1. Statistical Distributions

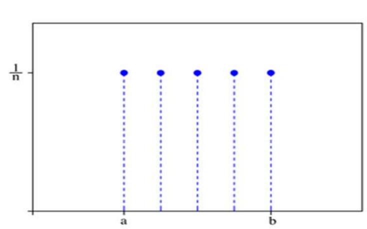
The distributions in the theory of statistics are classified mainly as discrete and continuous distributions.

Discrete Distributions		Continuous Distributions	
Uniform		Uniform	
Bernoulli		Normal & Standard Normal	
Binomial		t-Distribution	
Poisson		Chi-square	
Negative Binomial		F-Distribution	
Geometric		Exponential	
Hyper geometric		Gamma	
		Weibull	

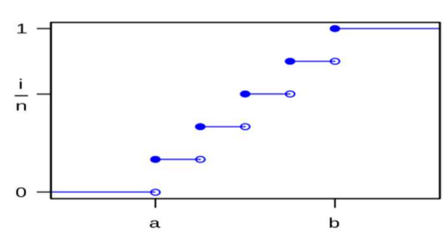
1.1 Discrete Distributions

Uniform Distribution

Probability Mass Function



Cummulative Distribution Function



Bernoulli and Binomial distributions

If an experiment has two possible outcomes, "success" and "failure", and their probabilities are, respectively, $\theta$and $(1-\theta)$, then the variable of number of successes(X), has a Bernoulli distribution with $pmf;\ f(x)=\theta^x(1-\theta)^{1-x}; x=0\ or\ 1.$

The experiment consists of n independent, repeated Bernoulli trials is said to be a binomial experiment. The variable of number of successes(X), then has a binomial distribution with $pmf;\ f(x)=\binom{n}{x}\theta^x(1-\theta)^{n-x}; x=0,1,\dots,n$

let $\theta = \frac{1}{2}$

$P(X=1) = {}^{5}C_1 \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^4$

$\overset{P(S)=\theta}{\underline{O}}\ \underline{X}\ \underline{X}\ \underline{X}\ \underline{X} \Rightarrow \theta \times (1-\theta)^4$

When repeating an experiment to get the desired number of successes,

If number of trials $= N$

If number of successes required $= n$

$P(S) = \theta$

then ${}^{N-1}C_{n-1} (\theta)^n (1-\theta)^{N-n}$

This distribution is the negative - Binomial distribution

ex:- $P(S) = 0.7$ $\quad r = 3$ $\quad x = 5$

$$P(X=5) = {}^{4}C_2 (0.7)^3 (0.3)^2$$

For multinomial

$$f\left(x_1, x_2, x_3; p_1, p_2, p_3\right) \quad \frac{n!}{x_1! \times x_2! \times x_3!} \times p_1^{x_1} p_2^{x_2} p_3^{x_3}$$

It can be proved that the mean and the standard deviation for a large sample from a binomial distribution are given by

$$\bar{x} = np \quad \text{and} \quad \sigma=\sqrt{npq};$$

where X is approximately distributed as Normal (Normal approximation to Binomial)

Poisson Distribution

In a Poisson experiment, the random occurrence of number of events over an interval (usually a time interval) is observed. In the same experiment if the time between two events is observed, the variable will theoretically follow a continuous distribution which will be discussed later.

The probability mass function of the r.v. *X*

$$f(x;\lambda) = \frac{e^{-\lambda}\lambda^{x}}{x!}; x = 0,1,2, \ldots$$

- λ – The mean number of counts in the interval (>0)
- E(X) = V(X) = λ

Negative Binomial distribution (Pascal distribution)

A **negative binomial experiment**

$$E(x) = \frac{r}{p}, V(x) = \frac{rq}{p^2}$$

- The experiment consists of *x* repeated trials.
- Each trial results only in two outcomes, one a success and the other, a failure.
- The probability of success, denoted by *p*, is the same on every trial.
- The trials are independent; that is, the outcome on one trial does not affect the outcome on other trials.
- The experiment continues until *r* successes are observed, where *r* is specified in advance.

Eg: Consider the statistical experiment of flipping a coin repeatedly and count the number of times the coin lands on heads. Continue flipping the coin until it has Head 5 times on top. Then the number of trials needed to have Head turned on 5 times (X), follows a negative binomial distribution.

X **:**	5	6	7	8	9	10	……………………
P(X) :	?	?	?	?	?	?	……………………

Exercises.

1) until 3 → "6" occure. $\theta = 1/6$

a) $P(v) = {}^{n-1}C_2 \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^{n-3}$

b) $\mu_x = \frac{r}{p} = 3 \times \frac{1}{\left(\frac{1}{6}\right)} = 18$

d) $P(V=10) = {}^{9}C_2 \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^7$ $\sigma_v = 9.48$

c) $Var(v) = \frac{rQ}{p^2} = \frac{3 \times 5/6}{1/36} = 90$

2) suppose $W \Rightarrow$ Neg. binomial $r = 2$ $p = 3/4$

a) $P(W=3) = {}^{2}C_1 \left(\frac{3}{4}\right)^2 \left(\frac{1}{4}\right)^1 = 2 \times \frac{9}{64} = \frac{9}{32}$

b) $E(W) = \mu_w = 2 \times \frac{4}{3} = \frac{8}{3} = 2.67$

c) $V(W) = \frac{rQ}{p^2} = \frac{r(1-p)}{p^2} = 2 \times 1/4 \times \frac{16}{9}$

$= \frac{8}{9}$

3) neg-binomial, $r = 3$, $p = 1/4$

$P(W \leq 2) = 0$

$E(W) = 12$

$Var(W) = \frac{r\theta}{p^2} = 3 \times 3/4 \times 16 = 36$

Notation and terminology

x : The number of trials required to produce *r* successes in a negative binomial experiment.

r :The number of successes in the negative binomial experiment.

P :The probability of success on an individual trial.

Q : The probability of failure on an individual trial, *1-P*.

^nC_r : The number of combinations of *n* things, taken *r* at a time.

Negative Binomial probability

b*(*x*; *r*, *P*) : - the probability that an *x*-trial negative binomial experiment results in the r^{th} success on the x^{th} trial, when the probability of success of an individual trial is P.

$$b^*(x; r, P) = {}^{x-1}C_{r-1} \cdot P^r \cdot (1 - P)^{x - r}$$

Mean of Negative Binomial distribution

$$\mu_x = \frac{r}{P}$$

Variance of Negative Binomial distribution

$$V_x = \frac{rQ}{P^2}$$

NB: When dealing with negative binomial distribution, check on how the negative binomial random variable is defined.

Alternative definitions can be:

- The negative binomial random variable is *R*, the number of successes before the binomial experiment results in *k* failures. The mean of *R* is $\mu_R = kP/Q$.
- The negative binomial random variable is *K*, the number of failures before the binomial experiment results in *r* successes. The mean of *K* is $\mu_K = rQ/P$.

Geometric Distribution (A special case of Negative Binomial)

This is a special case of the negative binomial distribution, where the variable of interest is the **number of trials required for a single success or the first success**. Thus, the geometric distribution is negative binomial distribution with the number of successes (r) is equal to 1.

An example of a geometric distribution would be asking for the probability that the first head occurs on the third flip. That probability is referred to as a **geometric probability** and is denoted by g(x; p). The formula for geometric probability is

$$\boxed{g(x;\, p) = p.q^{x-1}}$$

$g(x;p) = p.q^{x-1}$

Mean of Geometric distribution

$$\boxed{\mu_x = \frac{1}{P}}$$

Variance of Geometric distribution

$$\boxed{V_x = \frac{q}{P^2}}$$

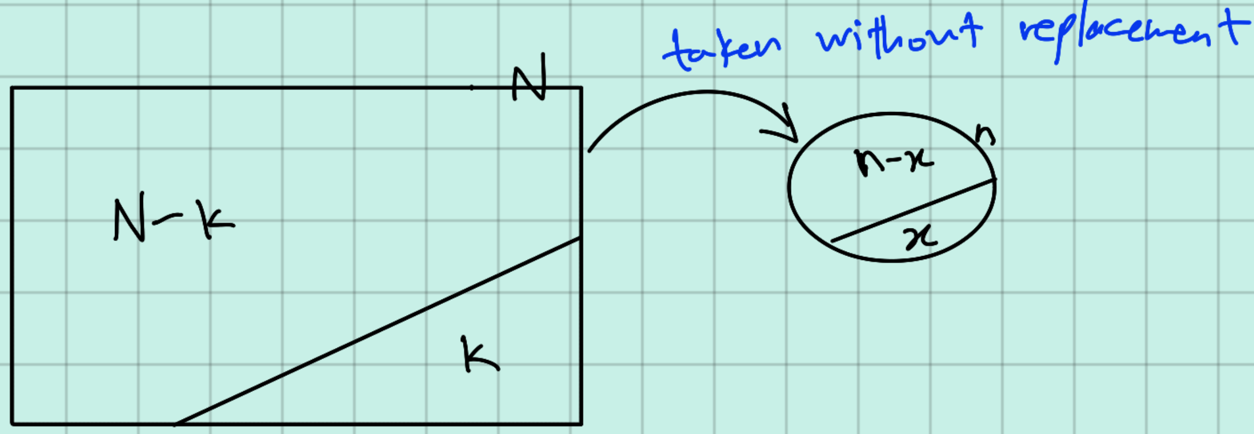
Hyper Geometric distribution

Hypergeometric Experiment

- A sample of size n is randomly selected without replacement from a population of N items.
- In the population, k items can be classified as successes, and N - k items can be classified as failures.

Eg: Consider the statistical experiment of randomly selecting 2 marbles without replacement from an urn of 10 marbles - 5 red and 5 green. The variable of interest is the number of red marbles selected. This is a hyper geometric experiment.

Note : As binomial experiment requires that the probability of success be constant on every trial, the above is not a binomial experiment. In the above experiment, the probability of a success changes on every trial. Further that if the marbles were selected with replacement, the probability of success would not change. It would be 5/10 on every trial. Then, this would be a binomial experiment.



$$P(X=x) = \frac{{}^{k}C_{x}\,{}^{N-k}C_{n-x}}{{}^{N}C_{n}}$$

if $k > n$,

$x = 0, 1, \ldots, n$

if $k < n$,

$x = 0, 1, \ldots, k$

ex!.-

$$P(X=1) = \frac{{}^{5}C_{1} \times {}^{5}C_{1}}{{}^{10}C_{2}}$$

$$P(X=0) = \frac{{}^{5}C_{2}}{{}^{10}C_{2}}$$

$$P(X=2) = \frac{{}^{5}C_{2}}{{}^{10}C_{2}}$$

Notations and terminology

N : The number of items in the population.

k : The number of items in the population that are classified as successes.

n : The number of items in the sample.

x : The number of items in the sample that are classified as successes.

^kC_x: The number of combinations of *k* things, taken *x* at a time.

Hypergeometric probability

h(*x*; *N*, *n*, *k*): - the probability that an *n*-trial hypergeometric experiment results in exactly*x* successes, when the population consists of *N* items, *k* of which are classified as successes.

$$h(x; N, n, k) = [\,^kC_x\,]\,[\,^{N-k}C_{n-x}\,] / [\,^NC_n\,]$$

Mean of the Hypergeometric distribution

$$\mu_x = nk / N$$

Variance of the Hypergeometric distribution

$$V_x = nk\,(N - k)(N - n) / [\,N^2\,(N - 1)\,]$$

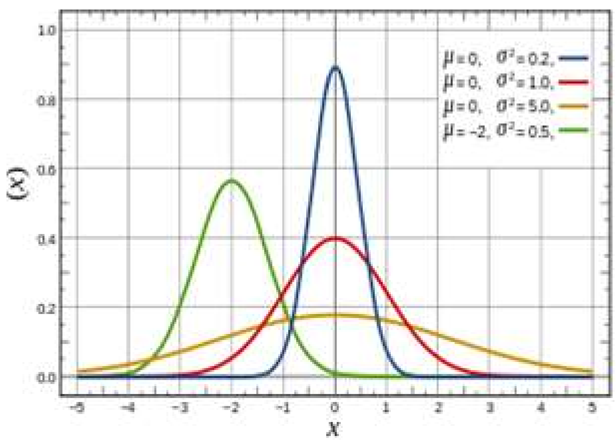
$$V_n = \frac{nk}{N} \cdot \frac{(N-k)(N-n)}{N(N-1)}$$

1.2 Continuous Distributions

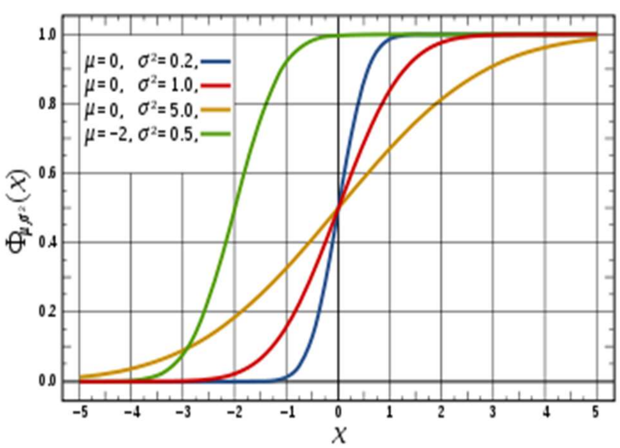
Normal Distribution and Standard Normal Distribution

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}$$;Normal *pdf*

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$;Standard Normal *pdf* $\mu = 0$, $\sigma^2 = 1$



Cummulative density function



Exponential Distribution

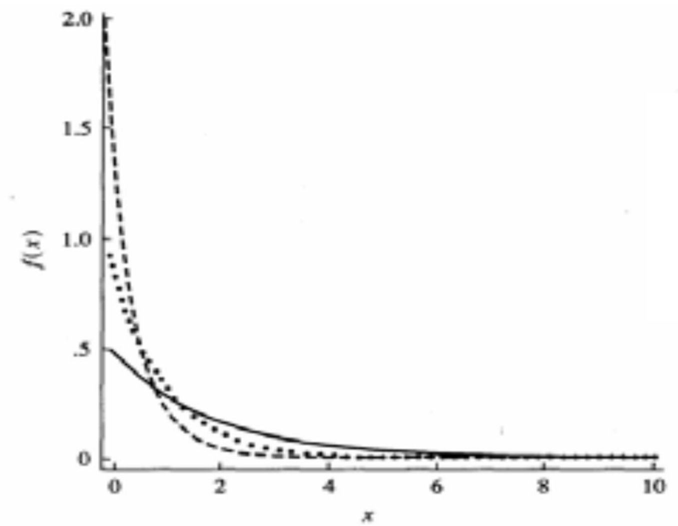
If a random variable X has the *pdf*

$$f(x) = \lambda e^{-\lambda x}, \qquad x \geq 0 \text{ and } \lambda > 0,$$

then it is said to have the exponential distribution with parameter λ and written as $X \sim Exp(\lambda)$.

The exponential distribution is often used to model the length of time until an event occurs. The exponential distribution can be thought of as the continuous analogue of the geometric distribution.

This parameter λ represents the **"mean number of events per unit time"** e.g. the rate of arrivals or the rate of failures as same as in Poisson distribution.



$\lambda = 0.5(solid)$

$\lambda = 1(dotted)$

$\lambda = 2(dashed)$

- Poisson distribution has Probability mass function. (discret)
- exponential distribution has probability density function. (Continuous)

Applications

- Model inter arrival times (time between arrivals) when arrivals are completely random; $\boldsymbol{\lambda}$ = arrivals / hour
- Model service times; $\boldsymbol{\lambda}$ = services / minute
- Model the lifetime of a component that fails catastrophically (i.e. light bulb); $\boldsymbol{\lambda} =$ failure rate

happens purely randomly.

Properties of the random variable X which has exponential distribution

1. It is closely related to the Poisson distribution – if X describes the time between two failures then the number of failures per unit time has the Poisson distribution with parameter $\boldsymbol{\lambda}$, the same.

2. The *cdf* is $\boldsymbol{F_X(x) = \lambda \int_0^x e^{-\lambda y}\ dy\ = 1 - e^{-\lambda x}}$

3. The 100(1 −∝)% percentile is $\boldsymbol{x_\propto = -\frac{1}{\lambda} \ln \propto}$

4. Mean $\boldsymbol{\mu_x = 1/\lambda}$

5. Variance $\boldsymbol{V_x = 1/\lambda^2}$

6. Moment Generating Function (*mgf*) $M_X(t) = \boldsymbol{\lambda/(\lambda - t)}$

7. "Memoryless" property

   For all s>= 0 and t >=0

   P(X > s + t | X > s) = P(X > t)

   Instance 1: If it is known that a component has survived s hours so far,the remaining amount of time that it survives follows the same distribution as the original distribution. It does not remember that it already has been used for s amount of time.

   *Instance 2:*This means that the distribution of the waiting time to the next event remains the same regardless of how long we have already been waiting. This only happens when events occur (or not) totally at random, i.e., independent of past history

Exercise : Suppose the life of an industrial lamp is exponentially distributed with failure rate $\boldsymbol{\lambda}$=1/3 (one failure every 3000 hours on the avg.) Determine the probability that

a) the lamp will last no longer than its mean life time. (constant for any $\boldsymbol{\lambda}$**)**
b) the lamp will last longer than its mean life time
c) the industrial lamp will last between 2000 and 3000 hours.
d) the lamp will last for another 1000 hours given that it is operating after 2500 hours.

Answer:

a) $P(X \leq 3)$=
b) $P(X > 3)$=
c) $P(2 \leq X \leq 3)$=

At a certain location in the highway the number of cars exceeding the speed limit by more than 10 kmh^{-1} in $\frac{1}{2}$ hour is a r.v. having a poisson distribution with $\lambda = 8.4$

1) What is the probability of a waiting time of less than 5 mins between cars exceeding the speed limit by more than 10 kmh^{-1}.

$$P\left(X \leqslant \frac{5}{30}\right) = \int_0^{1/6} 8.4\, e^{-8.4x}\, dx$$

$$= \frac{8.4}{8.4}\left[1 - e^{-1.4}\right]$$

exercise :-

$\lambda = \frac{1}{3}$ (1 failure every 3000 hours on avg.)

a) $P(X \leqslant 3) = 1 - e^{-\lambda x} = (1 - e^{-1})$

b) $P(X > 3) = 1 - P(X \leqslant 3)$

$= e^{-1}$

d) $P(X > 3.5 \mid X > 2.5) = P(X > 2.5 + 1 \mid x > 2.5)$

$= P(X > 1)$

d) $P(X > 3.5|X > 2.5)$=$P(X > 2.5 + 1|X > 2.5)$=$P(X > 1)$

[illegible] :X has an exponential distribution **iff** X is a positive continuous r.v. and [illegible]

Proof: *Omitted*

Gamma distribution

Gamma distribution is more suitable to describe some of the real world applications when they follow exponential patterns. The general command of a such probability density is given by

$$f(x) = \begin{cases} kx^{\alpha-1}e^{-x/\beta}; & for\ x > 0 \\ 0 & ;\ elsewhere \end{cases}$$

where $\alpha > 0$ and $\beta > 0$, and k must be such that the total area under the curve is equal to 1.

In evaluating k, using calculus theory, the ***Gamma function*** which only depends on α is derived:

$$\tau(\alpha) = \int_0^\infty y^{\alpha-1}e^{-y}dy \qquad for\ \alpha > 0$$

The ***Gamma function*** follows the recursion formula

$$\tau(\alpha) = (\alpha - 1)\,\tau(\alpha - 1);$$

$$\tau(\alpha) = (\alpha - 1)!$$

where $\tau(1) = \int_0^1 y^0e^{-y}dy = 1 \quad and \quad \tau(1/2) = \sqrt{\pi}$

α = Scale parameter
β = shape parameter

Thus $\int_0^\infty kx^{\alpha-1}e^{-x/\beta}dx = k\beta^\alpha\tau(\alpha) = 1$

A random variable X has a ***Gamma distribution*** has the probability density function

$$g(x;\alpha,\beta) = \begin{cases} \dfrac{1}{\beta^\alpha\tau(\alpha)}x^{\alpha-1}e^{-\frac{x}{\beta}}, & for\ x > 0 \\ 0\ , & elsewhere \end{cases}$$

Where $\alpha > 0$ and $\beta > 0$.

- The mean $\mu = \alpha\beta$ and $V(X) = \alpha\beta^2$
- Observe the graphs of gamma functions for different pairs of values for $\alpha$and β

Exercise: In a certain city, the daily consumption of electric power in millions of kilowatt-hours can be treated as a random variable having a Gamma distribution with $\alpha = 3$ and $\beta = 2$.

(i) What is the average consumption of electric power per day by the city?

(ii) If the power plant of this city has a daily capacity of 12 million kilowatt-hours, what is the probability that this power supply will be inadequate on any given day?

Answer:

(i) Average = $\alpha\beta = 3 * 2 = 6$

(ii) $P(\text{daily consumption of electric power } \geq 12) = \int_{12}^{\infty} \frac{1}{2^3 \tau(3)} x^{3-1} e^{-\frac{x}{2}} \ dx$

$$= 1 - \int_0^{12} \frac{1}{2^3 \tau(3)} x^{3-1} e^{-\frac{x}{2}} \ dx$$