Platon
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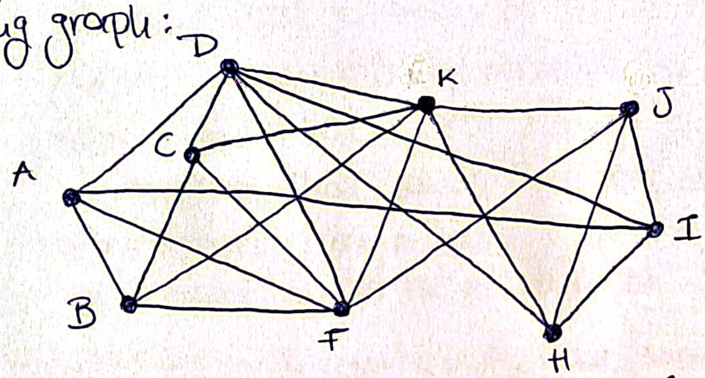
P3180068

1 Give an example of a simple incomplete Eulerian graph G that has its radius equal to its diameter and every vertex in G has a degree bigger than 2.

We need to construct a graph that follows the constraints set by the exercise, but first we must decode these constraints.

- The first one is stated clearly, every vertex must have a degree > 2.
- The second one, comes from the fact that the graph is Eulerian. Since its Eulerian, every vertex must have an <u>even</u> degree.
- Finally, we can deduce from the fact that the radius and the diameter of G are equal, that the minimum eccentricity in G must be equal to its maximum eccentricity.

Finally, we combine all the above constraints and we get the following graph:

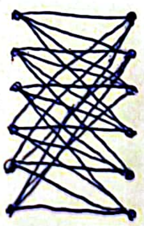


→ A is even with degree 4 and $\min\{e(A) \mid A \in V(G)\} = \max\{e(A) \mid A \in V(G)\} = 2$
→ B is even with degree 4 and $\min\{e(B) \mid B \in V(G)\} = \max\{e(B) \mid B \in V(G)\} = 2$
→ C is even with degree 4 and $\min\{e(C) \mid C \in V(G)\} = \max\{e(C) \mid C \in V(G)\} = 2$
→ D is even with degree 6 and $\min\{e(D) \mid D \in V(G)\} = \max\{e(D) \mid D \in V(G)\} = 2$
→ K is even with degree 6 and $\min\{e(K) \mid K \in V(G)\} = \max\{e(K) \mid K \in V(G)\} = 2$
→ F is even with degree 6 and $\min\{e(F) \mid F \in V(G)\} = \max\{e(F) \mid F \in V(G)\} = 2$
→ J is even with degree 4 and $\min\{e(J) \mid J \in V(G)\} = \max\{e(J) \mid J \in V(G)\} = 2$
→ H is even with degree 4 and $\min\{e(H) \mid H \in V(G)\} = \max\{e(H) \mid H \in V(G)\} = 2$
→ I is even with degree 4 and $\min\{e(I) \mid I \in V(G)\} = \max\{e(I) \mid I \in V(G)\} = 2$

So the graph follows exactly the rules set by the exercise and therefore it is a valid example.

Note

An additional example that fits in a specific graph family and that would fit in this case as well, is a matching with 6 vertices in each set. The eccentricities will be equal to 2 and each vertex will have an even degree equal to 4.



[2] Given a simple graph with $|V(G)| \geq 2$ and $\delta(G) \geq \frac{|V(G)|-1}{2}$ prove that $diam(G) \leq 2$.

Assume that we have the given graph G with $|V(G)| \geq 2$ and $\delta(G) \geq \frac{|V(G)|-1}{2}$ but the diameter is bigger than 2. Let's also assume that $diam(G) = 3$.

We must think which edges should be "forbidden" in the graph, because they would break the hypothesis we made, that $diam(G)=3$.

Let's denote V_1 as the vertex that starts the path with diameter 3 and V_k the other endpoint of the path.

We demand that:

- There must not be an edge e_1 connecting V_1, V_k.
- We know that V_1 must be the endpoint of at least $\frac{|V(G)|-1}{2}$ edges. For simplicity assume $m = \frac{|V(G)|-1}{2}$. We demand that there should not exist an edge, connecting any of these m vertices, (that constitute the other endpoint of the corresponding m edges) to V_k. If at least one of those existed then there would be a path of length 2.

We notice that both constraints have something in common, they restrict the available edges for V_k. The total vertices available for connection are $\underline{n-1}$ where $n \equiv V(G)$ and we also assumed that every vertex in G has at least $\frac{n-1}{2}$ degree. The total constraints "forbid" $m+1$ vertices.

So we need:

$$(n-1)-(m+1) \geq \frac{n-1}{2} \xRightarrow{m=\frac{n-1}{2}} n-\left(\frac{n-1}{2}\right)-2 \geq \frac{n-1}{2}$$

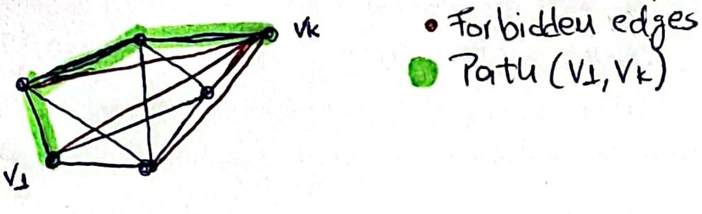
$$\Rightarrow \frac{n-1}{2}-2 \geq \frac{n-1}{2} \Rightarrow -2 \geq 0$$

This is obviously incorrect, so a diameter of length 3 is not possible since the constraints restrict the degree of V_k below $\frac{|V(G)|-1}{2}$.

If the constraints for a diameter $=3$ are too "heavy" for the given graph G then we can deduce that it is also impossible for diameter >3 since the constraints would be even stricter, restricting further the available vertices for V_k.

Therefore, we can say that the diameter is ≤ 2.

Example Graph



3 Given a simple 5-regular graph G, which is also Hamiltonian, prove that G contains an ℓ-regular spanning subgraph for every $1 \leq \ell \leq 5$.

- For $\ell = 5$

This is the simplest case, the graph G can be considered a subgraph of itself, and it is 5-regular.

- For $\ell = 1$

We know that the number of vertices is an even number from the handshake theorem. Since: $2|E(G)| = \sum_{x \in V(G)} d_G(x) = 5 \cdot x$, where x is the number of vertices in the graph.
Therefore, by connecting pairs of vertices, it is certain that we can get a 1-regular subgraph of G, as long as the vertices we connect are connected in G.

- For $\ell = 2$

This case is simple as well, all we have to do is create a Hamilton cycle using the vertices of G. It is certain that we can create a subgraph of G this way, since G is 5-regular. If we couldn't then we would violate the initial constraints of the exercise, since besides 5-regular it is also Hamiltonian. Therefore we create a 2-regular subgraph of G.

- For $\ell = 3$

In this case, all we have to do is get the initial graph G and delete the Hamilton cycle (or at least a Hamilton cycle if there are multiple). By removing the cycle, we know that the degree of each vertex will be reduced exactly by 2. Since we know that G is 5-regular, we know that the resulting graph must be 3-regular.

- For $\ell = 4$

Finally, to get a 4-regular subgraph of G, we separate the vertices of G in pairs (foreign pairs, meaning that no vertex can belong in more than one) and we remove the edge connecting them. Obviously, the pairs are chosen in a way that the edge which we delete, exists in the initial graph. The result is a 4-regular subgraph.

4 Given a simple graph G and u, v vertices $\in G$ s.t. $d_G(u) + d_G(v) \geq |V(G)|$ prove that if there is a $(u-v)$ Hamilton path in G, then G is Hamiltonian.

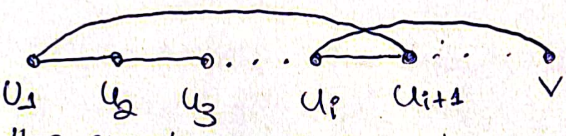
Case 1: u and v are adjacent

If u, v are adjacent, then it is obvious that G is Hamiltonian, since we have a path that starts with u and then visits every vertex, before reaching v and finally using the edge connecting them to complete the cycle.

Case 2: u and v are not adjacent

In this case we still have a path of maximum length, but the proof that we always have a cycle is not so trivial. We will use a similar approach as Newmann's proof (1958) of Dirac's theorem.

Consider G to be in this form:



We will separate the graph in 2 discrete sets:

$X = \{u_i \mid u_1 \text{ is adjacent to } u_{i+1}\}$

$Y = \{v_i \mid v_i \text{ is adjacent to } v\}$

From these sets we get:

- $|X| + |Y| = d_G(u) + d_G(v) \geq |V(G)|$ ①
- $v \notin |X \cup Y| \Rightarrow |X \cup Y| < |V(G)| - 1 \Rightarrow -|X \cup Y| \geq -(|V(G)| - 1)$ ②

Using the principle of inclusion/exclusion we have:

$|X \cap Y| = |X| + |Y| - |X \cup Y|$ ③

③ $\xRightarrow[②]{①}$ $|X \cap Y| \geq |V(G)| - (V(G| - 1) \Rightarrow \boxed{|X \cap Y| \geq 1}$

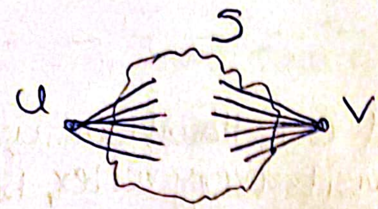
The result we get, states that if there is at least one endpoint that is common between u and v, then if we have a Hamilton path and also $d_G(u) + d_G(v) \geq |V(G)|$, there will be a Hamilton cycle in G.

Notes

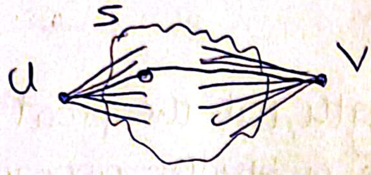
① If $|X \cap Y| = 0$, then $d_G(u) + d_G(v) < |V(G)|$.

② The same goes for $|X \cap Y| = 1$. It is not possible for the inequality to hold but this doesn't affect our solution which is valid iff the inequality is true. (View examples on the next

- $|X \cap Y| = 0$



- $|X \cap Y| = 1$



$$d_G(u) + d_G(v) \geq |V(G)|$$

$$k + l \geq k + l + 2 - x \quad (4)$$

where 2 corresponds to u, v and $k + l - x$ corresponds to the vertices in S and x contains the common endpoints between $d_G(u) = k$ and $d_G(v) = l$.

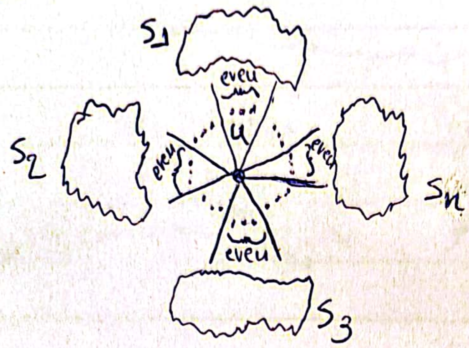
$(4) \Rightarrow \boxed{x \geq 2}$

This proves that the cases on the left ~~are~~ ~~are~~ do not concern us since they represent the reverse case where $\boxed{x < 2}$.

5 Assume that we have a graph (simple and connected) whose vertices are all of an even degree. Prove that for every vertex u of G:

$$\omega(G-\{u\}) \leq \frac{1}{2} d_G(u) \quad (1)$$

Since our graph G has all of its vertices with an even degree then we can say that it will have the following form:



An easy way to explain why this will be the general shape of G is the fact that it must contain an Euler path (since we know that all the vertices have an even degree). If u was connected with a component S_i with m edges where m is odd then we wouldn't have an Euler path in the graph.

Note

If we had an odd number of edges towards S_1 and S_2, but also S_1, S_2 where connected then we could still have an Euler path but then S_1, S_2 wouldn't be different components, it would be a single component. Hence this case is reduced to the above shape which supports our claim that its a valid representation.

To continue where we left off, now that we have proved that u is connected with an even number of edges with each component, it is easy to prove (1).

For every graph we can say:

$$\omega(G-\{u\}) \leq d_G(u)$$

Which means that by removing a vertex u from G we can't get more components than the degree of u.

In our case though, if we delete u we are guaranteed to get at most $\frac{1}{2} d_G(u)$ components, since we proved that u must be connected with each component, with at least 2 vertices (and with an even number k). If for example u had 4 edges towards each component, it would be $\omega(G-\{u\}) \leq \frac{1}{4} d_G(u)$, for 8 edges $\omega(G-\{u\}) \leq \frac{1}{8} d_G(u)$ etc.

So we have proved that if we delete u then:

$$\omega(G-\{u\}) \leq \frac{1}{2} d_G(u)$$, for any random $u \in V(G)$