Consider the free electromagnetic field with a non-zero θ-angle placed in a regular cube box with volume V. The Lagrangian is given by

$$\mathcal{L} = -\frac{1}{4e^2} F^{\mu\nu} F_{\mu\nu} + \frac{i\theta}{32\pi^2} \tilde{F}^{\mu\nu} F_{\mu\nu} \,,$$

where $\tilde{F}^{\mu\nu}$ is defined as

$$\tilde{F}^{\mu\nu} \equiv \frac{i}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} \,.$$

We impose the periodic boundary condition to the gauge potential A_μ along the three space directions, and choose the Coulomb gauge. Now answer the following questions:

a. Rewrite $F^{\mu\nu}$ in terms of electromagnetic field $\vec{E}$ and $\vec{B}$, and show the Lagrangian can be spelt as

$$\mathcal{L} = \frac{1}{2e^2}(\vec{E}^2 - \vec{B}^2) - \frac{\theta}{8\pi^2} \vec{E} \cdot \vec{B} \,;$$

b. Find the canonical momentum $\vec{\pi}$, and thus the Hamilton is given by

$$H = \frac{1}{2} \int \mathrm{d}x^3 \; e^2 \left(\vec{\pi} + \frac{\theta}{8\pi^2} \vec{B} \right)^2 + \frac{1}{e^2} \vec{B}^2 \,.$$

Further show that (hint: you should take $\vec{\pi}$ as independent variables)

$$\frac{\partial H}{\partial e^2} = \frac{1}{2e^4} \int \mathrm{d}x^3 \; \left(\vec{E}^2 - \vec{B}^2 \right) \,, \quad \text{and} \quad \frac{\partial H}{\partial \theta} = \frac{1}{8\pi^2} \int \mathrm{d}x^3 \; \vec{E} \cdot \vec{B} \,;$$

c. Use canonical quantization to find that

$$\vec{A}(t, \vec{x}) = \sum_{\vec{k}} \sum_{s=\pm 1} \sqrt{\frac{e^2}{2\omega_k V}} \vec{\epsilon}_s(\vec{k}) \left(a_{\vec{k},\, s} e^{-ik\cdot x} + a^\dagger_{\vec{k},\, s} e^{ik\cdot x} \right) \,,$$

and specify ω_k and the 4-vector k^μ. Therefore show that

$$\frac{\partial H}{\partial e^2} = -\frac{1}{2e^4} \sum_{\vec{k}} \sum_{s=\pm 1} \omega_k \left(a_{\vec{k},\, s} a_{-\vec{k},\, s} + a^\dagger_{\vec{k},\, s} a^\dagger_{-\vec{k},\, s} \right) \,,$$

$$\frac{\partial H}{\partial \theta} = -\frac{ie^2}{16\pi^2} \sum_{\vec{k}} \sum_{s=\pm 1} s\, \omega_k \left(a_{\vec{k},\, s} a_{-\vec{k},\, s} - a^\dagger_{\vec{k},\, s} a^\dagger_{-\vec{k},\, s} \right) \,;$$

d. Define

$$O_{e^2} \equiv \frac{1}{4e^2} \sum_{\vec{k}} \sum_{s=\pm 1} \left(a_{\vec{k},\,s} a_{-\vec{k},\,s} + a^\dagger_{\vec{k},\,s} a^\dagger_{-\vec{k},\,s} \right) ,$$

$$O_\theta \equiv -\frac{ie^2}{32\pi^2} \sum_{\vec{k}} \sum_{s=\pm 1} s \left(a_{\vec{k},\,s} a_{-\vec{k},\,s} - a^\dagger_{\vec{k},\,s} a^\dagger_{-\vec{k},\,s} \right) .$$

Show that

$$\frac{\partial H}{\partial e^2} = [H, O_{e^2}] , \quad \text{and} \quad \frac{\partial H}{\partial \theta} = [H, O_\theta] ;$$

e. **Berry connection and curvature:** For a Hamilton $H(\lambda)$ dependent on a set of parameters $\{\lambda^i\}$, the associated eigenstates $|n(\lambda)\rangle$ and eigenvalues $E_n(\lambda)$ thus also depend on λ, given by

$$H(\lambda)|n(\lambda)\rangle = E_n(\lambda)|n(\lambda)\rangle ,$$

When λ^i changes adiabatically as $\lambda^i = \lambda^i_0 + \delta\lambda^i$, we expand,

$$H(\lambda) = H + \delta\lambda^i \frac{\partial H}{\partial \lambda^i} , \quad E_n(\lambda) = E_n + \delta\lambda^i \frac{\partial E_n}{\partial \lambda^i} , \quad \text{and} \quad |n(\lambda)\rangle = |n\rangle + \delta\lambda^i \frac{\partial |n\rangle}{\partial \lambda^i} ,$$

where H, $|n\rangle$ and E_n are the Hamilton, eigenstates and eigenvalues at λ_0 respectively. The Berry connection and curvature are defined as

$$\mathcal{A}^{(n)}_i \equiv \langle n|\partial_{\lambda^i}|n\rangle , \quad \text{and} \quad \mathcal{F}^{(n)}_{ij} \equiv \partial_{\lambda^i}\mathcal{A}^{(n)}_j - \partial_{\lambda^j}\mathcal{A}^{(n)}_i = \langle \partial_{\lambda^i} n|\partial_{\lambda^j} n\rangle - (i \leftrightarrow j) .$$

Show that the Berry curvature can be given by

$$\mathcal{F}^{(n)}_{ij} = \sum_{m \neq n} \frac{\langle n|\partial_{\lambda^i} H|m\rangle\langle m|\partial_{\lambda^j} H|n\rangle - (i \leftrightarrow j)}{(E_m - E_n)^2} ;$$

f. Compute the Berry curvature of the electromagnetic field for given states $|\vec{k}, s\rangle$ and $|\vec{k}', s'\rangle$

$$\mathcal{F}^{(\vec{k},s;\,\vec{k}',s')}_{e^2\theta} = \sum_{E_p \neq E_{k,k'}} \sum_{r=\pm 1} \frac{\langle \vec{k}, s|\partial_{e^2} H|\vec{p}, r\rangle\langle \vec{p}, r|\partial_\theta H|\vec{k}', s'\rangle - (e^2 \leftrightarrow \theta)}{(E_k - E_p)(E_p - E_{k'})} .$$

(Hint: you may use the result of question d to simplify your computation dramatically.)