THE PRICE OF STABILITY FOR NETWORK DESIGN WITH FAIR COST ALLOCATION

Elliot Anshelevich, Anirban Dasgupta, Jon M. Kleinberg, Eva Tardos, Tom Wexler, and Tim Roughgarden. SIAM J. Comput., 38(4):1602–1623, 2008.

A summary by Platon A. Karageorgis

John Nash's influential concept of "Nash Equilibrium" goes beyond theoretical concepts as it can have practical applications. One of the areas that it contributes significantly is network design, since it is crucial for the development of the network, to identify the strategic behavior and its ramifications. The Nash Equilibria in this case represent a set of stable solutions for a group of self-interested agents, whose goal is to construct a network of specific traits. In this problem the identification of a Nash Equilibrium is not enough, as practical applications demand quality results, but this has a direct conflict with the philosophy of the agents that act merely on personal benefit. The paper introduces the term "Price of Stability" which is the ratio of the top Nash Equilibrium in relation to the global best solution. This term aims to quantify the impact of this philosophy on the value of the final Nash Equilibrium and is applied to many examples throughout the paper. The authors also define a potential function Φ, which aims to prove that any move from the agents attempting to reduce their own costs will only decrease its value. Also, they guarantee that any kind of improving sequence will lead to a Nash Equilibrium, due to the finite number of states, and they consider every Nash Equilibrium to be pure, meaning that no agent will have a financial benefit by deviating from the current strategy.

Network design games provide a natural context for investigating the price of stability, due to the extensive research on methods to effectively share the cost of the designed networks. There are multiple ways to do that and the prime one is the Shapley cost mechanism. This method divides the cost equally to every agent without considering additional variables, and it is named after Shapley because it satisfies a set of axioms that can be crucial in a network design. The goal is to minimize the loss created by the strategies of the agents to a value ε. In this case, it is proved that for every instance with Shapley cost-sharing, the worst case is $H(k)$* OPT, where H represents the harmonic sum and the proof is followed by a tightness example. Simultaneously, they prove that any game strictly following Shapley's cost mechanism is a congestion game as defined by Rosenthal, which means that there will always be a deterministic equilibrium. These results lead to further generalization, where the cost seizes to be a constant function and it is scaled by the number of agents using the given edge. It is proved that as long as this function is concave, the previously mentioned bounds hold in this case as well. Furthermore, they use a reduction from 3D-Matching to prove that it is NP-Hard to determine if the defined game has a Nash Equilibrium with a cost at most c. In the subsequent sections of the paper, the authors introduce extensions to the current network designed game.

The first extension that the authors suggest adds the delay parameter, which has two separate components. On the one hand, the length of the total path that the agent has in the network, is a delay parameter. This is proven to be concave, as long as the basic cost function defined beforehand is concave. On the other hand, they define a function $d(x)$ that will represent the latency. This will count the congestion in every edge. In contrast with the previous cases, this will not be concave as latency tends to be convex, so they assume that both cost – as defined in the previous section – and latency are monotonically increasing, but only the cost function is concave. This leads to another theorem, which proves that, if the cost function is concave and the latency function is non-decreasing, then the price of stability is once again bounded, but this time the bound is more flexible, as it depends at the degree of the latency function. Moreover, for the rest of this section, the cost is redefined and it will be solely dependent on the latency function. The highlighted case will be the one where one specific edge is extremely popular between the agents and that causes congestion. The first proof in this context is comparing the cheapest Nash Equilibrium with the global optimum, in a case where there is the double amount of players, proving that the cost in the first case is a lower bound for the second one. Next, the authors attempt to tighten the bound established by Roughgarden in a single source fair connection game. In this game, all costs adhere to the previously defined form and the delay functions belong to a family D that contains only constant functions. They demonstrate that the upper bound is α(D), where α(D) represents the price of anarchy for non-atomic games containing delays from D.

Moving on, the paper refers to undirected graphs. The authors suggest that the price of stability cannot have a bound in this case to the best of their knowledge, and the problem remains open. Nevertheless, they prove that in the case of two players the bound can be reduced from 3/2 to 4/3. They consider each agent having two terminals with one terminal in common. They also attach an example where the tightness is proved. Subsequently, the next section is about the convergence of best response dynamics within the game. It is demonstrated that in general, the process of best response dynamics can take a significant amount of time to converge to a Nash Equilibrium. The authors construct a sequence of best responses that prove exponential convergence time towards an equilibrium.

The final extension introduces a new element to the problem by assigning a weight to each agent. In this modified version, agents will pay costs proportional to their individual weights, deviating from the characteristics of congestion games. To accommodate this change, a new type of potential function is defined based on the players' weights. The research proves that any attempt by an agent to improve the potential function will only decrease it, ensuring convergence to a Nash Equilibrium. Moreover, they show that two-player weighted games always possess a Nash Equilibrium, even though the price of stability can be weak in some cases. On the other hand, for games involving three or more agents, a pure Nash Equilibrium may not exist, and the stability bounds from unweighted games do not apply to the weighted version. They show that the price of stability for two players with weights 1 and w is upper-bounded by 1+ 1/(1+W). For games with k players, an example is provided where the price of stability is Θ(logW) and Θ(k), where w represents the total weight of all players.

To conclude, the paper mentions the open problems that can be a motivation for further exploration. The undirected case lacks a definitive bound for the price of stability, with a current lower bound of 12/7. In addition, the efficient computation of a Nash Equilibrium for games involving multiple players within polynomial time remains an open challenge, despite the potential utilization of best-response dynamics. As for the delays only case, so far there are promising outcomes in single examples only, and a generalized analysis of stability is yet to be found. Finally, in weighted fair connection games the question revolves around determining the price of stability for approximate Nash Equilibria, specifically when all players share a common terminal.