Set 1

Platon
Karageorgis
P3180068

① Prove that for every orthogonal matrix A $(A \in \mathbb{R}^{n,n}, A^{-1} = A^T)$ the following relations are true:

$$\|A\|_2 = 1, \quad \|A\|_1 \geq 1, \quad \|A\|_\infty \geq 1$$

$\boxed{\|A\|_2 = 1}$

The second norm comes from the dot product and as we know:
$A \cdot A^T = I$, because of orthogonality.
Therefore, $\det(A) \cdot \det(A^T) = \det(I) \Rightarrow \det(A) \cdot \det(A^T) = 1$
$\Rightarrow (\det(A))^2 = 1 \Rightarrow |\det(A)| = |\pm 1|$
So the determinant is ± 1 and we also know that the rest of the elements are 0's. From these facts we can deduce that norm 2 will be equal to the absolute value of the diagonal, which is 1.

$\boxed{\|A\|_1 \geq 1}$

Essentially in this case we must prove that it is impossible for $\|A\|_1 = 0$ to stand. This is obvious since if it was true then every element should have been 0 and then there wouldn't be linear independence and this means that we proved $\|A\|_1 \geq 1$

$\boxed{\|A\|_\infty \geq 1}$

The proof in this case should be similar to the previous one.

(2) Prove the following relations connecting the spectral radius:

(a) Two real, symmetric matrices $A, B \in \mathbb{R}^{n,n}$

$$\rho(A+B) \leq \rho(A) + \rho(B)$$

$$\rho(AB) \leq \rho(A) \cdot \rho(B)$$

(b) Two real orthogonal matrices $P, Q \in \mathbb{R}^{n,n}$

$$\rho(P+Q) \leq 2$$

$$\rho(P \cdot Q) \leq 1$$

a) To begin with, we know that $B \equiv A^T$. Moreover, we can say that $\rho(A) = \rho(A^T)$ since the spectral radius is equal to the maximum eigenvalue of the corresponding matrix and A, A^T have the same eigenvalues

This can be proved:

$$\det(A^T - \lambda I) = \det((A-\lambda I)^T) = \det(A - \lambda I)$$

since I is symmetric.

Therefore, $\rho(A+B) \leq \rho(A) + \rho(B)$

$$\rho(A+A^T) \leq \rho(A) + \rho(A^T)$$

$$\rho(A+A) \leq \rho(A) + \rho(A)$$

which is true since if λ an (the largest) eigenvalue of A then $A+A$ can't have an eigenvalue bigger than 2λ.

Similarly, we can prove that $\rho(A \cdot A^T) \leq \rho(A) \cdot \rho(A^T) \Rightarrow$
$\Rightarrow \rho(A \cdot A) \leq \rho(A) \cdot \rho(A)$. With the same pattern, $A \cdot A$ can't have an eigenvalue bigger than λ^2.

b) The spectral radius of an orthogonal matrix is equal to the maximum eigenvalue. The eigenvalues of any orthogonal matrix can't surpass 1.

$Ax = \lambda x \Rightarrow \|Ax\|^2 = \|\lambda x\|^2 \Rightarrow \|Ax\|^2 = |\lambda|^2 \cdot \|x\|^2$

Also, $\|Ax\|^2 = (\bar{A}x)^T \cdot (Ax) = (\bar{x})^T \cdot A^T \cdot A \cdot x = \bar{x}^T \cdot x = \|x\|^2$

So, $\|x\|^2 = |\lambda|^2 \cdot \|x\|^2 \Rightarrow \boxed{|\lambda| = 1}$ and that proves both relations

③ If $A \in \mathbb{C}^{n,n}$ a Hermitian, invertible matrix, prove that the condition number $\kappa_2(A)$ is given by:

$$\kappa_2(A) = \frac{\max_i |\lambda_i|}{\min_i |\lambda_i|}, \text{ where } \lambda_i = 1, \dots, n \text{ A's eigenvalues}$$

Since the matrix is Hermitian we can say from theory that the eigenvalues are real.

Also, we have $\|A\| \cdot \|x\| \geq \|Ax\| = \|\lambda x\| = |\lambda| \cdot \|x\|$

By keeping the supremum of the λ_i's we have:

$$\|A\| \geq |\lambda_{max}|$$

And since A is Hermitian we can say that the equality holds

$$\boxed{\|A\| = |\lambda_{max}|} \text{ (1)}$$

Similarly,

$\|A^{-1}\| \cdot \|A\| \cdot \|y\| \geq \|y\| \Rightarrow \|A^{-1}\| \geq \|A\|^{-1}$

Again due to A being Hermitian we get:

$$\boxed{\|A^{-1}\| = \|A\|^{-1}} \text{ (2)}$$

Finally, it holds that the biggest eigenvalue of A^{-1} is the inverse of A and we get:

$$\boxed{\|A\|^{-1} = \lambda_{max}(A^{-1}) = \frac{1}{\lambda_{min}(A)}} \text{ (3)}$$

From (1), (2), (3) we conclude:

$$\boxed{\kappa(A) = \frac{\lambda_{max}}{\lambda_{min}}}$$

④ Assume that the height of a rocket was measured 4 times and the following data came up:

$$(t,h) = (1,135), (2,265), (3,385), (4,485)$$

counted in seconds and meters. By using the method of the least squares adapt on the data the optimized model $\boxed{h = a + bt - 4.905 \cdot t^2}$ and using that calculate the maximum height that will be reached, as well as the arrival time on earth.

We have $y = a \cdot t + b$. Let's define

$$h^* = c_1 \cdot t + c_2, \quad h^* = h + \underbrace{4.905}_{c} t^2$$

- $c_1 \cdot 1 + c_2 = h^* \Rightarrow c_1 + c_2 = 135 + c$
- $c_1 \cdot 2 + c_2 = h^* \Rightarrow 2c_1 + c_2 = 265 + 4c$
- $c_1 \cdot 3 + c_2 = h^* \Rightarrow 3c_1 + c_2 = 385 + 9c$
- $c_1 \cdot 4 + c_2 = h^* \Rightarrow 4c_1 + c_2 = 485 + 16c$

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}}_{C} = \underbrace{\begin{bmatrix} 135 + c \\ 265 + 4c \\ 385 + 9c \\ 485 + 16c \end{bmatrix}}_{b}$$

We also have $A^T \cdot A \cdot C = A^T b$

$$\rightarrow A^T \cdot A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 30 & 10 \\ 10 & 4 \end{bmatrix}$$

$$\rightarrow A^T \cdot b = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 135 + c \\ 265 + 4c \\ 385 + 9c \\ 485 + 16c \end{bmatrix} = \begin{bmatrix} 3760 + 100c \\ 1270 + 30c \end{bmatrix}$$

$$\rightarrow \underbrace{\begin{bmatrix} 30 & 10 \\ 10 & 4 \end{bmatrix}}_{A^T A} \cdot \underbrace{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}}_{C} = \begin{bmatrix} 3760 + 100c \\ 1270 + 30c \end{bmatrix}$$

$$\left. \begin{array}{l} 30c_1 + 10c_2 = 3760 + 100c \\ 10c_1 + 4c_2 = 1270 + 30c \end{array} \right| \Rightarrow \boxed{\begin{array}{l} c_1 = 141.525 \\ c_2 = 0.475 \end{array}}$$

$h(t) = 141.525t + 0.475 - 4.905t^2, \quad t>0$

$h'(t) = -9.81t + 141.525$

$h'(t) = 0 \Rightarrow \boxed{t = 14.43 \text{ sec}}$

We can see that h is parabolic with a total maximum at $h(14.43) = 1021.34$ which will be the maximum height. The rocket will return back to earth at $t = t_0$ where $h(t_0) = 0 \Rightarrow 141.525 \cdot t + 0.475 - 4.905 \cdot t^2 = 0$ so at $\boxed{t = 28.85 \text{ sec}}$

(5) Assume the points:

$(x,y,z) = (0,0,3), (0,1,2), (1,0,3), (1,1,5), (1,2,6)$

Find ~~that~~ $\boxed{Z = C_0 + C_1 \cdot X + C_2 \cdot Y}$ that approaches optimally the points in $\mathbb{R}^3$ using the second norm.

We have,

- $C_0 + 0 + 0 = 3$
- $C_0 + 0 + C_2 = 2$
- $C_0 + C_1 = 3$
- $C_0 + C_1 + C_2 = 5$
- $C_0 + C_1 + 2C_2 = 6$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} C_0 \\ C_1 \\ C_2 \end{bmatrix}}_{C} = \underbrace{\begin{bmatrix} 3 \\ 2 \\ 3 \\ 5 \\ 6 \end{bmatrix}}_{b}$$

Given $A^T A C = A^T b$:

$$A^T A: \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 3 & 4 \\ 3 & 3 & 3 \\ 4 & 3 & 6 \end{bmatrix}$$

$$A^T b: \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 2 \\ 3 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 19 \\ 14 \\ 19 \end{bmatrix}$$

$$\begin{array}{l} 5C_0 + 3C_1 + 4C_2 = 19 \\ 3C_0 + 3C_1 + 3C_2 = 14 \\ 4C_0 + 3C_1 + 6C_2 = 19 \end{array} \Rightarrow \begin{array}{l} C_0 = 2 \\ C_1 = \frac{5}{3} \\ C_2 = 1 \end{array}$$

So, $\boxed{Z = \frac{5}{3} \cdot x + y + 2}$

⑥ Find the optimal polynomyms of degree 1 and 2 using the 2nd norm for the points (0,0), (1,3), (2,3), (5,6). Compare the 2 methods using their error.

→ $y = C_0 + C_1 \cdot x$

- $C_0 + 0 = 0$
- $C_1 + C_0 = 3$
- $2C_1 + C_0 = 3$
- $5C_1 + C_0 = 6$

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 5 \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} C_0 \\ C_1 \end{bmatrix}}_{c} = \underbrace{\begin{bmatrix} 0 \\ 3 \\ 3 \\ 6 \end{bmatrix}}_{b}$$

→ $A^T A$: $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 5 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ 8 & 30 \end{bmatrix}$

→ $A^T \cdot b$: $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 5 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 3 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 12 \\ 39 \end{bmatrix}$

$$\begin{bmatrix} 4 & 8 \\ 8 & 30 \end{bmatrix} \cdot \begin{bmatrix} C_0 \\ C_1 \end{bmatrix} = \begin{bmatrix} 12 \\ 39 \end{bmatrix}$$

$4C_0 + 8C_1 = 12$
$8C_0 + 30C_1 = 39$ $\Rightarrow$ $\boxed{C_0 = \frac{6}{7} \quad C_1 = \frac{15}{14}}$

$\varepsilon_1^2 = \left(0 - \frac{6}{7}\right)^2 = \frac{36}{49}$ $\qquad$ $\varepsilon_3^2 = \left(3 - \left(\frac{21}{7}\right)\right)^2 = 0$

$\varepsilon_2^2 = \left(3 - \left(\frac{15}{14} + \frac{12}{14}\right)\right)^2 = \frac{225}{196}$ $\qquad$ $\varepsilon_4^2 = \left(6 - \left(\frac{75}{14} + \frac{12}{14}\right)\right)^2 = \frac{9}{196}$

$\min\left(\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2 + \varepsilon_5^2\right) = \frac{144 + 9 + 225}{196} = 1.928$

→ $y = C_0 + C_1 \cdot x + C_2 \cdot x^2$

- $C_0 + 0 = 0$
- $C_1 + C_1 + C_2 = 3$
- $C_0 + 2C_1 + 4C_2 = 3$
- $C_0 + 5C_1 + 25C_2 = 6$

So we get, $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 5 & 25 \end{bmatrix} \cdot \begin{bmatrix} C_0 \\ C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 3 \\ 6 \end{bmatrix}$

$$A^T A: \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 5 \\ 0 & 1 & 4 & 25 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 5 & 25 \end{bmatrix} = \begin{bmatrix} 4 & 8 & 30 \\ 8 & 30 & 134 \\ 30 & 134 & 642 \end{bmatrix}$$

$$A^T b: \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 5 \\ 0 & 1 & 4 & 25 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 3 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 12 \\ 39 \\ 165 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 8 & 30 \\ 8 & 30 & 134 \\ 30 & 134 & 642 \end{bmatrix} \cdot \begin{bmatrix} C_0 \\ C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 12 \\ 39 \\ 165 \end{bmatrix}$$

$$\begin{aligned} 4C_0 + 8C_1 + 30C_2 &= 12 \\ 8C_0 + 30C_1 + 134C_2 &= 39 \\ 30C_0 + 134C_0 + 642C_2 &= 165 \end{aligned} \Rightarrow \boxed{C_0 = \frac{3384}{362}} \quad \boxed{C_1 = \frac{705}{362}} \quad \boxed{C_2 = -\frac{60}{362}}$$

$$\boxed{Y = \frac{3384}{362} + \frac{705}{362}x - \frac{60}{362} \cdot x^2}$$

$$\varepsilon_1^2 = \left(0 - \frac{3384}{362}\right)^2 = 87.38$$

$$\varepsilon_2^2 = \left(3 - \frac{4029}{362}\right)^2 = \left(\frac{1086 - 4029}{362}\right)^2 = 66.09$$

$$\varepsilon_3^2 = \left(3 - \frac{4554}{362}\right)^2 = 91.78$$

$$\varepsilon_4^2 = \left(6 - \frac{5409}{362}\right)^2 = 79.95$$

$$\text{Min}\left(\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2\right) = 325.17$$

⑦ Adapt the data: $\{(0,0), (\frac{1}{6},2), (\frac{1}{3},0), (\frac{1}{2},-1), (\frac{2}{3},1), (\frac{5}{6},1)\}$ to the periodic models:

- $F_3(t) = C_1 + C_2 \cdot \cos(2\pi t) + C_3 \cdot \sin(2\pi t)$
- $F_4(t) = C_1 + C_2 \cdot \cos(2\pi t) + C_3 \cdot \sin(2\pi t) + C_4 \cdot \cos(4\pi t)$

Find the error $\|e\|_2$ of the two models

$\boxed{\text{For the first model}}$:

- $C_1 + C_2 + 0 = 0$
- $C_1 + C_2(0.5) + C_3(0.86) = 2$
- $C_1 + C_2 \cdot (-0.5) + C_3 \cdot (0.86) = 0$
- $C_1 + 0 + C_3 = -1$
- $C_1 + C_2 \cdot (-0.5) + C_3 \cdot (-0.86) = 1$
- $C_1 + C_2 \cdot (0.5) + C_3 \cdot (-0.86) = 1$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0.5 & 0.86 \\ 1 & -0.5 & 0.86 \\ 1 & 0 & 1 \\ 1 & -0.5 & -0.86 \\ 1 & 0.5 & -0.86 \end{bmatrix} \cdot \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \\ -1 \\ 1 \\ 1 \end{bmatrix}$$

- $A^T A$: $$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0.5 & -0.5 & 0 & -0.5 & 0.5 \\ 0 & 0.86 & 0.86 & 1 & -0.86 & -0.86 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0.5 & 0.86 \\ 1 & -0.5 & 0.86 \\ 1 & 0 & 1 \\ 1 & -0.5 & -0.86 \\ 1 & 0.5 & -0.86 \end{bmatrix} = \begin{bmatrix} 6 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 3.96 \end{bmatrix}$$

- $A^T b$: $$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0.5 & -0.5 & 0 & -0.5 & 0.5 \\ 0 & 0.86 & 0.86 & 1 & -0.86 & -0.86 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 2 \\ 0 \\ -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$$

So we get,

$$\begin{bmatrix} 6 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 3.96 \end{bmatrix} \cdot \begin{bmatrix} C_0 \\ C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$$

$$\left.\begin{array}{l} 6C_0 + C_1 + C_2 = 3 \\ C_0 + 2C_1 = 1 \\ C_0 + 3.96\, C_2 = -1 \end{array}\right| \Rightarrow \boxed{\begin{array}{l} C_0 = 0.52 \\ C_1 = 0.24 \\ C_2 = -0.385 \end{array}}$$

$$F_3(t) = 0.52 + 0.24\cos(2\pi t) - 0.385 \cdot \sin(2\pi t)$$

$$\varepsilon_1^2 = (0 - 0.52)^2 = 0.27$$
$$\varepsilon_2^2 = (2 - (0.52 + 0.24 \cdot 0.5 - 0.385 \cdot 0.86)) = (1.69)^2 = 2.85$$
$$\varepsilon_3^2 = (0 - (0.52 + 0.24(-0.5) - 0.385 \cdot 0.86)) = (0.069)^2 = 0.004$$
$$\varepsilon_4^2 = (-1 - (0.52 + 0.24 \cdot (0) - 0.385 \cdot 1)) = (1.135)^2 = 1.29$$
$$\varepsilon_5^2 = (1 - (0.52 + 0.24 \cdot (-0.5) - 0.385 \cdot (-0.86)) = (0.27)^2 = 0.07$$
$$\varepsilon_6^2 = (1 - (0.52 + 0.24 \cdot (0.5) - 0.385 \cdot (-0.86)) = (0.028)^2 = -0.0008$$

$$\min(\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2 + \varepsilon_5^2 + \varepsilon_6^2) \simeq 4.484$$

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0.5 & 0.86 & -0.5 \\ 1 & -0.5 & 0.86 & -0.5 \\ 1 & 0 & 1 & 1 \\ 1 & -0.5 & -0.86 & -0.5 \\ 1 & 0.5 & -0.86 & -0.5 \end{bmatrix} \cdot \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \\ -1 \\ 1 \\ 1 \end{bmatrix}$$

$\boxed{\text{For the second model}}$:

- $C_1 + C_2 + 0 + C_4 = 0$
- $C_1 + C_2(0.5) + C_3 \cdot (0.86) + C_4((-0.5) = 2$
- $C_1 + C_2 \cdot (-0.5) + C_3 \cdot (0.86) + C_4 \cdot (-0.5) = 0$
- $C_1 + 0 + C_3 + C_4 = -1$
- $C_1 + C_2 \cdot (-0.5) + C_3 \cdot (-0.86) + C_4 \cdot (-0.5) = 1$
- $C_1 + C_2 \cdot (0.5) + C_3 \cdot (-0.86) + C_4 \cdot (-0.5) = 1$

- $A^T A$: $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0.5 & -0.5 & 0 & -0.5 & 0.5 \\ 0 & 0.86 & 0.86 & 1 & -0.86 & -0.86 \\ 1 & -0.5 & -0.5 & 1 & -0.5 & -0.5 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0.5 & 0.86 & -0.5 \\ 1 & -0.5 & 0.86 & -0.5 \\ 1 & 0 & 1 & 1 \\ 1 & -0.5 & -0.86 & -0.5 \\ 1 & 0.5 & -0.86 & -0.5 \end{bmatrix} = \begin{bmatrix} 6 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \\ 1 & 0 & 3.96 & 1 \\ 0 & 1 & 1 & 3 \end{bmatrix}$

- $A^T \cdot b$: $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0.5 & -0.5 & 0 & -0.5 & 0.5 \\ 0 & 0.86 & 0.86 & 1 & -0.86 & -0.86 \\ 1 & -0.5 & -0.5 & 1 & -0.5 & -0.5 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 2 \\ 0 \\ -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}$

$$\begin{bmatrix} 6 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \\ 1 & 0 & 3.96 & 1 \\ 0 & 1 & 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}$$

$6C_1 + C_2 + C_3 = 3$
$C_1 + 2C_2 + C_4 = 1$
$C_1 + 3.96\, C_3 + C_4 = -1$
$C_2 + C_3 + 3C_4 = 3$

$\Rightarrow$
$C_1 = 0.735$
$C_2 = -0.6$
$C_3 = -0.8$
$C_4 = 1.47$

$F_4(t) = 0.735 - 0.6\cdot\cos(2\pi t) - 0.8\sin(2\pi t) + 1.47\cos(4\pi t)$

$\varepsilon_1^2 = \left(0 - (0.735 - 0.6 + 1.47)\right) = 2.57$

$\varepsilon_2^2 = \left(2 - (0.735 - 0.6\cdot 0.5 - 0.8\cdot(0.86) + 1.47\cdot(-0.5))\right) = 8.93$

$\varepsilon_3^2 = \left(0 - (0.735 - 0.6\cdot(-0.5) - 0.8\cdot(0.86) + 1.47(-0.5))\right) = 0.15$

$\varepsilon_4^2 = \left(-1 - (0.735 - 0 - 0.8 + 1.47)\right) = 1.97$

$\varepsilon_5^2 = \left(1 - (0.735 - 0.6(-0.5) - 0.8\cdot(-0.86) + 1.47(-0.5))\right) = 0.0001$

$\varepsilon_6^2 = \left(1 - (0.735 - 0.6\cdot(0.5) - 0.8\cdot(-0.86) + 1.47(-0.5))\right) = 0.374$

$\min\left(\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2 + \varepsilon_5^2 + \varepsilon_6^2\right) \cong 13.99$

So finally, we can clearly see that the first model is better since the error is much smaller.

(8) Adapt the data $\{(-2,1),(0,2),(1,2),(2,5)\}$ to the exponential model $c_1\cdot e^{c_2 t}$. Find c_1, c_2 converting it to a least squares problem. For the values found, calculate the error using norm 2, of both models (the exponential and its linear version).

We have $f(t) = c_1\cdot e^{c_2 t}$ and the linear version is $g(t) = c_1 + c_2\cdot t$.

For the linear version we have:

- $c_1 + c_2\cdot(-2) = 1$
- $c_1 + c_2\cdot(0) = 2$
- $c_1 + c_2\cdot 1 = 2$
- $c_1 + c_2\cdot 2 = 5$

$$\begin{bmatrix} 1 & -2 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 5 \end{bmatrix}$$

• $A^T A$: $\begin{bmatrix} 1 & 1 & 1 & 1 \\ -2 & 0 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 1 & 9 \end{bmatrix}$

• $A^T b$: $\begin{bmatrix} 1 & 1 & 1 & 1 \\ -2 & 0 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$

$$\begin{bmatrix} 4 & 1 \\ 1 & 9 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$$

$$\begin{cases} 4c_1 + c_2 = 10 \\ c_1 + 9c_2 = 10 \end{cases} \Rightarrow \boxed{\begin{matrix} c_1 = \frac{16}{7} \\ c_2 = \frac{6}{7} \end{matrix}}$$

$$\varepsilon_1^2 = \left(1 - \left(\frac{16}{7} - \frac{12}{7}\right)\right)^2 = \left(\frac{3}{7}\right)^2 = \frac{9}{49}$$

$$\varepsilon_2^2 = \left(2 - \left(\frac{16}{7}\right)\right)^2 = \left(\frac{2}{7}\right)^2 = \frac{4}{49}$$

$$\varepsilon_3^2 = \left(2 - \left(\frac{16}{7} + \frac{6}{7}\right)\right)^2 = \left(\frac{-8}{7}\right)^2 = \frac{64}{49}$$

$$\varepsilon_4^2 = \left(5 - \left(\frac{16}{7} + \frac{12}{7}\right)\right)^2 = 1$$

$$\min\left(\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2\right) = \frac{9 + 4 + 64 + 49}{49} = 2.57$$

For the exponential model:

$$f(t) = c_1 \cdot e^{c_2 \cdot t} \Rightarrow \ln(f(t)) = \ln(c_1) + c_2 \cdot t$$

• $\ln(c_1) + c_2 = \ln(1) \Rightarrow \ln(c_1) + c_2 = 0$

• $\ln(c_1) + 0 = \ln(2) \Rightarrow \ln(c_1) + 0 = 0.693$

• $\ln(c_1) + c_2 = \ln(2) \Rightarrow \ln(c_1) + c_2 = 0.693$

• $\ln(c_1) + 2c_2 = \ln(5) \Rightarrow \ln(c_1) + 2c_2 = 1.609$

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} \ln C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.693 \\ 0.693 \\ 1.609 \end{bmatrix}$$

$$\cdot\ A^T A : \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 4 & 6 \end{bmatrix}$$

$$\cdot\ A^T b : \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0.693 \\ 0.693 \\ 1.609 \end{bmatrix} = \begin{bmatrix} 2.995 \\ 3.911 \end{bmatrix}$$

$$\cdot\ \begin{bmatrix} 4 & 4 \\ 4 & 6 \end{bmatrix} \cdot \begin{bmatrix} \ln C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 2.995 \\ 3.911 \end{bmatrix}$$

$$\left.\begin{aligned} 4\ln C_1 + 4C_2 &= 2.995 \\ 4\ln C_1 + 6C_2 &= 3.911 \end{aligned}\right| \Rightarrow \boxed{\begin{aligned} C_1 &= 1.336 \\ C_2 &= 0.458 \end{aligned}}$$

$$g(t) = 1.336 \cdot e^{0.458 \cdot t}$$

$$\varepsilon_1^2 = \left(1 - \left(1.336 \cdot e^{-0.916}\right)\right)^2 = (1 - 0.53)^2 = 0.22$$

$$\varepsilon_2^2 = \left(2 - \left(1.336 \cdot e^{0}\right)\right)^2 = (0.664)^2 = 0.44$$

$$\varepsilon_3^2 = \left(2 - \left(1.336 \cdot e^{0.458}\right)\right)^2 = (0.11)^2 = 0.01$$

$$\varepsilon_4^2 = \left(5 - \left(1.336 \cdot e^{0.916}\right)\right)^2 = (1.66)^2 = 2.75$$

$$\min\left(\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2\right) = 3.42$$

So finally we conclude that the linear version is a bit more precise.