Set 2

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(10) Given $A \in \mathbb{R}^{n,m}$ with SVD: $A = U\Sigma V^T$. Find the SVD analysis of the following matrices: $(A^TA)^{-1}$, $(A^TA)^{-1}A^T$, $A(A^TA)^{-1}$, $A(A^TA)^{-1}A^T$

- First we calculate $A^T = V\Sigma^T U^T = V\Sigma U^T$
  So $A^TA = V\Sigma U^T U\Sigma V^T = V\Sigma^2 V^T$
  Then, $(A^TA)^{-1} = (V\Sigma^2V^T)^{-1} = V(\Sigma^2)^{-1}V^T = V\Sigma^{-2}V^T$

- $(A^TA)^{-1} \cdot A^T = V\Sigma^{-2}V^T \cdot V\Sigma U^T = V\Sigma^{-1} \cdot \Sigma^{-1} \cdot \Sigma \cdot U^T = V\Sigma^{-1}U^T$
- $A(A^TA)^{-1} = U\Sigma V^T \cdot V \cdot \Sigma^{-2} \cdot V^T = U\Sigma^{-1} \cdot V^T$
- $A(A^TA)^{-1}A^T = U\Sigma^{-1}V^TV\Sigma U^T = UU^T = I$

(12) Given matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix}$ and a vector $b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$:

(i) Find the SVD analysis
(ii) Find the pseudoinverse of A
(iii) Find the minimum norm solution for the least squares

(i) To begin with, we need to calculate the eigenvalues.

$$A^TA = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 14 & 28 \\ 28 & 56 \end{bmatrix}$$

$$A^TA - \lambda I = \begin{bmatrix} 14-\lambda & 28 \\ 28 & 56-\lambda \end{bmatrix}$$

$$\det(A^TA - \lambda I) = (14-\lambda)(56-\lambda) - 28 \cdot 28 = \lambda^2 - 70\lambda$$

So our eigenvalues are: $\boxed{\lambda = 0}$ and $\boxed{\lambda = 70}$

Next we calculate the eigenvectors.

$\boxed{\text{For } \lambda=0}$:

$A^T A - 0 \cdot I = A^T A = \begin{bmatrix} 14 & 28 \\ 28 & 56 \end{bmatrix}$

$\begin{bmatrix} 14 & 28 \\ 28 & 56 \end{bmatrix} \cdot \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = 0 \overset{\text{Simplify}}{\Leftrightarrow} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = 0 \Leftrightarrow \boxed{V_1 = -2V_2}$

We set $V_2 = 1$ and so we have: $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

Finally, we divide by the sum of squares to make it unitary.

We divide by: $\sqrt{(-2)^2 + (1)^2} = \sqrt{5}$

So the first eigenvector is: $\begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$

$\boxed{\text{For } \lambda=70}$:

$A^T A - 70 I = \begin{bmatrix} -56 & 28 \\ 28 & -14 \end{bmatrix}$

$\begin{bmatrix} -56 & 28 \\ 28 & -14 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = 0 \overset{\text{Simplify}}{\Leftrightarrow} \begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = 0 \Leftrightarrow$

$\Leftrightarrow \boxed{V_1 = \frac{1}{2} V_2}$

We set $V_2 = 1$ and we get: $\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$

Once more we divide by: $\sqrt{(\frac{1}{2})^2 + 1^2} = \sqrt{\frac{5}{4}}$

So the second eigenvector is: $\begin{bmatrix} 1/2\sqrt{5/4} \\ 1/\sqrt{5/4} \end{bmatrix}$

Now we can write V as: $\begin{bmatrix} 1/2\sqrt{5/4} & -2/\sqrt{5} \\ 1/\sqrt{5/4} & 1/\sqrt{5} \end{bmatrix}$

(since $70 > 0$, the 2nd vector goes first)

And Σ should be: $\begin{bmatrix} \sqrt{70} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

since the singular values are the square root of the eigenvalues. Finally we calculate $U = A \cdot V \cdot \Sigma^{-1}$

$$A \cdot V = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix} \cdot \begin{bmatrix} 1/2\sqrt{5/4} & -2/\sqrt{5} \\ 1/\sqrt{5/4} & 1/\sqrt{5} \end{bmatrix} = \begin{bmatrix} \frac{5}{2\sqrt{5/4}} & 0 \\ \frac{5}{\sqrt{5/4}} & 0 \\ \frac{15}{2\sqrt{5/4}} & 0 \end{bmatrix}$$

And Σ^{-1} should be the right Moore-Penrose which is simple in this case:

$$A \cdot V \cdot \Sigma^{-1} = \begin{bmatrix} \frac{5}{2\sqrt{5/4}} & 0 \\ \frac{5}{\sqrt{5/4}} & 0 \\ \frac{15}{2\sqrt{5/4}} & 0 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{\sqrt{70}} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{5}{2\cdot\sqrt{70}\cdot\sqrt{5/4}} & 0 & 0 \\ \frac{5}{\sqrt{70}\cdot\sqrt{5/4}} & 0 & 0 \\ \frac{15}{2\cdot\sqrt{70}\cdot\sqrt{5/4}} & 0 & 0 \end{bmatrix}$$

So finally U is :

$$U = \begin{bmatrix} \frac{5}{2\sqrt{70}\cdot\sqrt{5/4}} & 0 & 0 \\ \frac{5}{\sqrt{70}\cdot\sqrt{5/4}} & 0 & 0 \\ \frac{15}{2\sqrt{70}\cdot\sqrt{5/4}} & 0 & 0 \end{bmatrix}$$

(ii) Next, we want to calculate the pseudoinverse of A

$$A^{+} = V \cdot \Sigma^{-1} \cdot U^{T} = \begin{bmatrix} 1/2\sqrt{5/4} & -2/\sqrt{5} \\ 1/\sqrt{5/4} & 1/\sqrt{5} \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{\sqrt{70}} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \frac{5}{2\sqrt{70}\sqrt{5/4}} & \frac{5}{\sqrt{70}\cdot\sqrt{\frac{5}{4}}} & \frac{15}{2\sqrt{70}\sqrt{\frac{5}{4}}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(4)

$$V \cdot \Sigma^{-1} = \begin{bmatrix} \frac{1}{2\cdot\sqrt{70}\sqrt{\frac{5}{4}}} & 0 & 0 \\ \frac{1}{\sqrt{70}\cdot\sqrt{\frac{5}{4}}} & 0 & 0 \end{bmatrix}$$

$$V \cdot \Sigma^{-1} \cdot U^{T} = \begin{bmatrix} \frac{1}{2\sqrt{70}\cdot\sqrt{\frac{5}{4}}} & 0 & 0 \\ \frac{1}{\sqrt{70}\cdot\sqrt{\frac{5}{4}}} & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \frac{5}{2\sqrt{70}\sqrt{\frac{5}{4}}} & \frac{5}{\sqrt{70}\cdot\sqrt{\frac{5}{4}}} & \frac{15}{2\cdot\sqrt{70}\cdot\sqrt{\frac{5}{4}}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} 0.014 & 0.028 & 0.042 \\ 0.028 & 0.056 & 0.084 \end{bmatrix}$$ Moore-Penrose pseudoinverse

(iii) We can use the pseudoinverse we just calculated.

$$Ax = b \Rightarrow x = A^+ b \Rightarrow x = \begin{bmatrix} 0.014 & 0.028 & 0.042 \\ 0.028 & 0.056 & 0.084 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \boxed{x = \begin{bmatrix} 0.084 \\ 0.168 \end{bmatrix}}$$ which is the solution that minimizes the norm

(11)(i) $trace(A^T A) = \sum_{i=1}^{n} (A^T A)_{ii} = \sum_i \left(\sum_j A^T_{ij} A_{ji} \right) = \sum_{i,j} A_{ij}^2$ (1)

$\|A\|_F = \left(\sum_{i,j} |A_{i,j}|^2 \right)^{1/2} = \sqrt{\sum_{i,j} (A_{ij})^2} = \sum_{i,j} A_{ij}$ (2)

From (1),(2) $\Rightarrow \|A\|_F^2 = trace(A^T A)$

(ii) We proved $\|A\|_F^2 = trace(A^T A)$

But it also stands that: $trace(A^T A) = trace(AA^T)$

Proof

$trace(A^T A) = \sum_{i=1}^{m} (A^T A)_{ii} = \sum_{i=1}^{m} \sum_{j=1}^{n} A^T_{ij} A_{ji} = \sum_{j=1}^{n} \sum_{i=1}^{m} A_{ji} A^T_{ij} =$

$= \sum_{j=1}^{n} (AA^T)_{jj} = trace(AA^T)$

So $\|A\|_F = trace(AA^T) = trace(I)$ where I is $n \times n$

$\left\{ \begin{array}{l} trace(I_{n\times n}) = n \\ \|A\|_F^2 = trace(AA^T) \end{array} \right. \Rightarrow \boxed{\|A\|_F = \sqrt{n}}$

(iii) The Frobenius norm is always greater or equal than the Euclidean norm. The Euclidean norm is equal to the largest eigenvalue λ_i and the Frobenius norm is equal to the sum of squares of all the eigenvalues, let's say $\sqrt{\lambda_i^2 + k_i^2}$ where $(k_i)^2$ stands for all the rest eigenvalues.

Then, we solve: $\lambda_i \geq \sqrt{\lambda_i^2 + k_i^2} \Rightarrow \boxed{k_i^2 \leq 0}$ (where k_i^2 obviously ≥ 0)

Which means that the only case that there is equality is the case where λ_i is the only eigenvalue i.e. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(13) • The left singular ~~eigen~~ vectors of A are A^TA eigenvectors
→ This is $\boxed{\text{False}}$ since $\begin{cases} A^TA = V\Sigma^2V^T \\ AA^T = U\Sigma^2 \end{cases}$ which shows that they are the right eigenvectors

• For every matrix A, AA^T and A^TA have the same $\neq 0$ ~~singular~~ eigenvalues
→ This is $\boxed{\text{False}}$ since they do have the same eigenvalues but they can also be 0.

• If S is a symmetric matrix, then the $\neq 0$ eigenvalues of S are also its singular values
→ This is $\boxed{\text{False}}$. $S = U\Sigma V^T$ and $s_i = |\lambda_i|$ since the eigenvalues come from $\sqrt{\Sigma^2}$ and therefore can't be negative. So if the matrix has at least 1 negative eigenvalue the statement is proven to be wrong

• If A is not full-rank then 0 is a singular value of A
→ This is $\boxed{\text{True}}$ since the determinant will be 0 and at least one eigenvalue will be 0.

• Given matrix A, invertible with $\sigma_1, \sigma_2 \dots \sigma_n$. For every $k \geq 0$ the singular values of $A + k \cdot I_n$ are $\sigma_1 + k, \sigma_2 + k, \dots, \sigma_n + k$.
→ Without loss of generality, A is symmetric. Then $\sigma_i = |\lambda_i|$ and if $\lambda_i = -5$, $\sigma_i = 5$. We multiply $k=7$ to I and add to A. The eigenvalue λ_i we said before will now be $\lambda_i' = -5+7 = 2$. Similarly $s_i' = 2$. If the statement was true then s_i' should be $s_i + k$ which is $5+7=12$. But $s_i' = 2$ so the statement is not true for <u>every</u> invertible A matrix so $\boxed{\text{False}}$.

• The right singular vectors of A are orthogonal to $A(\text{Null}(A))$
→ The right singular vectors are coming from $V\Sigma^2V^T$ and specifically V is the right singular vector. So a simplified version of the statement is: The right eigenvector V of A doesn't belong to the nullspace of A or $\ker(A)$. If it did then we would have $A \cdot V = 0$. But $A \cdot V$ since V is A's eigenvector must be a linear transformation of A's columns, not 0. So, we should have $V=0$ which is not possible since it would not be a valid eigenvector. So the statement is $\boxed{\text{True}}$.