SimCalc Math Reference Sheet

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Introduction

This document is intended as a reference of the calculations done in the SimCalc application, so I can get rid of the giant stack of papers sitting on my desk.

Terminology

CTC lever: The most common type of actuator for a budget motion simulator. Consists of a lever arm attached to a high torque gearmotor

DOF: Degrees-of-freedom, describes the number of independant axes of motion of a simulator. Typical budget simulators are 2-DOF, meaning that they can move in two independant directions

Pitch: A rotation in the YZ plane

Roll: A rotation in the XZ plane

Shoulder Mount: A typical style of simulator construction, where the actuators connect to the moving platform near the shoulders of the occupant

DIY: Do-it-yourself

Universal Joint (Cardan Joint): A mechanical connection which restricta all movement except for rotation along two axes.

CTC levers

Below is a picture of a typical 2DOF shoulder mount simulator. The seat sits on top of a universal joint to allow for pitch and roll rotation, and two rods with ball joints on either end join the actuators to the seat frame.

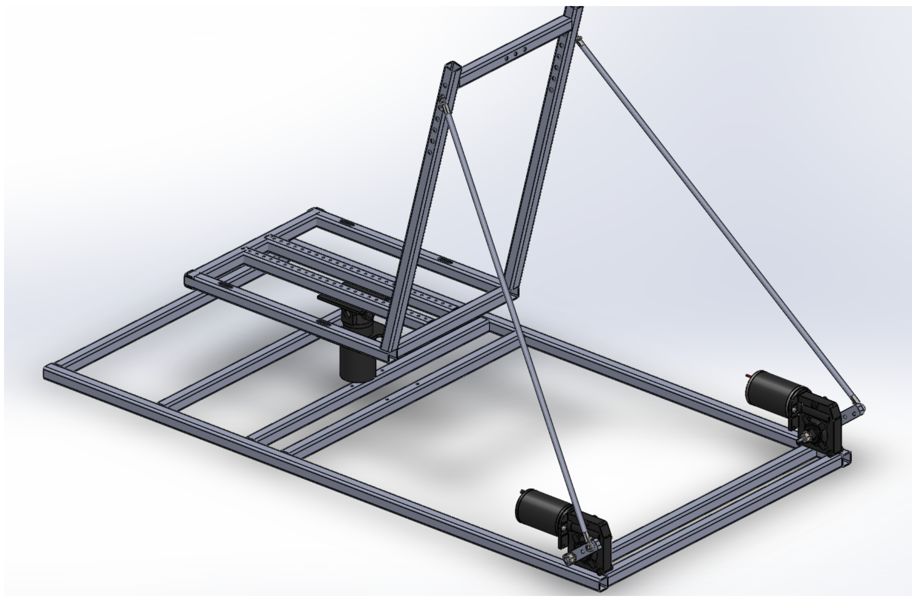

Figure 1: Typical 2DOF shoulder mount simulator

The most common form of actuator for DIY simulators is the combination of a high torque motor, with an arm attached to the shaft. The arm works somewhat similarly to an backwards crank and slider mechanism to convert the rotational motion of shaft into (somewhat) linear motion of the rod. The main equations governing the performance of these actuators are as follows:

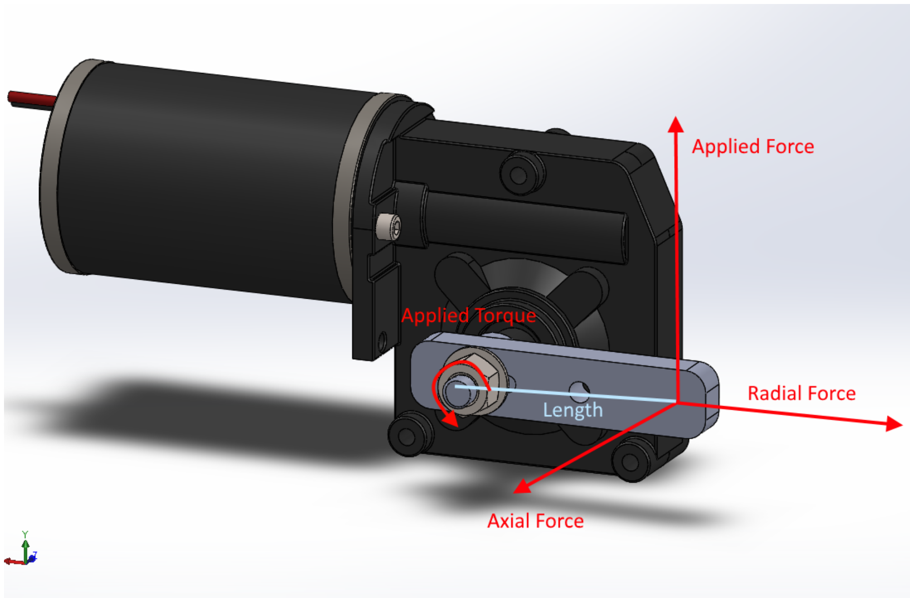

Figure 2: Forces acting on a "CTC Lever" actuator

The applied force of the actuator will be perpendicular to the lever, and its magnitude is governed by the following equation:

$$\vec{F}_{app} = \tau_M * l_{lever}$$

where τ_m is the motor torque, and l_{lever} is the length of the lever arm. The axial component $\vec{F}_{ax}$ and radial component $\vec{F}_{rad}$ of the actuator force depend on the mounting geometry.

The equivalent "linear speed" of the actuator can be calculated with the following:

$$v = l_{lever} * \omega_m * 0.10472$$

where ω_m is the motor speed, in RPM. Something to note here, is that as the lever rotates through its range of travel (+/- 90^o max), the apparent speed and force will seem to drop off. This is because the direction of the linear motion, as well as the direction vector for the applied force will always be perpendicular to the lever arm. If we assume that we only care about the speed and force output in one direction, we can approximate the loss in speed and force with the equations below:

$$\vec{F'}_{app} = \vec{F}_{app} * cos(\theta_d)$$
$$v' = v * cos(\theta_d)$$

where θ_d is the angle of deflection from the rest position of the lever. To minimize these losses when building a simulator, care should be taken to:

a) Make sure the resting position of the lever is near or at the position that would yield the maximum force and speed output at the frame

b) Limit the total travel of the lever (empirically +/-30^o were found to be good limits, as this keeps the reduction to no more than 86.6% of what the original values were). This serves to keep the force/speed outputs at the frame to an acceptable level, as well as to keep the output of the motors approximately linear over the full range of travel.

Simulator Travel

The motion of a typical 2DOF simulator is constrained by a few key dimensions, as shown in the following diagram:

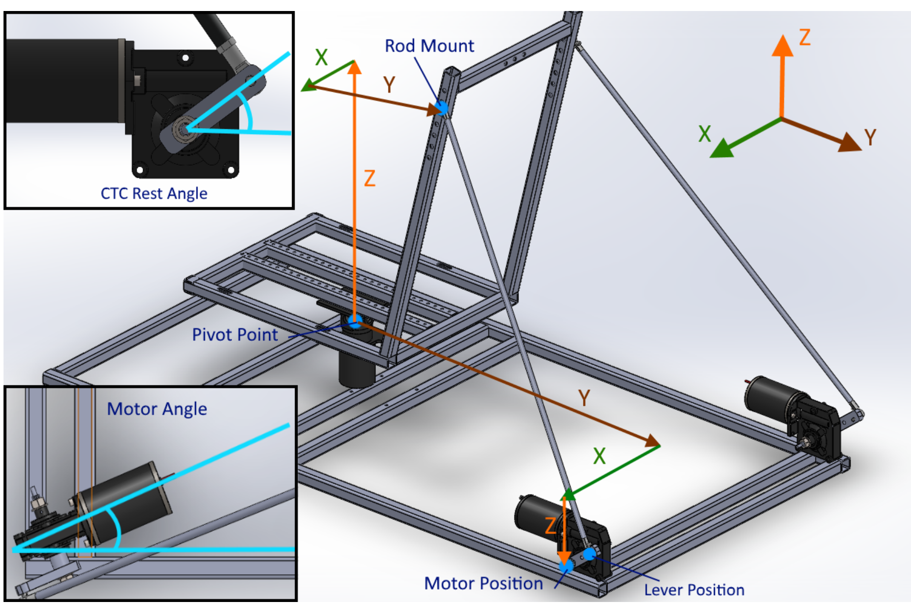

Figure 3: 2DOF simulator geometry

We take the pivot point of the system to be the origin of the coordinate system. The seat frame sits on top of this pivot, and is allowed to rotate in the pitch and roll directions, but yaw is constrained. Two connecting points (labelled "Rod Mount" in the diagram), connect the moving seat frame to a pair of rods. These rods, with length d, have ball-joint connections on either end, and link the seat frame to the actuators. The actuators consist of a lever of length l connected to the shaft of a motor, as described before. The end point of the lever is noted as "Lever Position" in the diagram, and the other end of the connecting rod mounts to this point. The simulator is assumed to be symmetric across the YZ plane (symmetric in the X direction).

To move the seat frame, one or both of the motor shafts rotates. When this happens, the lever position changes, which in turn moves the rod, which pushes the frame. Moving both motors in the same direction results in a pitch rotation of the seat frame, whereas moving both levers in opposite directions results in a roll rotation.

When designing a simulator, it is useful to know how much total rotation in the pitch and roll directions your frame will be capable of. This can be calculated with the math outlined below.

The seat frame is constrained in all types of motion except for pitch rotation and roll rotation. If we subtract the cartesian coordinates of one of the Rod Mount points ($\vec{P}_{RM}$) (it is assumed that they are symmetrical) from the origin, and take the length of that vector, this gives us the distance between the pivot point and the rod mount:

$$r = |\vec{P}_{RM}(x, y, z) - (0, 0, 0)|$$

Now we have the radius of the sphere $S1$ that contains all possible positions that the rod mount point can occupy if the seat frame is rotated in the pitch or roll directions. This sphere is centred on the origin.

Next we need to find a way to calculate the cartesian coordinates of the lever position, relative to the pivot point. The lever has length l is rotated by an angle θ around the motor shaft. Additionally, the motor may be rotated in the XY plane by an angle ϕ_m (labelled "Motor Angle" in the diagram). Knowing this, we can calculate the cartesian coordinates of the lever position (relative to the center of rotation) as follows:

$$x_l = sin(\phi_m)cos(\theta) * l$$
$$y_l = cos(\phi_m)cos(\theta) * l$$
$$z_l = sin(\theta) * l$$

To translate these relative coordinates to absolute ones, we add these coordinates to the position vector of the center of rotation for the lever, $\vec{P}_{Motor}$ (the coordinates of the motor shaft, labelled Motor Position in the diagram):

$$\vec{P}_{lever} = (x_l, y_l, z_l) + \vec{P}_{Motor}$$

Now we know the cartesian coordinates of the end of the lever ($\vec{P}_{lever}$). The lever is connected to a rod via a ball joint, and this rod also has fixed length d. This means that if unconstrained, the other end of the rod can occupy any point in space that lies on a sphere $S2$ of radius d, centered on the xyz coordinates of the current lever position.

Now, since we know that the other end of the rod has to be attached to the rod mount point on the seat frame, we can constrain the possible positions of the end of the rod to the intersection of spheres $S1$ and $S2$. This intersection will be a circle. The mathematical process for finding the intersection is well described at: http://paulbourke.net/geometry/circlesphere/, but the basic steps for doing so will be reiterated here:

Firt, temporarily set the centre of sphere 1 (P_0) as the origin of the system. Subtract the centre of sphere 2 (P_1) to get a vector going from the centre of the first sphere to the centre of the second. Let this vector be called $\vec{d}$

$$\vec{d} = P_1 - P_0$$

and the corresponding direction vector will be:

$$\hat{d} = \frac{\vec{d}}{|\vec{d}|}$$

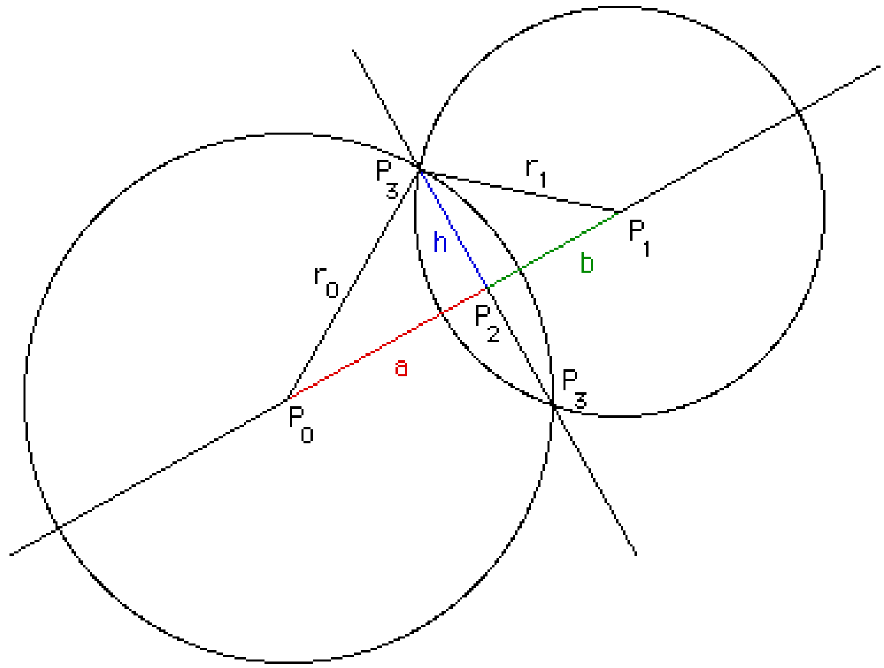

Figure 4: Intersection of Two Spheres (section view, credit: http://paulbourke.net/geometry/circlesphere/)

If we scale the direction vector of $\vec{d}$ ($\hat{d}$) by a magnitude of a, we will reach point P_2. P_2 is can be located on a plane defined as containing point P_2, with normal vector $\hat{d}$. If we draw a circle on this plane, centered on P_2, with radius h, the resulting circle will be the intersection of the two spheres.

First solving for a. From the diagram:

$$a^2 + r_0^2 = h^2$$
$$b^2 + r_1^2 = h^2$$

Knowing that $|\vec{d}| = a + b$, we can rearrange the previous equations to get:

$$a = \frac{r_0^2 - r_1^2 + |\vec{d}|^2}{2|\vec{d}|}$$

From there, we can find h:

$$h = \sqrt{r_0^2 + a^2}$$

Now we can find point P_2 by scaling the direction vector for $\vec{d}$

$$P_2 = \hat{d} * a$$

Finally, from this we can conclude that the intersection of these two spheres will be a circle, centered on point P_2, situated in the plane defined as containing P_2, and having normal vector $\vec{d}$. Now, to find a set of equations that describes this circle:

First, we find an orthonormal basis for the plane containing the circle. We define this new basis as having unit vectors $\hat{u}$, and $\hat{v}$, such that the following properties hold (note, "$\cdot$" and "$\times$" indicate vector dot and cross products):

$$\hat{u} \cdot \hat{d} = 0$$

$$\hat{v} \cdot \hat{d} = 0$$

$$\hat{u} \cdot \hat{v} = 0$$

To generate these vectors with a computer program, one can use the following procedure: let:

$$\hat{d} = (d_x, d_y, d_z)$$

$$\vec{r} = (-d_x, d_y, 0)$$

then let:

$$\vec{u} = \vec{r} \times \hat{d}$$

$$\hat{u} = \frac{\vec{u}}{|\vec{u}|}$$

$$\vec{v} = \hat{u} \times \hat{d}$$

$$\hat{v} = \frac{\vec{v}}{|\vec{v}|}$$

Knowing this, we can now create a parametric equation defining our circle using the unit vectors $\hat{u}$, and $\hat{v}$, and parameter ϕ, which will be the angle of rotation about the circle:. Let $C(\phi)$ be the equation defining the circle:

$$C(x, y, z) = C(\phi) = P_2 + h * cos(\phi) * \hat{u} + h * sin(\phi) * \hat{v}$$

Now going back to the simulator geometry. Recall originally we were trying to find the circle of intersection of the two spheres because it represented all possible positions in space that the rod mount point of the simulator could occupy based on a given lever position. The original goal of all of the math was to determine the maximum pitch and roll travel extents of a 2DOF simulator based on its geometry.

Looking at the pitch case only, if we were to pitch the simulator forward or backwards, the only coordinates of the rod mount position that would change would be the Y coordinate, and the Z coordinate, since pitch rotation takes place in the YZ plane. We also know that the simulator will be at its negative-most pitch position when the lever position is at one extreme of its travel, and it will be at its positive-most position when the lever position is at the other extreme of travel. We can use the equations we derived earlier to find the lever positions for a minimum and maxmum angular deflection of the motor:

Let θ_{min} be the minimum angle of the lever, and θ_{max} be the maximum angle of the lever, and l be the length of the lever:

$$P_{MaxLever} = \Big([sin(\phi_m)cos(\theta_{max}) * l], [cos(\phi_m)cos(\theta_{max}) * l], [sin(\theta_{max}) * l]\Big)$$

$$P_{MinLever} = \Big([sin(\phi_m)cos(\theta_{min}) * l], [cos(\phi_m)cos(\theta_{min}) * l], [sin(\theta_{min}) * l]\Big)$$

Finally: since the initial rod mount position will always be known (by design), we can simply constrain our set of solutions to $C(\phi)$ to ones that have the same x values as the initial position, since we know that for a pitch rotation of the frame, the x values will never change. To find the minimum and maximum positions in the pitch direction, we apply the process we just discussed, using the maximum lever position as the centre of the second sphere. Once you've derived an equation for the values of $C(\phi)$ for the maximum value of lever position, simply constrain the set of solutions such that the value of the x coordinate is the same as the initial rod mount position. This will yield the coordinates of the rod mount point for the maximum pitch position. Do the same, except this time use the minimum lever postition, with the same constraint. Now you should have both the maximum and minimum pitch positions. From here, calculating the angle of rotation or linear displacement is trivial.

For the roll case, the same process is applied, except instead of constraining the solutions of $C(\phi)$ such that the x coordinate remains unchanged from the rest position, we now make the constraint that the y coordinate is the one that does not change.

Force Calculations

Another parameter of interest in designing a simulator is the maximum force that the motors can apply to the frame of the simulator, which will limit the overall weight of the platform (eg. weight of the occupant, what materials the moving frame must be made of, how many peripherals may be attached). With standard linear actuators, this calculation can be made easily enough with linear actuators, but using the aforementioned lever actuator makes the calculation slightly more complicated:

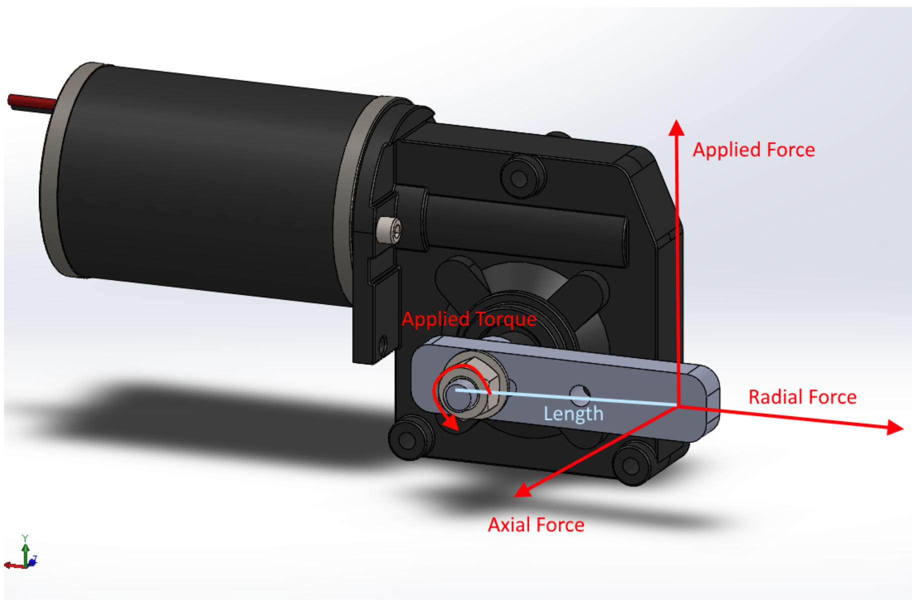

Figure 5: Forces acting on a "CTC Lever" actuator

As explained previously, the actuator applies a force that is perpendicular to the lever. The magnitude of the force is determined by the torque of the motor, divided by the length of the lever:

$$\vec{F}_{app} = \tau_M * l_{lever}$$

In addition to the applied force, the joint at the end of the lever can also experience a radial force $\vec{F}_{rad}$, which is in-line with the lever, and an axial force $\vec{F}_{ax}$, which is parallel to the motor's shaft. The values of these latter two forces in the simulator system will change based on the mounting geometry of the motors, as well as where the joint that attaches the connecting rod to the top frame is located. Since the rod is connected to the frame with a ball-joint at one end, and the lever with a ball-joint at the other end, the reaction forces on either end of the rod will be the same, and the resultant force will be in-line with the axis of the rod. Knowing this, if we know the $\vec{F}_{app}$, we can then find $\vec{F}_{rad}$ and $\vec{F}_{ax}$ by using static force analysis. Something to note here, is that for all of the following calculations, we assume that the rod mounting joint will be held statically, and that all components are perfectly rigid. This analysis will be made much easier by a slight change in reference coordinates. Previously for all calculations, we've been using standard cartesian basis, with unit vectors

$$\hat{\imath} = (1, 0, 0)$$
$$\hat{\jmath} = (0, 1, 0)$$
$$\hat{k} = (0, 0, 1)$$

Noticing that the three force vectors are all linearly independant, it would make the following static analysis much easier if we made a change of basis to a new coordinate system B':

$$\hat{\imath}' = \hat{F}_{ax}$$
$$\hat{\jmath}' = \hat{F}_{rad}$$
$$\hat{k}' = \hat{F}_{app}$$

Where $\hat{F}_{ax}$, $\hat{F}_{rad}$, and $\hat{F}_{app}$ are unit vectors along the directions of their respective forces. Now, if we write out the vectors explicitly:

$$\hat{F}_{ax} = (x_{ax}, y_{ax}, z_{ax})$$
$$\hat{F}_{rad} = (x_{rad}, y_{rad}, z_{rad})$$
$$\hat{F}_{app} = (x_{app}, y_{app}, z_{app})$$

Let:

$$M = \begin{bmatrix} x_{ax} & x_{rad} & x_{app} \\ y_{ax} & y_{rad} & y_{rad} \\ z_{ax} & z_{rad} & z_{rad} \end{bmatrix}$$

This will be the transformation matrix to take a vector from our new basis $B' \to R2$. If we invert this matrix, we will get the transformation matrix from $R2 \to B'$. Or, for some generic vector $\vec{V}$:

$$\vec{V}_{R2} = M\vec{V}_B$$

$$\vec{V}_B = M^{-1}\vec{V}_{R2}$$

Now that we have the transformation matrix, we look at the two important points for the mechanical system: $P_{RodMount}$, which is the coordinate point of the joint that attaches the connecting rod to the moving platform, and P_{lever}, which is the coordinate point of the joint that attaches the lever of the actuator to the connecting rod. First, transform these coordinate points to the new basis:

$$P'_{RodMount} = M^{-1}P_{RodMount}$$
$$P'_{Lever} = M^{-1}P_{Lever}$$

Now we have the following diagram:

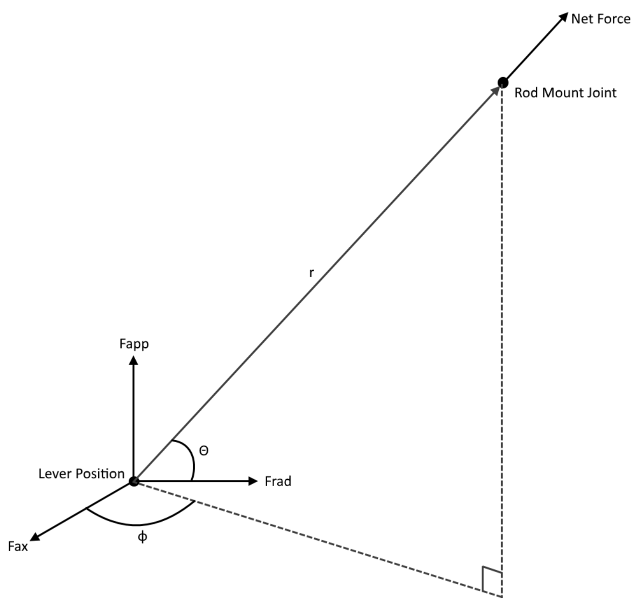

Figure 6: Vector Forces in Transformed System

At the lever, we have three forces, one of which is known. Because the rod is connected by ball joints at each end, the resultant force vector of these three forces must be in-line with the axis of the rod. We create a vector $\vec{r}$ such that:

$$\vec{r} = P'_{RodMount} - P'_{Lever}$$

with corresponding unit vector $\hat{r}$. This unit vector $\hat{r}$ must be the direction vector for the final resultant force obtained by summing our three forces. We know F_{app}, so this leaves us with a system of equations that can be solved. Define our resultant force vector as simply $\vec{F'}$

$$\vec{F} = (\vec{F}_{ax}, \vec{F}_{rad}, \vec{F}_{app})$$

Given that:

$$\hat{r} = (r_{i'}, r_{j'}, r_{k'})$$
$$\hat{r} = (a * \hat{F}_{ax}, b * \hat{F}_{rad}, c * \hat{F}_{app})$$
$$\hat{r} = (a * \hat{i}', b * \hat{j}', c * \hat{k}')$$

Where a, b, and c are known constants. Using the scaling property of vectors, we can use the fact that if we now know the magnitude of $\vec{F}_{app}$, we know that the other components of the vector will be scaled by the same amount, thus:

$$\vec{F'} = (a * |\vec{F}_{app}|\hat{i}', b * |\vec{F}_{app}|\hat{j}', c * |\vec{F}_{app}|\hat{k}')$$
$$\vec{F'} = |\vec{F}_{app}|\hat{r}$$

Finally, since we're interested in a cartesian coordinate solution, we transform this vector back to our original basis using the transformation matrix:

$$\vec{F} = M\vec{F'}$$

This is our final resultant force at the actuator joint. We'll only goes so far as to consider static force analysis for the moment, so to accomplish this, we let the rod mount joint be fixed, and observe the resultant force at the joint if the motor applies a certain amount of torque to the lever. This will be the reverse of the resultant force vector we obtained previously. The last step is simply transforming this resultant force into a pitch or roll force, as these are the relevant parameters we set out to acquire in the first place.

Now, we have the reaction force at the other end of the rod. It has magnitude $\vec{F}$, and direction $\hat{F}$. However, not all of this force will necessarily be used to move the platform. The platform sits atop a universal (cardan) joint, which restricts its movements to rotation in the pitch direction, and rotation in the roll direction.

Create a new vector $\vec{p}$ which goes from the universal joint to the rod-mount joint on the moving platform.

$$\vec{p} = P_{RodMount} - P_{PivotJoint}$$
$$\vec{p} = P_{RodMount} - (0,0,0)$$

We're interested in finding the pitch and roll forces respectively, however it makes more sense to find the equivalent torque at the pivot joint. Since the platform is sitting on a rotating joint, the applied force at the rod mount will create a moment at the pivot joint. If the applied force is not perfectly perpendicular to the vector $\vec{r}$, some of the applied force will be wasted applying tension to the frame, rather than rotating the frame (recall $\tau = \vec{F}sin(\theta)$). We only care about the component of the force that is used to rotate the frame.

Pitch first:

If we take the YZ projection of our vector P_{yz}, as well as the YZ projection of our force vector $\vec{F}_{yz}$, we're left with the following diagram:

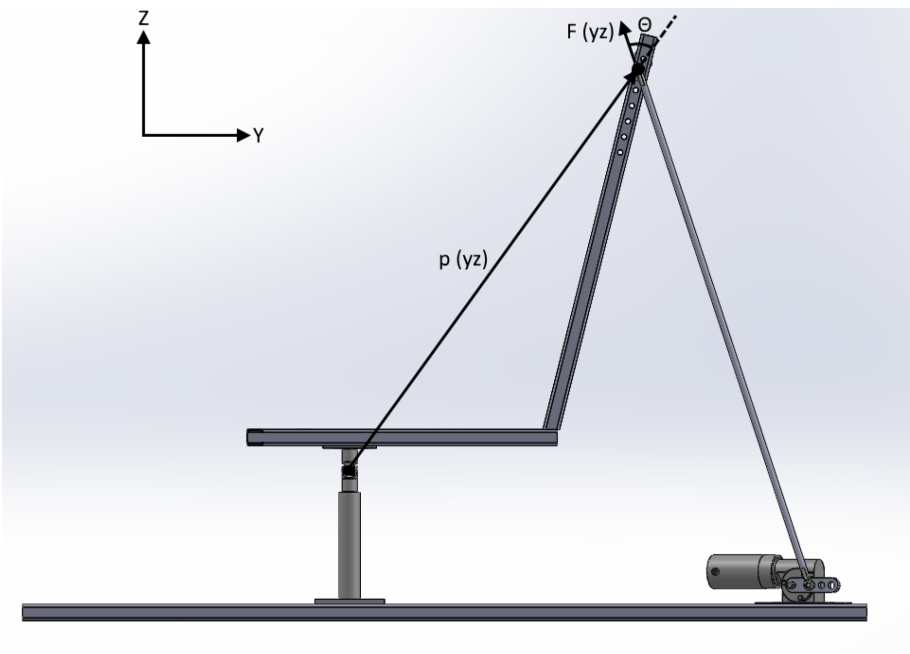

Figure 7: Equivalent Pitch Force

From this, recalling the formula for torque, we can see that:

$$\vec{F}_{Pitch} = \vec{F}sin(\theta_{yz})$$

Note, sometimes it is also useful to know the equivalent torque at the pivot point, which is simply:

$$\tau_{Pitch} = |\vec{F}_{Pitch}||\vec{P}_{yz}|$$

The remaining component of this force is simply wasted applying tension or compression to the top part of the frame, so an efficient design will generally try to aim for θ to be as close to 90^o as possible. The process for finding the roll force is the same, but using the XZ projection instead of the YZ projection:

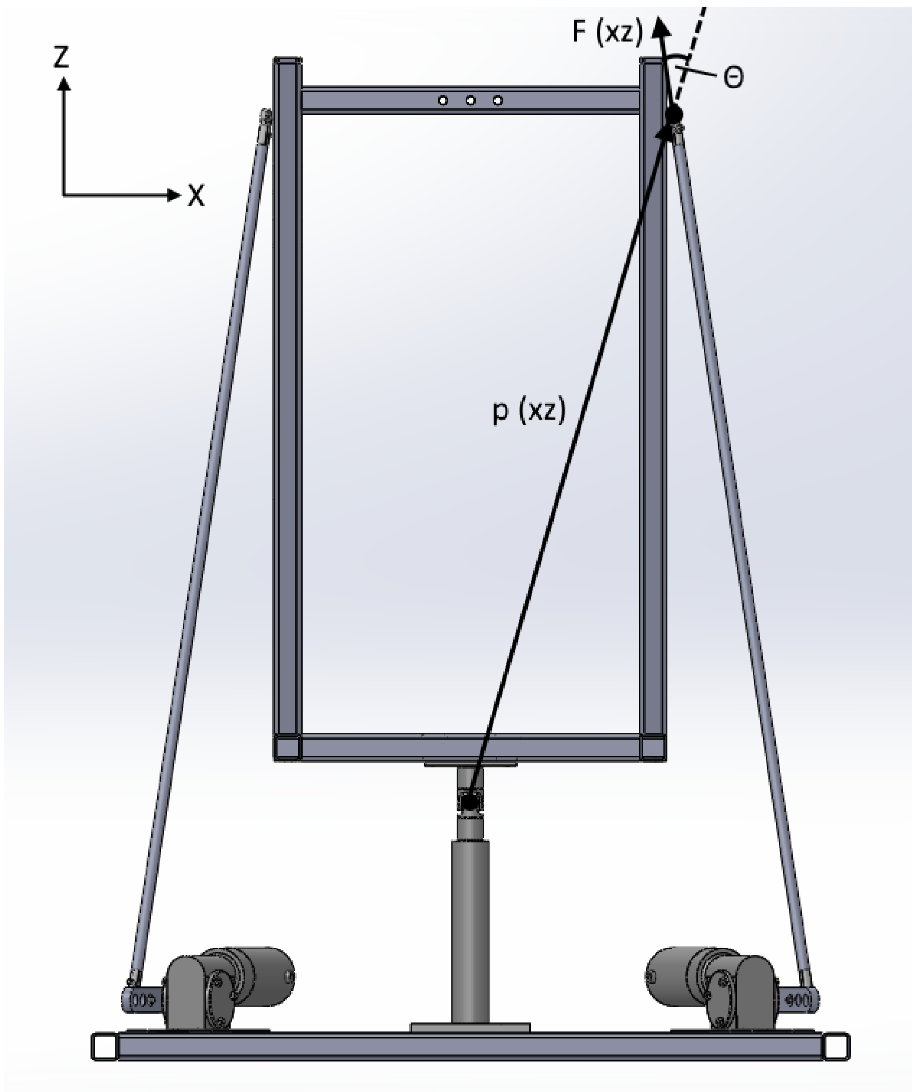

Figure 8: Equivalent Roll Force

$$\vec{F}_{Roll} = \vec{F} sin(\theta_{xz})$$

$$\tau_{Roll} = |\vec{F}_{Roll}||\vec{P}_{xz}|$$