loaded at $x = L/4$ by weight W is given by,

$$EI \frac{d^4 y}{dx^4} = W\delta(x - L/4)$$

find the equation of deflection curve given that

$$y = \frac{dy}{dx} = 0 \quad \text{at} \quad x = 0 \quad \text{and} \quad x = L$$

where,

E = young's modulus of beam material.

I = moment of inertia of beam cross section.

$$EI\, y^4(x) = W.\delta(x - L/4)$$

$$EI.\, \mathcal{L}\{y^4(x)\} = W.\, \mathcal{L}\{\delta(x - L/4)\}$$

$$EI.\left\{ s^4 Y(s) - \cancel{s^3 y(0)}_{0} - \cancel{s^2 y'(0)}_{0} - s\underbrace{y''(0)}_{A} - \underbrace{y'''(0)}_{B} \right\} = We^{-\frac{L}{4}s}$$

$$s^4 Y(s) - As - B = \frac{W}{EI} e^{-\frac{L}{4}s}$$

$$Y(s) = \frac{A}{s^3} + \frac{B}{s^4} + \frac{W}{EI} \cdot \frac{e^{-\frac{L}{4}s}}{s^4}$$

$$y(x) = L^{-1}\{Y(s)\}$$

$$y(x) = \frac{Ax^2}{2!} + \frac{Bx^3}{3!} + \frac{W}{EI} \cdot \frac{(x-\frac{L}{4})^3 \cdot u(x-\frac{L}{4})}{3!}$$

$x=0,\ y=0$

and

$x=L,\ y=0$

$$0 = \frac{AL^2}{2!} + \frac{BL^3}{3!} + \frac{W}{EI} \cdot \frac{(\frac{3L}{4})^3 \cdot (1)}{3!}$$