Simulation

Assignment 1

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(1) i) $f(x) = \begin{cases} \frac{4}{\pi^2}x, & 0 \leq x \leq \pi/2 \\ \frac{4}{\pi} - \frac{4}{\pi^2}x, & \pi/2 \leq x \leq \pi \\ 0, & \text{other} \end{cases}$

We proceed to calculate the CDF:

- $\int_0^x \frac{4}{\pi^2} \cdot t \, dt = \left[\frac{2t^2}{\pi^2} \right]_0^x = \frac{2x^2}{\pi^2}$

- $\int_0^{\pi/2} \frac{4}{\pi^2} \cdot t \, dt + \int_{\pi/2}^x \frac{4}{\pi} - \frac{4}{\pi^2} t \, dt = \left[\frac{2t^2}{\pi^2} \right]_0^{\pi/2} + \left[\frac{4t}{\pi} - \frac{2t^2}{\pi^2} \right]_{\pi/2}^x =$

$= \frac{2 \cdot (\frac{\pi}{2})^2}{\pi^2} + \frac{4x}{\pi} - \frac{2x^2}{\pi^2} - \frac{\cancel{4}}{\cancel{\pi}} \cdot \frac{\cancel{\pi}}{\cancel{2}} + \frac{2(\frac{\pi}{2})^2}{\cancel{\pi^2}} = \frac{1}{2} + \frac{4x}{\pi} - \frac{2x^2}{\pi^2} - \frac{3}{2}$

$= \frac{4x}{\pi} - \frac{2x^2}{\pi^2} - 1$

So, $F(x) = \begin{cases} \frac{2x^2}{\pi^2}, & 0 \leq x \leq \pi/2 \\ \frac{4x}{\pi} - \frac{2x^2}{\pi^2} - 1, & \pi/2 \leq x \leq \pi \end{cases}$

Moving on, we will calculate the inverse:

- $U = \frac{2x^2}{\pi^2} \Rightarrow X = \sqrt{\frac{U}{2}} \cdot \pi$

- $U = \frac{4x}{\pi} - \frac{2x^2}{\pi^2} - 1 \Rightarrow \pi^2 \cdot U = 4x \cdot \pi - 2x^2 - \pi^2 \Rightarrow \boxed{2x^2 - 4\pi x + \pi^2(U+1) = 0}$

$X_{1,2} = \frac{4\pi \pm \sqrt{16\pi^2 - 8U\pi^2 - 8\pi^2}}{4} = \frac{4\pi \pm \sqrt{8\pi^2 - 8U\pi^2}}{4} = \pi \left(\frac{2 \pm \sqrt{2 - 2U}}{2} \right)^{(*)}$

(*) Keep in mind that since F^{-1} is "1-1" in $[\frac{\pi}{2}, \pi]$ we must accept exactly one X. Also, the set of values for F(x) is [1/2, 1] which means that u will have values that will always lie in $[1/2, 1]$. This is very convenient for this particular case since F^{-1} in $[0, \pi/2]$ has a set of values equal to $[0, 1/2]$ which means that we can directly "feed" our to-be-generated "u" directly to these formulas.

So the algorithm will be:

repeat 1000 times $\Big\{$
1. Generate $U \sim U(0,1)$
2. If $u > 1/2$ then set $X = \dfrac{\pi(2 \pm \sqrt{2-2U})}{2}$ (only on of the two will be valid)
   else:
   set $X = \sqrt{\frac{U}{2}} \cdot \pi$

ii) We begin by calculating the supremum for each interval:

- $\lim\limits_{x \to \frac{\pi}{2}^-} \left(\dfrac{f(x)}{g(x)}\right) = \lim\limits_{x \to \frac{\pi}{2}^-} \left(\dfrac{\frac{4 \cdot x}{\pi^2}}{\frac{\sin x}{2}}\right) = \lim\limits_{x \to \frac{\pi}{2}^-} \left(\dfrac{8x}{\pi^2 \cdot \sin x}\right) = \dfrac{8 \cdot \frac{\pi}{2}}{\pi^2 \cdot \sin(\frac{\pi}{2})} = \dfrac{4}{\pi}$

Since f is an increasing function in $[0, \frac{\pi}{2}]$ this is sufficient

- $\lim\limits_{x \to \frac{\pi}{2}^+} \left(\dfrac{f(x)}{g(x)}\right) = \lim\limits_{x \to \frac{\pi}{2}^+} \left(\dfrac{\frac{4}{\pi} - \frac{4x}{\pi^2}}{\frac{\sin x}{2}}\right) = \lim\limits_{x \to \frac{\pi}{2}^+} \left(\dfrac{8\pi - 8x}{\sin x \cdot \pi^2}\right) = \dfrac{4}{\pi}$

Similarly, since f is a decreasing function in $[\frac{\pi}{2}, \pi]$ this is sufficient

So now we have the supremum which means we obtained M (since $f(x) \leq M g(x)$)

Moving on, we will get a sample from g with the method of inversion.

- $\displaystyle\int_0^x \frac{\sin t}{2} dt = \left[-\frac{\cos t}{2}\right]_0^x = -\frac{\cos x}{2} + \frac{1}{2}$

$u = -\dfrac{\cos x}{2} + \dfrac{1}{2} \Rightarrow 2U = -\cos x + 1 \Rightarrow \boxed{X = \arccos(1 - 2U)}$

repeat $\frac{1}{M} = \frac{\pi}{4}$ times

So the algorithm will be:

1. Generate a Y from $g(y)$ (using $G^{-1}(x) = \arccos(1-2x)$ prev. calculated
2. Generate $U \sim U(0,1) = u$
3. Generate $U_2 \sim U(0,1)$ (it will be the deciding variable to choose which branch will be used, its more correct to have a separate variable for this)

If $U_2 > 1/2$ then:

If $u \leq \frac{f(y)}{g(y) \cdot \frac{4}{\pi}}$ then accept $\left(\text{where } f(y) = -\frac{4 \cdot y}{\pi^2} + \frac{4}{\pi}\right)$

Else go back to (1)

If $U_2 \leq 1/2$ then:

If $u \leq \frac{f(y)}{g(y) \cdot \frac{4}{\pi}}$ then accept $\left(\text{where } f(y) = \frac{4y}{\pi^2}\right)$

Else go back to (1)

(2) i) $h(x) \propto \frac{4}{\pi^2} \cdot x \quad , 0 \leq x \leq \pi/2$

We set $h^{*}(x) = \frac{4}{\pi^2} \cdot x$ and we proceed to find the inverse H^{-1}. The method will be identical with (1)(i) with the only difference being the fact that in (ii) we will not be able to predict the expectation of the number of trials since c is unknown $\left(h(x) = c \cdot \frac{4 \cdot x}{\pi^2}\right)$

- $\int_0^x \frac{4 \cdot t}{\pi^2} \, dt \underset{\text{as before}}{\overset{\text{same}}{=}} \frac{2x^2}{\pi^2}$

$$H^{*}(x) = \frac{2x^2}{\pi^2} \quad , 0 \leq x \leq \pi/2$$

$$u = \frac{2x^2}{\pi^2} \Rightarrow x = \sqrt{\frac{u}{2}} \cdot \pi$$

Before continuing on writing down the algorithm let's think forward. We are going to have to ~~[illegible]~~

~~[illegible]~~ get a sample from f but we have only the part that lies in $[0, \pi/2]$ (since $h(x) \equiv f(x)$, where $x \in [0, \frac{\pi}{2}]$). So, we will take advantage of the fact that f is symmetrical, and set $(1-U)$ when we are in the case $u > 1/2$. Also we are going to subtract the result from π since ~~[illegible]~~ $H^{-1}(x) = \sqrt{\frac{U}{2}} \cdot \pi$ can't overcome $\frac{\pi}{2}$ and will throw all the values exactly on the interval that we want them.

So finally:

repeat 1000 times

1. Generate $U \sim U(0,1)$
2. If $U \leq 1/2$ then set $x = \sqrt{\frac{U}{2}} \cdot \pi$
   else:
   set $x = \pi - \sqrt{\frac{(1-U)}{2}} \cdot \pi$

ii) Like before we will begin by computing the supremum but first we will have to alter the envelope function. Firstly, we fix accordingly the interval of x. So now $g(x) = \frac{\sin x}{2}$ takes values from $x \in [0, \frac{\pi}{2}]$. But, an envelope function must also have area $= 1$ so we will remove $\frac{1}{2}$ from $\sin x$.

The final $g(x)$ is:

$$g(x) = \sin x, \quad x \in [0, \tfrac{\pi}{2}]$$

$$\lim_{x \to \frac{\pi}{2}^-} \left(\frac{f(x)}{g(x)} \right) = \lim_{x \to \frac{\pi}{2}} \left(\frac{\frac{4x}{\pi^2}}{\sin x} \right) = \left(\frac{\frac{4}{\pi}}{2} \right) = \frac{2}{\pi}$$

Now we will get the sample from g:

- $\int_0^x \sin t \, dt = [-\cos t]_0^x = -\cos x + 1$

- $u = -\cos x + 1 \Rightarrow \boxed{x = \arccos(1-U)}$

Moreover, in order to perform the check $u_1 < \frac{f(x)}{M \cdot g(x)}$ for the branch in $[\frac{\pi}{2}, \pi]$ we will have to use again the attribute of symmetry. Before setting Y equal to $g(y)$ we wait the outcome of u_2. If $u_2 \geq 1/2$ then we ~~reverse~~ set $(1-u)$ and again subtract it from π.

The final algorithm:

repeat $1000 \cdot \left(\frac{1}{M} = \frac{\pi}{2}\right)$ times

1. Generate a Y from $g(y)$ (using $G^{-1}(x) = \arccos(1-x)$)
2. Generate $U \sim U(0,1) = u$
3. Generate $U_2 \sim U(0,1)$

   If $U_2 \geq \frac{1}{2}$ then ~~accept~~ set $Y = \pi - \arccos(\cancel{1-}\,y)$

   If $u \leq \frac{h(y)}{M \cdot g(y)}$ then accept

   Else go back to 1

   If $U_2 < \frac{1}{2}$ then set $Y = \arccos(1-y)$

   If $u \leq \frac{h(y)}{M \cdot g(y)}$ then accept

   Else go back to 1