Simulation

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Assignment 5

Assume that a random variable $\Lambda$ follows the logarithmic normal distribution with parameters $\mu, \sigma$, $\Lambda \sim LN(\mu, \sigma)$ with:

$$p_\Lambda(\lambda; \mu, \sigma) = \frac{1}{\lambda\sqrt{2\pi\sigma^2}} \exp\left(-(\log(\lambda)-\mu)^2/2\sigma^2\right)$$

Given $\Lambda = \lambda$, a random variable $N$ follows the Poisson distribution with parameter $\lambda$, $N \sim Poisson(\lambda)$. Assume that we observe $N = n$. An estimator of $(\mu, \sigma)$ is given by maximizing the likelyhood function $L(\mu, \sigma; n) = \mathbb{P}(N = n \mid \mu, \sigma)$.

① Prove that $L(\mu, \sigma; n) \propto E_\Lambda(\Lambda^n \exp(-\Lambda) \mid \mu, \sigma)$

② Define an algorithm that takes 1000 values from the normal distribution $N(\mu, \sigma^2)$ with known $\mu, \sigma$. The algorithm will convert them to 1000 values from $\Lambda \sim LN(\mu, \sigma)$ and using these values it creates an estimator of $L(\mu, \sigma; n)$.

⑤ Experiment with different variance reduction methods

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① We have $L(\mu, \sigma; n) = P(N = n \mid \mu, \sigma)$, $N \sim Poisson(\lambda)$ with $\Lambda = \lambda$ so we get:

$$\mathbb{P}(N = n \mid \mu, \sigma) = \frac{\lambda^n \exp(-\lambda)}{n!} = \frac{\Lambda^n \exp(-\Lambda)}{n!}$$

Moreover, we have: $L(\mu, \sigma; n) \propto \Lambda^n \exp(-\Lambda) \mid \mu, \sigma$ ④

Since $n$ is considered a constant and therefore we can omit it.

Finally, since the likelihood function $L(\mu, \sigma; n)$ is proportional to the probability density function of Poisson($\lambda$), we can deduce that it will be proportional to the expected value of Poisson($\lambda$)

This means that (1) $\Rightarrow$

$$P(N=n \mid \mu, \sigma) = \frac{\lambda^n \exp(-\lambda)}{n!} = E_\lambda\left(\frac{\lambda^n \exp(-\lambda)}{n!}\right)$$

(2) The algorithm will have the following steps:

(1) We get 1000 samples from the Normal distribution and we set $\mu=0$ and $\sigma=1$.

(2) We convert these samples to the Logarithmic Normal distribution using $Y \sim e^X$ where $x$ denotes the samples that we got from the normal distribution. This stands, because log-normal distribution means the logarithm of the random variable is normally distributed, so in our case, if $X$ has a normal distribution then $Y=e^X$ is log-normally distributed.

(Optional) (3) Prove that the samples are indeed following the lognormal distribution.

(4) Calculate the mu, sigma of $Y \sim LN$ since we expect them to be different from $\mu=0, \sigma=1$ set for the Normal distribution.

(5) Set $LN \sim (mu, sigma)$ as $f$ and $x^n \cdot e^{-x}$ as $\varphi$ and simulate the integral $\int_0^{+\infty} x^n \cdot e^{-x} \cdot \frac{1}{x\, sigma \sqrt{2\pi}} \cdot e^{-\frac{(\log x - mu)^2}{2}} dx$

(6) Confirm that the mean is correct and calculate the variance of $\hat{\theta}$

(5) For this part we just have to follow the theory. We need to find an $h(x) \simeq \varphi(x)$ where $E[h(x)]$ is known. If we choose $h(x) = \text{Gamma}(\theta=1, k=n)$ then we will get:

$$\boxed{h(x) = \frac{x^n \cdot e^{-x}}{(n+1)!}}$$

We also know that the mean of this distribution is x and the plot shows that there is indeed a close relation between them. We get:

$$\hat{\theta}_B = \frac{1}{n}\left[\sum_{i=1}^{n}\varphi(x_i) + \beta\left(\frac{1}{n}\sum_{i=1}^{n}\frac{x_i^n \cdot e^{-x_i}}{(n+1)!} - x_i\right)\right.$$

and we need to minimize:

$$\left(\frac{\sum_{i=1}^{N}\left(x^n \cdot e^{-x} - \beta\left(\frac{x^n e^{-x}}{(n+1)!} - x\right)\right)}{N}\right)$$

The way that I chose $\beta$ was to run the above algorithm for $i=10$ samples and minimize the function in terms of $\beta$ each time. Afterwards, I set $\beta$ equal to the mean value of the generated b's.

After calculating the above sum, we divide by $n$ where $n$ is the number of samples (1000) and return the variance.

Note → The algorithm achieves a quite good variance without this method for small values of $n$, but as $n$ gets bigger the variance skyrockets and we can really see the benefit of this method. To be precise, for $n = 1, 2, 3$ the initial variance is less than the variance using control variates for $n=4$ there seems to be a balance and for $n \geq 5$ this method is by far the best.