Consider a $1+1$D massless Dirac fermion with Lagrangian

$$\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi\,.$$

on a circle of perimeter L. Recall in the class that, we can choose the 2D Gamma matrices as $\gamma^\mu = (\sigma_2,\, i\sigma_1)$, therefore the 2D Dirac fermion is a two-component spinor. Further, for fermions, we need to impose anti-perodic boundary condition as

$$\psi(t, x+L) = -\psi(t,x)\,.$$

Now answer the following questions:

a. Show that the solution to the Dirac equation can be written as

$$\psi(t,x) = \frac{1}{\sqrt{L}}\sum_{n=0}^{+\infty}\begin{pmatrix} c_n e^{-i\omega_n(t-x)} \\ d_n^\dagger e^{i\omega_n(t+x)}\end{pmatrix}\,,$$

and specify the value of ω_n. Further use canonical quantization to diagonalize the Hamilton $\hat{H}$ in terms of c_n and d_n. Can you interpret the Dirac spinor ψ as a 2D electron-positron system? Explain your reason;

b. The Lagrangian admits a $U(1)$-symmetry respect to $\psi \to e^{i\alpha}\psi$, with α a constant phase factor. The associated $U(1)$-charge $\hat{N}$ is called fermionic number. Find the fermionic number $\hat{N}$ in terms of c_n and d_n, and show that it can be interpreted as the difference between the numbers of positrons and electrons;

c. Define $q = e^{-\omega/T}$ and $z = e^{\mu/T}$ for the energy and the fermionic number fugacities, then the grand canonical partition function is defined as

$$Z(q,z) = \mathrm{Tr}\left(e^{-\hat{H}/T + \mu\hat{N}/T}\right) = \sum_n \left\langle n|e^{-\hat{H}/T+\mu\hat{N}/T}|n\right\rangle\,.$$

Show that the partition function is given by

$$Z(q,z) = \prod_{n=1}^{\infty}\left(1 + z\,q^{n-\frac{1}{2}}\right)\left(1 + z^{-1}q^{n-\frac{1}{2}}\right)\,;$$

d. From an old perspective of "Dirac sea" (The vacuum is the state in which all negative energy states are filled. Therefore an electron is a state created above the vacuum, while a positron is a "hole" state where all negative energy states are occupied except one), the partition function can be recast as

$$Z(q,z) = \sum_{\substack{\text{fermion}\\ \text{occupations}}} e^{-E/T+\mu N/T} = \sum_{N=-\infty}^{+\infty} z^N Z_N(q)\,,$$

where $Z_N(q)$ is the partition function counting states for a given fermionic number N. Show that

$$Z_0(q) = \prod_{n=1}^{\infty} \frac{1}{1-q^n}\,,$$

and thus

$$Z_N(q) = q^{\frac{N^2}{2}} Z_0(q)\,.$$

Now combining the result of (c), you can give a physical proof of the famous Jacobi triple identity

$$\prod_{n=1}^{\infty} (1-q^n)\left(1+z\,q^{n-\frac{1}{2}}\right)\left(1+z^{-1}q^{n-\frac{1}{2}}\right) = \sum_{N=-\infty}^{+\infty} z^N q^{\frac{N^2}{2}}\,.$$