PROJECT 2

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Exercise 1

Given the following game:

	W	X	Y	Z
A	2, 42	13, 40	9, 23	0, 23
B	2, 19	2, 14	5, 13	1, 0
C	20, 7	20, 5	11, 3	1, 2
D	20, 45	3, 11	3, 5	1, 2

Discuss the following claims shortly:

1. B is strictly dominated by A
2. X strictly dominates Y
3. D weakly dominates B
4. Z is weakly dominated by W
5. There is no dominant strategy in the game for Player 1
6. There is a dominant strategy in the game for Player 2
7. Z is a best response on A
8. C is a best response on W
9. There exists a Nash Equilibrium with social welfare less than 27
10. There exists a Nash Equilibrium with social welfare bigger than 34

Solution

1. This is **false** because in the first column “W” strategy A offers an identical profit for Player 1, which is not possible in terms of strict dominance. Also, in the last column “Z” strategy A offers an even better profit than B.
2. This is **true** because for every strategy chosen by Player 1, strategy X results in better profits than strategy Y.

3. This is **false** since in the third column “Y” strategy B offers a better alternative than strategy D.
4. This is **true** since strategy Z is strictly dominated by W. By recalling the definition of weak and strict dominance, we can say that if a strategy strictly dominates another strategy, then it also weakly dominates it.
5. This is **false** because strategy C is a valid dominant strategy in the given game.
6. This is **true** there is indeed a dominant strategy in the game for Player 2 and that is strategy W. In fact, it is strictly dominant.
7. This is **false** because the best response on A is W, since Player 2 has by far the biggest profit in that case.
8. This is **true** since there is no better alternative. Strategy D is equal to C but that doesn’t change the fact that C is still a best response.
9. This is **false** since the Nash Equilibrium with the smallest possible social welfare in this game is 27. That would be CW.
10. This is **true**, DW has social welfare equal to 65.

Exercise 2

Assume the following game portrays the prisoner dilemma with {C, D} being the strategies for each player.

	C	D
C	3, 3	0, 4
D	4, 0	2, 2

We will define a new game based on the game above, where the final utility of each player depends on the utility of the other player in the initial game. This modification attempts to add a level of connection between the choices of the two players, they are not entirely independent anymore. If for instance (s, t) is the strategy pair chosen by both players then player 1 will have the following utility:

$$u1'(s,t) = u1(s,t) + a * u2(s,t)$$

Similarly for player 2:

$$u2'(s,t) = u2(s,t) + a * u1(s,t)$$

where $\alpha \in [0, 1]$

(i) Show in a matrix form the game that we will get if we set $\alpha = \frac{3}{4}$. Is the new game a valid representation of prisoner’s dilemma?

(ii) Find for which values of α the new game is still portraying the prisoner's dilemma.

(iii) Find the range of values of α (if it exists) such that (C, C) is a Nash Equilibrium in the new game.

Solution

(i) If we set, $\alpha = \frac{3}{4}$ then we get the following game:

	C	D
C	$\frac{21}{4}, \frac{21}{4}$	3, 4
D	4, 3	$\frac{7}{2}, \frac{7}{2}$

This new representation is not a valid representation of prisoner's dilemma, since in this game (C, C) is a Nash Equilibrium and in the original case it is not.

(ii) First let's rewrite the above matrix with α as a parameter:

	C	D
C	$3 + \alpha * 3, \quad 3 + \alpha * 3$	$4\alpha, \quad 4$
D	$4, \quad 4\alpha$	$2 + \alpha * 2, \quad 2 + \alpha * 2$

If we want this modified version to be a valid representation of prisoner's dilemma, we need the strategy pair (D, D) to be the only Nash Equilibrium. So, we need to solve the following system:

$$\begin{cases} 3 + \alpha * 3 < 4 \\ 2 + \alpha * 2 > 4 * \alpha \end{cases} \rightarrow \begin{cases} \alpha < 1/3 \\ \alpha < 1 \end{cases}$$

Ultimately, α needs to be less than 1/3 so for $\alpha \in [0, \frac{1}{3})$ the modified game is a valid representation of prisoner's dilemma.

(iii) This comes directly from the previous part of the exercise, if $a \in [\frac{1}{3}, 1]$ then (C, C) will be a Nash Equilibrium, since it will be greater than or equal to the other options and therefore the players will not be motivated to change their strategies.

Exercise 3

Assume the congestion game in road networks described below. Each player must choose between 3 paths to go from an initial point to their destination. The latency depends on the number of players that chose the same path. Find all the Nash Equilibria with a pure strategy if there are any. Explain and analyze the game.

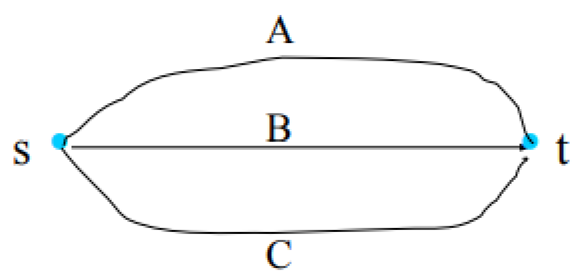

Figure 1: The congestion game discussed in lecture 2

- 3 possible routes A, B, C
- 16 players
- Latency functions: $dA(x) = 5 * x,\ dB(x) = 7.5 * x,\ dC(x) = 10 * x$

Solution

A state of the above problem will be in a Nash Equilibrium if and only if none of the players is motivated to change their path, to reduce their own latency cost. For example, if a player leaves his path for another, the cost on his new path must not be less than the cost the player paid before leaving his path. This can be mathematically formulated via a set of inequalities:

- $5 * XA \leq 7.5 * (XB + 1)$
- $5 * XA \leq 10 * (XC + 1)$
- $7.5 * XB \leq 5 * (XA + 1)$
- $7.5 * XB \leq 10 * (XC + 1)$

- $10 * XC \leq 5 * (XA + 1)$
- $10 * XC \leq 7.5 * (XB + 1)$

Where XA represents the number of players that chose path A and similarly for XB, XC. We will create a plane that will contain all the valid solutions. Firstly, define:

- $XA = x$
- $XB = y$
- $XC = z$

So, the inequations will be:

- $5 * x \leq 7.5 * (y + 1)$
- $5 * x \leq 10 * (z + 1)$
- $7.5 * y \leq 5 * (x + 1)$
- $7.5 * y \leq 10 * (z + 1)$
- $10 * z \leq 5 * (x + 1)$
- $10 * z \leq 7.5 * (y + 1)$

Then, we can replace z:

- $z = 16 - x - y$

Once again, the inequations will be:

- $5 * x \leq 7.5 * (y + 1)$
- $5 * x \leq 10 * ((16 - x - y) + 1)$
- $7.5 * y \leq 5 * (x + 1)$
- $7.5 * y \leq 10 * ((16 - x - y) + 1)$
- $10 * (16 - x - y) \leq 5 * (x + 1)$
- $10 * (16 - x - y) \leq 7.5 * (y + 1)$

Moreover, we will solve in terms of y every inequation:

- $y \geq \frac{2}{3} * x - 1$
- $y \leq -1.5 * x + 17$
- $y \leq \frac{2}{3} * x + \frac{2}{3}$
- $y \leq -\frac{4}{7} * x + \frac{68}{7}$
- $y \geq -1.5 * x + 15.5$
- $y \geq -\frac{4}{7} * x + \frac{61}{7}$

Finally, we draw each one of these lines. Then, we consider only the upper or lower plane, and we figure out which one by the inequality. For example, for the first equation we will consider only the upper plane, since y is larger than the rightmost part of the inequality. The set of solutions will be the overlapped region of all the inequalities.

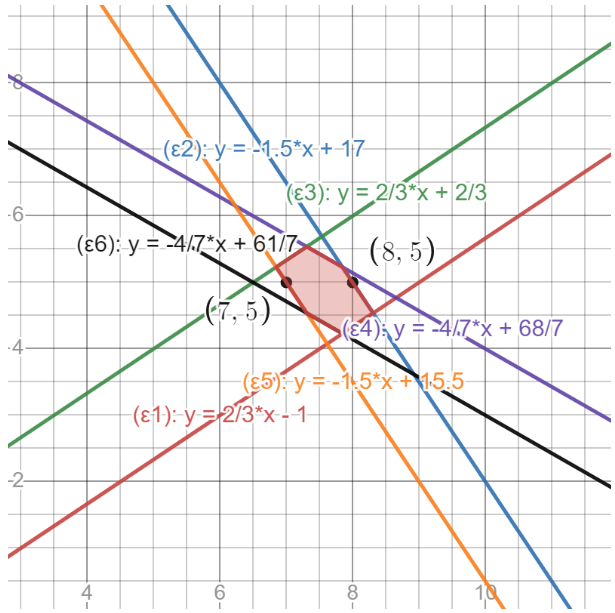

Figure 2: The set of solutions that fulfills all the inequalities.

We can see that the shaded red part is the overlapped region and therefore if we have any solutions, they will be there. The only solutions that come up from the shaded area are:

- $(7,5)$ $where\ x = 7,\ y = 5\ and\ z = 16 - 7 - 5\ \rightarrow z = 4$
- $(8,5)$ $where\ x = 8,\ y = 5\ and\ z = 16 - 8 - 5\ \rightarrow z = 3$

These points are accepted even though they are exactly on the border of the overlapped region, because all the inequalities included equality in their solutions. There are no other possible solutions since the results should be integers and these are the only integer solutions in the overlapped region.

So, the final Nash Equilibria are:

- $(XA = 7,\ XB = 5,\ XC = 4)$
- $(XA = 8,\ XB = 5,\ XC = 3)$

Exercise 4

This exercise refers merely to 0-sum games.

(i) Assume the following game:

$$\begin{bmatrix} \alpha & 9 \\ 12 & d \end{bmatrix}$$

For the parameters α, d it stands that $0 \leq \alpha \leq 12$ and $d > 12$. Can we always have a Nash Equilibrium with pure strategies? If not, give a specific example with α, d where there is no equilibrium. Otherwise, show that for every $\alpha \in [0, 12)$ and $d \in (12, \infty)$, with pure strategies, there must be a Nash Equilibrium.

(ii) Find all the Nash Equilibria for the 0-sum game below (with pure and mixed strategies). Use the method we saw in class for $2 \; x \; n$ 0-sum games.

$$\begin{bmatrix} -2 & 5 & 1 & 0 & -4 \\ 4 & -3 & -1 & 2 & 7 \end{bmatrix}$$

Solution

(i) The bottom left cell with benefit 12 will always be a Nash Equilibrium. Even if we maximize the possible values of α and minimize those of d, the bottom left cell will still be superior.
First, we analyze Player 1. He will choose the first row, since it has the best worse value. That will be $v1 = \max(9, 12)$, so $v1 = 12$. So, if we are in the bottom left cell state, it will be a Nash Equilibrium in terms of Player 1, because Player 1 won't be motivated to change his strategy, since their benefit will not increase.
Similarly, the second player will choose the worse best value. So $v2 = \min\,(12, d)$, and we get $v2 = 12$ as well. Once again, if we assume that we are in the bottom left state then Player 2 is not motivated to move because d is worse than 12. So, the bottom left state must be a Nash Equilibrium, and that is also supported by theory since $v1 = v2$. Ultimately, for every possible value of α, d there will be a Nash Equilibrium in this game using pure strategies.

(ii) Let's first check the case where the strategies are pure. It is certain that in this case we will never reach an equilibrium. Assume for example that Player 1 is first. Player 1 will choose the row that has the best worse value. The worse value in the first row is -4, while the worse value in the second row is -3. Therefore, we get that $v1 = -3$. Subsequently, if Player 2 is first, then they will choose the column that has the worse best value. In our example that will be $v2 = \min\,(4,\ 5,\ 1,\ 2,\ 7)$ and so, $v2 = 1$.

Since $v1 < v2$ we know from theory that we will never reach a Nash Equilibrium in this case.
Now let us see the case where the strategies are mixed. We will first analyze Player 1:

- $w1 = max_p min_q u1(p, q)$
- $w1 = max_p \min \{u1(p, e^1), u1(p, e^2), u1(p, e^3), u1(p, e^4), u1(p, e^5)\}$
- $w1 = max_p \min \{-2 * p1 + 4 * (1 - p1), 5 * p1 - 3 * (1 - p1), p1 - (1 - p1), 2 * (1 - p1), -4 * p1 + 7 * (1 - p1)\}$
- $w1 = max_p \min \{-6 * p1 + 4, 8 * p1 - 3, 2 * p1 - 1, -2 * p1 + 2, -11 * p1 + 7\}$
- $w1 = max_p \min\big(f1(p1), f2(p1), f3(p1), f4(p1), f5(p1)\big)$ where:
  - $f1(p1) = -6 * p1 + 4$
  - $f2(p1) = 8 * p1 - 3$
  - $f3(p1) = 2 * p1 - 1$
  - $f4(p1) = -2 * p1 + 2$
  - $f5(p1) = -11 * p1 + 7$

Now that we have generated the 5 functions, we draw the lines:

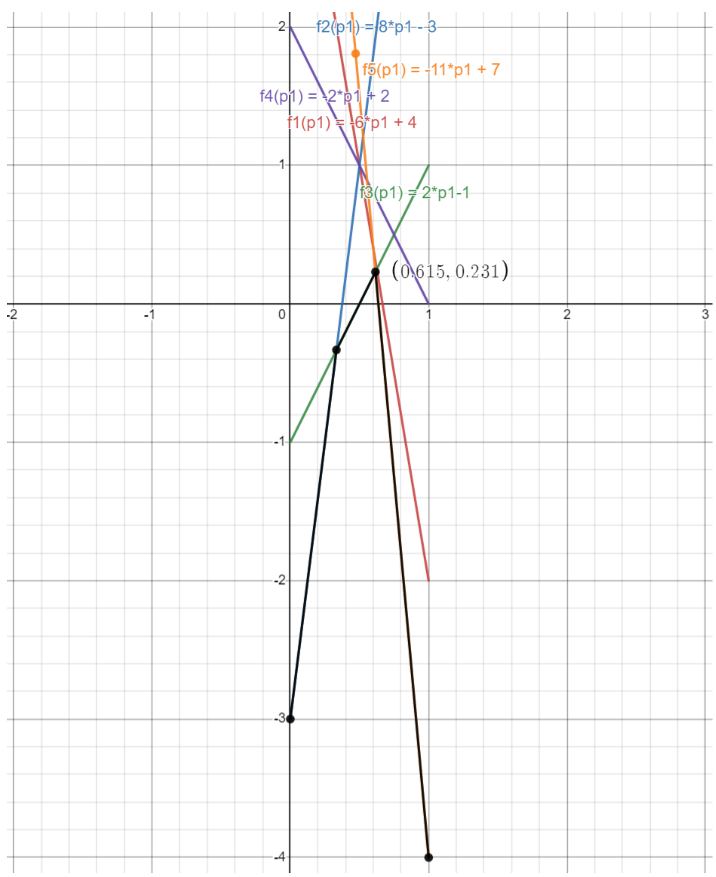

Figure 3: The maximum point after the minimization of the functions

So, we get:

- $w1 = f3(p1) = f5(p1) = 0.231 \rightarrow p1 = 0.615$

Moving on for Player 2 we get:

- $w2 = min_q max_p u1(p,q)$
- $w2 = min_{q2} max\{u1(e^1, (0,0,q2,0,1-q2)), u1(e^2, (0,0,q2,0,1-q2))\}$
- $w2 = min_{q2} \max\{q2 - 4 * (1-q2), -q2 + 7 * (1-q2)\}$
- $w2 = min_{q2} \max\{5 * q2 - 4, -8 * q2 + 7\}$

Using von Neumann's theorem, we can say that $w1 = w2$ and therefore:

- $5 * q2 - 4 = 0.231$
- $q2 = 0.8462$

So ultimately, the strategies are:

- $((0.615, 0.385), (0, 0, 0.8462, 0, 0.1538))$

Exercise 5

(i) Assume the following 2 x 2 game. Find a value for x such that the profile $(s1, t1)$ is a Nash Equilibrium (if there are many choose just one).

For the price of x that you chose, find if there is a Nash Equilibrium with mixed strategies, where both players play with positive probability both of their pure strategies.

	$t1$	$t2$
$s1$	$x^2, \quad x+2$	$2, \quad x^2$
$s2$	$x+2, \quad 2$	$x^2, \quad 5$

(ii) In the following game, find all the Nash Equilibria with pure <u>and</u> mixed strategies.

$$\begin{bmatrix} 0,0 & 5,2 & 3,4 & 6,5 \\ 2,6 & 3,5 & 5,3 & 1,0 \end{bmatrix}$$

Solution

(i) Since we want (s1, t1) to be a Nash Equilibrium the following conditions must be met:

$$\begin{cases} x^2 \geq x+2 \\ x+2 \geq x^2 \end{cases}$$

So obviously we can deduce that it must stand:

- $x^2 - x - 2 = 0$

And that means that $x = -1 \ or \ x = 2$. Both values make (s1, t1) a valid Nash Equilibrium strategy. The exercise asks to choose one, so we will choose $x = 2$.

The game after this shift will be:

	$t1$	$t2$
$s1$	4, 4	2, 4
$s2$	4, 2	4, 5

Moving on, we will search for Nash Equilibrium points, where both players choose to play with mixed strategies that have positive probabilities assigned in both pure strategies. Essentially this means that we want to find at least one valid Nash Equilibrium point using mixed strategies, but the strategy of that point, should not be a pure strategy -which can be considered as a sub-category of mixed strategies where probability is 1 in one case and 0 otherwise- as it should have positive probabilities in both cases.

We start by defining a mixed strategy for Player 2:

- $q = (q1, \ 1 - q1)$

And we must calculate the best response of Player 1. The point (p, q) will be a Nash Equilibrium point if and only if, p is the best response of q and q is the best response of p.

Player 1 will respond to q in the following way:

- $Player \ 1 \ chooses \ s1 \rightarrow u1(s1, q) = 4 * q1 + 2 * (1 - q1) = 2 * q1 + 2$
- $Player \ 1 \ chooses \ s2 \rightarrow u1(s2, q) = 4 * q1 + 4 * (1 - q1) = 4$

We plot the results:

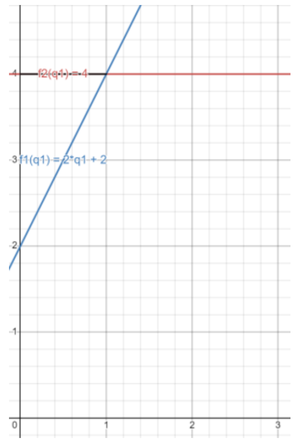

Figure 4: s2 seems to be a better strategy than s1.

From the plot above we can define the best response function of Player 1:

- $B1(q) = (0, 1),\ \forall\, q1\ \in [0, 1]$

The best response function states that Player 1 will always choose the second row and that comes from the fact that $y = 4$ is above $y = 2 * x + 2$ for every probability in (0, 1). This means that we can already answer the exercise's question without continuing the analysis. Player 1 will never decide to play both strategies with a positive probability, and even if they did it would not be an equilibrium, because there will always be a motive to change to a better strategy, which is described by B1. Therefore, the answer is no, there can not be an equilibrium where both players play with positive probability both of their pure strategies.

(ii) We will begin with the pure strategies, and we will exhaustively check every point and see if it is in both players' interests to maintain their current strategies.

- $(0, 0)$ is not a valid Nash Equilibrium point since both players are motivated to change their strategies. For example, Player 1 prefers to change their strategy and choose the second row since 2>0 and Player 2 would prefer any of the rest of the options.
- $(5, 2)$ is not a valid Nash Equilibrium point since Player 2 prefers both $(3, 4), (6, 5)$.
- $(3, 4)$ is not a valid Nash Equilibrium point since both players are motivated to change their strategies.
- $(6, 5)$ is a Nash Equilibrium point since neither player has an alternative option that would increase their personal benefit.
- $(2, 6)$ is a Nash Equilibrium point since neither player has an alternative option that would increase their personal benefit.
- $(3, 5)$ is not a valid Nash Equilibrium point since both players are motivated to change their strategies.

- $(5, 3)$ is not a valid Nash Equilibrium point since Player 2 prefers both $(2, 6), (3, 5)$.
- $(1, 0)$ is not a valid Nash Equilibrium point since both players are motivated to change their strategies.

Ultimately, the 2 Nash Equilibrium points where both players choose pure strategies are:

- $(6, 5)$
- $(2, 6)$

Subsequently, we will search for Nash Equilibrium points where both players choose mixed strategies.

First, assume that Player 1 chooses a strategy:

- $p = (p1,\ p2) = (p1,\ 1 - p1)$

Then, we calculate the utility of Player 2 for p1:

- $u2(p, e^1) = f1(p1) = \ 0 * p1 + 6 * (1 - p1) = \ -6 * p1 + 6$
- $u2(p, e^2) = f2(p1) = \ 2 * p1 + 5 * (1 - p1) = \ -3 * p1 + 5$
- $u2(p, e^3) = f3(p1) = 4 * p1 + 3 * (1 - p1) = p1 + 3$
- $u2(p, e^4) = f4(p1) = 5 * p1 + 0 * (1 - p1) = \ 5 * p1$

We once again plot the results:

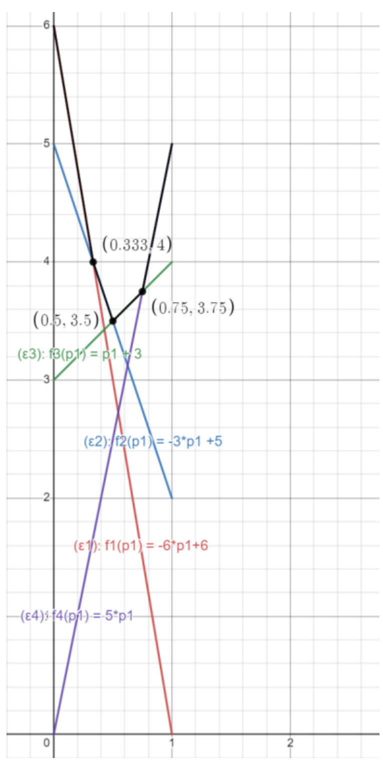

Figure 5: The 3 candidate points

The black line demonstrates how Player 1 will respond optimally and from this plot we generate 3 possible equilibrium points:

- $\left(\frac{1}{3}, \frac{2}{3}\right), which\ is\ where\ f1\ and\ f2\ meet$
- $\left(\frac{1}{2}, \frac{1}{2}\right), which\ is\ where\ f2\ and\ f3\ meet$
- $\left(\frac{3}{4}, \frac{1}{4}\right), which\ is\ where\ f3\ and\ f4\ meet$

Now we need to check each one and see if it's valid. For $\left(\frac{1}{3}, \frac{2}{3}\right)$, since the point is generated where $f1, f2$ meet and these are the utility functions of Player 2, it is expected that Player 2 will assign probability equal to 0 for $f3, f4$. Therefore, Player 2 will choose a strategy:

- $(q2, 1 - q2,\ 0,\ 0)$

From the support theorem we know that $u1(e^1, q) = u1(e^2, q)$ must stand. If it didn't then this could not be an equilibrium because Player 1 would be motivated to change when Player 2 chose strategy q. So, we check the relation:

- $0 + 5 * (1 - q1) = 2 * q1 + 3 * (1 - q1)\ \leftrightarrow q1 = \frac{1}{2}$

The result is a valid probability for q1 which means that $(\left(\frac{1}{3},\ \frac{2}{3}\right), \left(\frac{1}{2},\ \frac{1}{2},\ 0,\ 0\right))$ is a Nash Equilibrium point.

For $\left(\frac{1}{2}, \frac{1}{2}\right)$, since the point is generated where $f2, f3$ meet, it is expected that Player 2 will assign probability equal to 0 for $f1, f4$. Therefore, Player 2 will choose a strategy:

- $(0,\ q2,\ 1 - q2,\ 0)$

Following the same steps, we know that $u1(e^2, q) = u1(e^3, q)$ must stand. So, we check the relation:

- $5 * q2 + 3 * (1 - q2) = 3 * q2 + 5 * (1 - q2)\ \leftrightarrow q2 = \frac{1}{2}$

The result is a valid probability for q2 which means that $(\left(\frac{1}{2},\ \frac{1}{2}\right), \left(0,\ \frac{1}{2},\ \frac{1}{2}, 0\right))$ is a Nash Equilibrium point.

Finally, for $\left(\frac{3}{4}, \frac{1}{4}\right)$, since the point is generated where $f3, f4$ meet, it is expected that Player 2 will assign probability equal to 0 for $f1, f2$. Therefore, Player 2 will choose a strategy:

- $(0,\ 0,\ q3,\ 1 - q3)$

Following the same steps, we know that $u1(e^3, q) = u1(e^4, q)$ must stand. So, we check the relation:

- $3 * q3 + 6 * (1 - q3) = 5 * q3 + (1 - q3) \leftrightarrow q3 = \frac{5}{7}$

The result is a valid probability for q2 which means that $(\left(\frac{3}{4}, \frac{1}{4}\right), \left(0, 0, \frac{5}{7}, \frac{2}{7}\right))$ is a Nash Equilibrium point.

To sum up, using mixed strategies there are 3 Nash Equilibrium points:

- $(\left(\frac{1}{3}, \frac{2}{3}\right), \left(\frac{1}{2}, \frac{1}{2}, 0, 0\right))$
- $(\left(\frac{1}{2}, \frac{1}{2}\right), \left(0, \frac{1}{2}, \frac{1}{2}, 0\right))$
- $(\left(\frac{3}{4}, \frac{1}{4}\right), \left(0, 0, \frac{5}{7}, \frac{2}{7}\right))$

Exercise 6

(i) For the game of Exercise 5 (ii), calculate the Price of Anarchy for the equilibrium points of the pure strategies (if any). Then, calculate the Price of Anarchy again using the equilibrium points of the mixed strategies.

(ii) Construct a $3\ x\ 3$ game with positive utilities, were the price of anarchy for pure strategies is 3.14. Justify your answer.

(iii) Suggest the least possible values that could take place on the game that you suggested to significantly reduce the price of anarchy.

Solution

(i) We will start with the points of the pure strategies:

In Exercise 5 we found that $(2, 6), (6, 5)$ are the two Nash Equilibrium points when the players use pure strategies. The Price of Anarchy is the global optimum divided by the worst Nash Equilibrium. The global optimum in this game is spotted at the point $(6, 5)$ and naturally it is 11, while the worst Nash Equilibrium is obviously $(2, 6)$ where the sum is 8. Consequently, the Price of Anarchy in this case is:

- $PoA = \frac{11}{8} = 1.375$

Subsequently, we continue for the points of the mixed strategies:

In Exercise 5 we found that $(\left(\frac{1}{3}, \frac{2}{3}\right), \left(\frac{1}{2}, \frac{1}{2}, 0, 0\right))$, $(\left(\frac{1}{2}, \frac{1}{2}\right), \left(0, \frac{1}{2}, \frac{1}{2}, 0\right))$ and $(\left(\frac{3}{4}, \frac{1}{4}\right), \left(0, 0, \frac{5}{7}, \frac{2}{7}\right))$ are the three Nash Equilibrium points. First, we need to calculate the social welfare of each point to figure out which one is the worst.

For $(\left(\frac{1}{3}, \frac{2}{3}\right), \left(\frac{1}{2}, \frac{1}{2}, 0, 0\right))$ we get:

- $\frac{1}{3} * \frac{1}{2} * (0 + 0) + \frac{2}{3} * \frac{1}{2} * (2 + 6) + \frac{1}{3} * \frac{1}{2} * (5 + 2) + \frac{2}{3} * \frac{1}{2} * (3 + 5) = 6.5$

For $(\left(\frac{1}{2}, \frac{1}{2}\right), \left(0, \frac{1}{2}, \frac{1}{2}, 0\right))$ we get:

- $\frac{1}{2} * \frac{1}{2} * (5 + 2) + \frac{1}{2} * \frac{1}{2} * (3 + 5) + \frac{1}{2} * \frac{1}{2} * (3 + 4) + \frac{1}{2} * \frac{1}{2} * (5 + 3) = 7.5$

For $(\left(\frac{3}{4}, \frac{1}{4}\right), \left(0, 0, \frac{5}{7}, \frac{2}{7}\right))$ we get:

- $\frac{3}{4} * \frac{5}{7} * (3 + 4) + \frac{1}{4} * \frac{5}{7} * (5 + 3) + \frac{3}{4} * \frac{2}{7} * (6 + 5) + \frac{1}{4} * \frac{2}{7} * (1 + 0) = \frac{213}{28} = 7.6$

So, the worst Nash Equilibrium is $(\left(\frac{1}{3}, \frac{2}{3}\right), \left(\frac{1}{2}, \frac{1}{2}, 0, 0\right))$ and the Price of Anarchy is:

- $PoA = \frac{11}{\frac{13}{2}} = 1.692$

(ii) The $3\ x\ 3$ game that fulfills everything that the exercise asks for is:

	$t1$	$t2$	$t3$
$s1$	$\frac{1}{2}, \frac{1}{2}$	$\frac{1}{2}, \frac{1}{2}$	$\frac{1}{2}, \frac{1}{2}$
$s2$	$\frac{1}{2}, \frac{1}{2}$	$\frac{1}{2}, \frac{1}{2}$	$\frac{1}{2}, \frac{1}{2}$
$s3$	$\frac{1}{2}, \frac{1}{2}$	$\frac{1}{2}, \frac{1}{2}$	3.14, 0

In this game:

- The global optimum is $(s3, t3)$ with social welfare equal to 3.14.
- The worst Nash Equilibrium is any one of $(s1, t1), (s2, t1), (s1, t2), (s2, t2)$. They all have social welfare equal to 1.

So, the Price of Anarchy is:

- $PoA = \frac{3.14}{1} = 3.14$

(iii) If we substitute 3.14 with 1 then:

- We don't alter the dynamics of the game.
- We obtain the minimum PoA possible, which is 1.
- We have made the least possible number of changes.

So, this will be our suggestion for this part of the exercise.

Exercise 7

Perform the VCG mechanism in an auction with 3 players and 3 identical goods, where the players have symmetric submodular utility functions as follows:

1. $v1(1) = 4,\ v1(2) = 6,\ v1(3) = 6$
2. $v2(1) = 3,\ v2(2) = 6,\ v2(3) = 9$
3. $v3(1) = 2,\ v3(2) = 3,\ v3(3) = 4$

Since the goods are identical, the only information we need is the value of each player for every possible number of goods that they can take.

Find which is the assignment of the goods that the mechanism performs, and which are the payments of each player.

Solution

The assignment that the mechanism will perform is the following:

	First Item	Second Item	Third Item
Player 1	4	2	2
Player 2	3	3	3
Player 3	2	2	2

In the above table the values marked in red demonstrate the assignment of the mechanism. When the first item is auctioned, the value of Player 1 is the biggest, so Player 1 takes the item. Then, when the second item is auctioned, the value of Player 1 is not 4 anymore, it's 2 since they got the first one. So, the second item goes to Player 2 that has the biggest value. Finally, when the third item is auctioned the value of Player 1 is still 2, and Player 2 has again value equal to 3 even though they got the second item, since this is the behavior described by the exercise $(6 - 3 = 3)$. Therefore, Player 2 takes the third item as well.

As for the payments of the Players they obviously follow the above assignment and the formula that is used to generate them is:

- $pi = SW_{-i} - \sum_{j \neq i} v_j(S_j)$

So, using this formula we get:

- For Player 1: $p1 = SW_{-1} - \sum_{j \neq 1} v_j\left(S_j\right) = (3+3+3) - (3+3) = 3$
- For Player 2: $p2 = SW_{-2} - \sum_{j \neq 2} v_j\left(S_j\right) = (4+2+2) - 4 = 4$
- For Player 3: $p3 = SW_{-3} - \sum_{j \neq 3} v_j\left(S_j\right) = (4+3+3) - (4+3+3) = 0$

Exercise 8

Assume the following game which is commonly called "first-price all-pay" auction. There are 2 players A and B and a good for sale, where both players consider the same value $K > 0$ for the item. The 2 players submit their offers for the item in a shielded envelope and they can be any real number in $[0,\ K]$. The winner in the auction is the player that calls the biggest bid. If the bids are identical then player A wins. Show that this game does not have equilibrium points with pure strategies. To do that, express mathematically the utility functions for both players, $u1(b1, b2)$ and $u2(b1, b2)$ where $u1$ is the utility function of A when they call a bid $b1$ and $u2$ is the utility function of B when they call a bid $b2$.

Solution

To begin with, let's describe the utility functions of the players:

- $u1(b1, b2) = \begin{cases} K - b1, & b1 \geq b2 \\ -b1, & b1 < b2 \end{cases}$
- $u2(b1, b2) = \begin{cases} K - b2, & b1 < b2 \\ -b2, & b1 \geq b2 \end{cases}$

Assume that the auction took place and player A won. Firstly, Player A does not have a motive to increase their bid $b1$, on the contrary someone could say that there is a motive to decrease it. That is because Player A would ideally want to win with the smallest possible difference ε from Player B's bid $b2$ to maximize their utility. But this part is not significant since the behavior of Player B will answer the question on why there can't be an equilibrium. Since Player A is winning, Player B is motivated to increase their bid by a value ε. After this change, there are two possible scenarios. On the one hand, the new bid of Player B, $b2 + \varepsilon$ could be again less (or equal) than $b1$. But that would mean that once again Player B is motivated to increase their bid, so we end up exactly on the same situation. On the other hand, the new bid of Player B, $b2 + \varepsilon$ could surpass $b1$. This would mean though that Player A is now motivated to increase their bid, which makes the whole process start over with opposite roles for the two players. The way that this game is described, there will never be a state where none of the players is motivated to change their bid and therefore there can't be any equilibrium whilst both players choose pure strategies.

Exercise 9

Assume the following game with consecutive moves:

In a vertical bar we have placed n disks. There are two players, Player 1 and Player 2 and each player, when their turn comes, they can remove 1, 2 or 3 disks from the bar. The player that will remove the final disk wins 1 euro which must be paid by the other player. Assume that player 1 starts first.

(i) For n = 4, design the tree that the game is describing and find all the subgame perfect equilibrium points. Does any of the two players have a strategy that wins the game no matter the strategy that the other player chooses?

(ii) If n is a multiple of 4, does any of the two players have a strategy that wins the game no matter the strategy that the other player chooses? Justify your answer.

(iii) If n is **not** a multiple of 4, does any of the two players have a strategy that wins the game no matter the strategy that the other player chooses? Justify your answer.

Solution

(i) First let's design the tree:

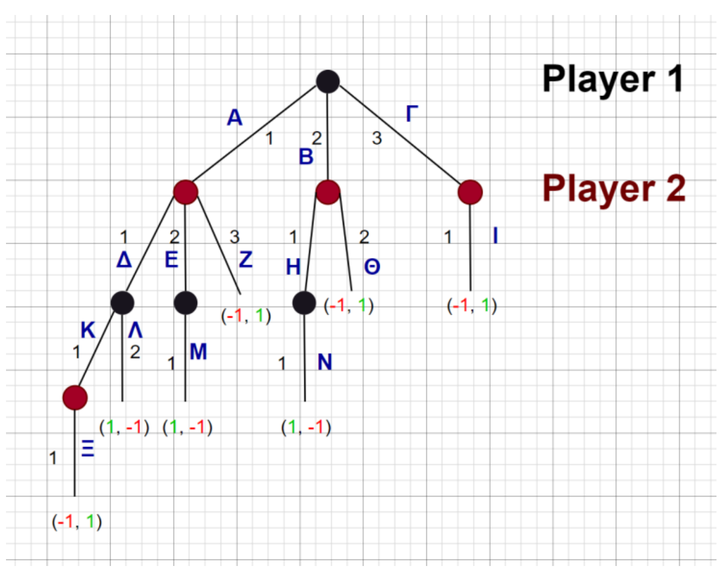

Figure 6: The tree showing the possible moves of each player.

The tree contains all the possible choices for each player. We cut the branches that are unnecessary in terms of the rules of the game, meaning that for example Player 2 can't remove 3 disks when Player 1 has already discarded 3, since the total number is 4.
We will first analyze the strategies of the two players:

- Strategies for Player 1:
  $\{(A, K, M, N), (A, \Lambda, M, N), (B, K, M, N), (B, \Lambda, M, N), (\Gamma, K, M, N), (\Gamma, \Lambda, M, N)\}$
- Strategies for Player 2:
  $\{(\Delta, H, I, \Xi), (\Delta, \Theta, I, \Xi), (E, H, I, \Xi), (E, \Theta, I, \Xi), (Z, H, I, \Xi), (Z, \Theta, I, \Xi)\}$

Now, we need to see if any of these strategies is winning independently of the other player's choice. It is easy to deduce that Player 1 will not have a strategy of this type, because Player 2 can finish the game in their first step no matter what Player 1 decides in their first move. On the other hand, the strategy we just described is a winning strategy and in fact, the only winning strategy for Player 2. That would be (Z, Θ, I, Ξ) and it states that Player 2 instantly finishes the game in every subtree that is created after the first move of Player 1. So, yes, Player 2 has a strategy that wins the game no matter the strategy that the other player chooses. On the other hand, Player 1 does not have this privilege.
Let's now find all the possible equilibrium points. We will start by analyzing every subtree of the game.
For the subgames that have length 1 there is no need for further analysis since all of them have only 1 option, to discard 1 disk. So, we start the analysis from the subgames with length equal to 2:

- <u>For Player 1:</u>

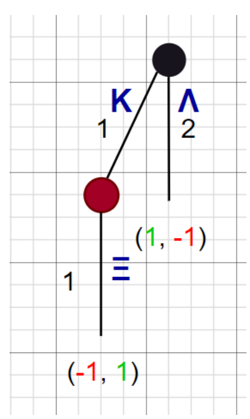

Figure 7: Subgame of length 2 for Player 1

- ➢ Player 1 will choose the strategy "Λ" since strategy "K" will lead to defeat.
- ➢ This is the only subgame of length 2 for Player 1.

- <u>For Player 2:</u>

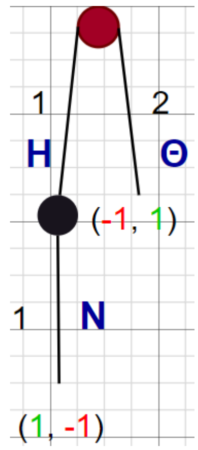

Figure 8: Subgame of length 2 for Player 2

- ➢ Player 2 will choose the strategy "Θ" since strategy "H" will lead to defeat.
- ➢ This is the only subgame of length 2 for Player 2 as well.

We move on with subgames of length 3:

- The only subgame of length 3 is the following and it concerns Player 2:

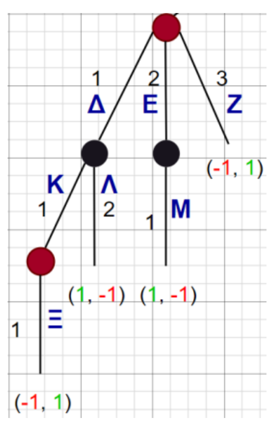

Figure 9: Subgame of length 3 for Player 2

- ➢ Player 2 will choose the strategy "Z" since strategy "E" leads to defeat, and strategy "Δ" will lead to defeat as well since we know that Player 1 will choose "Λ".

Finally, the last subgame is the game itself, and essentially what it shows is that Player 1 will not have a motive to choose any of the strategies A, B, Γ, since they lose in every case and the outcome is the same, that is $(-1, 1)$. On the other hand, Player 2 has an optimal strategy (Z, Θ, I, Ξ) that wins in every case, so every time Player 2 has this strategy there will not be a motive to shift to another. If Player 2 chooses another strategy, there will be a motive to shift to (Z, Θ, I, Ξ). So, we will have many SPE points, and they will be all the

combinations of Player 1's strategies where Player 1 does not have a preference, with (Z, Θ, I, Ξ).
So, the SPE equilibrium points are:

- $((A, \Lambda, M, N), (Z, \Theta, I, \Xi))$
- $((B, \Lambda, M, N), (Z, \Theta, I, \Xi))$
- $((\Gamma, \Lambda, M, N), (Z, \Theta, I, \Xi))$

We skip the rest of Player 1's strategies that contain "K" as they will never be selected in the case that the game reaches that state.

(ii) If the game has the same formulation and the same rules, but n is a multiple of 4 and not exactly 4, then Player 2 will still have a winning strategy like in (i), no matter the value of the multiplier. The reason is that Player 1 will still start the game and will remove 1, 2 or 3 disks, but then Player 2 will have the opportunity to redirect Player 1 to the subtree where n is again a multiple of 4. For example, if n = 64 then Player 1 chooses to remove 1 disk, then Player 2 will just remove 3 and then the game is "restarted" with n being a multiple of 4 and Player 1 forced to make their move. The same will happen with the other cases where Player 1 removes 2 or 3 disks, as Player 2 will just remove 2 and 1 in those cases. Eventually, n will be equal to 4 and we know from the analysis that we previously made, that Player 2 always has a winning strategy in this case. Therefore, Player 2 still has a winning strategy and Player 1 doesn't, because this case can always with the help of Player 2 be "redirected" to the original case where n = 4.

(iii) If the game has the same formulation and the same rules again, but n is not a multiple of 4, then there are two cases but in both Player 1 will now have a winning strategy. The first case is that n is 1, 2 or 3. In that case Player 1 can win immediately. The other case, which is the main one, is the case where n > 4. Player 1 will be able to redirect this case to the subtree where the disks are a multiple of 4, but this time Player 2 will be playing first, which will mean that we will have an identical case with (ii), but this time the roles are reversed. For example, if n = 67 then Player 1 will just remove 3 disks, if n = 66 then they will remove 2 disks and if n = 65 then they will remove 1 disk. There is no case in this scenario that Player 1 does not have a winning strategy. Therefore, we can confidently say that Player 1 will always have a winning strategy in this case and Player 2 won't.