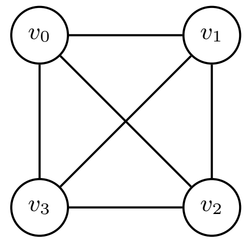
Homework 4

Due: May 7th, 2020

1. Consider the following graph:



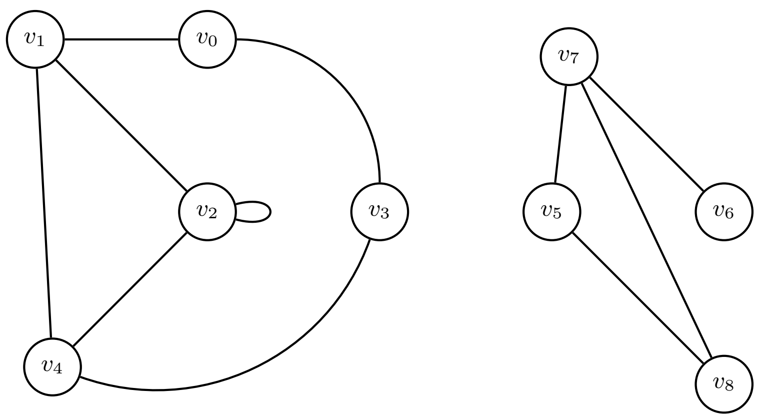
   (a) Give a walk in this graph from v_3 to v_1 of length 2.

   (b) Give a trail in this graph from v_0 to v_3 of length 5.

   (c) Give a circuit starting and ending at v_0 of length 4.

   (d) Is it possible to make a path of length 5 in this graph? If so, give one. If not, briefly explain why.

2. Consider the following graph:



   (a) What is the highest degree vertex, and what is its degree?

   (b) Give a circuit of length 6 in the graph.

   (c) Is it possible to make a cycle of length 5 in this graph?

3. Let $G = (V, E)$ be a graph. Prove that, if G contains an Eulerian circuit C, then C passes through every vertex in G.

4. Let $m, n \in \mathbb{Z}$. Prove that, if nm is even, then n or m is even.

Definition 1. Let $a, b \in \mathbb{Z}$ and $k \in \mathbb{N}^+$. k is the **greatest common divisor** of a and b if the following are true:

- $k \mid a$ and $k \mid b$
- for $l \in \mathbb{N}^+$ such that $l \mid a$ and $l \mid b$, $k \geq l$.

This can be written as $\gcd(a, b) = k$

5. Let $p, n \in \mathbb{Z}$. Show that, if p is prime, then $\gcd(p, n) = 1$ or $\gcd(p, n) = p$ using...
   (a) Proof by contraposition
   (b) Proof by contradiction
6. Let A be a set. Prove that $\emptyset \times A = \emptyset$
7. Consider the following "proof"

   Theorem 1. All horses in a set of n horses are the same color for $n > 0$.

   Proof. Proceed with induction over n:
   Base case: Consider $n = 1$. Clearly, a set containing one horse contains only horses of the same color.
   Inductive hypothesis: Suppose the statement holds for a set of k horses.
   Inductive step: We will show that the hypothesis implies that the statement holds for a set of $k = 1$ horses. Let $H = \{h_1, h_2, ..., h_k, h_{k+1}\}$ be a set of $k + 1$ horses. Further let $H_1 = \{h_1, h_2, ..., h_k\}$ and $H_2 = \{h_2, ..., h_k, h_{k+1}\}$. $H_1, H_2 \subseteq H$ and $|H_1| = |H_2| = k$. By the hypothesis, H_1 and H_2 contain only horses of the same color.
   Note that $h_2 \in H_1$ and $h_2 \in H_2$. Then h_2 must be the same color as all other horses in H_1 and H_2. h_2 cannot be two different colors at once. Then all of the horse in H_1 and H_2 must all have the same color. Thus, H contains only horses of the same color.
   Therefore, by the principle of mathematical induction, all horses in as set of n horses have the same color for $n \in \mathbb{Z}^+$. ■

   What is wrong with this "proof"?
8. Let $P(n)$ be the statement "$1^2 + 2^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$"
   (a) What is the statement $P(1)$?
   (b) Show $P(1)$ is true, completing the basis step of the proof
   (c) What is the inductive hypothesis?
   (d) Complete the inductive step, identifying where you use the inductive hypothesis
9. Prove that $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n + 1)! - 1$ for all positive integers n
10. Prove that 3 divides $n^3 + 2n$ for all positive integers n

Definition 2. Let $A_1, A_2, ..., A_n$ be sets.

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n$$

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \cdots \cap A_n$$

11. Let $A, B_1, B_2, ..., B_n$ be sets. Prove the following theorems [1] for $n \in \mathbb{Z}_{\geq 2}$. You may use the distributive laws previously given to you as a base case.
   (a)
   $$A \cup \bigcap_{i=1}^{n} B_i = \bigcap_{i=1}^{n} (A \cup B_i)$$
   (b)
   $$A \cap \bigcup_{i=1}^{n} B_i = \bigcup_{i=1}^{n} (A \cap B_i)$$

[1] Generalized Distributive Laws for Sets