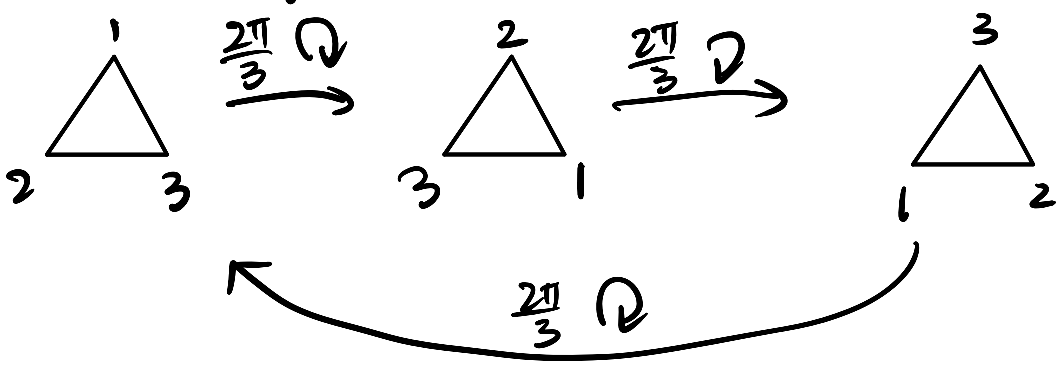
— A Bit Discussion On Groups

* Group, very roughly, can be used to characterize the Symmetry of a Geometric Object.

e.g. regular triangle



We call $a \equiv \frac{2\pi}{3} \circlearrowright$ an operation on the triangle.

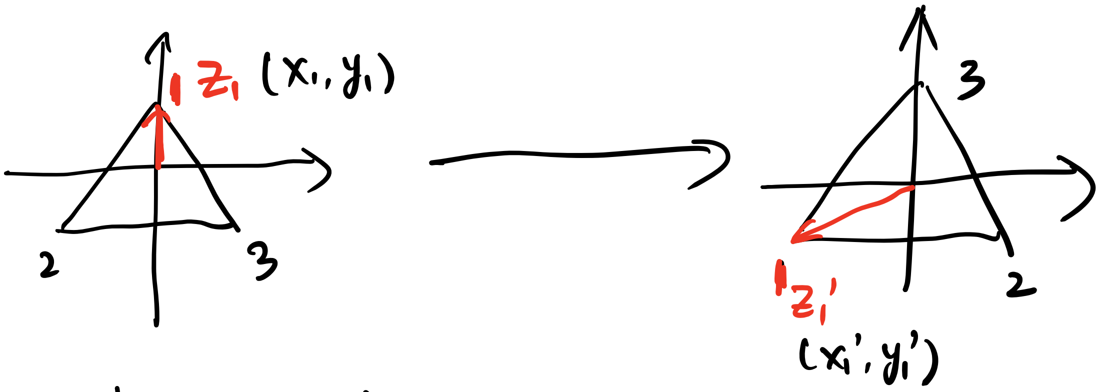
then $a^2 \equiv a \circ a = \frac{4\pi}{3} \circlearrowright \quad a^3 \equiv a \circ a^2 \equiv \frac{6\pi}{3} = 2\pi \circlearrowright$
= No action.

$\mathbb{Z}_3 \equiv \{a \mid a^3 = \mathbb{1}\}$ collects rotation symmetric actions of $\triangle$

Rmk: "a" is an <u>Operation</u>, multiplication "$\circ$" is a <u>Composition</u> of operations. "$\mathbb{1}$" is Identity Operation, i.e. Do NOTHING.

But if we place the triangle on the xy-plane, we

Can express the rotation operation a in terms of coordinates



$$\begin{pmatrix} x_1' \\ y_1' \end{pmatrix} = R(a) \cdot \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

$R(a)$ now is a 2×2 matrix, called a Representation of $\mathbb{Z}_3$.

↑ Real Rep.

Apperantly, $R(a) = \begin{pmatrix} \cos\frac{2\pi}{3}, & -\sin\frac{2\pi}{3} \\ \sin\frac{2\pi}{3} & \cos\frac{2\pi}{3} \end{pmatrix}$

In fact, it's much simple to introduce complex number

i.e. $z_1 = x_1 + i y_1$ $\quad z_1' = x_1' + i y_1'$

then $z_1' = R_c(a) \cdot z_1 = e^{\frac{2\pi}{3} i} z_1$

check:

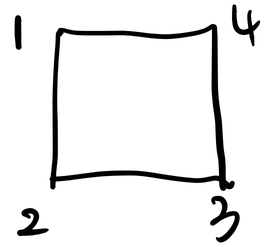
$$x_1' + i y_1' = \left(\cos\frac{2\pi}{3} + i \sin\frac{2\pi}{3}\right)(x_1 + i y_1) = \left(\cos\frac{2\pi}{3} x_1 - \sin\frac{2\pi}{3} y_1\right) + i\left(\sin\frac{2\pi}{3} x_1 + \cos\frac{2\pi}{3} y_1\right)$$

i.e. $$\begin{pmatrix} x_1' \\ y_1' \end{pmatrix} = \begin{pmatrix} \cos\frac{2\pi}{3}, & -\sin\frac{2\pi}{3} \\ \sin\frac{2\pi}{3}, & \cos\frac{2\pi}{3} \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

Thus, $R_c(a)$ is a <u>complex</u> Rep. of $\mathbb{Z}_3$.

Now we use Complex Rep. and simply call $R(a)$ as a.

Consider a square. now

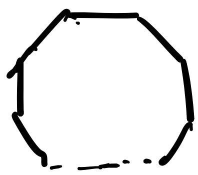


Collections of rotation symmetric action.

$\mathbb{Z}_4 = \{a \mid a^4 = 1\}$. "$a$" is $\frac{2\pi}{4} = \frac{\pi}{2}$ ↺

In terms of Rep. $a = e^{\frac{2\pi i}{4}}$.

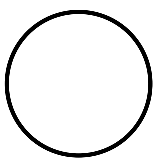
One can simply genalize it to a N-polygon.



Sym. Group is $\mathbb{Z}_N = \{a \mid a^N = 1\}$, $a = e^{\frac{2\pi i}{N}}$.

Now. let $N \to \infty$ we go from N-polygon. to S^1

$N \to \infty$

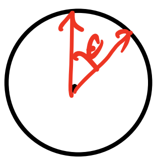


ANY arbitary Rotation by angle θ is a symmetric action of the circle S^1

The sym. of S^1 is called $U(1) = \{e^{i\theta} \mid \theta \in \mathbb{R}\}$.

The sym group of S^1 itself is a circle of radius 1, which is called a Lie Group. $\begin{cases} \text{Group structure} \\ \text{geometric structure.} \end{cases}$

On the other hand, in Real Rep.



$$\vec{r}' = \begin{pmatrix} x' \\ y' \end{pmatrix} = R(\alpha) \cdot \vec{r} = R(\alpha) \begin{pmatrix} x \\ y \end{pmatrix}$$ is a sym. action

iff $|\vec{r}'|^2 = |\vec{r}|^2$ i.e. $\vec{r}'^T \vec{r}' = \vec{r}^T \cdot R(\alpha)^T R(\alpha) \cdot \vec{r}$
$$= \vec{r}^T \cdot \vec{r}$$

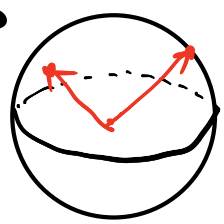
$$\Rightarrow \quad R(\alpha)^T R(\alpha) = \mathbb{1}_2.$$

$$SO(2) = \{ g \mid g^T g = \mathbb{1}_2 \} \rightsquigarrow g = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Rmk: $U(1) = SO(2)$

Analogously, consider two-sphere S^2, what's the sym. group of S^2?

S^2



$$\vec{r}' = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = R(\alpha) \cdot \vec{r} = R(\alpha) \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

iff $|\vec{r}'|^2 = |\vec{r}|^2$ i.e. $R(\alpha)^T R(\alpha) = \mathbb{1}_2$

$$\Rightarrow SO(3) = \{ g \mid g^T g = \mathbb{1}_3 \}$$

Similarly, For a S^{N-1}-sphere in $\mathbb{R}^N$, its sym.

Group is called $SO(N) = \{ g \mid g^T g = \mathbb{1}_N, \ \mathrm{Det}\, g = 1 \}$

(Rmk: The sym. Group of S^{N-1} is actually $O(N)$, but I have no time to explain it. $O(N) \xrightarrow{\det g = 1} SO(N)$)

* Lie Algebra.

For Lie Group, because they are also manifold, one can study them by considering a infinetismal transformation, that is called a generaton of the Lie Group.

The collections of generators $\{T_i\}$ form an Algebra. called <u>Lie algebra</u>.

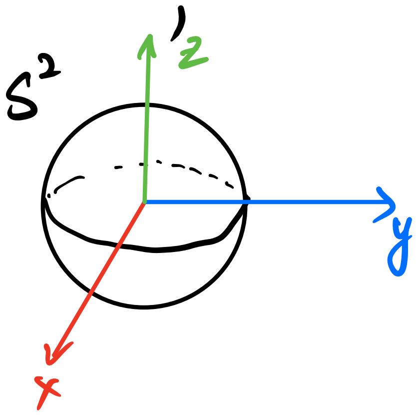
e.g. $U(1) = \{ e^{i\theta} \mid \theta \in \mathbb{R} \}$ $\quad e^{i\theta} = 1 + i\theta + O(\theta^2)$

the generator $T_{\mathbb{C}} = i$. Complex Rep. of Lie Alg. $U(1)$.

or for $SO(2)$ $\quad g = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \mathbb{1}_2 + \theta \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

the generator. $T_{\mathbb{R}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Real Rep. of Lie Alg. $U(1)$.

e.g.2. $SO(3) = \{ g \mid g^T g = 1 \}$



One can rotate along x, y, z-axis.

$$R_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix} = \mathbb{1}_3 + \alpha \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}}_{= T_1}$$

$$R_y = \begin{pmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{pmatrix} = \mathbb{1}_3 + \beta \underbrace{\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}}_{= T_2}$$

$$R_z = \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbb{1}_3 + \gamma \underbrace{\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{= T_3}$$

There are three generators $\{T_1, T_2, T_3\}$.

one can check the generators satisfy the commutation relation $[T_i, T_j] = \epsilon_{ijk} T_k$.

II. FREE FIELDS & THEIR QUANTIZATION.

A. Lorentz Symmetry

– Lorentz Group.

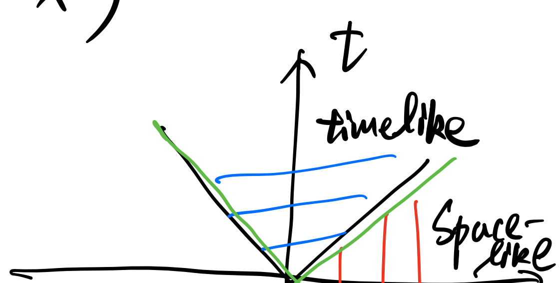
The law of NATURE is relativistic, i.e. time and space are on the equal footing. We collect time "t" and space coordinates "$\vec{x}$" together as.

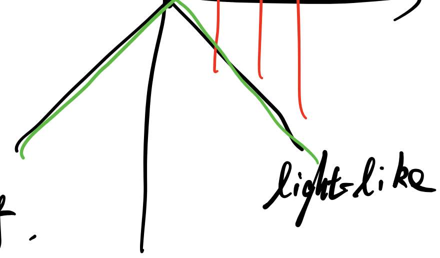
$$x^\mu = (t,\ x,\ y,\ z)$$

* Space time interval:

$$d(x', x) = \eta_{\mu\nu}(x'^\mu - x^\mu)(x'^\nu - x^\nu)$$

with $\eta_{\mu\nu} = \mathrm{diag}\{1, -1, -1, -1\}$.





* 4-momentum $p^{\mu} = (E, \vec{p})$

$$p^2 = \eta_{\mu\nu} p^{\mu} p^{\nu} = E^2 - \vec{p}^{\,2} = m^2$$ rest mass L.I. quantity.

Lorentz sym. is a transformation (Λ)

that keeps the spacetime interval Invariant

$$SO(1,3) = \{ \Lambda \mid \Lambda^T \eta \Lambda = \eta,\ \text{Det}\Lambda = 1^{①},\ (\Lambda x)^0 > 0 \text{ if } x^0 > 0^{②} \}$$

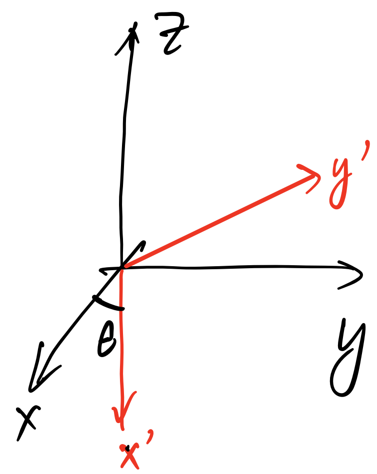
① preserve orientation

② proper boost that keeps time arrow.

* A special rotation (along z-axis)

$$x^{\mu} \to x^{\mu'} = \Lambda_{r\ \nu}^{\ \mu} x^{\nu}$$

$$\Lambda_{r\ \nu}^{\ \mu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



All special rotations along x, y, z-axis form a

group $SO(3) = \{ g \mid g^T g = \mathbb{1}_3,\ \det g = 1 \}$

Special orthogonal group (Lie Group)

特殊 正交 群

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = e^{\theta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}} = \mathbb{1}_2 + \theta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \mathcal{O}(\theta^2)$$

$$g = e^{\theta^i T_i} = \mathbb{1} + \theta^i T_i + \mathcal{O}(\theta^{i2})$$

But. $g^T g = \mathbb{1} = (\mathbb{1} + \theta^i T_i^T)(\mathbb{1} + \theta^i T_i) + \cdots$

$$= \mathbb{1} + \theta^i (T_i^T + T_i) + \mathcal{O}(\theta^2) = \mathbb{1}$$

$$\Rightarrow T_i^T = -T_i$$

$$T_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad T_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$[T_i, T_j] = \epsilon_{ijk} T_k \qquad \epsilon_{ijk} = \begin{cases} +1 & (123) \\ -1 & (213) \\ 0 & \text{etc.} \end{cases}$$

(column heading: ijk)

$$J_i = i T_i$$

$[J_i, J_j] = i\epsilon_{ijk} J_k$. (usual commutation relation)

* A Lorentz Boost. between t and x with velocity $v<1$.

$$x^\mu \longrightarrow x'^\mu = \Lambda_{b}{}^{\mu}{}_{\nu}\, x^\nu$$

$$\Lambda_{b}{}^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & \gamma v & 0 & 0 \\ \gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \gamma = \frac{1}{\sqrt{1-v^2}}$$

Notice $\gamma^2 - (\gamma v)^2 = 1$

set $\gamma \equiv \cosh\beta$ $\qquad$ $\gamma v \equiv \sinh\beta$

$$= \frac{e^\beta + e^{-\beta}}{2} \qquad\qquad = \frac{e^\beta - e^{-\beta}}{2}$$

$$= 1 + \mathcal{O}(\beta^2) \qquad\qquad = \beta + \mathcal{O}(\beta^3).$$

$$\begin{pmatrix} \gamma & \gamma v \\ \gamma v & \gamma \end{pmatrix} = \begin{pmatrix} \cosh\beta & \sinh\beta \\ \sinh\beta & \cosh\beta \end{pmatrix} \simeq \mathbb{1}_2 + \beta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$K_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad K_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$K_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad [K_i, K_j] = \epsilon_{ijk} T_k$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}$$

and $[K_i, T_j] = \epsilon_{ijk} K_k$

$[T_i, T_j] = \epsilon_{ijk} T_k$ $\qquad M_i = (T_i + K_i)/2$

$[K_i, K_j] = \epsilon_{ijk} T_k$ $\qquad N_i = (T_i - K_i)/2$

$[K_i, T_j] = \epsilon_{ijk} K_k$.

$$\Rightarrow \begin{array}{l} [M_i, M_j] = \epsilon_{ijk} M_k. \\ [N_i, N_j] = \epsilon_{ijk} N_k \end{array} \quad \Rightarrow SO(1,3) \simeq SO(3) \times SO(3) \simeq SU(2) \times SU(2)$$

We know that Rep. of $SU(2)$ are labeled by spin s.

e.g. $s = 1/2$ 2-dim. Rep. $\qquad (2s+1)$-dim Rep.

Now, a relativistic particle/Field needs two labels.

(s_1, s_2)

e.g. $(0,0)$ Scalar field. complex

$(1/2, 0)$ 2-component Spinor (left-handed.) (Weyl

$(0,\frac{1}{2})$ - - - - - - - - (Right ......) Spiner.

$(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ 4-component complex Spinor (Both L/R) Dirac Spiner.

$(\frac{1}{2},\frac{1}{2})$ 4-component Lorentz vector. (photon)

They are off-shell D.O.F. (Not physical).

Rmk: A field $\phi(x)$ has infinitely many D.O.F. as claimed in the beginning. Here D.O.F. we mean is to count the internal D.O.F. of a field $\phi(x)$ at a fixed timespace spot "x".

— Little Group.

Reps of $SO(1,3)$ count off-shell D.O.F.

From physics perspective, we are more interested in on-shell D.O.F., i.e. those D.O.F. really propagate, or say those satisfying the E.O.M.

e.g. We use 4-vector potential $A_\mu=(\phi,\vec{A})$ to describe photon. (Recall $\vec{E} = \vec{\nabla}\phi$, $\vec{B} = \vec{\nabla}\times\vec{A}$)

there are $\underline{4}$ D.O.F off-shell, But photon only has $\underline{2}$ physical D.O.F (2 polarization)

To count D.O.F on-shell, we introduce the concept of "little group." That is:

We consider a <u>massive</u> particle at its rest frame under the reduced Lorentz group (little group) that keeps the particle in the frame, and thus count its D.O.F.; a <u>massless</u> particle ······ light-cone frame ·-------------------

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* massive case: $p^\mu = (E, \vec{P})$ $E^2 - |\vec{P}|^2 = M^2$

rest frame: $P^\mu_{rest} = (M, \underbrace{0, 0, 0}_{SO(3)})$

the little group $\Lambda_\ell \cdot P = P. \Rightarrow \Lambda_\ell \in SO(3) \simeq SU(2)$

		D.O.F.	SU(2)
$\Rightarrow$ For	scalar	1	0 Rep
	Dirac Spinor	4 (2 complex)	½ Rep.
	massive vector ($W, Z^{\pm}$)	3	1 Rep.
	massive spin 2 particle	5	2 Rep.

* massless case: $p^\mu = (E, \vec{p})$ $\quad E^2 - |\vec{p}|^2 = 0$

light cone frame : $P^\mu_{l.c.} = (E, \underbrace{0, 0,}_{SO(2)} \pm E)$

		D.O.F.	massive.
For	scalar	1	1
	Weyl Spinor	2	$4 = 2_L + 2_R$
	photon (massless vec.)	2	$3 = \underset{\{-1,+1\}}{2} + \underset{\{0\}}{1}$
I will explain right away.	graviton (massless spin 2)	2	$5 = \underset{\{\pm 2\}}{2} + \underset{\cancel{\{\pm 1\}}}{2} + \underset{\cancel{\{0\}}}{1}$

* Generic case in D-dimensional Spacetime.

Lorentz Group $SO(1, D-1)$ Scalar too simple

Spinor kind of intricate.

massive little group $SO(D-1)$ D.O.F.

Vec. field A_μ $\quad D-1$

Rank-2 traceless $g_{\mu\nu}$ $\quad \frac{(D-1)D}{2}-1 = \frac{(D-2)(D+1)}{2}$

massless little group $SO(D-2)$ D.O.F.

Vec. field A_μ $\quad D-2$

Rank-2 traceless $g_{\mu\nu}$ $\quad \frac{(D-2)(D-1)}{2}-1 = \frac{D(D-3)}{2}$

2D gauge theory has no D.O.F. $\simeq$ Topological Field Theory.

3D gravity has no D.O.F.

$\simeq$

3D Chern-Simon Theory.

— Infinetesmal Transformation under Lorentz Symmetry (Poincaré)

Consider Lorentz Group

$$SO(1,3) = \left\{ \Lambda \,\middle|\, \Lambda^T \eta \Lambda = \eta,\ \operatorname{Det}\Lambda = 1,\ (\Lambda x)^0 > 0 \text{ if } x^0 > 0 \right\}$$

① ②

$X^\mu \to X'^\mu = \Lambda^\mu{}_\nu X^\nu$ or a Vector $V^\mu \to V'^\mu = \Lambda^\mu{}_\nu V^\nu$

$$X_\mu X^\mu = X^\rho \eta_{\rho\mu} X^\mu \to \Lambda^\rho{}_\sigma \eta_{\rho\mu} \Lambda^\mu{}_\nu x^\sigma x^\nu = \eta_{\sigma\nu} x^\sigma x^\nu$$

$\Rightarrow \Lambda^\rho \ \eta \ \Lambda^\mu \ \eta$

$$\Rightarrow \Lambda^\rho{}_\alpha \eta_{\rho\mu} \Lambda^\mu{}_\nu = \eta_{\alpha\nu}$$

use $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$ ω is infinetesmal.

$$(\delta^\rho{}_\alpha + \omega^\rho{}_\alpha)\eta_{\rho\mu}(\delta^\mu{}_\nu + \omega^\mu{}_\nu) = \eta_{\alpha\nu} + \omega_{\nu\alpha} + \omega_{\alpha\nu} = \eta_{\alpha\nu}$$
$$+ \mathcal{O}(\omega^2)$$

$\Rightarrow \omega_{\nu\alpha} = -\omega_{\alpha\nu}$ anti-symmetric matrix..

one can check how many independent matrices in 4-dim.

$$\binom{4}{2} = \frac{4\times 3}{2} = 6 = \underset{\text{rotations}}{\underset{\uparrow}{3}} + \underset{\text{boosts}}{\underset{\uparrow}{3}}$$

Find A basis for the 6 AS-matrice.

$$\boxed{(M^{\rho\alpha})^{\mu\nu} = \eta^{\rho\mu}\eta^{\alpha\nu} - \eta^{\rho\nu}\eta^{\alpha\mu}}$$

$(M^{\rho\alpha})^{\mu\nu}$ — $\rho\alpha$: label 6 M's. $\mu\nu$: entries of $M^{\rho\alpha}$

one can check the pairs (ρ,α) - AS. (μ,ν) - AS.

use "η" to lower the entries.

$$(M^{\rho\alpha})^\mu{}_\nu = (M^{\rho\alpha})^{\mu\tau}\eta_{\tau\nu} = \eta^{\rho\mu}\delta^\alpha{}_\nu - \eta^{\alpha\mu}\delta^\rho{}_\nu$$

one can check. e.g. $(M^{01})^\mu{}_\nu = \eta^{0\mu}\delta^1{}_\nu - \eta^{1\mu}\delta^0{}_\nu$

$$= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \cdots & & & \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = K_1$$

$$(M^{12})^\mu{}_\nu = \eta^{1\mu}\delta^2_\nu - \eta^{2\mu}\delta^1_\nu$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = -T_3$$

One can verify the commutation relation:

$$[M^{\mu\nu}, M^{\rho\sigma}] = -\eta^{\mu\rho}M^{\nu\sigma} + \eta^{\mu\sigma}M^{\nu\rho} + \eta^{\nu\rho}M^{\mu\sigma} - \eta^{\nu\sigma}M^{\mu\rho}$$

A finite Lorentz transformation

$$\Lambda^\mu{}_\nu = \exp\left\{\frac{1}{2}\theta_{\rho\sigma}M^{\rho\sigma}\right\}^\mu{}_\nu$$

$$= \delta^\mu{}_\nu + \frac{1}{2}\theta_{\rho\sigma}(M^{\rho\sigma})^\mu{}_\nu$$

$$+ \frac{1}{2}\left(\frac{1}{2}\right)^2 \theta_{\rho\sigma}\theta_{\tau\lambda}(M^{\rho\sigma})^\mu{}_\kappa (M^{\tau\lambda})^\kappa{}_\nu$$

$$+ \cdots O(\theta^3)$$

A infinetesmal one $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$

with $\omega^{\mu}{}_{\nu} = \frac{1}{2}\epsilon_{\rho\sigma}(M^{\rho\sigma})^{\mu}{}_{\nu}$

$$\delta x^{\mu} = (x')^{\mu} - x^{\mu} = \left(\delta^{\mu}{}_{\nu} + \frac{1}{2}\epsilon_{\rho\sigma}(M^{\rho\sigma})^{\mu}{}_{\nu}\right)x^{\nu} - x^{\mu}$$

$$= \frac{1}{2}\epsilon_{\rho\sigma}(M^{\rho\sigma})^{\mu}{}_{\nu}\, x^{\nu}$$

Besides, there are also translation symmetry.

$$x^{\mu} \to (x')^{\mu} = x^{\mu} + a^{\mu} \quad \rightsquigarrow 4 \text{ sym.}$$

parametrized by a^{μ}. $a^{\mu} \in \mathbb{R}^4$

Infinetesmally, $\delta x^{\mu} = (x')^{\mu} - x^{\mu}$

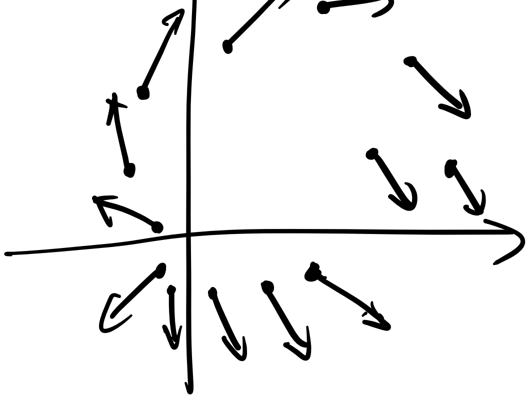
$$= (x^{\mu} + \epsilon^{\mu}) - x^{\mu} = \epsilon^{\mu}.$$

the Full Sym. of a QFT is

"$SO(1,3) \ltimes \mathbb{R}^4$" the Poincaré Group

*Now we turn to Lorentz (Poincaré) transformations of Fields

$V^{\mu}(x)$ assigns a vector for



each spot of x,

So when taking a Lorentz transformation

$$V^\mu(x) \to \underset{\substack{\uparrow \\ \text{vector transformed}}}{V'^\mu}(\underset{\substack{\uparrow \\ \text{coordinate transformed}}}{x'}) = \Lambda^\mu{}_\nu V^\nu(x)$$

$V^\mu(x) \longrightarrow V'^\mu(x')$ is called "Passive transformation"

that is the coordinate also changes from x to x'

It's therefore equivalent to rotate/boost the coordinate,

and thus it's similar to just rotate/boost the vector. V^μ

i.e. $V'^\mu(x') = \Lambda^\mu{}_\nu V^\nu(x)$

More generically, Consider a Manifold $\mathcal{M}$. on which there is a vector field. $V = V^\mu(x) \frac{\partial}{\partial x^\mu}$

a diffeomorphism $x \to x' = f(x)$

changes the vector field to $V = V'^{\mu}(x') \frac{\partial}{\partial x'^{\mu}}$

$$= V'^{\mu}(x') \frac{\partial x^{\nu}}{\partial x'^{\mu}} \frac{\partial}{\partial x^{\nu}}$$

$$= V^{\nu}(x) \frac{\partial}{\partial x^{\nu}}$$

$$\Rightarrow \quad V^{\nu}(x) = V'^{\mu}(x') \frac{\partial x^{\nu}}{\partial x'^{\mu}}$$

$$\text{or} \quad V'^{\mu}(x') = \underbrace{\frac{\partial x'^{\mu}}{\partial x^{\nu}}}_{\Lambda^{\mu}{}_{\nu}} V^{\nu}(x)$$

$\Lambda^{\mu}{}_{\nu}$ if the diffeomorphism is Lorentz.

i.e. $x' = f(x) = \Lambda^{\mu}{}_{\nu} \cdot x^{\nu}$

However, the "passive transformation" won't tell us ANYTHING about the symmetry of our system. It simply shows how a field transform under an action (diffeomorphism, group action...)

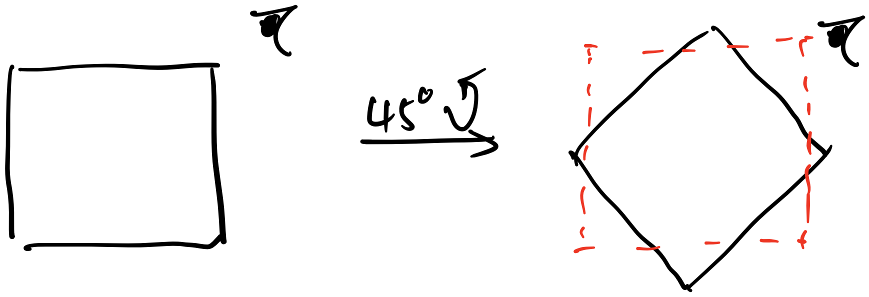
What we really need is an "active transformation" that tells us the difference of the field at the SAME spot before and after the transformation.

e.g.

passitive.

Blind.

45° ↻ is a sym. of square?



active? No.

An infinitesimal "active transformation" is to compare

$$\delta\,\text{Field}(x) = \text{Field}'(x) - \text{Field}(x)$$

e.g. Scalar $\phi(x)$.

passive: $\phi(x) \longrightarrow \phi'(x') = \phi(x)$

active: $\delta\phi(x) = \phi'(x) - \phi(x)$

$$= \phi'(x' - \delta x) - \phi(x) = -\delta x^\mu \partial_\mu \phi$$

translation: $\delta x^\mu = \epsilon^\mu$ $\quad \delta_\epsilon \phi(x) = -\epsilon^\mu \partial_\mu \phi(x)$

rotation/boost: $\delta x^\mu = \frac{1}{2}\epsilon_{\rho\sigma}(M^{\rho\sigma})^\mu{}_\nu x^\nu$ $\quad \delta_L \phi(x) = -\frac{1}{2}\epsilon_{\rho\sigma}(M^{\rho\sigma})^\mu{}_\nu x^\nu \partial_\mu \phi$

Recall $(M^{\rho\sigma})^\mu{}_\nu = \eta^{\rho\mu}\delta^\sigma{}_\nu - \delta^\rho{}_\nu \eta^{\sigma\mu}$

$$\delta_L \phi(x) = -\frac{1}{2}\epsilon_{\rho\sigma}(\eta^{\rho\mu}x^\sigma - x^\rho\eta^{\sigma\mu})\partial_\mu\phi$$

$$= \frac{1}{2}\epsilon_{\rho\sigma}\underbrace{(x^\rho\partial^\sigma - x^\sigma\partial^\rho)}_{L^{\rho\sigma}}\phi$$

$L^{\rho\sigma}$ ← orbit angular momentum.

$$= \frac{1}{2}\epsilon_{\rho\sigma}L^{\rho\sigma}\phi.$$

e.g. A vector field $V^\mu(x)$

$$V'^\mu(x) - V^\mu(x) = V'^\mu(x'+x-x') - V^\mu(x)$$

$$= V'^\mu(x') - \delta x^\nu\partial_\nu V'^\mu(x') - V^\mu(x)$$

$$= \Lambda^\mu{}_\nu V^\nu(x) - V^\mu(x) - \delta x^\nu\partial_\nu(\Lambda^\mu{}_\rho V^\rho(x))$$

$$= (\delta^\mu{}_\nu + \omega^\mu{}_\nu)V^\nu - V^\mu(x) - \delta x^\nu\partial_\nu V^\mu$$

$$= -\omega^\nu{}_\lambda x^\lambda\partial_\nu V^\mu + \omega^\mu{}_\nu V^\nu$$

$$\omega^\nu{}_\lambda x^\lambda\partial_\nu = \frac{1}{2}\epsilon_{\rho\sigma}(M^{\rho\sigma})^\nu{}_\lambda x^\lambda\partial_\nu$$

$$= \frac{1}{2}\epsilon_{\rho\sigma}(\eta^{\rho\nu}\delta^\sigma{}_\lambda - \delta^\rho{}_\lambda\eta^{\sigma\nu})x^\lambda\partial_\nu$$

$$= \frac{1}{2}(\epsilon_{\rho\lambda}x^\lambda\partial^\rho - \epsilon_{\lambda\sigma}x^\lambda\partial^\sigma)$$

$$= -\frac{1}{2}\epsilon_{\lambda\rho}\left(x^\lambda\partial^\rho - x^\rho\partial^\lambda\right)$$

$$= -\frac{1}{2}\epsilon_{\lambda\rho} L^{\lambda\rho}$$

$$= \frac{1}{2}\epsilon_{\lambda\rho}\left(L^{\lambda\rho}\delta^\mu_\nu + (M^{\lambda\rho})^\mu{}_\nu\right)V^\nu$$

orbit angular momentum. Spin. angular momentum.

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e.g. Consider $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)$

translation inv. $\delta\phi = \epsilon^\mu\partial_\mu\phi \Rightarrow X_\mu = \partial_\mu\phi$.

Further, $\mathcal{L} = \mathcal{L}(x)$ is a scalar resp. to x

$$\Rightarrow \delta\mathcal{L} = \epsilon^\mu\partial_\mu\mathcal{L}$$

$$= \frac{\delta\mathcal{L}}{\delta\phi}\delta\phi + \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}\partial_\mu\delta\phi.$$

$$= \left(\frac{\delta\mathcal{L}}{\delta\phi} - \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}\right)\right)\delta\phi + \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}\delta\phi\right)$$

$$\Rightarrow \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}\delta\phi - \epsilon^\mu\mathcal{L}\right) = 0$$

i.e. $\partial_\mu\left(\frac{\delta \mathcal{L}}{\delta \partial_\mu \phi}\epsilon^\nu \partial_\nu \phi - \epsilon^\nu \delta_\nu{}^\mu \mathcal{L}\right) = 0$

$$\epsilon^\nu \partial_\mu\left(\frac{\delta \mathcal{L}}{\delta \partial_\mu \phi}\partial_\nu \phi - \delta_\nu{}^\mu \mathcal{L}\right) = 0$$

$T_\nu{}^\mu = \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi}\partial_\nu \phi - \delta_\nu{}^\mu \mathcal{L}$ is conserved

or $T^{\mu\nu} = \frac{\delta \mathcal{L}}{\delta \partial_\nu \phi}\partial^\mu \phi - \eta^{\mu\nu}\mathcal{L}$

$= \partial^\mu \phi \partial^\nu \phi - \eta^{\mu\nu}\mathcal{L}$. (symmetric tenser)

the conserved charges. $P^\mu = \int d^3x\, T^{\mu 0}(t, \vec{x})$

$\frac{dP^\mu}{dt} = 0 \Rightarrow$ 4-momentum conserved. $\bar{\mathbb{R}}$ translation inv.

Lorentz inv.

$\delta\phi = \frac{1}{2}\epsilon^{\rho\sigma} L_{\rho\sigma}\phi = \epsilon^{\rho\sigma} x_\rho \partial_\sigma \phi$., $L_{\rho\sigma} = x_\rho \partial_\sigma - x_\sigma \partial_\rho$.

$\delta \mathcal{L} = \epsilon^{\rho\sigma} x_\rho \partial_\sigma \mathcal{L} = \epsilon^{\rho\sigma}\partial_\sigma(x_\rho \mathcal{L}) - \underbrace{\epsilon^{\rho\sigma}\eta_{\sigma\rho}}_{0}\mathcal{L}$

$= \partial_\mu(\epsilon^{\rho\mu} x_\rho \mathcal{L})$

$$\Rightarrow \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}\delta\phi - \epsilon^{\rho\mu}x_\rho\mathcal{L}\right)$$

$\epsilon^{\rho\nu}\delta_\nu{}^\mu$

$$= \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}\epsilon^{\rho\nu}x_\rho\partial_\nu\phi - \epsilon^{\rho\mu}x_\rho\mathcal{L}\right)$$

$$= \partial_\mu\epsilon^{\rho\nu}\left(\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}x_\rho\partial_\nu\phi - \delta_\nu{}^\mu x_\rho\mathcal{L}\right)$$

$$= \partial_\mu\epsilon^{\rho\nu}\left(x_\rho\partial^\mu\phi\partial_\nu\phi - \delta_\nu{}^\mu\mathcal{L}\right)$$

$$= \partial_\mu\epsilon^{\rho\nu}\left(x_\rho T^\mu{}_\nu\right) = \frac{1}{2}\partial_\mu\epsilon^{\rho\nu}\left(x_\rho T^\mu{}_\nu - x_\nu T^\mu{}_\rho\right)$$

$$= \partial^\mu\epsilon^{\rho\nu}\frac{1}{2}\left(x_\rho T_{\mu\nu} - x_\nu T_{\mu\rho}\right)$$

$$\Rightarrow m_{\mu\nu\rho} = \frac{1}{2}\left(T_{\mu\nu}x_\rho - T_{\mu\rho}x_\nu\right)$$

charges of angular momentum.

$$M_{\nu\rho} = \int d^3x \; m^0{}_{\nu\rho}.$$