


<p>COPYING</p> <p><i>Flash Cards for the Book:</i></p> <p><i>“Representations and Characters of Groups”</i></p> <p><i>by Gordon James and Martin Liebeck</i></p> <p>REPRESENTATION THEORY</p>	<p>DEFINITION</p> <p><i>group</i></p> <p>REPRESENTATION THEORY</p>
<p>DEFINITION</p> <p><i>subgroup</i></p> <p>REPRESENTATION THEORY</p>	<p>DEFINITION</p> <p><i>dihedral group D_{2n}</i></p> <p>REPRESENTATION THEORY</p>
<p>DEFINITION</p> <p><i>cyclic group C_n</i></p> <p>REPRESENTATION THEORY</p>	<p>DEFINITION</p> <p><i>quaternion group Q_8</i></p> <p>REPRESENTATION THEORY</p>
<p>DEFINITION</p> <p><i>alternating group A_n</i></p> <p>REPRESENTATION THEORY</p>	<p>DEFINITION</p> <p><i>direct product</i></p> <p>REPRESENTATION THEORY</p>
<p>DEFINITION</p> <p><i>homomorphism / isomorphism</i></p> <p>REPRESENTATION THEORY</p>	<p>DEFINITION</p> <p><i>coset</i></p> <p>REPRESENTATION THEORY</p>

<p>A group consists of a set G, together with a rule for combining any two elements $g, h \in G$ to form another element of G satisfying:</p> <ol style="list-style-type: none"> 1. $\forall g, h, k \in G, (gh)k = g(hk)$ 2. $\exists e \in G$ such that $\forall g \in G, eg = ge = g$ 3. $\forall g \in G, \exists g^{-1} \in G$ such that $gg^{-1} = g^{-1}g = e$ 	<p>© 2017 Jason Underdown</p> <p>These flash cards are licensed under the:</p> <p>Creative Commons Attribution 4.0 International License</p> <p></p> <p>https://creativecommons.org/licenses/by/4.0/</p>
$D_{2n} = \langle a, b : a^n = 1, b^2 = 1, b^{-1}ab = a^{-1} \rangle$	<p>Let G be a group. A subset H of G is a subgroup if H is itself a group under the operation inherited from G.</p> $H \leq G$
$Q_8 = \langle a, b : a^4 = 1, a^2 = b^2, b^{-1}ab = a^{-1} \rangle$	$C_n = \{1, a, a^2, \dots, a^{n-1}\}$ $C_n = \langle a : a^n = 1 \rangle$
<p>Let G and H be groups, consider</p> $G \times H = \{(g, h) : g \in G \text{ and } h \in H\}.$ <p>Define a product operation on $G \times H$ by</p> $(g, h)(g', h') = (gg', hh').$ <p>The group $G \times H$ is called the direct product of G and H.</p>	$A_n = \{g \in S_n : g \text{ is an even permutation}\}$ <p>Recall that every permutation $g \in S_n$ can be expressed as a product of transpositions. An even permutation has an even number of transpositions, and an odd permutation has an odd number of transpositions.</p>
<p>Let G be a group and H a subgroup of G. For $x \in G$, the subset</p> $Hx = \{hx : h \in H\}$ <p>of G is called a right coset of H in G. The distinct right cosets of G partition G.</p>	<p>If G and H are groups, then a homomorphism from G to H is a map $\vartheta : G \rightarrow H$, which for all $g_1, g_2 \in G$ satisfies:</p> $(g_1g_2)\vartheta = (g_1\vartheta)(g_2\vartheta).$ <p>If ϑ is also invertible, then ϑ is called an isomorphism.</p>

<p>THEOREM</p> <p><i>Lagrange's theorem</i></p> <p>REPRESENTATION THEORY</p>	<p>DEFINITION</p> <p><i>index</i></p> <p>REPRESENTATION THEORY</p>
<p>DEFINITION</p> <p><i>normal subgroup</i></p> <p>REPRESENTATION THEORY</p>	<p>DEFINITION</p> <p><i>factor group</i></p> <p>REPRESENTATION THEORY</p>
<p>DEFINITION</p> <p><i>simple group</i></p> <p>REPRESENTATION THEORY</p>	<p>DEFINITION</p> <p><i>kernel / image</i></p> <p>REPRESENTATION THEORY</p>
<p>THEOREM</p> <p><i>first isomorphism theorem</i></p> <p>REPRESENTATION THEORY</p>	<p>DEFINITION</p> <p><i>vector space</i></p> <p>REPRESENTATION THEORY</p>
<p>DEFINITION</p> <p><i>linear dependence / linear independence</i></p> <p>REPRESENTATION THEORY</p>	<p>DEFINITION</p> <p><i>linear combination / span</i></p> <p>REPRESENTATION THEORY</p>

<p>Suppose H is a subgroup of G. The number of distinct right cosets of H in G is written as $G : H$. If G is finite, then</p> $ G : H = G / H $ <p>by Lagrange's theorem.</p>	<p>If G is a finite group and H is a subgroup of G, then H divides G.</p>
<p>If $N \triangleleft G$, then define G/N to be the set of right cosets of N in G. This set is made into a group via the multiplication operation:</p> $(Ng)(Nh) = Ngh \quad \forall g, h \in G.$ <p>This operation makes G/N into a group called the factor group of G by N.</p>	<p>A subgroup N of a group G is said to be a normal subgroup of G if $g^{-1}Ng = N$ for all $g \in G$, where</p> $g^{-1}Ng = \{g^{-1}ng : n \in N\}.$ <p>We indicate that N is a normal subgroup of G by writing:</p> $N \triangleleft G.$
<p>Let G and H be groups. Suppose that</p> $\vartheta : G \rightarrow H$ <p>is a homomorphism then the kernel of ϑ and image of ϑ are defined to be:</p> $\begin{aligned} \text{Ker } \vartheta &= \{g \in G : g\vartheta = 1\} & \text{Ker } \vartheta &\triangleleft G \\ \text{Im } \vartheta &= \{g\vartheta : g \in G\} & \text{Im } \vartheta &\leq H \end{aligned}$	<p>A group G is said to be simple if $G \neq \{1\}$ and the only normal subgroups of G are $\{1\}$ and G.</p>
<p>A vector space over a field F is a set V, equipped with addition and scalar multiplication satisfying:</p> <ol style="list-style-type: none"> 1. V is an abelian group under addition; 2. $\forall u, v \in V$ and $\forall \lambda, \mu \in F$, <ol style="list-style-type: none"> (a) $\lambda(u + v) = \lambda u + \lambda v$ (b) $(\lambda + \mu)v = \lambda v + \mu v$ (c) $(\lambda\mu)v = \lambda(\mu v)$ (d) $1v = v$ 	<p>Suppose that G and H are groups and let $\vartheta : G \rightarrow H$ be a homomorphism. Then</p> $G / \text{Ker } \vartheta \cong \text{Im } \vartheta.$ <p>An isomorphism is given by the function</p> $Kg \rightarrow g\vartheta \quad (g \in G)$ <p>where $K = \text{Ker } \vartheta$.</p>
<p>Let v_1, \dots, v_n be vectors in a vector space V over F. A vector v in V is a linear combination of v_1, \dots, v_n if</p> $v = \lambda_1 v_1 + \dots + \lambda_n v_n$ <p>for some $\lambda_1, \dots, \lambda_n \in F$.</p> <p>The vectors v_1, \dots, v_n span V if every vector in V is a linear combination of v_1, \dots, v_n.</p>	<p>We say that v_1, \dots, v_n are linearly dependent if</p> $\lambda_1 v_1 + \dots + \lambda_n v_n = 0$ <p>for some $\lambda_1, \dots, \lambda_n \in F$ not all zero, otherwise the vectors v_1, \dots, v_n are linearly independent.</p>

<p>DEFINITION</p> <p><i>basis</i></p> <p>REPRESENTATION THEORY</p>	<p>DEFINITION/THEOREM</p> <p><i>subspace / conditions for a subspace</i></p> <p>REPRESENTATION THEORY</p>
<p>DEFINITION</p> <p><i>sum / direct sum</i></p> <p>REPRESENTATION THEORY</p>	<p>THEOREM</p> <p><i>conditions for a direct sum</i></p> <p>REPRESENTATION THEORY</p>
<p>THEOREM</p> <p><i>direct sum of direct sums</i></p> <p>REPRESENTATION THEORY</p>	<p>DEFINITION</p> <p><i>external direct sum</i></p> <p>REPRESENTATION THEORY</p>
<p>DEFINITION</p> <p><i>linear transformation</i></p> <p>REPRESENTATION THEORY</p>	<p>THEOREM</p> <p><i>rank–nullity theorem</i></p> <p>REPRESENTATION THEORY</p>
<p>THEOREM</p> <p><i>invertibility of linear transformations</i></p> <p>REPRESENTATION THEORY</p>	<p>DEFINITION</p> <p><i>endomorphism</i></p> <p>REPRESENTATION THEORY</p>

<p>A subspace of a vector space V over F is a subset of V which is itself a vector space under the operations inherited from V.</p> <p>A subset U of a vector space V is a subspace iff</p> <ol style="list-style-type: none"> 1. $0 \in U$; 2. if $u, v \in U$ then $u + v \in U$; 3. if $\lambda \in F$ and $u \in U$ then $\lambda u \in U$. 	<p>The vectors $v_1, \dots, v_n \in V$ form a basis of V if they</p> <ol style="list-style-type: none"> 1. <i>span</i> V, and are 2. <i>linearly independent</i>.
<p>Suppose that $V = U + W$, with u_1, \dots, u_r a basis of U and w_1, \dots, w_s a basis of W, then the following three conditions are equivalent:</p> <ol style="list-style-type: none"> 1. $V = U \oplus W$, 2. $u_1, \dots, u_r, w_1, \dots, w_s$ is a basis of V, 3. $U \cap W = \{0\}$. 	<p>If U_1, \dots, U_r are subspaces of a vector space V, then define the sum of subspaces to be</p> $U_1 + \dots + U_r = \{u_1 + \dots + u_r : u_i \in U_i \text{ for } 1 \leq i \leq r\}.$ <p>If every element in $U_1 + \dots + U_r$ can be written in a unique way as $u_1 + \dots + u_r$ with $u_i \in U_i$ for $1 \leq i \leq r$, then the sum is called a direct sum and is denoted:</p> $U_1 \oplus \dots \oplus U_r$
<p>Let U_1, \dots, U_r be vector spaces over F, and let</p> $V = \{(u_1, \dots, u_r) : u_i \in U_i \text{ for } 1 \leq i \leq r\},$ $U'_i = \{(0, \dots, u_i, \dots, 0) : u_i \in U_i\}.$ <p>Then $V = U'_1 \oplus \dots \oplus U'_r$ is a vector space. Abusing notation slightly, we write</p> $V = U_1 \oplus \dots \oplus U_r$ <p>and call it the external direct sum of U_1, \dots, U_r.</p>	<p>Suppose $U, W, U_1, \dots, U_a, W_1, \dots, W_b$ are subspaces of the vector space V. If $V = U \oplus W$ and also</p> $U = U_1 \oplus \dots \oplus U_a$ $W = W_1 \oplus \dots \oplus W_b$ <p>then</p> $V = U_1 \oplus \dots \oplus U_a \oplus W_1 \oplus \dots \oplus W_b.$
<p>Suppose V and W are vector spaces and</p> $\vartheta : V \rightarrow W$ <p>is a linear transformation, then</p> $\dim V = \dim(\text{Ker } \vartheta) + \dim(\text{Im } \vartheta)$	<p>Let V and W be vector spaces over F. A linear transformation from V to W is a function</p> $\vartheta : V \rightarrow W$ <p>which satisfies</p> <ol style="list-style-type: none"> 1. $(u + v)\vartheta = u\vartheta + v\vartheta$ for all $u, v \in V$, and 2. $(\lambda u)\vartheta = \lambda(v\vartheta)$ for all $\lambda \in F$ and $v \in V$.
<p>A linear transformation from a vector space V to itself is called an endomorphism of V.</p>	<p>Let ϑ be a linear transformation from V to itself, then the following conditions are equivalent:</p> <ol style="list-style-type: none"> 1. ϑ is invertible, 2. $\text{Ker } \vartheta = \{0\}$, 3. $\text{Im } \vartheta = V$.

<p>DEFINITION</p> <p><i>matrix of an endomorphism</i> $[\vartheta]_{\mathcal{B}}$</p> <p>REPRESENTATION THEORY</p>	<p>DEFINITION</p> <p><i>endomorphism algebra</i> $End(V)$</p> <p>REPRESENTATION THEORY</p>
<p>THEOREM</p> <p>$\vartheta \rightarrow [\vartheta]_{\mathcal{B}}$ <i>is an algebra homomorphism</i></p> <p>REPRESENTATION THEORY</p>	<p>DEFINITION</p> <p><i>change of basis matrix</i></p> <p>REPRESENTATION THEORY</p>
<p>THEOREM</p> <p><i>change of basis</i></p> <p>REPRESENTATION THEORY</p>	<p>PROPOSITION</p> <p><i>direct sums induce projections</i></p> <p>REPRESENTATION THEORY</p>
<p>DEFINITION</p> <p><i>projection</i></p> <p>REPRESENTATION THEORY</p>	<p>PROPOSITION</p> <p><i>projections induce direct sum decomposition</i></p> <p>REPRESENTATION THEORY</p>
<p>DEFINITION</p> <p><i>representation of a group / degree</i></p> <p>REPRESENTATION THEORY</p>	<p>DEFINITION</p> <p><i>equivalent representations</i></p> <p>REPRESENTATION THEORY</p>

<p>If V is a vector space over F, then the set of endomorphisms of V denoted $\text{End}(V)$ form an algebra. Suppose $\vartheta, \phi \in \text{End}(V)$ and $\lambda \in F$, then we define the functions $\vartheta + \phi$, $\vartheta\phi$ and $\lambda\vartheta$ from V to V by</p> $\begin{aligned} v(\vartheta + \phi) &= v\vartheta + v\phi, \\ v(\vartheta\phi) &= (v\vartheta)\phi, \\ v(\lambda\vartheta) &= \lambda(v\vartheta), \end{aligned}$ <p>for all $v \in V$. Then $\vartheta + \phi$, $\vartheta\phi$ and $\lambda\vartheta$ are endomorphisms of V.</p>	<p>Let V be a vector space over F, and let ϑ be an endomorphism of V. Once a basis $\mathcal{B} = \{v_1, \dots, v_n\}$ for V is chosen, then there are n^2 scalars $a_{ij} \in F$ ($1 \leq i, j \leq n$) such that for all i:</p> $v_i\vartheta = a_{i1}v_1 + \dots + a_{in}v_n.$ <p>The $n \times n$ matrix (a_{ij}) is called the matrix of ϑ relative to the basis \mathcal{B}, and is denoted by $[\vartheta]_{\mathcal{B}}$.</p>
<p>Let $\mathcal{B} = \{v_1, \dots, v_n\}$ be a basis of the vector space V, and let $\mathcal{B}' = \{v'_1, \dots, v'_n\}$ be another basis of V. Then for $1 \leq i \leq n$,</p> $v'_i = t_{i1}v_1 + \dots + t_{in}v_n$ <p>for certain scalars t_{ij}. The $n \times n$ matrix $T = (t_{ij})$ is invertible and is called the change of basis matrix from \mathcal{B} to \mathcal{B}'.</p>	<p>Suppose that \mathcal{B} is a basis of the vector space V, and ϑ and ϕ are endomorphisms of V, then</p> $\begin{aligned} [\vartheta + \phi]_{\mathcal{B}} &= [\vartheta]_{\mathcal{B}} + [\phi]_{\mathcal{B}} \\ [\vartheta\phi]_{\mathcal{B}} &= [\vartheta]_{\mathcal{B}}[\phi]_{\mathcal{B}} \\ [\lambda\vartheta]_{\mathcal{B}} &= \lambda[\vartheta]_{\mathcal{B}} \end{aligned}$
<p>Suppose that $V = U \oplus W$. Define $\pi : V \rightarrow V$ by</p> $(u + w)\pi = u \quad \text{for all } u \in U, w \in W.$ <p>Then π is an endomorphism of V. Further</p> $\text{Im } \pi = U, \quad \text{Ker } \pi = W, \quad \text{and } \pi^2 = \pi.$	<p>If \mathcal{B} and \mathcal{B}' are bases of V and ϑ is an endomorphism of V, then</p> $[\vartheta]_{\mathcal{B}} = T^{-1}[\vartheta]_{\mathcal{B}'}T,$ <p>where T is the change of basis matrix from \mathcal{B} to \mathcal{B}'.</p>
<p>Suppose that π is a projection of a vector space V. Then</p> $V = \text{Im } \pi \oplus \text{Ker } \pi$	<p>An endomorphism π of a vector space V satisfying $\pi^2 = \pi$ is called a projection of V.</p>
<p>Let $\rho : G \rightarrow GL(m, F)$ and $\sigma : G \rightarrow GL(n, F)$ be representations of G over F. We say that ρ is equivalent to σ if $n = m$ and there exists an invertible $n \times n$ matrix T such that for all $g \in G$,</p> $g\sigma = T^{-1}(g\rho)T.$ <p>Equivalence of representations is an equivalence relation.</p>	<p>A representation of G over F is a homomorphism</p> $\rho : G \rightarrow GL(n, F) \quad \text{for some } n.$ <p>The degree of ρ is the integer n.</p>

<p>DEFINITION</p> <p><i>trivial representation</i></p> <p>REPRESENTATION THEORY</p>	<p>DEFINITION</p> <p><i>faithful representation</i></p> <p>REPRESENTATION THEORY</p>
<p>PROPOSITION</p> <p>$\rho \text{ faithful} \Leftrightarrow \text{Im } \rho \cong G$</p> <p>REPRESENTATION THEORY</p>	<p>DEFINITION</p> <p><i>FG-module</i></p> <p>REPRESENTATION THEORY</p>
<p>DEFINITION</p> <p><i>matrix of an endomorphism</i></p> <p>REPRESENTATION THEORY</p>	<p>THEOREM 4.4</p> <p><i>representations induce FG-modules and vice versa</i></p> <p>REPRESENTATION THEORY</p>
<p>PROPOSITION 4.6</p> <p><i>defining the action of G on a basis of V induces an FG-module</i></p> <p>REPRESENTATION THEORY</p>	<p>DEFINITION</p> <p><i>the trivial FG-module</i></p> <p>REPRESENTATION THEORY</p>
<p>DEFINITION</p> <p><i>faithful FG-module</i></p> <p>REPRESENTATION THEORY</p>	<p>DEFINITION</p> <p><i>permutation module</i></p> <p>REPRESENTATION THEORY</p>

<p>A representation $\rho : G \rightarrow \text{GL}(n, F)$ is said to be faithful if $\text{Ker } \rho = \{1\}$; that is, if the identity element of G is the only element g for which $g\rho = I_n$.</p>	<p>The representation $\rho : G \rightarrow \text{GL}(1, F)$ defined by</p> $g\rho = (1) \quad \text{for all } g \in G,$ <p>is called the trivial representation of G.</p>
<p>Let V be a vector space over F and G a group. Then V is an FG-module if a multiplication vg is defined and satisfies, for all $u, v \in V, \lambda \in F$ and $g, h \in G$:</p> <ol style="list-style-type: none"> 1. $vg \in V$ 2. $v(gh) = (vg)h$ 3. $v1 = v$ 4. $(\lambda v)g = \lambda(vg)$ 5. $(u + v)g = ug + vg$ 	<p>A representation ρ of a finite group is faithful if and only if $\text{Im } \rho$ is isomorphic to G.</p>
<ol style="list-style-type: none"> 1. If $\rho : G \rightarrow \text{GL}(n, F)$ is a representation of G over $F, V = F^n$, then V becomes an FG-module by defining the multiplication to be $vg = v(g\rho)$. Moreover there exists a basis \mathcal{B} of V such that $g\rho = [g]_{\mathcal{B}}$. 2. If V is an FG-module and \mathcal{B} a basis of V, then $\rho : g \mapsto [g]_{\mathcal{B}}$ is a representation of G over F. 	<p>Let V be an FG-module, and let \mathcal{B} be a basis of V. For each $g \in G$, let</p> $[g]_{\mathcal{B}}$ <p>denote the matrix of the endomorphism $v \mapsto vg$ of V, relative to the basis \mathcal{B}.</p>
<p>The trivial FG-module is the 1-dimensional vector space V over F with</p> $vg = v \quad \text{for all } v \in V, g \in G.$	<p>Let $\mathcal{B} = \{v_1, \dots, v_n\}$ be a basis for a vector space V over F. If $v_i g$ is defined for all $v_i \in \mathcal{B}$ and for all $g \in G$ and satisfies $\forall g, h \in G$, and $\forall \lambda_1, \dots, \lambda_n \in F$:</p> <ol style="list-style-type: none"> 1. $v_i g \in V$ 2. $v_i(gh) = (v_i g)h$ 3. $v_i 1 = v_i$ 4. $(\lambda_1 v_1 + \dots + \lambda_n v_n)g = \lambda_1(v_1 g) + \dots + \lambda_n(v_n g)$ <p>Then V is an FG-module.</p>
<p>Let G be a subgroup of S_n. The FG-module V with basis v_1, \dots, v_n such that</p> $v_i g = v_{ig} \quad \text{for all } i, \text{ and all } g \in G,$ <p>is called the permutation module for G over F. We call v_1, \dots, v_n the natural basis of V.</p>	<p>An FG-module V is faithful if the identity element of G is the only element of G for which</p> $vg = v \quad \text{for all } v \in V.$