Consider an anharmonic oscillator with Hamilton

$$\hat{H} = \frac{1}{2}\hat{p}^2 + \frac{\omega^2}{2}\hat{q}^2 + \frac{\lambda}{24}\hat{q}^4 + \frac{\kappa}{720}\hat{q}^6 \,.$$

where $\lambda/\omega^3,\ \kappa/\omega^4 \ll 1$ are assumed. The two non-quadartic interaction terms can be treated as perturbations to the usual harmonic oscillator. Therefore the eigenvalues of $\hat{H}$, $E_n(\lambda, \kappa)$, can be computed perturbatively respect to those of harmonic oscillator as the expansion of the couplings of λ and κ

$$E_n(\lambda,\, \kappa) = \left(n + \frac{1}{2}\right)\omega + \mathcal{O}(\lambda,\, \kappa)$$

We focus on computing the ground state energy up to the second order of λ and κ,

$$E_0(\lambda,\, \kappa) = \frac{\omega}{2} + \sum_{m+n\leq 2} a_{m,n}(\omega)\lambda^m\kappa^n + \cdots$$

where $a_{m,n}(\omega)$ are some unknown coefficients to be determined.

a. Use the perturbation approach in your QM class to compute $a_{m,n}(\omega)$.

b. Formulate the problem in the path integral appraoch, and analytically continue the system to Euclidean time $t \to -i\tau$. In the low temperature limit $\beta \to \infty$, write down the Feynman rules for the propagator and the interaction terms.

c. Compute the free energy W with source $J = 0$, say the vacuum bubble diagrams, up to the second order of λ and κ. Compare it to the result of your question a, and explain if they are consistent with each other.

   (Hint: It is easier to perform your computation directly in the coordinate space, i.e. the τ-coordinate. But I encourage you to try it also in the momentum space to get familar with the art of Feynman diagram calculation.)