$\boxed{\text{Set 4}}$

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Exercise 1 (5/pg 89)

Given a matrix $A=\begin{bmatrix}1 & a & a\\ a & 1 & 0\\ a & 0 & 1\end{bmatrix}$ find all the possible values of $a\in\mathbb{R}$ s.t. :

(a) Jacobi converges

(b) Gauss-Seidel converges

(a) $A = D - L - U$

$$\begin{pmatrix}1 & a & a\\ a & 1 & 0\\ a & 0 & 1\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}-\begin{pmatrix}0 & 0 & 0\\ -a & 0 & 0\\ -a & 0 & 0\end{pmatrix}-\begin{pmatrix}0 & -a & -a\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

$M = D$

$N = L + U$

$$T_J = D^{-1}\cdot(L+U)=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\cdot\begin{pmatrix}0 & -a & -a\\ -a & 0 & 0\\ -a & 0 & 0\end{pmatrix}=\begin{pmatrix}0 & -a & -a\\ -a & 0 & 0\\ -a & 0 & 0\end{pmatrix}$$

$$\det(T_J-\lambda I)=\begin{vmatrix}0-\lambda & -a & -a\\ -a & 0-\lambda & 0-\lambda\\ -a & 0 & 0\end{vmatrix}= -\lambda^3+2a^2\lambda$$

$$\max|\lambda_i| = \rho(T_J)=|\sqrt{2}\cdot a| = \sqrt{2}\cdot|a|$$

$$\sqrt{2}\cdot|a|<1 \Rightarrow |a|<\frac{1}{\sqrt{2}} \Rightarrow \boxed{-\frac{\sqrt{2}}{2}<a<\frac{\sqrt{2}}{2}}$$

(b) $M = D - L$

$N = U$

$$D-L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ -a & 0 & 0 \\ -a & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ a & 0 & 1 \end{pmatrix}$$

$$(D-L)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ -a & 0 & 1 \end{pmatrix}$$

$$T_G = (D-L)^{-1} \cdot U = \begin{pmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ -a & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & -a & -a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -a & -a \\ 0 & a^2 & a^2 \\ 0 & a^2 & a^2 \end{pmatrix}$$

$$\det(T_G - \lambda I) = \begin{vmatrix} 0-\lambda & -a & -a \\ 0 & a^2-\lambda & a^2 \\ 0 & a^2 & a^2-\lambda \end{vmatrix} = -\lambda^3 + 2a^2\lambda^2$$

$$\max|\lambda_i| = \rho(T_G) = 2a^2$$

$$2a^2 < 1 \Rightarrow a^2 < \frac{1}{2} \Rightarrow \boxed{-\frac{\sqrt{2}}{2} < a < \frac{\sqrt{2}}{2}}$$

Exercise 2 (7/pg 90)

(a) Prove that the Gauss-Seidel iterative matrix has at least one eigenvalue equal to 0

(b) Prove that if the Gauss-Seidel iterative matrix has an eigenvalue equal to 1, A is ~~[illegible]~~ not invertible.

(a) Assume $A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1i} \\ \vdots & \ddots & & \\ a_{i1} & & & a_{ii} \end{pmatrix}$

Then,

$$A = \begin{pmatrix} a_{11} & 0 & \dots & 0 \\ 0 & \ddots & & \\ \vdots & & & \\ 0 & & & a_{ii} \end{pmatrix} - \begin{pmatrix} 0 & \dots & & 0 \\ -a_{21} & \ddots & & \\ \vdots & & & 0 \\ -a_{i1} & \dots & & \end{pmatrix} - \begin{pmatrix} 0 & -a_{12} & \dots & -a_{1i} \\ 0 & \ddots & & \\ \vdots & & & 0 \\ 0 & \dots & & \end{pmatrix}$$

And $\begin{cases} M = D - L \\ N = U \end{cases}$

$$(D-L) = \begin{pmatrix} a_{11} & 0 & \dots & 0 \\ a_{21} & \ddots & & \vdots \\ \vdots & & & \\ a_{i1} & \dots & & a_{ii} \end{pmatrix} \xrightarrow{(D-L)^{-1}} \begin{pmatrix} \frac{1}{a_{11}} & 0 & \dots & 0 \\ b & & \ddots & \\ \vdots & & & \frac{1}{a_{ii}} \\ c & \dots & & \end{pmatrix}$$

$$T_G = (D-L)^{-1} U = \begin{pmatrix} \frac{1}{a_{11}} & 0 & \dots & 0 \\ b & \ddots & & \\ \vdots & & & \\ c & & & \frac{1}{a_{ii}} \end{pmatrix} \cdot \begin{pmatrix} 0 & -a_{12} & \dots & a_{1i} \\ \vdots & \ddots & & \\ 0 & & & 0 \end{pmatrix}$$

$$G = \begin{pmatrix} 0 & f & \dots & j \\ b' & d & & \\ \vdots & & \ddots & \\ c' & & & e \end{pmatrix}$$

We can't tell if we will have more than 1 zero on the main diagonal but we definitely have 1. Therefore, we have at least 1 eigenvalue equal to 0.

(b) Assume that T_G has an eigenvalue $\lambda = 1$

Then,

$$\det(T_G - \lambda I) = 0 \Leftrightarrow \det(T_G - I) = 0 \Leftrightarrow$$

$$\Leftrightarrow \det((D-L)^{-1}U - I) = 0 \Leftrightarrow \det((D-L)^{-1}U - (D-L)^{-1}(D-L)) = 0$$

$$\Leftrightarrow \det((D-L)^{-1}) \cdot \det(U - D + L) = 0$$

$$\overset{(*)}{\Leftrightarrow} \det(U - D + L) = 0$$

(*) $\det((D-L)^{-1}) \neq 0$, if it was 0 then $D-L$ wouldn't be invertible

Then, $\det(-(D-L-U)) = 0 \Leftrightarrow -\det(A) = 0 \Leftrightarrow \det(A) = 0$

Which means that A is not invertible

Exercise 3 (5/Pg 90)

We have the iterative method:

$$x^{(k+1)} = (I-A)\cdot x^k + b, \quad k=0,1,2,\dots \quad (1)$$

and

$$x^{(k+1)} = (I-\omega A)\cdot x^k + \omega b, \quad k=0,1,2,\dots \quad (2)$$

(i) A has $-10, -5$ as eigenvalues
(ii) A has $-5, 5$ as eigenvalues
(iii) A has $5, 10$ as eigenvalues

(a) Prove that (1) diverges in every case
(b) Find $\omega \in \mathbb{R}$ such that (2) converges

(a) $M = I$
$N = I - A$

T_ω will be equal to: $T_\omega = I^{-1}(I-A) = I-A$
since M is the identity matrix

(i) If $\lambda_1 = -10$ and $\lambda_2 = -5$
Then $I-A$ will have $|1-(-10)| = 11$ and $|1-(-5)| = 6$ as eigenvalues
$|\lambda_{max}| = 11 = \rho(T_\omega) > 1$, so it diverges

(ii) If $\lambda_1 = -5$ and $\lambda_2 = 5$
Then $I-A$ will have $|1-(-5)| = 6$ and $|1-5| = 4$ as eigenvalues
$|\lambda_{max}| = 6 = \rho(T_\omega) > 1$, so it diverges

(iii) If $\lambda_1 = 5$ and $\lambda_2 = 10$
Then $I-A$ will have $|1-5| = 4$ and $|1-10| = 9$ as eigenvalues
$|\lambda_{max}| = 9 = \rho(T_\omega) > 1$, so it diverges

(6) (i) We have $|\lambda_{max}| = |1-\omega\cdot(-10)| < 1$

$$-1 < 1+\omega\cdot 10 < 1$$
$$-2 < 10\cdot\omega < 0$$
$$-0.2 < \omega < 0$$

So $\omega \in (-\frac{1}{5}, 0)$ and it converges

(ii) We have $|\lambda_{max}| = |1-\omega\cdot(-5)| < 1$

$$-1 < 1+5\omega < 1$$
$$-2 < 5\omega < 0$$
$$-0.4 < \omega < 0$$

So $\omega \in \left(-\frac{2}{5}, 0\right)$ and it converges

(iii) We have $|\lambda_{max}| = |1-\omega\cdot 9| < 1$

$$-1 < 1-\omega\cdot 9 < 1$$
$$-2 < -\omega\cdot 9 < 0$$
$$\frac{2}{9} > \omega > 0$$

So $\omega \in \left(0, \frac{2}{9}\right)$ and it converges

Exercise 4 (pg 101/6)

Given linear system $Ax=b$ the coefficient matrix is $A=\text{trid}(1,2,1)\in\mathbb{R}^{4,4}$

ⓐ find Jacobi eigenvalues

ⓑ Why does SOR converge for every $\omega\in(0,2)$?

ⓒ Why can we find ω_{OPT}? Find it along with the spectral radius of SOR's iterative matrix.

ⓐ $A = D - L - U$

$$M = D = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \quad D^{-1} = \begin{pmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 \end{pmatrix}$$

$$L = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$J = D^{-1}(L+U) = \begin{pmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & -1/2 & 0 & 0 \\ -1/2 & 0 & -1/2 & 0 \\ 0 & -1/2 & 0 & -1/2 \\ 0 & 0 & -1/2 & 0 \end{pmatrix}$$

$$\det(J-\lambda I) = \begin{vmatrix} 0-\lambda & -1/2 & 0 & 0 \\ -1/2 & 0-\lambda & -1/2 & 0 \\ 0 & -1/2 & 0-\lambda & -1/2 \\ 0 & 0 & -1/2 & 0-\lambda \end{vmatrix} = \lambda^4 - 12\lambda^2 + 16 \dots$$

$$|\lambda_{max}| = \frac{\sqrt{5}+1}{4}$$

(b) A is symmetric and positive definite so according to ROV theorem SOR converges for every $\omega \in (0,2)$

(c) Since A is circulant.

$$\omega_{OPT} = \frac{2}{1+\sqrt{1-\rho^2}} = \frac{2}{1+\sqrt{1-\left(\frac{\sqrt{3}+1}{4}\right)^2}} = \frac{2}{1+\sqrt{\frac{10-2\sqrt{3}}{16}}} =$$

$$\simeq \frac{2}{1+\sqrt{0.36}} = 1.25$$

$$\rho(T_{SOR}) = \omega_{OPT} - 1 = 0.25$$

Exercise 5 (9/PJ 102)

Given the linear system $Ax=b$ with $A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

(a) Do Jacobi, G-S converge?

(b) Find ω_{OPT} for SOR

(c) Compare the three methods in terms of convergence speed

— — —

(a) A is strictly diagonally dominant so both Jacobi and Gauss - Seidel should converge

(b) $A = D - L - U$

$M = D$

$N = L + U$

$$M = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \quad L = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$U = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad M^{-1} = \begin{pmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 0 \\ 0 & 0 & 1/4 & 0 \\ 0 & 0 & 0 & 1/2 \end{pmatrix}$$

$$T_J = M^{-1} N = D^{-1}\cdot(L+U) = \begin{pmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 0 \\ 0 & 0 & 1/4 & 0 \\ 0 & 0 & 0 & 1/2 \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & -1/2 & 0 & 0 \\ -1/4 & 0 & -1/4 & 0 \\ 0 & -1/4 & 0 & -1/4 \\ 0 & 0 & -1/2 & 0 \end{pmatrix}$$

$$\det(T_J - \lambda I) = \begin{vmatrix} 0-\lambda & -1/2 & 0 & 0 \\ -1/4 & 0-\lambda & -1/4 & 0 \\ 0 & -1/4 & 0-\lambda & -1/4 \\ 0 & 0 & -1/2 & 0-\lambda \end{vmatrix} = \lambda^4 - \frac{5}{16}\lambda^2 + \frac{1}{64} \Leftarrow \ldots$$

$$\rho(T_J) = |\lambda_{max}| = 1/2 = \beta$$

So, $\omega_{OPT} = \dfrac{2}{1+\sqrt{1-\beta^2}} = \dfrac{1}{1+\sqrt{\frac{3}{4}}} = \dfrac{4}{2+\sqrt{3}}$

(c) Since A is circulant and Jacobi converges then from our theory we draw that G-S is x2 faster than Jacobi. Simultaneously, SOR for $\omega = 1$ is identical to G-S but $\omega_{OPT} \neq 1$. Therefore, SOR is the fastest, followed by G-S and lastly Jacobi.

Exercise 5 6 (q2/pg 103)

If $A \in \mathbb{C}^{4,4}$ is strictly diagonally dominant (in terms of lines) prove that SOR will converge for every $\omega \in (0,1]$

This is similar to the proof of Kahan's theorem.
Every eigenvalue should be < 1.
Therefore, $\prod_{i=1}^{4} |\lambda_i| < 1 \Rightarrow |\det(T_{SOR})| < 1$
But,

$$|\det(T_{SOR})| = \det(D-\omega L)^{-1} \cdot \det((1-\omega) \cdot D + \omega U)$$

$$= \frac{1}{\det(D)} \cdot (1-\omega)^4 \cdot \det(D) = |1-\omega|^4 \Rightarrow |1-\omega| < 1$$

So $\rho(T_{SOR}) \geq |\omega - 1| < 1$ (1) and for $\omega \in (0,1]$ the method converges according to the above proof.
This can be confirmed by another theorem stating that if A is strictly dominant, G-S will converge and the same will happen for SOR when $\omega = 1$. (2)

From (1), (2) we can deduce that SOR will converge for every $\omega \in (0,1]$.