Simulation

Assignment 4

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① We will prove that θ_1, θ_2 are unbiased.

$\boxed{\text{For } \theta_1}$

$$E[\theta_1] = E\left[\frac{1}{n}\cdot\sum_{i=1}^{n}\varphi(y_i\cdot\sigma)\right] = \frac{1}{\cancel{n}}\cdot\cancel{n}\cdot E[\varphi(\sigma y_i)] =$$

$$= \int \varphi(\sigma\cdot y)\cdot f(y)\,dy = \int \varphi(\sigma y)\cdot\frac{e^{\frac{-y^2}{2}}}{\sqrt{2\pi}}\,dy$$

Set $x = y\cdot\sigma \Rightarrow dx = \sigma\cdot dy \Rightarrow dy = \frac{dx}{\sigma}$

And we get, $\int \varphi(x)\cdot e^{\frac{-x^2}{2\sigma^2}}\cdot\frac{1}{\sigma\sqrt{2\pi}}\,dx = E[\varphi(x)] = \mu$ ✓

$\boxed{\text{For } \theta_2}$

$$E[\theta_2] = E\left[\frac{1}{n\sigma}\cdot\sum_{i=1}^{n}\left(e^{-y^2\left(\frac{1}{2\sigma^2}-\frac{1}{2}\right)}\cdot\varphi(y)\right)\right] =$$

$$= \frac{1}{\cancel{n}\sigma}\cdot\cancel{n}\cdot E\left[e^{-y^2\cdot\left(\frac{1}{2\sigma^2}-\frac{1}{2}\right)}\cdot\varphi(y)\right] =$$

$$= \frac{1}{\sigma}\cdot\int e^{-y^2\left(\frac{1}{2\sigma^2}-\frac{1}{2}\right)}\cdot\varphi(y)\cdot e^{\frac{-y^2}{2}}\cdot\frac{1}{\sqrt{2\pi}}\,dy =$$

$$= \frac{1}{\sigma}\cdot\frac{1}{\sqrt{2\pi}}\cdot\int e^{\frac{-y^2}{2\sigma^2}+\frac{y^2}{2}}\cdot\varphi(y)\cdot e^{\frac{-y^2}{2}}\,dy = \frac{1}{\sigma}\cdot\frac{1}{\sqrt{2\pi}}\cdot\int e^{\frac{-y^2}{2\sigma^2}}\cdot\varphi(y)\,dy$$

$$= E[\varphi(x)] = \mu$$ ✓

Now that we proved that both estimators are in fact unbiased, we will calculate the variances.

$\boxed{\text{For } \Theta_1}$

$$V(\Theta_1) = V\left(\frac{1}{n}\,\varphi(\sigma Y_i)\right) = \frac{1}{n}\cdot V(\varphi(\sigma Y_i)) =$$

$$= \frac{1}{n}\cdot\int |\varphi(\sigma y) - \mu|^2 \cdot e^{-\frac{y^2}{2}}\cdot\frac{1}{\sqrt{2\pi}}\,dy \Rightarrow$$

$\boxed{\text{Set } x = \sigma y}$ $\Rightarrow$ $$\boxed{V(\Theta_1) = \frac{1}{n}\int |\varphi(x) - \mu|^2 \cdot e^{-\frac{x^2}{2\sigma^2}}\cdot\frac{1}{\sigma\cdot\sqrt{2\pi}}}$$

$\boxed{\text{For } \Theta_2}$

$$V(\Theta_2) = V\left(\frac{1}{n\sigma}\cdot e^{-y^2\left(\frac{1}{2\sigma^2}-\frac{1}{2}\right)}\cdot\varphi(Y)\right) =$$

$$= \frac{1}{n}\cdot\int \left|e^{-y^2\left(\frac{1}{2\sigma^2}-\frac{1}{2}\right)}\cdot\varphi(Y) - \mu\right|^2\cdot e^{-\frac{y^2}{2}}\cdot\frac{1}{\sqrt{2\pi}}\,dy =$$

$$= \frac{1}{n}\cdot\int \left|\frac{e^{-y^2\left(\frac{1}{2\sigma^2}-\frac{1}{2}\right)}}{\sigma}\cdot\varphi(Y) - \mu\right|^2\cdot e^{-y^2/2}\cdot\frac{1}{\sqrt{2\pi}}\,dy$$

$dx = dy \Rightarrow$ $$\boxed{V(\Theta_2) = \frac{1}{n}\cdot\int \left|\frac{1}{\sigma}\cdot e^{-x^2\left(\frac{1}{2\sigma^2}-\frac{1}{2}\right)}\cdot\varphi(x) - \mu\right|^2\cdot e^{-x^2/2}\cdot\frac{1}{\sqrt{2\pi}}\,dx}$$

(2) In order to prove for which values of "σ" $V(\Theta_2) \to \infty$ we need to calculate the $\boxed{\lim_{\sigma\to 0} V(\Theta_2)}$ and $\boxed{\lim_{\sigma\to +\infty} V(\Theta_2)}$ since $\sigma \in [0, +\infty)$.

First we calculate $\lim_{\sigma\to 0} V(\Theta_2)$, but we don't need to keep the entire thing for our calculations.

$$\lim_{\sigma\to 0}\left(\left|\frac{1}{\sigma}\cdot e^{-x^2\left(\frac{1}{2\sigma^2}-\frac{1}{2}\right)}\cdot\varphi(x) - \mu\right|^2\cdot e^{-\frac{x^2}{2}}\cdot\frac{1}{\sqrt{2\pi}}\right) \quad (*)$$

We will calculate a simplified version that will also do the job for us.

(1) $\lim_{\sigma \to 0}\left(\int \frac{1}{\sigma} \cdot e^{-\frac{x^2}{2\sigma^2}}\right)$

Using the dominated convergence theorem:

$$\int \lim_{\sigma \to 0}\left(\frac{1}{\sigma} \cdot e^{-\frac{x^2}{2\sigma^2}}\right) = \int \lim_{\sigma \to 0}\left(\frac{1}{\sigma \cdot e^{x^2/2\sigma^2}}\right) \text{ L'Hospital}$$

$$= \int\left(\lim_{\sigma \to 0}\left(\frac{1}{e^{x^2/2\sigma^2}}\right) \to 0 + \lim_{\sigma \to 0}\left(\frac{-\sigma^3}{e^{x^2/2\sigma^2}}\right) \to 0\right)$$

$$= \int 0 = C$$

So for $\sigma \to 0$, $V(\theta_2)$ is finite. We move on with $\lim_{\sigma \to +\infty} V(\theta_2)$.

$$\lim_{\sigma \to +\infty} V(\theta_2) = \lim_{\sigma \to +\infty}\left(\int \frac{1}{\sigma} \cdot e^{-\frac{x^2}{2\sigma^2}}\right) = \int\left(\lim_{\sigma \to +\infty} \frac{1}{\sigma} \cdot e^{-\frac{x^2}{2\sigma^2}}\right) =$$

$$= \lim_{\sigma \to +\infty}\left(\frac{1}{\sigma}\right) \to 0 \cdot \lim_{\sigma \to +\infty}\left(e^{-\frac{x^2}{2\sigma^2}}\right) \to 0 = \int 0 = C$$

$V(\theta_2)$ seems to be finite in this case as well, but since "σ" will be constant on the actual calculations and will have a finite value, when x^2 becomes larger this limit will go to infinity. Therefore, for very large values of $x \gg \sigma$, $V(\theta_2)$ will not be finite.

③ Let's compare $V(\Theta_1)$ and $V(\Theta_2)$:

$$\begin{cases} \frac{1}{n} \cdot \int |\varphi(x) - \mu|^2 \cdot e^{\frac{-x^2}{2\sigma^2}} \cdot \frac{1}{\sigma\sqrt{2\pi}} \\ \frac{1}{n} \cdot \int \left| \frac{1}{\sigma} e^{-x^2\left(\frac{1}{2\sigma^2} - \frac{1}{2}\right)} \cdot \varphi(x) - \mu \right|^2 \cdot e^{-x^2/2} \cdot \frac{1}{\sqrt{2\pi}} \, dx \end{cases}$$

$$\begin{cases} \int |\varphi(x) - \mu|^2 \cdot e^{\frac{-x^2}{2\sigma^2}} \\ \int \left| \frac{1}{\sigma} e^{-x^2\left(\frac{1}{2\sigma^2} - \frac{1}{2}\right)} \cdot \varphi(x) - \mu \right|^2 \cdot e^{-\frac{x^2}{2}} \end{cases}$$

$$\xRightarrow{I[x>1]} \begin{cases} \int | I[x>1] - \mu |^2 \cdot e^{\frac{-x^2}{2\sigma^2}} \\ \int \left| \frac{1}{\sigma} \cdot e^{-x^2\left(\frac{1}{2\sigma^2} - \frac{1}{2}\right)} \cdot I[x>1] - \mu \right|^2 \cdot e^{-\frac{x^2}{2}} \end{cases}$$

- $I[x>1] - \mu$: This number will be the smallest possible when $I[x>1] \sim \mu$. In that case,
- $\frac{1}{\sigma} \cdot e^{-x^2 \cdot \left(\frac{1}{2\sigma^2} - \frac{1}{2}\right)}$ needs to be as close to 1 as possible in order for the "error" to be increased to the minimum possible degree when it multiplies $I[x>1]$. The "error" will be without a doubt larger in Θ_2 but outside the absolute, Θ_1 is multiplied by $e^{\frac{-x^2}{2\sigma^2}}$ which is larger than $e^{-\frac{x^2}{2}}$ when $\sigma << 1$

The real question that we end up to is: For the smallest possible values of σ, is $\boxed{\frac{1}{\sigma} e^{-x^2\left(\frac{1}{2\sigma^2} - \frac{1}{2}\right)}}$ or $\boxed{e^{\frac{-x^2}{2\sigma^2}}}$ more dominant?

As we will also see in ④ which has $\varphi(x) = I[x>1]$ as σ decreases $V(\Theta_2)$ becomes smaller than $V(\Theta_1)$ which actively proves that $e^{-x^2/2\sigma^2}$ is more influential and therefore causes $V(\Theta_1)$ to converge slower than $V(\Theta_2)$.

So to sum up, yes there are values for "σ" for which $V(\Theta_2) < V(\Theta_1)$.

(4) For this part we set $\phi(x) = I[X>1]$ and proceed on calculating θ_1, θ_2. The function has "σ" as an attribute so it is easy to run experiments. The code also calculates the variances and finally performs a quick t-test that validates the fact that θ_1, θ_2 are indeed valid estimators for μ and are also unbiased.

- When we set $\sigma << 1$ we can see the "proof" of (3)
- For $\sigma >> 1$ like we explained in (2) $V(\theta_2)$ loses its precision.