Computational Intelligence WS24/25

Exercise Sheet 3 (Solution) — November 21st, 2024

Thomas Gabor, Maximilian Zorn, Claudia Linnhoff-Popien

1 Three-Valued Logic

Some logicians have felt that the two truth values used in classic Boolean algebra are not sufficient to express every part of everyday life that one might want to reason about. We now consider a logic based on the three truth values $\{true, unknown, false\}$.

For this logic, we might define the following functions...

x	NOT x
true	*false*
unknown	*unknown*
false	*true*

x	y	x OR y
true	*true*	*true*
true	*unknown*	*true*
true	*false*	*true*
unknown	*true*	*true*
unknown	*unknown*	*unknown*
unknown	*false*	*unknown*
false	*true*	*true*
false	*unknown*	*unknown*
false	*false*	*false*

(i) "To be or not to be." The formula $b \vee \neg b$ can always be reduced to *true* in Boolean two-valued logic. How does the formula behave in our three-valued logic?

$b \vee \neg b$ in Boolean logic is always true because (a) *true* $\vee$ *false* $=$ *true* and (b) *false* $\vee$ *true* $=$ *true.*
For three-valued logic (a) and (b) still hold, but *unknown* $\vee \neg$*unknown* $=$ *unknown*, so $b \vee \neg b$ can also assume the value *unknown* and is no longer always *true.*

(ii) Give a reasonable definition that is in line with the logical functions defined above for a three-valued logical AND.

x	y	x AND y
true	*true*	*true*
true	*unknown*	*unknown*
true	*false*	*false*
unknown	*true*	*unknown*
unknown	*unknown*	*unknown*
unknown	*false*	*false*
false	*true*	*false*
false	*unknown*	*false*
false	*false*	*false*

(iii) Let $\mathbb{K} = \{-1, 0, +1\}$. Let us assume a mapping of truth values to numbers from $\mathbb{K}$, i.e., *true* $\mapsto +1$, *unknown* $\mapsto 0$, *false* $\mapsto -1$. Give concise mathematical definitions that perform the logical operations NOT, OR, and AND on $\mathbb{K}$.

$\text{NOT} : \mathbb{K} \to \mathbb{K}$
$\text{NOT}(x) = -x$

$\text{OR} : \mathbb{K} \times \mathbb{K} \to \mathbb{K}$
$\text{OR}(x, y) = \max\{x, y\}$

$\text{AND} : \mathbb{K} \times \mathbb{K} \to \mathbb{K}$
$\text{AND}(x, y) = \min\{x, y\}$

2 Fuzzy Logic

Around 1988 Lotfi Zadeh famously introduced *fuzzy logic*.[1] Fuzzy logic is based on a continuous set of truth values $\mathbb{F} = [0; 1] \subseteq \mathbb{R}$. This allows to give propositions a scalar truth value telling *how* true they are, i.e., *raining* $= 0.99$ might mean that it is most

[1] https://www.scss.tcd.ie/Khurshid.Ahmad/Teaching/Lectures_on_Fuzzy_Logic/00000053.pdf

definitely raining while *raining* $= 0.3$ might encode that it is not that clear if the current weather is accurately described as “raining”.

(i) The logical function NOT on $\mathbb{F}$ is then written as follows:

$$\begin{aligned} \text{NOT} &: \mathbb{F} \to \mathbb{F} \\ \text{NOT}(x) &= 1 - x \end{aligned}$$

We can see that when we call this function on the truth values of $\mathbb{B} = \{0, 1\}$ the results are consistent with classical Boolean logic as shown in the truth table below:

x	classical NOT x	fuzzy NOT x
0	1	$1 - 0 = 1$
1	0	$1 - 1 = 0$

Similarly to task 1(iii) on this exercise sheet, give a reasonable mathematical definition for the logical functions OR and AND. Use the truth table below to show that the fuzzy logical function are consistent with the classical Boolean functions when called on the classical Boolean truth values $\mathbb{B} = \{0, 1\}$.

$\text{OR} : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$
$\text{OR}(x, y) = \max(x, y)$

$\text{AND} : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$
$\text{AND}(x, y) = \min(x, y)$

x	y	classical x OR y	fuzzy x OR y	classical x AND y	fuzzy x AND y
0	0	0	$\max(0, 0) = 0$	0	$\min(0, 0) = 0$
0	1	1	$\max(0, 1) = 1$	0	$\min(0, 1) = 0$
1	0	1	$\max(1, 0) = 1$	0	$\min(1, 0) = 0$
1	1	1	$\max(1, 1) = 1$	1	$\min(1, 1) = 1$

(ii) One of the main uses of fuzzy logic is in expert systems, where it allows to encode knowledge in seemingly natural language. For example, we might encode rules like...

- If it is raining and very windy, take your umbrella and your coat.
- If it is somewhat raining and not very windy, take your umbrella but not your coat.
- If it is raining very much or extremely windy, stay at home.

All of these have a precise logical interpretation in fuzzy logic, if we define the used modifiers accordingly. For example, given a fuzzy truth value $\mathit{raining} \in \mathbb{F}$, $\mathit{somewhat}(\mathit{raining})$ should possibly be more true than *raining*. We could thus define

$$\begin{aligned} \mathit{somewhat} &: \mathbb{F} \to \mathbb{F}, \\ \mathit{somewhat}(x) &= \min(2x, 1) \end{aligned}$$

if we like. The modifier *very* is used often in fuzzy logic and is usually given by

$$\begin{aligned} \mathit{very} &: \mathbb{F} \to \mathbb{F}, \\ \mathit{very}(x) &= x^2. \end{aligned}$$

Within this line of definitions, give a reasonable definition for the modifier *extremely*.

$$\begin{aligned} \mathit{extremely} &: \mathbb{F} \to \mathbb{F} \\ \mathit{extremely}(x) &= \mathit{very}(\mathit{very}(x)) = x^4 \end{aligned}$$

Assume $\mathit{raining} = 0.7$ and $\mathit{windy} = 0.2$. What is the truth value of...?

- raining and very windy

  $\mathit{raining} = 0.7$
  $\mathit{very(windy)} = 0.2^2 = 0.04$
  $\mathit{raining} \wedge \mathit{very(windy)} = \min(0.7, 0.04) = 0.04$

- somewhat raining and not very windy

  $\mathit{somewhat(raining)} = \min(2 \cdot 0.7, 1) = 1$
  $\mathit{not(very(windy))} = 1 - 0.2^2 = 0.96$
  $\mathit{somewhat(raining)} \wedge \mathit{not(very(windy))} = \min(1,\ 0.96) = 0.96$

- very raining or extremely windy

$very(raining) = 0.7^2 = 0.49$
$extremely(windy) = 0.2^4 = 0.0016$
$very(raining) \vee extremely(windy) = \max(0.49, 0.0016) = 0.49$

(iii) A fuzzy logic engine usually translates observations into multiple fuzzy variables, whose meaning can overlap. For example, our sensor might measure 1mm of rain per square meter per hour and our logic engine might translate that to $heavy\_rain = 0.3$, $light\_rain = 0.8$, $dry\_weather = 0.05$. Then a fuzzy inference system can apply fuzzy rules like the ones shown in task 2(ii) to compute truth values for action propositions like $take\_umbrella = 0.8, take\_coat = 0.2$. Lastly, these need to be translated back into concrete action points (by a so-called defuzzifier) as our agent can probably not take "most of an umbrella" but needs to decide to take or not take the umbrella in a discrete manner. See Figure 1 for an overview of that process.

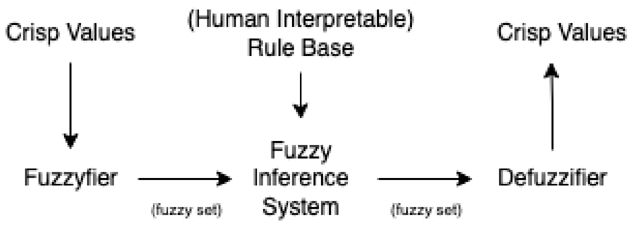


Figure 1: The fuzzy logic engine. Inspired by the explanation of this video.

Give mathematical functions or programs for

- a fuzzifier determining the truth value of $light\_rain \in \mathbb{F}$ based on a rain sensor and/or other suitable sensors

We assume as sensor *rainfall* that returns the measured rainfall in mm.

$$light\_rain = \begin{cases} rainfall/1.25\text{mm} & \text{if } rainfall < 1.25\text{mm}, \\ 1.25\text{mm}/rainfall & \text{otherwise.} \end{cases}$$

- a defuzzifier that decides to take or not take and umbrella based on fuzzy variables $take\_umbrella, stay\_home \in \mathbb{F}$ and/or other suitable fuzzy variables.

We return the decision variable $really\_take\_umbrella \in \mathbb{B}$ defined as follows:

$$agent\_gets\_umbrella = \begin{cases} true & \text{if } take\_umbrella > 0.5 \text{ and } stay\_home < 0.65, \\ false & \text{otherwise.} \end{cases}$$

(iv) Consider the following everyday situations you may want to reason about:

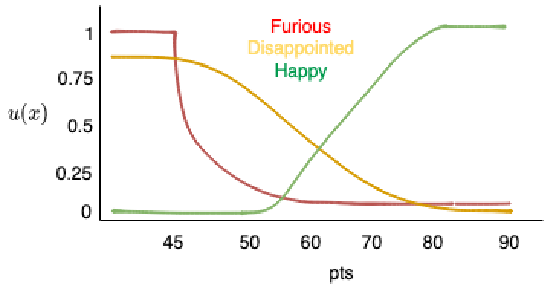
- A meal measured in how much you like to eat it.
- Your satisfaction with your grade/pts on the final CoIn exam.
- How the speed of driving in a/your car feels.
- A traffic light measured in what color is on.
- The heat level your stove should have to cook your favorite meal.
- The size of your shoes depending on how well they fit you.

Various qualities observed in these scenarios might be encoded via fuzzy variables. In each case, suggest some fuzzy variables corresponding to the possible observations made in the respective situations. For which of these six cases is the use of a fuzzy variables not really necessary? Why?

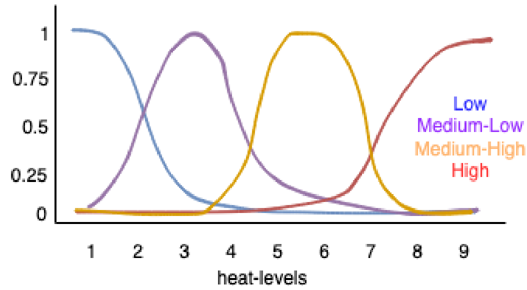
- meal: variables *disgusting*, *okay*, *delicious*, ...
- satisfaction: variables *furious*, *disappointed*, *happy*, ...
- speed: variables *sluggish*, *slow*, *reasonable*, *fast*, *dangerous*, ...
- traffic light: variables *green*, *yellow*, *red*, ...
- heat: variables *low*, *medium-low*, *medium-high*, *high*, ...
- shoe size: variables *hurts*, *tight fit*, *perfect fit*, *too big*, ...

It is not necessary to use fuzzy representation for a traffic light. The reason for that is that the values for variables *green*, *yellow*, *red* for a (correctly working) traffic light are truly binary and we do not need to consider intermediate states. Fuzzy variables are intended to use when we really have to consider ‘blurred’ states.

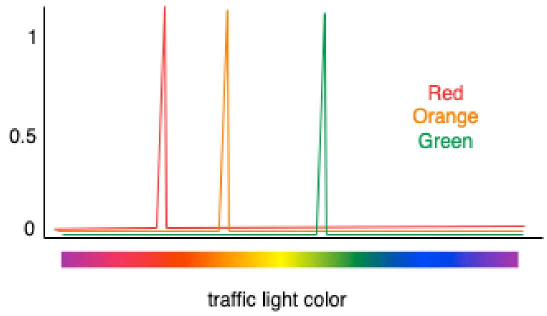
(v) For two of those fuzzy variables you just defined, draw the membership functions for the degree of truth of your variables in the templates of Fig. 2.



(a)



(b)



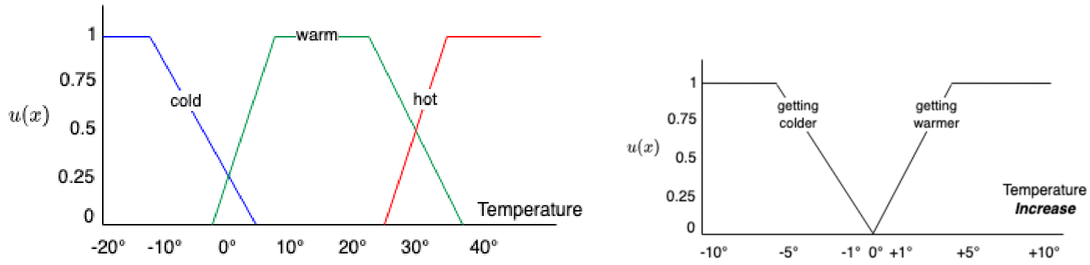
(c)

Figure 2: Fuzzy Logic Solutions

Some membership function examples shown in Figure 2. Remember we construct the fuzzy variables ourselves and then collect information about (a group's / the collective / etc.) believed 'membership' of the factual measurements to a variable's function (e.g., asking how many percent of 100 people consider 'stove setting 1 is *low* heat' to be true? $\rightarrow$ 100/100, therefore mark function *low* at $x = 1$ as 1.0 in the y-axis, and so on.)

(vi) Consider the following rules for an expert system recommendation on wearing sunscreen protection:

Rule	Condition	Action	Confidence
R1:	IF temp is hot	THEN sunscreen	M = 0.8
R2:	IF temp is hot AND getting warmer	THEN sunscreen	M = 0.6
R3:	IF temp is warm OR getting colder	THEN no sunscreen	M = 0.6
R4:	IF temp is cold	THEN no sunscreen	M = 0.8



(a) (b)

Figure 3: Fill in these membership functions

From this information, can you infer what the missing membership function needs to model? You can suggest the degrees of membership.

Model for the *increase in temperature*. Cf. Fig 3(b).

(vii) What is the value of the fuzzy variables *cold*, *warm*, and *hot* for the temperature values of 0° and 30°?

We can read these values directly from the graph in Fig 3(a) to get a degree of truth for *cold, warm, hot*. In this case:
$0^\circ \implies M(cold) = 0.25,\ M(warm) = 0.25,\ M(hot) = 0$
$30^\circ \implies M(cold) = 0,\ M(warm) = 0.5,\ M(hot) = 0.5$

(viii) Based on the membership functions you defined, how much is it *getting hotter* or *getting colder* if the temperature changes +1°?

Depending on how the $\{getting\ colder, getting\ hotter\}$ system was modeled, we can read from the graph:
$+1^\circ \implies \quad M(getting\ colder) \approx 0,\ M(getting\ hotter) \approx 0.1$

(ix) Using the membership values from the/your membership graphs and the confidences of the rules in the table above, calculate the degree of confidence that you may want to wear sunscreen for a temperature of 30° while it is *getting colder* by −1°.

The formula to calculate the degree of confidence of each rule is given by

$$M(\text{action}) = M(\text{condition}) \cdot M(\text{rule})$$

In our case we have two final conclusions x on wearing sunscreen (wear sunscreen or do not wear sunscreen). You can use the following formula for combining memberships of two conclusions:

$$M(x) = M_1(x) + M_2(x) - M_1(x) \cdot M_2(x)$$

Membership values of individual rules for state $s = (\text{temp} = 30°, \text{temp\_inc} = -1°)$:
$s \implies M(\text{temp is hot}) = 0.5$
$s \implies M(\text{temp is getting warmer}) = 0$
$s \implies M(\text{temp is warm}) = 0.5$
$s \implies M(\text{temp is getting colder}) = 0.1$
$s \implies M(\text{temp is cold}) = 0$

Membership values of combined rules:
$s \implies M(\text{temp is hot AND getting warmer}) = \min\{0.5, 0\} = 0$
$s \implies M(\text{temp is warm OR getting colder}) = \max\{0.5, 0.1\} = 0.5$

Degree of confidence by rule:
$s, R1 \implies M_1(\text{sunscreen}) = 0.5 \cdot 0.8 = 0.4$
$s, R2 \implies M_2(\text{sunscreen}) = 0 \cdot 0.6 = 0$
$s, R3 \implies M_3(\text{no sunscreen}) = 0.5 \cdot 0.6 = 0.3$
$s, R4 \implies M_4(\text{no sunscreen}) = 0 \cdot 0.8 = 0$

Combined Conclusions:
$M_1, M_2 \implies M(\text{sunscreen}) = 0.4 + 0 - 0.4 \cdot 0 = 0.4$
$M_3, M_4 \implies M(\text{no sunscreen}) = 0.3 + 0 - 0.3 \cdot 0 = 0.3$

Note: How to decide on the sunscreen issue is now the defuzzifier's problem.