Consider a Lagrangian consisted of N real scalar fields ϕ_i,

$$\mathcal{L} = \frac{1}{2}\sum_{i=1}^{N} \partial_\mu \phi_i \partial^\mu \phi_i - V(\phi^2)\,,$$

where $\phi^2 \equiv \sum_{i=1}^{N} \phi_i \phi_i$. Answering the following questions:

- Show that the Lagrangian enjoys a $SO(N)$ global symmetry.
  (Hint: It means that you need to find the infinitesimal transformations of the $SO(N)$ symmery on the fields ϕ_i, verify the Lagrangian is indeed invariant respect to these transformations, and compute the commutation relation of these generators. It would be helpful to first understand how many independent transformations you could have, and thus find proper basis to express them. Recall what I have shown to you the generators of $SO(1,3)$ in the class.);
- Write down the currents as well as the charges respect to the $SO(N)$ symmetry;
- Use canonical quantization to quantize the system, spell out the Hamiltonian, and verify that the charge operators found in the previous subquestion indeed commute with the Hamilton;
- Further verify that the charge operators themselves satisfy the same commutation relation of the $SO(N)$ symmetry as they should be.

Solution: Consider the transformation

$$\phi_i \longrightarrow \phi_i' = \sum_{j=1}^{N} R_{ij}\phi_j \qquad \text{with} \quad R \in SO(N)\,. \tag{1}$$

Obviously, the kinetic term and potential are invariant respect to (1). Or infinitesimally, we expand R-matrix to be

$$R_{ij} = \delta_{ij} + \frac{1}{2}\epsilon^{ab}\left(T_{ab}\right)_{ij}\,, \tag{2}$$

where T_{ab} are the basis of $\mathfrak{so}(N)$ Lie algebra, given by

$$\left(T_{ab}\right)_{ij} = \delta_{ai}\delta_{bj} - \delta_{aj}\delta_{bi} \tag{3}$$

Therefore we have

$$\delta_\epsilon \phi_i = \phi_i' - \phi_i = \sum_{j=1}^{N} \frac{1}{2} \epsilon^{ab} \left(T_{ab}\right)_{ij} \phi_j \,. \tag{4}$$

Notice that, for $SO(N)$, we have $\frac{N(N-1)}{2}$ independent generators labeled by the anti-symmetrized indice "ab". In each direction, we denote

$$\delta_{ab} \phi_i = \sum_{j=1}^{N} \left(T_{ab}\right)_{ij} \phi_j \,. \tag{5}$$

Using (4), one can easily check the Lagrangian is invariant respect to the $SO(N)$-symmetry,

$$\begin{aligned}\delta_\epsilon \mathcal{L} &= \sum_{i=1}^{N} \partial_\mu \left(\sum_{j=1}^{N} \frac{1}{2} \epsilon^{ab} \left(T_{ab}\right)_{ij} \phi_j \right) \partial^\mu \phi_i + \sum_{i=1}^{N} \frac{\partial V}{\partial \phi^2} \times (2\phi_i) \times \left(\sum_{j=1}^{N} \frac{1}{2} \epsilon^{ab} \left(T_{ab}\right)_{ij} \phi_j \right) \\ &= \sum_{i,j} \frac{1}{2} \epsilon^{ab} \left(T_{ab}\right)_{ij} \left(\partial_\mu \phi_i \partial^\mu \phi_j + 2 \frac{\partial V}{\partial \phi^2} \phi_i \phi_j \right) = 0 \,, \end{aligned} \tag{6}$$

where we have used the anti-symmetric property of T_{ab}, i.e.

$$\left(T_{ab}\right)_{ij} = - \left(T_{ab}\right)_{ji} \,. \tag{7}$$

Now, with these preparations, we can spell down the currents respect to $SO(N)$-symmetry via the standard Neother procedure,

$$J^\mu_{ab} = \sum_i \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \phi_i\right)} \delta_{ab} \phi_i = \sum_{i,j} \left(T_{ab}\right)_{ij} \phi_j \partial^\mu \phi_i \,, \tag{8}$$

and correspondingly the charges

$$Q_{ab} = \int \mathrm{d}\vec{x}^{\,3} J^0_{ab} = \int \mathrm{d}\vec{x}^{\,3} \sum_{i,j} \left(T_{ab}\right)_{ij} \dot{\phi}_i \, \phi_j \tag{9}$$

In canonical quantization, one finds

$$\pi_i = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_i} = \dot{\phi}_i \,, \tag{10}$$

and imposes the commutation relation as

$$[\phi_i(t,\vec{x}),\ \ \pi_j(t,\vec{y})] = i\delta_{ij}\delta^{(3)}(\vec{x}-\vec{y})\,. \tag{11}$$

The Hamilton and the charges are thus spelt as

$$\begin{aligned} H &= \int \mathrm{d}\vec{x}^{\,3}\,\frac{1}{2}\sum_i\left(\pi_i^2 + \vec{\nabla}\phi_i\cdot\vec{\nabla}\phi_i\right) + V(\phi^2)\,, \\ Q_{ab} &= \int \mathrm{d}\vec{x}^{\,3}\sum_{i,j}\left(T_{ab}\right)_{ij}\pi_i\,\phi_j\,. \end{aligned} \tag{12}$$

Using the anti-symmetric property of the indice "ij" of $(T_{ab})_{ij}$, it's not hard to show that, e.g.

$$\begin{aligned} \left[Q_{ab},\ \ V\left(\phi^2(t,\vec{y})\right)\right] &= \int \mathrm{d}\vec{x}^{\,3}\sum_{i,j}\left(T_{ab}\right)_{ij}\phi_j(t,\vec{x})\left[\pi_i(t,\vec{x}),\ \ V\left(\phi^2(t,\vec{y})\right)\right] \\ &= \int \mathrm{d}\vec{x}^{\,3}\sum_{i,j,k}\left(T_{ab}\right)_{ij}\phi_j(t,\vec{x})\times\left(2\frac{\partial V}{\partial\phi^2}\phi_k(t,\vec{y})\right)\left[\pi_i(t,\vec{x}),\ \ \phi_k(t,\vec{y})\right] \\ &= i\sum_{i,j}\left(T_{ab}\right)_{ij}\phi_j(t,\vec{y})\left(2\frac{\partial V}{\partial\phi^2}\right)\phi_i(t,\vec{y}) = 0\,. \end{aligned} \tag{13}$$

Therefore, one can easily show that the Hamilton H and Q_{ab} are commuting, i.e.

$$[H,\ \ Q_{ab}] = 0\,. \tag{14}$$

At last, let's compute

$$\begin{aligned} [Q_{ab},\ \ Q_{cd}] &= \int \mathrm{d}\vec{x}^{\,3}\mathrm{d}\vec{y}^{\,3}\sum_{i,j,k,l}\left(T_{ab}\right)_{ij}\left(T_{cd}\right)_{kl}\left[\pi_i\,\phi_j(t,\vec{x}),\ \ \pi_k\,\phi_l(t,\vec{y})\right] \\ &= i\int \mathrm{d}\vec{x}^{\,3}\sum_{i,j,k,l}\left(T_{ab}\right)_{ij}\left(T_{cd}\right)_{kl}\left(\pi_i\phi_l\delta_{jk} - \delta_{il}\pi_k\phi_j\right) \\ &= i\int \mathrm{d}\vec{x}^{\,3}\sum_{i,l}\left[T_{ab},\ \ T_{cd}\right]_{il}\pi_i\phi_l\,, \end{aligned} \tag{15}$$

where, to get the third equality, the indices of the second term in the second line has been reshuffled. Applying the commutation relation of $\mathfrak{so}(N)$ Lie algebra,

$$[T_{ab},\ \ T_{cd}] = -\delta_{ac}T_{bd} + \delta_{ad}T_{bc} + \delta_{bc}T_{ad} - \delta_{bd}T_{ac}\,, \tag{16}$$

we finally find

$$[Q_{ab},\ \ Q_{cd}] = -i\delta_{ac}Q_{bd} + i\delta_{ad}Q_{bc} + i\delta_{bc}Q_{ad} - i\delta_{bd}Q_{ac}\,. \tag{17}$$