Simulation

Assignment 3

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① Calculate: $\int_1^{\infty} e^{\frac{-x^2}{4}} \cdot \sin(2\pi x)\,dx$

We need to find an f and a φ function where f is a PDF in $(-\infty, 1)$. We will set f to be the Rayleigh distribution

Rayleigh Distribution: $f(x;\sigma) = \frac{x}{\sigma^2} \cdot e^{\frac{-x^2}{2\sigma^2}}$

Since we need the exponent of e to be "$\frac{-x^2}{4}$" we will set $\sigma = \sqrt{2}$.

Now that we have found a fitting distribution for the integral calculation, we need to normalize it. The normalization factor is equal to $e^{1/4}$

So our f is: $\boxed{f(x) = e^{\frac{-x^2}{4}} \cdot \frac{x}{2} \cdot e^{1/4}}$

and this means our $\boxed{\varphi(x) = \sin(2\pi x) \cdot \frac{2}{x} \cdot e^{-1/4}}$

Algorithm

1. Find a fitting f
2. Normalize it
3. Find φ
4. Get a sample from f
5. Calculate $\hat{\theta}$ and the variance

② Calculate $\int_0^{\infty} x^2 \cdot e^{-\frac{x^2}{2}} \cdot \sin(\pi x)\, dx$

For this problem I have 2 solutions:

Method 1

Using again the Rayleigh distribution, we set $\sigma = 1$ and we get:

$$\boxed{f(x) = x \cdot e^{-\frac{x^2}{2}}}$$

This is already integrating to 1 and fits our problem so there is no need for normalization this time.
Our $\varphi(x)$:

$$\boxed{\varphi(x) = \sin(\pi \cdot x) \cdot x}$$

Note: The "issue" with this method is that if we look at the plot (which I included in a file), $\varphi(x)$ doesn't converge to the initial fuction and we get ~~the~~ a good $\hat{\theta}$ because the integrals of $\sin(\pi \cdot x)$ cancel each other while oscillating to infinity around 0. I wasn't sure if this is entirely correct or if its just a trick for $\hat{\theta}$, so I implemented one more method.

Method 2

On this method we choose the Gamma distribution to be the PDF.

Gamma Distribution: $f(x) = \frac{1}{\Gamma(k) \cdot \theta^k} \cdot x^{(k-1)} \cdot e^{-\frac{x}{\theta}}$

We set $k=3$ and we get:

$$\boxed{f(x) = \frac{x^2 \cdot e^{-x}}{2}}$$ (Which also doesn't need normalization)

And, $$\boxed{\varphi(x) = 2 \cdot e^{\left(\frac{-x^2+2x}{2}\right)} \cdot \sin(\pi x)}$$

$\boxed{\text{Note}}$: Both methods are great but the first one is better

The algorithm is identical as in ①.

③ Calculate $\int_{-1}^{1} \sqrt{1-x^2}\, dx$

For this problem I will first alter the range $(-1,1)$.

$$\int_{-1}^{1} \sqrt{1-x^2}\, dx = \int_{-1}^{0} \sqrt{1-x^2}\, dx + \int_{0}^{1} \sqrt{1-x^2}\, dx \quad (1)$$

But they are symmetrical so we can say:

$$(1) \Rightarrow 2 \cdot \int_0^1 \sqrt{1-x^2}\, dx$$

So for this altered version we can use $\boxed{f(x) = 1}$ which has an integral equal to 1 in $(0,1)$ and is a proper PDF.

So, $\boxed{\varphi(x) = \sqrt{1-x^2}}$

In Python, we get N samples from the uniform distribution in $(0,1)$ and then we create N extra samples that are identical to the previous N but with an opposite sign. The $\hat{\theta}$ we get will not require any extra work. For the variance though, we calculate the variance just for $(0,1)$ and then we have to multiply it by 2 since the values over the twice as big interval have a bigger distance from the mean.

The algorithm is identical as in ①.