Introduction

In this chapter we begin our study in earnest. The first order of business is to build up enough machinery to give a proper definition of manifolds. The chief problem with the provisional definition given in Chapter 1 is that it depends on having an "ambient Euclidean space" in which our *n*-manifold lives. This introduces a great deal of extraneous structure that is irrelevant to our purposes. Instead, we would like to view a manifold as a mathematical object in its own right, not as a subset of some larger space. The key concept that makes this possible is that of a topological space, which is the main topic of this chapter.

1 Topological Spaces

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1.1 Local Compactness and Paracompactness

The next topological property of manifolds that we need is local compactness (see Appendix A for the definition).

Proposition 1.1 (Manifolds Are Locally Compact). Every topological manifold is locally compact.

Another key topological property possessed by manifolds is called *paracompactness*. It is a consequence of local compactness and second-countability, and in fact is one of the main reasons why second-countability is included in the definition of manifolds.

Let M be a topological space. A collection \mathcal{X} of subsets of M is said to be **locally finite** if each point of M has a neighborhood that intersects at most finitely many of the sets in X. Given a cover \mathcal{U} of M, another cover \mathcal{V} is called a **refinement of \mathcal{U}** if for each $V \in \mathcal{V}$ there exists some $U \in \mathcal{U}$ such that $V \subseteq U$. We say that M is **paracompact** if every open cover of M admits an open, locally finite refinement.

Theorem 1.2 (The Residue Theorem). Suppose f is a meromorphic function in Ω . Let A be the set of points in Ω at which f has poles dQ. If Γ is a cycle in $\Omega \setminus A$ such that

$$\operatorname{Ind}_{\Gamma}(\alpha) = 0 \quad \text{for all} \quad \alpha \notin \Omega,$$
 (1)

then

$$\frac{1}{2\pi i} \int_{\Gamma} f(z) \, \mathrm{d}z = \sum_{a \in A} \mathrm{Res}(f; a) \, \mathrm{Ind}_{\Gamma}(a). \tag{2}$$

PROOF. Let $B = \{a \in A \mid \operatorname{Ind}_{\Gamma}(a) \neq 0\}$. Let W be the complement of Γ^* . Then $\operatorname{Ind}_{\Gamma}(z)$ is constant in each component of V in W. If V is unbounded, or if V intersects Ω^c , (1) implies that $\operatorname{Ind}_{\Gamma}(z) = 0$ for every $z \in V$. Since A has no limit point in Ω , we conclude that B is a finite set.

The sum in (2), though formally infinite, is therefore actually finite.

Let a_1, \ldots, a_n be the points of B, let Q_1, \ldots, Q_n be the principal parts of f at a_1, \ldots, a_n , and put $g = f - (Q_1 + \cdots + Q_n)$. Put $\Omega_0 = \Omega \setminus (A \setminus B)$. Since g has singularities at a_1, \ldots, a_n , Proposition 1.1, applied to the function g and the open set Ω_0 , shows that (1)

$$\int_{\Gamma} g(z) \, \mathrm{d}z = 0. \tag{3}$$

Hence

$$\frac{1}{2\pi i} \int_{\Gamma} f(z) \, dz = \sum_{k=1}^{n} \frac{1}{2\pi i} \int_{\Gamma} Q_k(z) \, dz = \sum_{k=1}^{n} \text{Res}(Q_k; a_k) \, \text{Ind}_{\Gamma}(a_k) \,, \tag{4}$$

and since f and Q_k have the same residue at a_k , we obtain (2).

This proof may be found in [2]. Observe that the crucial ingredient in the proof of the Riesz-Fischer theorem is the fact that L^2 is complete. This is so well recognized that the name "Riesz-Fisher theorem" is sometimes given to the theorem which asserts the completeness of L^2 , or even of any L^p .

Theorem 1.3. For a linear transformation Λ of a normed linear space X into a normed linear space Y, each of the following three conditions implies the other two:

- (a) Λ is bounded.
- (b) Λ is continuous.
- (c) Λ is continuous at one point of X.

PROOF OF THEOREM 1.3(a). Since $\|\Lambda(x_1 - x_2)\| \le \|\Lambda\| \|x_1 - x_2\|$, it is clear that (a) implies (b), and (b) implies (c) trivially. Suppose Λ is continuous at x_0 . To each $\epsilon > 0$ one can then find a $\delta > 0$ so that $\|x - x_0\| < \delta$ implies $\|\Lambda x - \Lambda x_0\| < \epsilon$. In other words, $\|x\| < \delta$ implies

$$\|\Lambda(x_0+x)-\Lambda x_0\|<\epsilon$$
.

But then linearity of Λ shows that $\|\Gamma x\| < \epsilon$. Hence $\|\Lambda\| \le \epsilon/\delta$, and (c).

¹If $B = \emptyset$, a possibility which is not excluded, then g = f.

1.2 Elementary Properties of Measures

Definition 1.4.

(a) A **positive measure** is a function μ , defined on a σ -algebra \mathfrak{M} , whose range is in $[0, \infty]$ and which is **countable additive**. This means that if $\{A_i\}$ is a *disjoint* countable collection of members of \mathfrak{M} , then

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i). \tag{5}$$

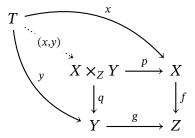
To avoid trivialities, we shall also assume that $\mu(A) < \infty$ for at least one $A \in \mathfrak{M}$.

- (b) A *measure space* is a measurable space which has a positive measure defined on the σ -algebra of its measurable sets.
- (c) A *complex measure* is a complex-valued countably additive function defined on a σ -algebra.

Hey look its a quantum state! Theorem 1.3(a)

 $|\phi\rangle$

Here is a commutative diagram! $f: X \to Y$



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