COPYING *Flash Cards for the Book:* *"Representations and Characters of Groups"* *by Gordon James and Martin Liebeck* REPRESENTATION THEORY	DEFINITION *group* REPRESENTATION THEORY
DEFINITION *subgroup* REPRESENTATION THEORY	DEFINITION *dihedral group* D_{2n} REPRESENTATION THEORY
DEFINITION *cyclic group* C_n REPRESENTATION THEORY	DEFINITION *quaternion group* Q_8 REPRESENTATION THEORY
DEFINITION *alternating group* A_n REPRESENTATION THEORY	DEFINITION *direct product* REPRESENTATION THEORY
DEFINITION *homomorphism / isomorphism* REPRESENTATION THEORY	DEFINITION *coset* REPRESENTATION THEORY

A **group** consists of a set G, together with a rule for combining any two elements $g, h \in G$ to form another element of G satisfying:

1. $\forall g, h, k \in G, (gh)k = g(hk)$
2. $\exists e \in G$ such that $\forall g \in G, eg = ge = g$
3. $\forall g \in G, \exists g^{-1} \in G$ such that $gg^{-1} = g^{-1}g = e$

$$D_{2n} = \left\langle a, b \,:\, a^n = 1, b^2 = 1, b^{-1}ab = a^{-1} \right\rangle$$

Let G be a group. A subset H of G is a **subgroup** if H is itself a group under the operation inherited from G.

$$H \leqslant G$$

$$Q_8 = \left\langle a, b \,:\, a^4 = 1, a^2 = b^2, b^{-1}ab = a^{-1} \right\rangle$$

$$C_n = \left\{1, a, a^2, \ldots, a^{n-1}\right\}$$
$$C_n = \langle a \,:\, a^n = 1 \rangle$$

Let G and H be groups, consider

$$G \times H = \{(g, h) \,:\, g \in G \text{ and } h \in H\}.$$

Define a product operation on $G \times H$ by

$$(g, h)(g', h') = (gg', hh').$$

The group $G \times H$ is called the **direct product** of G and H.

$$A_n = \{g \in S_n : g \text{ is an even permutation}\}$$

Recall that every permutation $g \in S_n$ can be expressed as a product of transpositions. An **even** permutation has an even number of transpositions, and an **odd** permutation has an odd number of transpositions.

Let G be a group and H a subgroup of G. For $x \in G$, the subset

$$Hx = \{hx \,:\, h \in H\}$$

of G is called a **right coset** of H in G. The distinct right cosets of G partition G.

If G and H are groups, then a **homomorphism** from G to H is a map $\vartheta : G \to H$, which for all $g_1, g_2 \in G$ satisfies:

$$(g_1 g_2)\vartheta = (g_1\vartheta)(g_2\vartheta).$$

If ϑ is also invertible, then ϑ is called an **isomorphism**.

THEOREM *Lagrange's theorem* REPRESENTATION THEORY	DEFINITION *index* REPRESENTATION THEORY
DEFINITION *normal subgroup* REPRESENTATION THEORY	DEFINITION *factor group* REPRESENTATION THEORY
DEFINITION *simple group* REPRESENTATION THEORY	DEFINITION *kernel / image* REPRESENTATION THEORY
THEOREM *first isomorphism theorem* REPRESENTATION THEORY	DEFINITION *vector space* REPRESENTATION THEORY
DEFINITION *linear dependence / linear independence* REPRESENTATION THEORY	DEFINITION *linear combination / span* REPRESENTATION THEORY

Suppose H is a subgroup of G. The number of distinct right cosets of H in G is written as $|G : H|$. If G is finite, then

$$|G : H| = |G| \,/\, |H|$$

by Lagrange's theorem.

If G is a finite group and H is a subgroup of G, then $|H|$ divides $|G|$.

If $N \lhd G$, then define G/N to be the set of right cosets of N in G. This set is made into a group via the multiplication operation:

$$(Ng)(Nh) = Ngh \qquad \forall g, h \in G.$$

This operation makes G/N into a group called the **factor group** of G by N.

A subgroup N of a group G is said to be a **normal** subgroup of G if $g^{-1}Ng = N$ for all $g \in G$, where

$$g^{-1}Ng = \left\{ g^{-1}ng \,:\, n \in N \right\}.$$

We indicate that N is a normal subgroup of G by writing:

$$N \lhd G.$$

Let G and H be groups. Suppose that

$$\vartheta : G \to H$$

is a homomorphism then the **kernel** of ϑ and **image** of ϑ are defined to be:

$$\begin{array}{lll} \operatorname{Ker}\vartheta = \{g \in G \,:\, g\vartheta = 1\} & & \operatorname{Ker}\vartheta \lhd G \\ \operatorname{Im}\vartheta \ = \{g\vartheta \,:\, g \in G\} & & \operatorname{Im}\vartheta \ \leqslant H \end{array}$$

A group G is said to be **simple** if $G \neq \{1\}$ and the only normal subgroups of G are $\{1\}$ and G.

A **vector space** over a field F is a set V, equipped with addition and scalar multiplication satisfying:

1. V is an abelian group under addition;
2. $\forall\, u, v \in V$ and $\forall\, \lambda, \mu \in F$,
   (a) $\lambda(u + v) = \lambda u + \lambda v$
   (b) $(\lambda + \mu)v = \lambda v + \mu v$
   (c) $(\lambda\mu)v = \lambda(\mu v)$
   (d) $1v = v$

Suppose that G and H are groups and let $\vartheta : G \to H$ be a homomorphism. Then

$$G/\operatorname{Ker}\vartheta \cong \operatorname{Im}\vartheta.$$

An isomorphism is given by the function

$$Kg \to g\vartheta \quad (g \in G)$$

where $K = \operatorname{Ker}\vartheta$.

Let $v_1, \ldots, v_n$ be vectors in a vector space V over F. A vector v in V is a **linear combination** of $v_1, \ldots, v_n$ if

$$v = \lambda_1 v_1 + \cdots + \lambda_n v_n$$

for some $\lambda_1, \ldots, \lambda_n \in F$.

The vectors $v_1, \ldots, v_n$ **span** V if every vector in V is a linear combination of $v_1, \ldots, v_n$.

We say that $v_1, \ldots, v_n$ are **linearly dependent** if

$$\lambda_1 v_1 + \cdots + \lambda_n v_n = 0$$

for some $\lambda_1, \ldots, \lambda_n \in F$ not all zero, otherwise the vectors $v_1, \ldots, v_n$ are **linearly independent**.

Definition *basis* Representation Theory	Definition/Theorem *subspace / conditions for a subspace* Representation Theory
Definition *sum / direct sum* Representation Theory	Theorem *conditions for a direct sum* Representation Theory
Theorem *direct sum of direct sums* Representation Theory	Definition *external direct sum* Representation Theory
Definition *linear transformation* Representation Theory	Theorem *rank–nullity theorem* Representation Theory
Theorem *invertibility of linear transformations* Representation Theory	Definition *endomorphism* Representation Theory

A **subspace** of a vector space V over F is a subset of V which is itself a vector space under the operations inherited from V.

A subset U of a vector space V is a subspace iff

1. $0 \in U$;
2. if $u, v \in U$ then $u + v \in U$;
3. if $\lambda \in F$ and $u \in U$ then $\lambda u \in U$.

The vectors $v_1, \ldots, v_n \in V$ form a **basis** of V if they

1. *span* V, and are
2. *linearly independent*.

Suppose that $V = U + W$, with $u_1, \ldots, u_r$ a basis of U and $w_1, \ldots, w_s$ a basis of W, then the following three conditions are equivalent:

1. $V = U \oplus W$,
2. $u_1, \ldots, u_r, w_1, \ldots, w_s$ is a basis of V,
3. $U \cap W = \{0\}$.

If $U_1, \ldots, U_r$ are subspaces of a vector space V, then define the **sum of subspaces** to be

$$U_1 + \cdots + U_r = \{u_1 + \cdots + u_r \,:\, u_i \in U_i \text{ for } 1 \leqslant i \leqslant r\}.$$

If every element in $U_1 + \cdots + U_r$ can be written in a unique way as $u_1 + \cdots + u_r$ with $u_i \in U_i$ for $1 \leqslant i \leqslant r$, then the sum is called a **direct sum** and is denoted:

$$U_1 \oplus \cdots \oplus U_r$$

Let $U_1, \ldots, U_r$ be vector spaces over F, and let

$$\begin{aligned} V &= \{(u_1, \ldots, u_r) \,:\, u_i \in U_i \text{ for } 1 \leqslant i \leqslant r\}, \\ U_i' &= \{(0, \ldots, u_i, \ldots, 0) \,:\, u_i \in U_i\}. \end{aligned}$$

Then $V = U_1' \oplus \cdots \oplus U_r'$ is a vector space. Abusing notation slightly, we write

$$V = U_1 \oplus \cdots \oplus U_r$$

and call it the **external direct sum** of $U_1, \ldots, U_r$.

Suppose $U, W, U_1, \ldots, U_a, W_1, \ldots, W_b$ are subspaces of the vector space V. If $V = U \oplus W$ and also

$$\begin{aligned} U &= U_1 \oplus \cdots \oplus U_a \\ W &= W_1 \oplus \cdots \oplus W_b \end{aligned}$$

then

$$V = U_1 \oplus \cdots \oplus U_a \oplus W_1 \oplus \cdots \oplus W_b.$$

Suppose V and W are vector spaces and

$$\vartheta : V \to W$$

is a linear transformation, then

$$\dim V = \dim(\operatorname{Ker} \vartheta) + \dim(\operatorname{Im} \vartheta)$$

Let V and W be vector spaces over F. A **linear transformation** from V to W is a function

$$\vartheta : V \to W$$

which satisfies

1. $(u + v)\vartheta = u\vartheta + v\vartheta$ for all $u, v \in V$, and
2. $(\lambda u)\vartheta = \lambda(v\vartheta)$ for all $\lambda \in F$ and $v \in V$.

A linear transformation from a vector space V to itself is called an **endomorphism** of V.

Let ϑ be a linear transformation from V to itself, then the following conditions are equivalent:

1. ϑ is invertible,
2. $\operatorname{Ker} \vartheta = \{0\}$,
3. $\operatorname{Im} \vartheta = V$.

Definition *matrix of an endomorphism* $[\vartheta]_{\mathcal{B}}$ Representation Theory	Definition *endomorphism algebra* *End*(V) Representation Theory
Theorem $\vartheta \to [\vartheta]_{\mathcal{B}}$ *is an algebra homomorphism* Representation Theory	Definition *change of basis matrix* Representation Theory
Theorem *change of basis* Representation Theory	Proposition *direct sums induce projections* Representation Theory
Definition *projection* Representation Theory	Proposition *projections induce direct sum decomposition* Representation Theory
Definition *representation of a group / degree* Representation Theory	Definition *equivalent representations* Representation Theory

If V is a vector space over F, then the set of endomorphisms of V denoted $\mathrm{End}(V)$ form an algebra. Suppose $\vartheta, \phi \in \mathrm{End}(V)$ and $\lambda \in F$, then we define the functions $\vartheta + \phi$, $\vartheta\phi$ and $\lambda\vartheta$ from V to V by

$$\begin{aligned} v(\vartheta + \phi) &= v\vartheta + v\phi, \\ v(\vartheta\phi) &= (v\vartheta)\phi, \\ v(\lambda\vartheta) &= \lambda(v\vartheta), \end{aligned}$$

for all $v \in V$. Then $\vartheta + \phi$, $\vartheta\phi$ and $\lambda\vartheta$ are endomorphisms of V.

Let V be a vector space over F, and let ϑ be an endomorphism of V. Once a basis $\mathcal{B} = \{v_1, \ldots, v_n\}$ for V is chosen, then there are n^2 scalars $a_{ij} \in F$ $(1 \leqslant i, j \leqslant n)$ such that for all i:

$$v_i\vartheta = a_{i1}v_1 + \cdots + a_{in}v_n.$$

The $n \times n$ matrix (a_{ij}) is called the **matrix of ϑ relative to the basis** $\mathcal{B}$, and is denoted by $[\vartheta]_{\mathcal{B}}$.

Let $\mathcal{B} = \{v_1, \ldots, v_n\}$ be a basis of the vector space V, and let $\mathcal{B}' = \{v'_1, \ldots, v'_n\}$ be another basis of V. Then for $1 \leqslant i \leqslant n$,

$$v'_i = t_{i1}v_1 + \cdots + t_{in}v_n$$

for certain scalars t_{ij}. The $n \times n$ matrix $T = (t_{ij})$ is invertible and is called the **change of basis matrix** from $\mathcal{B}$ to $\mathcal{B}'$.

Suppose that $\mathcal{B}$ is a basis of the vector space V, and ϑ and ϕ are endomorphisms of V, then

$$\begin{aligned} [\vartheta + \phi]_{\mathcal{B}} &= [\vartheta]_{\mathcal{B}} + [\phi]_{\mathcal{B}} \\ [\vartheta\phi]_{\mathcal{B}} &= [\vartheta]_{\mathcal{B}}[\phi]_{\mathcal{B}} \\ [\lambda\vartheta]_{\mathcal{B}} &= \lambda[\vartheta]_{\mathcal{B}} \end{aligned}$$

Suppose that $V = U \oplus W$. Define $\pi : V \to V$ by

$$(u + w)\pi = u \qquad \text{for all } u \in U, w \in W.$$

Then π is an endomorphism of V. Further

$$\mathrm{Im}\,\pi = U, \quad \mathrm{Ker}\,\pi = W, \text{ and } \pi^2 = \pi.$$

If $\mathcal{B}$ and $\mathcal{B}'$ are bases of V and ϑ is an endomorphism of V, then

$$[\vartheta]_{\mathcal{B}} = T^{-1}[\vartheta]_{\mathcal{B}'}T,$$

where T is the change of basis matrix from $\mathcal{B}$ to $\mathcal{B}'$.

Suppose that π is a projection of a vector space V. Then

$$V = \mathrm{Im}\,\pi \oplus \mathrm{Ker}\,\pi$$

An endomorphism π of a vector space V satisfying $\pi^2 = \pi$ is called a **projection** of V.

Let $\rho : G \to GL(m, F)$ and $\sigma : G \to GL(n, F)$ be representations of G over F. We say that ρ is **equivalent** to σ if $n = m$ and there exists an invertible $n \times n$ matrix T such that for all $g \in G$,

$$g\sigma = T^{-1}(g\rho)T.$$

Equivalence of representations is an equivalence relation.

A **representation** of G over F is a homomorphism

$$\rho : G \to GL(n, F) \qquad \text{for some } n.$$

The **degree** of ρ is the integer n.

Definition

trivial representation

Representation Theory

Definition

faithful representation

Representation Theory

Proposition

ρ *faithful* $\Leftrightarrow \operatorname{Im} \rho \cong G$

Representation Theory

Definition

FG*-module*

Representation Theory

Definition

matrix of an endomorphism

Representation Theory

Theorem 4.4

representations induce FG*-modules and vice versa*

Representation Theory

Proposition 4.6

defining the action of G *on a basis of* V *induces an* FG*-module*

Representation Theory

Definition

the trivial FG*-module*

Representation Theory

Definition

faithful FG*-module*

Representation Theory

Definition

permutation module

Representation Theory

A representation $\rho : G \to \mathrm{GL}\,(n\,,\,F)$ is said to be **faithful** if $\operatorname{Ker}\rho = \{1\}$; that is, if the identity element of G is the only element g for which $g\rho = I_n$.

The representation $\rho : G \to \mathrm{GL}\,(1\,,\,F)$ defined by

$$g\rho = (1) \qquad \text{for all } g \in G,$$

is called the **trivial representation** of G.

Let V be a vector space over F and G a group. Then V is an **FG-module** if a multiplication vg is defined and satisfies, for all $u, v \in V, \lambda \in F$ and $g, h \in G$:

1. $vg \in V$
2. $v(gh) = (vg)h$
3. $v1 = v$
4. $(\lambda v)g = \lambda(vg)$
5. $(u + v)g = ug + vg$

A representation ρ of a finite group is faithful if and only if $\operatorname{Im}\rho$ is isomorphic to G.

1. If $\rho : G \to \mathrm{GL}\,(n\,,\,F)$ is a representation of G over F, $V = F^n$, then V becomes an FG-module by defining the multiplication to be $vg = v(g\rho)$. Moreover there exists a basis $\mathcal{B}$ of V such that $g\rho = [g]_{\mathcal{B}}$.
2. If V is an FG-module and $\mathcal{B}$ a basis of V, then $\rho : g \mapsto [g]_{\mathcal{B}}$ is a representation of G over F.

Let V be an FG-module, and let $\mathcal{B}$ be a basis of V. For each $g \in G$, let

$$[g]_{\mathcal{B}}$$

denote the matrix of the endomorphism $v \mapsto vg$ of V, relative to the basis $\mathcal{B}$.

The **trivial** FG-module is the 1-dimensional vector space V over F with

$$vg = v \quad \text{for all} \quad v \in V,\ g \in G.$$

Let $\mathcal{B} = \{v_1, \ldots, v_n\}$ be a basis for a vector space V over F. If $v_i g$ is defined for all $v_i \in \mathcal{B}$ and for all $g \in G$ and satisfies $\forall\, g, h \in G$, and $\forall\, \lambda_1, \ldots, \lambda_n \in F$:

1. $v_i g \in V$
2. $v_i(gh) = (v_i g)h$
3. $v_i 1 = v_i$
4. $(\lambda_1 v_1 + \cdots + \lambda_n v_n)g = \lambda_1(v_1 g) + \cdots + \lambda_n(v_n g)$

Then V is an FG-module.

Let G be a subgroup of S_n. The FG-module V with basis $v_1, \ldots, v_n$ such that

$$v_i g = v_{ig} \quad \text{for all } i, \text{ and all } g \in G,$$

is called the **permutation module** for G over F. We call $v_1, \ldots, v_n$ the **natural basis** of V.

An FG-module V is **faithful** if the identity element of G is the only element of G for which

$$vg = v \quad \text{for all} \quad v \in V.$$

DEFINITION *permutation matrix* REPRESENTATION THEORY	THEOREM 4.12 *FG-modules and equivalent representations* REPRESENTATION THEORY
DEFINITION *FG-submodule* REPRESENTATION THEORY	IRREDUCIBLE FG-MODULE REPRESENTATION THEORY
DEFINITION *the vector space FG* REPRESENTATION THEORY	DEFINITION *group algebra multiplication* REPRESENTATION THEORY
DEFINITION *group algebra* REPRESENTATION THEORY	

Suppose that V is an FG-module with basis $\mathcal{B}$ and $\rho : g \mapsto [g]_{\mathcal{B}}$ is a representation of G over F.

1. If $\mathcal{B}'$ is a basis of V, then the representation $\phi : g \mapsto [g]_{\mathcal{B}'}$ is equivalent to ρ.
2. If σ is a representation of G, equivalent to ρ, then there is a basis, $\mathcal{B}''$ of V such that: $\sigma : g \mapsto [g]_{\mathcal{B}''}$.

A **permutation matrix** is any square matrix which has precisely one nonzero entry in each row and each column and that entry is 1.

An FG-module V is said to be **irreducible** if it is non–zero and it has no FG-submodules apart from $\{0\}$ and V.

If V has an FG-submodule W with W not equal to $\{0\}$ or V, then V is **reducible**.

Let V be an FG-module. A subset W of V is said to be an FG**-submodule** of V if W is a subspace and $wg \in W$ for all $w \in W$ and for all $g \in G$.

FG carries more structure than just that of a vector space—we can use the product of G to define multiplication in FG:

$$\left(\sum_{g\in G} \lambda_g g\right)\left(\sum_{h\in G} \mu_h h\right) = \sum_{g,h\in G} \lambda_g \mu_h (gh) = \sum_{g\in G}\sum_{h\in G} (\lambda_g \mu_{h^{-1}g}) g$$

Let G be a finite group whose elements are $g_1, \dots, g_n$ and let F be $\mathbb{R}$ or $\mathbb{C}$. We define the **vector space** FG over F with basis, $\{g_1, \dots, g_n\}$. The elements of FG are all expressions of the form:

$$\lambda_1 g_1 + \cdots + \lambda_n g_n \qquad (\lambda_i \in F)$$

The rules for addition and scalar multiplication are the natural ones.

The vector space FG, with multiplication defined by

$$\left(\sum_{g\in G} \lambda_g g\right)\left(\sum_{h\in G} \mu_h h\right) = \sum_{g,h\in G} \lambda_g \mu_h (gh)$$

$(\lambda_g, \mu_h \in F)$, is called the **group algebra** of G over F.