Pions, $\pi^{0,\pm}$, are three spin 0 bosons that play the role of a strong force mediator between hardrons. They are described by 3 real scalar fields that enjoy a $SO(3)$ symmetery, denoted as "isospin" (because such symmetery is not for spin, but an internal symmetey of the pion theory). The Lagrangian of the pion theory is given by

$$\mathcal{L} = \frac{1}{2}\sum_{i=1}^{3}\left(\partial_\mu\phi_i\partial^\mu\phi_i \,-m^2\phi_i\phi_i\right) + \cdots\,,$$

where the "$\cdots$" are the interaction terms between pions and other hardrons that won't bother us so far. We want to consider some baby versions of the pion theory. First, let's forget about the interaction terms, and reduce the spacetime dimension from $1+3$D to $1+0$D, i.e. a quantum mechanical problem with Lagarange

$$L = \frac{1}{2}\sum_{i=1}^{3}\left(\dot{q}_i\dot{q}_i, -m^2 q_i q_i\right).$$

- Using canonical quantization, show that the Hamilton can be diagonalized in terms of creation and annihilation operators a_i and $a_i^\dagger$, i.e.,

  $$\hat{H} = \omega\sum_{i=1}^{3}\left(a_i^\dagger a_i + \frac{1}{2}\right)\,,$$

  and specify the value of ω;

- The Lagrange has a $SO(3)$-symmetey, or say the isospin, and thus conserved charges $\hat{I}_i$. Write them in terms of a_i and $a_i^\dagger$, and verify their $\mathfrak{so}(3)$ commutation relations;

- Since the Lagrange admits the isospin symmetry, all its quantum states are classified by the representations of it, i.e. $\mathfrak{so}(3)$. Explain the ground state, the first and second excited states are in which representations of $\mathfrak{so}(3)$. (Hint: you need to choose a proper basis, and find the eigenvalues of $\hat{I}_3$ for these states);

- Define $q = e^{-\omega/T}$ and $z = e^{\mu/T}$ as the fugacities for the energy and the isospin $\hat{I}_3$, then the grand canonical partition function is given by

  $$Z(q,z) = q^{-\frac{3}{2}}\mathrm{Tr}\left(e^{-\hat{H}/T+\mu\hat{I}_3/T}\right) = q^{-\frac{3}{2}}\sum_n\left\langle n|e^{-\hat{H}/T+\mu\hat{I}_3/T}|n\right\rangle\,,$$

where n runs over all quantum states, and the prefactor $q^{-\frac{3}{2}}$ is used to offset the zero point energy. Find $Z(q,z)$ and expand it to the order of $\mathcal{O}(q^2)$, and check consistency with your result in the previous question. (Hint: you may find useful to first write down the partition function of a single harmonic oscillator);

- Now we lift the theory to $1+1$D, but for simplicity put it in a box of length L and set mass $m=0$. Then the theory has a Lagrangian

  $$\mathcal{L} = \frac{1}{2}\sum_{i=1}^{3} \partial_\mu \phi_i \partial^\mu \phi_i \, ,$$

  with a fixed Dirichlet boundary condition along the spatial dierction,

  $$\phi(t,0) = \phi(t,L) = 0$$

  Repeat the previous procedures and write down the partition function of this theory. Can you classify the third excited states are in which representations of $\mathfrak{so}(3)$?

Solution:

1. The first question is trivial, $\omega = m$.

2. From the first homework, you should know how to write down the charges in terms of q_i and $\dot{q}_i$. Further in the first question, you should find

$$q_i(t) = \frac{1}{\sqrt{2\omega}}\left(a_i e^{-i\omega t} + a_i^\dagger e^{i\omega t}\right)$$

Therefore, you can write down

$$\hat{I}_1 = a_2^\dagger a_3 - a_3^\dagger a_2, \quad \hat{I}_2 = a_3^\dagger a_1 - a_1^\dagger a_3, \quad \text{and} \quad \hat{I}_3 = a_1^\dagger a_2 - a_2^\dagger a_1 \, .$$

Using $[a_i,\, a_j^\dagger] = \delta_{ij}$, we find

$$[\hat{I}_i,\, \hat{I}_j] = -\epsilon_{ijk}\, \hat{I}_k \, .$$

Redefining $I_i \to -iI_i$, one recovers the standard $\mathfrak{so}(3)$ algebra,

$$[\hat{I}_i,\, \hat{I}_j] = i\epsilon_{ijk}\, \hat{I}_k \, .$$

3. Notice that a representation of $\mathfrak{so}(3)$, denoted by R_s, is labled by an half integer s, with dimension $\dim R_s = 2s+1$. Choosing $\hat{I}_3$, the third generator of $\mathfrak{so}(3)$, any states in R_s can be labeled by the $\hat{I}_3$-charge in the range of $\{-s, \dots, +s\}$.

With these preparetion, let's first check the vacuum state $|0\rangle$. Obviously

$$\hat{I}_i|0\rangle = 0 \quad \text{for} \quad i = 1, 2, 3\,.$$

Therefore the vacuum state is in the trivial Rep of $\mathfrak{so}(3)$, say R_0. Next let's check excitation states. For convenience, we define a new basis

$$\alpha_+ \equiv \frac{1}{\sqrt{2}}\left(a_1^\dagger + i a_2^\dagger\right)\,, \quad \alpha_- \equiv \frac{1}{\sqrt{2}}\left(a_1^\dagger - i a_2^\dagger\right)\,, \quad \text{and} \quad \alpha_3 \equiv a_3^\dagger\,,$$

in terms of which, one can check

$$[\hat{I}_3,\, \alpha_\pm] = \pm\alpha_\pm\,, \quad \text{and} \quad [\hat{I}_3,\, \alpha_3] = 0\,.$$

Now let's scan the first excitation states, which is consisted of three states

$$|\alpha_\pm\rangle = \alpha_\pm|0\rangle\,, \quad \text{and} \quad |\alpha_3\rangle = \alpha_3|0\rangle\,.$$

Acting $\hat{I}_3$ on them, we have

$$\hat{I}_3|\alpha_\pm\rangle = \pm|\alpha_\pm\rangle\,, \quad \text{and} \quad \hat{I}_3|\alpha_3\rangle = 0\,,$$

i.e. the three states have $\hat{I}_3$-charges $\{-1, 0, +1\}$, and thus furnish a vector Rep of $\mathfrak{so}(3)$, R_1. Further, one collects all 6 second excitation states by acting $\alpha_\pm$ and α_0 on the vacuum subsequently,

$$\alpha_+\alpha_+|0\rangle, \quad \alpha_+\alpha_3|0\rangle, \quad \alpha_+\alpha_-|0\rangle, \quad \alpha_3\alpha_3|0\rangle, \quad \alpha_3\alpha_-|0\rangle, \quad \text{and} \quad \alpha_-\alpha_-|0\rangle\,,$$

with the $\hat{I}_3$-charges {2,1,0,0,-1,-2} respectively. Therefore the only possible way to furnish Rep's of $\mathfrak{so}(3)$ are $\{-2, -1, 0, +1, +2\}$ and $\{0\}$, i.e R_2 and R_0.

Remark: If you know more on group representation, you can easily realize that, because $[a_i^\dagger, a_j^\dagger] = 0$, the second excitation states $|ij\rangle \equiv a_i^\dagger a_j^\dagger|0\rangle$ are symmetric respect to indices i, j. Therefore these states furnish a 2-symmetric traceless Rep, i.e. R_2, and a singlet trace state, R_0.

4. First let us review the partition function of a single harmonic oscillator. Notice that each state, labeled by $|n\rangle$, has energy

$$\hat{H}|n\rangle = \left(n + \frac{1}{2}\right)\omega|n\rangle$$

Therefore, the partition function is given by

$$Z(q) = q^{-\frac{1}{2}} \sum_n \left\langle n|e^{-\hat{H}/T}|n\right\rangle = 1 + q + q^2 + \cdots = \frac{1}{1-q}\,,$$

where we have multiplied a prefactor $q^{-\frac{1}{2}}$ to offset the vacuum energy.

Now let's go back to our problem. We have seen that all states can be generated by acting $\alpha_\pm$ and α_3 on the vacuum successively. Since the three α's commute with each other, we can thus decompse the full partition function to

$$Z(q,z) = Z_{\alpha_+}(q,z) \cdot Z_{\alpha_-}(q,z) \cdot Z_{\alpha_3}(q,z)\,,$$

where $Z_{\alpha_{\pm,3}}$ denote the partition function respect to the creation operator $\alpha_\pm$ and α_3 respectively. Let's first compute Z_{α_+}. Recall that $[\hat{I}_3,\, \alpha_+] = +\alpha_+$, we thus have

$$\hat{I}_3 \cdot \alpha_+^n|0\rangle \equiv \hat{I}_3|n,+\rangle = n|n,+\rangle$$

Therefore

$$\begin{aligned} Z_{\alpha_+}(q,z) &= q^{-\frac{1}{2}} \sum_n \left\langle n,+|\, e^{-\hat{H}/T+\mu\hat{I}_3/T}|n,+\right\rangle \\ &= 1 + zq + z^2q^2 + \cdots = \frac{1}{1-zq}\,. \end{aligned}$$

Simlarly, for $Z_{\alpha_-}(q,z)$ and $Z_{\alpha_3}(q,z)$, we have

$$Z_{\alpha_+}(q,z) = \frac{1}{1-z^{-1}q}\,, \quad \text{and} \quad Z_{\alpha_3}(q,z) = \frac{1}{1-q}\,.$$

Therefore the full partition function is given by

$$Z(q,z) = \frac{1}{(1-q)(1-zq)(1-z^{-1}q)}$$

Expand it to $\mathcal{O}(q^2)$, we have

$$Z(q,z) = 1 + (z^{-1} + 1 + z)q + \left(\left(z^{-2} + z^{-1} + 1 + z + z^2\right) + 1\right) q^2 + \mathcal{O}(q^3)\,.$$

From the partition function, you can clearly see the $\hat{I}_3$-charges of each states as the powers of variable z, and thus verify yourselves that they furnish which representation of the $\mathfrak{so}(3)$ algebra.

5. For the $1+1$D massless real scalar theory, since it is fixed at the ends of spatial direction of length L with the boundary condition,

$$\phi(t,0) = \phi(t,L) = 0\,,$$

we can spell out the ansatz for $\phi_i(t,x)$ as

$$\phi_i(t,x) = \sum_{n=1}^{\infty} A_{n,i}(t) \sin\left(\frac{\pi n}{L} x\right)\,.$$

Therefore, solving the equation of motion for ϕ_i

$$\partial_t^2 \phi_i - \partial_x^2 \phi_i = 0\,,$$

we must require

$$\ddot{A}_{n,i}(t) + \omega_n^2 A_{n,i}(t) = 0\,,$$

with $\omega_n = \frac{\pi n}{L}$. We thus solve ϕ_i as

$$\phi_i(t,x) = \sum_{n}^{\infty} \frac{1}{\sqrt{\omega_n L}} \sin\left(\frac{\pi n}{L} x\right) \left(a_{n,i} e^{-i\omega_n t} + a_{n,i}^{\dagger} e^{i\omega_n t}\right)\,,$$

where we choose the normalization $\frac{1}{\sqrt{\omega_n L}}$ so that $a_{n,i}$ and $a_{n,i}^{\dagger}$ are dimensionless. Through the standard canonical quantization, one can find

$$[a_{m,i},\, a_{n,j}^{\dagger}] = \delta_{ij}\delta_{mn}\,,$$

and further the Hamilton is given by

$$H = \frac{1}{2}\sum_{i=1}^{3} \int_0^L \mathrm{d}x\, : \pi_i^2 + (\partial_x \phi_i)^2 := \sum_{i=1}^{3}\sum_{n=1}^{\infty} \omega_n a_{n,i}^{\dagger} a_{n,i}\,.$$

On the other hand, you should find the three isospin charges $\hat{I}_i$ are given by

$$\hat{I}_i = -\frac{i}{2} \sum_{n=1}^{\infty} \sum_{j,k=1}^{3} \epsilon_{ijk} a^\dagger_{n,j} a_{n,k} \, .$$

Because of the commutation relation of $a_{m,i}$ and $a^\dagger_{n,j}$, the system can be regarded as infinite number sets of independent harmonic oscillators made of $a_{n,i}$ and $a^\dagger_{n,i}$. Similar to the quantum mechanics case, we re-group these oscillators as

$$\alpha_{n,\pm} \equiv \frac{1}{\sqrt{2}} \left(a^\dagger_{n,1} \pm i a^\dagger_{n,2} \right) \, , \quad \text{and} \quad \alpha_{n,3} \equiv a^\dagger_{n,3} \, .$$

One finds that, for each set of $\alpha_{n,\pm}$ and $\alpha_{n,3}$,

$$[\hat{H}, \, \alpha_{n,\pm}] = \omega_n \alpha_{n,\pm} \, , \qquad [\hat{H}, \, \alpha_{n,3}] = \omega_n \alpha_{n,3} \, ,$$

and

$$[\hat{I}_3, \, \alpha_{n,\pm}] = \pm \alpha_{n,\pm} \, , \quad \text{and} \quad [\hat{I}_3, \, \alpha_{n,3}] = 0 \, .$$

Therefore, you should write down the partition function for the set of $\alpha_{n,\pm}$ and $\alpha_{n,3}$ as

$$Z_n(z,q) = \frac{1}{(1-q^n)(1-zq^n)(1-z^{-1}q^n)} \, ,$$

where $q = e^{-\frac{\pi}{LT}}$. Therefore, overall, the full partition function is given by

$$Z(q,z) = \prod_{n=1}^{\infty} Z_n(z,q) = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)(1-zq^n)(1-z^{-1}q^n)} \, .$$

To see the third excitation states, we focus on the order of q^3 in $Z(q,z)$. Read off the coefficient of q^3:

$$\begin{aligned} \mathcal{O}(q^3): \quad & \left(z^{-3} + z^{-2} + z^{-1} + 1 + z + z^2 + z^3\right) + \left(z^{-2} + z^{-1} + 1 + z + z^2\right) \\ & + 3\left(z^{-1} + 1 + z\right) + 1 \end{aligned}$$

It tells us that, at the third excitation states, there are 22 states, corresponding to the representations of isospin: one R_3, one R_2, three R_1 and one singlet R_0.