Set 3

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Exercise 2

Nearest vector with a given average. Let a be an n-vector and β a scalar. How would you find the n-vector x that is closest to a among all vectors that have average value β? Give a formula for x and describe it in English.

We need to minimize $\|x-a\|_2^2$

$$\text{s.t. } \frac{x_1+x_2+\dots+x_n}{n} = \beta \Rightarrow \boxed{c^T \cdot \vec{x} = n \cdot \beta}$$

Also, $L(x,\lambda) = \|Ix-a\|^2 + \lambda(c^Tx - n\beta)$

- $\frac{\partial L(x,\lambda)}{\partial x} = 2I^TIx + c\lambda - 2I^Ta = 0 = 2Ix + c\lambda - 2a = 0$
- $\frac{\partial L(x,\lambda)}{\partial \lambda} = c^Tx - n\beta = 0$

Given the above we define:

$$\begin{bmatrix} 2I & c \\ c^T & 0 \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} = \begin{bmatrix} 2a \\ n\beta \end{bmatrix}$$

- $2Ix + c\lambda = 2a \Rightarrow \boxed{x = a - \frac{\lambda}{2}c}$ (1)
- $c^Tx = n\beta \overset{(1)}{\Rightarrow} c^Ta - c^T\frac{\lambda}{2}c = n\beta$

$$\Rightarrow c^Ta - \cancel{c^T}\cancel{c}\frac{\lambda}{2} = n\beta \Rightarrow c^Ta = n\beta + \frac{n\lambda}{2}$$

$$\Rightarrow \boxed{\lambda = 2\left(\frac{c^Ta}{n} - \beta\right)} \text{ (2)}$$

$$(1),(2) \Rightarrow \boxed{x = a - \left(\frac{c^Ta}{n} - \beta\right)\cdot c}$$

Exercise 3

Smoothest force sequence to move a mass. We consider the same setup as the example given on page 343, where the 10-vector f represents a sequence of forces applied to a unit mass over 10 1-second intervals. As in the example, we wish to find a force sequence f that achieves zero final velocity and final position one. In the example on page 343, we chose the smallest f, as measured by its norm (squared). Here though, we want the smoothest force sequence i.e. the one that minimizes:

$$f_1^2 + (f_2 - f_1)^2 + \ldots + (f_{10} - f_9)^2 + f_{10}^2$$

(This is the sum of the squares of the differences assuming that $f_0 = 0$ and $f_{11} = 0$). Explain how to find this force sequence. Plot it, and give a brief comparison with the force sequence found in the example on page 343

The constraints are identical to the ones of the book exercise.
We have : $\boxed{Cf = d}$

- $u^f = f_1 + f_2 + \ldots + f_{10}$ ①
- $p^f = \frac{19}{2} f_1 + \frac{17}{2} f_2 + \ldots + \frac{1}{2} f_{10}$ ②

Also, $C = \begin{bmatrix} 1 & 1 & \ldots & 1 & 1 \\ \frac{19}{2} & \frac{17}{2} & \ldots & \frac{3}{2} & \frac{1}{2} \end{bmatrix}$, $d = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

which are generated from ①, ②

And we get:

$$\min\left(f_1^2+(f_2-f_1)^2+\dots+(f_{10}-f_9)^2+f_{10}^2\right) \qquad \left(\|Af\|^2\right)$$

$$\begin{array}{cccccccccc}
f_1 & f_2 & f_3 & f_4 & f_5 & f_6 & f_7 & f_8 & f_9 & f_{10} \\
\end{array}$$
$$\begin{bmatrix}
1 & 0 & & & & & & & & \\
-1 & 1 & 0 & & & & & & & \\
 & -1 & 1 & 0 & & & & & & \\
 & & -1 & 1 & 0 & & & & & \\
 & & & -1 & 1 & 0 & & & & \\
 & & & & -1 & 1 & 0 & & & \\
 & & & & & -1 & 1 & 0 & & \\
 & & & & & & -1 & 1 & 0 & \\
 & & & & & & & -1 & 1 & 0 \\
 & & & & & & & & -1 & 1 \\
 & & & & & & & & & 1
\end{bmatrix}$$

$$\min\left(\|Af\|^2\right)$$

$$\text{s.t. } Cf = d$$

Finally, $L(f,\lambda) = \|Af-b\|^2 + \lambda_1(c_1^T f - d_1) + \dots + \lambda_f(c_n^T f - d_n)$

$$\begin{bmatrix} 2A^TA & C^T \\ C & 0 \end{bmatrix} \begin{bmatrix} f \\ \lambda \end{bmatrix} = \begin{bmatrix} 2A^Tb \\ d \end{bmatrix}$$