



LMU Munich winter term 2024/2025

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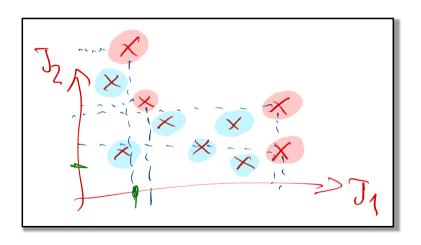
reading exercise 2024-11-28



"early exam" 2025-02-06

18:30h on maín campus

Multi-Objective Optimization



Definition 6 (multi-objective optimization). Let $\mathcal{E} = (\mathcal{X}, \mathcal{T}, \tau, E, \langle X_u \rangle_{0 \leq u \leq t})$ be an optimization process. \mathcal{E} is a multi-objective optimization process iff the target space \mathcal{T} has the form $\mathcal{T} = \mathcal{T}_0 \times \cdots \times \mathcal{T}_{N-1}$ for some $N \in \mathbb{N}$ with \leq_i being a total order on \mathcal{T}_i for any $i \in [0; N-1] \subset \mathbb{N}$. Unless stated otherwise, we assume that no single total order on \mathcal{T} is available. However, we can construct a partial order \preceq so that

$$(x_0, ..., x_{N-1}) \leq (x'_0, ..., x'_{N-1}) \iff \forall i \in [0; N-1] \subset \mathbb{N} : x_i \leq x'_i,$$

which is sufficient to adapt many standard optimization algorithms.

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Definition 7 (Pareto front for optimization). Let $\mathcal{E} = (\mathcal{X}, \mathcal{T}, \tau, E, \langle X_u \rangle_{0 \leq u \leq t})$ be a multi-objective optimization process with \leq being a partial order on the multi-objective target space \mathcal{T} .

- A solution candidate x Pareto-dominates a solution candidate x' (assuming minimization) iff $x \leq x'$.
- A solution candidate x is Pareto-optimal if there exists no other solution candidate $x' \in \mathcal{X}$ so that $x' \prec x$.
- The set of all Pareto-optimal solution candidates in \mathcal{X} is called the Pareto front of \mathcal{X} (w.r.t. \preceq).

Example: Multi-Objective Evolutionary Algorithm

The Goal Class Hierarchy

Goal Class 5: State Values Goal Class 4: Rewards and Costs Goal Class 3: Goal Direction Goal Class 2: Goal Valuation Goal Class 2.5: Multiple Goal Valuations Goal Class 1: Goal Predicate Goal Class 1.5: Multiple Goal Predicates Goal Class 0: No Goals

Goal Class 3: Goal Direction

"I know which way it's getting better!"

Gradients of Target Functions

Derivation

Algorithm 5 (gradient descent). Let $\mathcal{D} = (\mathcal{X}, \mathcal{T}, \tau, e, \langle x_u \rangle_{0 \leq u \leq t})$ be an optimization process. Let \mathcal{T} be continuous $(\mathcal{T} = \mathbb{R}, \text{e.g.})$ and let $\tau' : \mathcal{X} \to \mathcal{T}$ be the first derivative of τ . The process \mathcal{D} continues via gradient descent (with update rate $\alpha \in \mathbb{R}^+$) if e is of the form

$$e(\langle x_u \rangle_{0 \le u \le t}, \tau) = x_{t+1} = x_t - \alpha \cdot \tau'(x_t).$$

The learning rate α can also be given as a function, usually $\alpha : \mathbb{N} \to \mathbb{R}$ so that $e(\langle x_u \rangle_{0 \le u \le t}, \tau) = x_{t+1} = x_t - \alpha(t) \cdot \tau'(x_t)$.

If τ is stochastic, this process is called stochastic gradient descent (SGD).

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Algorithm 6 (gradient descent (policy)). Let π_{θ} be a policy π that depends on vector of continuous parameters $\theta \in \Theta$ such that usually $\Theta = \mathbb{R}^N$ for some N. Let $\tau : \Theta \to \mathcal{T}$ be a target function on the parameters θ of a policy π_{θ} . Let \mathcal{T} be continuous ($\mathcal{T} = \mathbb{R}$, e.g.) and let $\tau' : \Theta \to \mathcal{T}$ be the first derivative of τ , i.e., $\tau'(\theta) = \frac{\partial \tau(\theta)}{\partial \theta}$. If $\mathcal{D} = (\Theta, \mathcal{T}, \tau, e, \langle x_u \rangle_{0 \le u \le t})$ is an optimization process that continues via gradient descent, \mathcal{D} is a process of policy optimization via gradient descent.

Many Variants of Gradient Descent

while θ_t not converged do $t \leftarrow t+1$

return θ_t (Resulting parameters)

end while

source:

return x+

https://optimization.cbe.cornell.edu/ index.php?title=AdaGrad

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 \begin{aligned} & \textbf{Algorithm 1:} \text{ AdaGrad general algorithm} \\ & \eta \text{: Stepsize }; \\ & f(x) \text{: Stochastic objective function }; \\ & x_1 \text{: Initial parameter vector}; \\ & \textbf{for } t = 1 \text{ to } T \textbf{ do} \\ & \text{Evaluate } f_t(x_t) \text{ ;} \\ & \text{Get and save } g_t \text{ ;} \\ & G_t \leftarrow \sum_{\tau=1}^t g_\tau g_\tau^\top \text{ ;} \\ & x_{t+1} \leftarrow x_t - \eta G_t^{-1/2} g_t \text{ ;} \end{aligned}
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Algorithm 1: Adam, our proposed algorithm for stochastic optimization. See section 2 for details, and for a slightly more efficient (but less clear) order of computation. g_t^2 indicates the elementwise square g_t \odot g_t. Good default settings for the tested machine learning problems are \alpha = 0.001, \beta_1 = 0.9, \beta_2 = 0.999 and \epsilon = 10^{-8}. All operations on vectors are element-wise. With \beta_1^t and \beta_2^t we denote \beta_1 and \beta_2 to the power t.

Require: \alpha: Stepsize
Require: \beta_1, \beta_2 \in [0, 1): Exponential decay rates for the moment estimates
Require: \theta_0: Initial parameter vector
m_0 \leftarrow 0 (Initialize 1^{st} moment vector)
v_0 \leftarrow 0 (Initialize 2^{ut} moment vector)
t \leftarrow 0 (Initialize 1^{ut} moment vector)
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 $g_t \leftarrow \nabla_{\theta} f_t(\theta_{t-1})$ (Get gradients w.r.t. stochastic objective at timestep t) $m_t \leftarrow \beta_1 \cdot m_{t-1} + (1-\beta_1) \cdot g_t$ (Update biased first moment estimate) $v_t \leftarrow \beta_2 \cdot v_{t-1} + (1-\beta_2) \cdot g_t^2$ (Update biased second raw moment estimate) $\widehat{m}_t \leftarrow m_t/(1-\beta_1^t)$ (Compute bias-corrected first moment estimate) $\widehat{v}_t \leftarrow v_t/(1-\beta_1^t)$ (Compute bias-corrected second raw moment estimate)

 $\theta_t \leftarrow \theta_{t-1} - \alpha \cdot \widehat{m}_t / (\sqrt{\widehat{v}_t} + \epsilon)$ (Update parameters)

source: https://arxiv.ora/pdf/1412.6980.pdf%5D