Introduction

In this chapter we begin our study in earnest. The first order of business is to build up enough machinery to give a proper definition of manifolds. The chief problem with the provisional definition given in Chapter 1 is that it depends on having an "ambient Euclidean space" in which our n-manifold lives. This introduces a great deal of extraneous structure that is irrelevant to our purposes. Instead, we would like to view a manifold as a mathematical object in its own right, not as a subset of some larger space. The key concept that makes this possible is that of a topological space, which is the main topic of this chapter.

1 Topological Spaces

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1.1 Local Compactness and Paracompactness

The next topological property of manifolds that we need is local compactness (see Appendix A for the definition).

Proposition 1.1 (Manifolds Are Locally Compact). *Every topological manifold is locally compact.*

PROOF. See [1, p. 47]. □

Another key topological property possessed by manifolds is called *paracompactness*. It is a consequence of local compactness and second-countability, and in fact is one of the main reasons why second-countability is included in the definition of manifolds.

Let M be a topological space. A collection $\mathcal{X}$ of subsets of M is said to be ***locally finite*** if each point of M has a neighborhood that intersects at most finitely many of the sets in X. Given a cover $\mathcal{U}$ of M, another cover $\mathcal{V}$ is called a ***refinement of $\mathcal{U}$*** if for each $V \in \mathcal{V}$ there exists some $U \in \mathcal{U}$ such that $V \subseteq U$. We say that M is ***paracompact*** if every open cover of M admits an open, locally finite refinement.

Theorem 1.2 (The Residue Theorem). *Suppose f is a meromorphic function in Ω. Let A be the set of points in Ω at which f has poles* đQ. *If Γ is a cycle in $\Omega \setminus A$ such that*

$$\operatorname{Ind}_{\Gamma}(\alpha) = 0 \quad \textit{for all} \quad \alpha \notin \Omega\,, \tag{1}$$

then

$$\frac{1}{2\pi i}\int_\Gamma f(z)\,\mathrm{d}z = \sum_{a\in A} \operatorname{Res}(f;a)\,\operatorname{Ind}_\Gamma(a)\,. \tag{2}$$

PROOF. Let $B = \{a \in A \mid \operatorname{Ind}_\Gamma(a) \neq 0\}$. Let W be the complement of Γ^*. Then $\operatorname{Ind}_\Gamma(z)$ is constant in each component of V in W. If V is unbounded, or if V intersects Ω^c, (1) implies that $\operatorname{Ind}_\Gamma(z) = 0$ for every $z \in V$. Since A has no limit point in Ω, we conclude that B is a finite set.

The sum in (2), though formally infinite, is therefore actually finite.

Let $a_1, \dots, a_n$ be the points of B, let $Q_1, \dots, Q_n$ be the principal parts of f at $a_1, \dots, a_n$, and put $g = f - (Q_1 + \cdots + Q_n)$.[1] Put $\Omega_0 = \Omega \setminus (A \setminus B)$. Since g has singularities at $a_1, \dots, a_n$, Proposition 1.1, applied to the function g and the open set Ω_0, shows that (1)

$$\int_\Gamma g(z)\,\mathrm{d}z = 0\,. \tag{3}$$

Hence

$$\frac{1}{2\pi i}\int_\Gamma f(z)\,\mathrm{d}z = \sum_{k=1}^{n} \frac{1}{2\pi i}\int_\Gamma Q_k(z)\,\mathrm{d}z = \sum_{k=1}^{n} \operatorname{Res}(Q_k;a_k)\,\operatorname{Ind}_\Gamma(a_k)\,, \tag{4}$$

and since f and Q_k have the same residue at a_k, we obtain (2). ☐

This proof may be found in [2]. Observe that the crucial ingredient in the proof of the Riesz-Fischer theorem is the fact that L^2 is complete. This is so well recognized that the name “Riesz-Fisher theorem” is sometimes given to the theorem which asserts the completeness of L^2, or even of any L^p.

Theorem 1.3. *For a linear transformation Λ of a normed linear space X into a normed linear space Y, each of the following three conditions implies the other two:*

(a) *Λ is bounded.*

(b) *Λ is continuous.*

(c) *Λ is continuous at one point of X.*

PROOF OF THEOREM 1.3(a). Since $\|\Lambda(x_1 - x_2)\| \le \|\Lambda\|\,\|x_1 - x_2\|$, it is clear that (a) implies (b), and (b) implies (c) trivially. Suppose Λ is continuous at x_0. To each $\epsilon > 0$ one can then find a $\delta > 0$ so that $\|x - x_0\| < \delta$ implies $\|\Lambda x - \Lambda x_0\| < \epsilon$. In other words, $\|x\| < \delta$ implies

$$\|\Lambda(x_0 + x) - \Lambda x_0\| < \epsilon\,.$$

But then linearity of Λ shows that $\|\Gamma x\| < \epsilon$. Hence $\|\Lambda\| \le \epsilon/\delta$, and (c). ☐

[1] If $B = \varnothing$, a possibility which is not excluded, then $g = f$.

1.2 Elementary Properties of Measures

Definition 1.4.

(a) A ***positive measure*** is a function μ, defined on a σ-algebra $\mathfrak{M}$, whose range is in $[0, \infty]$ and which is ***countable additive***. This means that if $\{A_i\}$ is a *disjoint* countable collection of members of $\mathfrak{M}$, then

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)\,. \tag{5}$$

To avoid trivialities, we shall also assume that $\mu(A) < \infty$ for at least one $A \in \mathfrak{M}$.

(b) A ***measure space*** is a measurable space which has a positive measure defined on the σ-algebra of its measurable sets.

(c) A ***complex measure*** is a complex-valued countably additive function defined on a σ-algebra.

Hey look its a quantum state! Theorem 1.3(a)

$$|\phi\rangle$$

Here is a commutative diagram! $f\colon X \to Y$

$$\begin{tikzcd}
T \arrow[drr, bend left, "x"] \arrow[ddr, bend right, "y"'] \arrow[dr, dotted, "{(x,y)}"] & & \\
& X \times_Z Y \arrow[r, "p"] \arrow[d, "q"] & X \arrow[d, "f"] \\
& Y \arrow[r, "g"] & Z
\end{tikzcd}$$

References

[1] J. M. Lee, *Introduction to Smooth Manifolds*, 2nd ed., Graduate Texts in Mathematics 218 (Springer-Verlag, 2013).

[2] W. Rudin, *Real and Complex Analysis*, 3rd ed. (McGraw-Hill, 1986).

[3] R. Blumenhagen and E. Plauschinn, *Introduction to Conformal Field Theory: With Applications to String Theory*, 1st ed., Lecture Notes in Physics 779 (Springer-Verlag, 2009).

[4] J. M. Lee, *Introduction to Topological Manifolds*, 2nd ed., Graduate Texts in Mathematics 202 (Springer-Verlag, 2011).

[5] J. Polchinski, *String Theory*, Vol. 1: *An Introduction to the Bosonic String*, Cambridge Monographs on Mathematical Physics (Cambridge University Press, 1998).

[6] J. Polchinski, *String Theory*, Vol. 2: *Superstring Theory and Beyond*, Cambridge Monographs on Mathematical Physics (Cambridge University Press, 1998).

[7] M. Schottenloher, *A Mathematical Introduction to Conformal Field Theory*, 2nd ed., Lecture Notes in Physics 759 (Springer-Verlag, 2008).

[8] M. Talagrand, *What Is a Quantum Field Theory? A First Introduction for Mathematicians* (Cambridge University Press, 2022).