Pions, $\pi^{0,\pm}$, are three spin 0 bosons that play the role of a strong force mediator between hardrons. They are described by 3 real scalar fields that enjoy a $SO(3)$ symmetery, denoted as "isospin" (because such symmetery is not for spin, but an internal symmetey of the pion theory). The Lagrangian of the pion theory is given by

$$\mathcal{L} = \frac{1}{2}\sum_{i=1}^{3}\left(\partial_\mu\phi_i\partial^\mu\phi_i \; - m^2\phi_i\phi_i\right) + \cdots \, ,$$

where the "$\cdots$" are the interaction terms between pions and other hardrons that won't bother us so far. We want to consider some baby versions of the pion theory. First, let's forget about the interaction terms, and reduce the spacetime dimension from $1+3$D to $1+0$D, i.e. a quantum mechanical problem with Lagarange

$$L = \frac{1}{2}\sum_{i=1}^{3}\left(\dot{q}_i\dot{q}_i, -m^2 q_i q_i\right) .$$

- Using canonical quantization, show that the Hamilton can be diagonalized in terms of creation and annihilation operators a_i and $a_i^\dagger$, i.e.,

  $$\hat{H} = \omega\sum_{i=1}^{3}\left(a_i^\dagger a_i + \frac{1}{2}\right) ,$$

  and specify the value of ω;

- The Lagrange has a $SO(3)$-symmetey, or say the isospin, and thus conserved charges $\hat{I}_i$. Write them in terms of a_i and $a_i^\dagger$, and verify their $\mathfrak{so}(3)$ commutation relations;

- Since the Lagrange admits the isospin symmetry, all its quantum states are classified by the representations of it, i.e. $\mathfrak{so}(3)$. Explain the ground state, the first and second excited states are in which representations of $\mathfrak{so}(3)$. (Hint: you need to choose a proper basis, and find the eigenvalues of $\hat{I}_3$ for these states);

- Define $q = e^{-\omega/T}$ and $z = e^{\mu/T}$ as the fugacities for the energy and the isospin $\hat{I}_3$, then the grand canonical partition function is given by

  $$Z(q,z) = q^{-\frac{3}{2}}\mathrm{Tr}\left(e^{-\hat{H}/T+\mu\hat{I}_3/T}\right) = q^{-\frac{3}{2}}\sum_n \left\langle n|e^{-\hat{H}/T+\mu\hat{I}_3/T}|n\right\rangle ,$$

where n runs over all quantum states, and the prefactor $q^{-\frac{3}{2}}$ is used to offset the zero point energy. Find $Z(q, z)$ and expand it to the order of $\mathcal{O}(q^2)$, and check consistency with your result in the previous question. (Hint: you may find useful to first write down the partition function of a single harmonic oscillator);

- Now we lift the theory to $1 + 1$D, but for simplicity put it in a box of length L and set mass $m = 0$. Then the theory has a Lagrangian

  $$\mathcal{L} = \frac{1}{2} \sum_{i=1}^{3} \partial_\mu \phi_i \partial^\mu \phi_i \, ,$$

  with a fixed Dirichlet boundary condition along the spatial dierction,

  $$\phi(t, 0) = \phi(t, L) = 0$$

  Repeat the previous procedures and write down the partition function of this theory. Can you classify the third excited states are in which representations of $\mathfrak{so}(3)$?