CSC 348 – Homework #5

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Contents

1 Questions

1. $f(n) = n^2 + 1$ is not 1-1.

   Proof. Consider $-1, 1 \in \mathbb{R}$. Note that $1 \neq -1$, but that $f(-1) = f(1) = 1^2 + 1 = 2$. Thus, f is not 1-1 by definition of 1-1. `d(^_^)>`

2. $f(x) = n^5$ is 1-1.

   Proof. Suppose for some $x, y \in \mathbb{R}$, $f(x) = f(y)$.

   $$\begin{aligned} x^5 &= y^5 \\ \sqrt[5]{x^5} &= \sqrt[5]{y^5} \qquad (1) \\ x &= y \end{aligned}$$

   Thus, by definition of 1-1, f is 1-1. `d(^_^)>`

3. $f(n) = n - 1$ is onto.

   Proof. Let $y \in \mathbb{R}$. Note that $y = (y+1) - 1 = f(y+1)$. Since $(y+1) - 1 \in \mathbb{R}$, $f(x)$ is onto by definition of onto. `d(^_^)>`

4. $f(n) = n^2 + 1$ is not onto.

   Proof. Consider $0 \in \mathbb{R}$. Note that $x = \sqrt{0-1} \notin \mathbb{R}$. Because of this, no $x \in \mathbb{R}$ such that $f(x) = x^2 + 1 = 0$ is possible. Therefore, by definition of onto, f is not onto. `d(^_^)>`

5. $f(n) = \sqrt[5]{n}$ is onto.

Proof. Let $y \in \mathbb{R}$. Note that $y = \left(\sqrt[5]{y}\right)^5 = f\left(\sqrt[5]{y}\right)$. Since $(\sqrt[5]{y})^5 \in \mathbb{R}$, $f(x)$ is onto by definition of onto. `d(^_^)>`

6. $f : \mathbb{N} \to \mathbb{N}; f(n) = 2n$ is injective, but not surjective.

Proof of injectivity. Suppose $n = 3$ `d(^_^)>`

Proof of non-surjectivity. Suppose `d(^_^)>`

7. $f : \mathbb{N} \to \mathbb{N}; f(n) = \sqrt{n}$ is surjective, but not injective.

Proof of non-injectivity. Suppose `d(^_^)>`

Proof of surjectivity. Suppose `d(^_^)>`

8. $f : \mathbb{N} \to \mathbb{N}; f(n) = (-1)^n + n$ is one-to-one and onto.

Proof of injectivity. Suppose $a, b \in \mathbb{N}$ and $f(a) = f(b)$.

By the definition of those functions,

$$(-1)^a + a = (-1)^b + b \tag{2}$$

Note that the terms $(-1)^a$ and $(-1)^b$ will either be 1 or -1, which are both odd. Thus, $(-1)^a$ and $(-1)^b$ are always odd.

Case 1. $f(a) = f(b)$ is even.

By Lemma 1, since this value is even, its addends have the same sign. Thus, a and b must be odd.

By definition of odd, there exists some $i, j \in \mathbb{Z}$ such that $a = 2i + 1$ and $b = 2j + 1$.

$$\begin{aligned} (-1)^{2i+1} + a &= (-1)^{2j+1} + b \\ ((-1)^2)^i \cdot (-1)^1 + a &= ((-1)^2)^j \cdot (-1)^1 + b \\ -1 + a &= -1 + b \\ a &= b \end{aligned} \tag{3}$$

Thus, $f(n)$ is 1-1 for even n.

Case 2. $f(a) = f(b)$ is odd.

By Lemma 1, since this value is even, its addends have the same sign. Thus, a and b must be odd.

By definition of odd, there exists some $i, j \in \mathbb{Z}$ such that $a = 2i$ and $b = 2j$.

$$\begin{aligned}
(-1)^{2i} + a &= (-1)^{2j} + b \\
((-1)^2)^i + a &= ((-1)^2)^j + b \\
1 + a &= 1 + b \\
a &= b
\end{aligned} \tag{4}$$

Thus, $f(n)$ is 1-1 for even n.

Since both cases are true, $f(n)$ is a 1-1 function. `d(^_^)>`

Proof of surjectivity. Suppose $y \in \mathbb{N}$. We will prove that there always exists some $n \in \mathbb{N}$ such that $f(n) = y$.

Case 1. y is even.

By definition of even, for some $i \in \mathbb{N}$, $y = 2i$.

Consider $f(y+1)$.

$$\begin{aligned}
f(y+1) &= (-1)^{y+1} + (y+1) \\
&= (-1)^{2i+1} + y + 1 \\
&= ((-1)^2)^i \cdot (-1)^1 + y + 1 \\
&= 1 \cdot (-1) + y + 1 \\
&= y
\end{aligned} \tag{5}$$

Thus, if y is even, then $f(y+1) = y$.

Case 2. y is odd.

By definition of odd, for some $i \in \mathbb{N}$, $y = 2i + 1$.

Consider $f(y-1)$.

$$\begin{aligned}
f(y-1) &= (-1)^{y-1} + (y-1) \\
&= (-1)^{(2i+1)-1} + y - 1 \\
&= (-1)^{2i} + y - 1 \\
&= ((-1)^2)^i + y - 1 \\
&= 1 + y - 1 \\
&= y
\end{aligned} \tag{6}$$

Thus, if y is odd, then $f(y-1) = y$.

Therefore, for all $y \in \mathbb{N}$, there exists a value $n \in \mathbb{N}$ such that $f(n) = y$. Thus, f is onto. `d(^_^)>`

9. (a) $(f \circ g)(x) = (e^x)^2 + 1 = e^{2x} + 1$

   (b) $(g \circ f)(x) = e^{x^2+1}$

10.

Theorem 1. *Let $f : B \to C$ and $g : A \to B$. If f and g are onto, then $f \circ g$ is onto.*

Proof. Suppose $c \in C$. By definition of f as an onto function, there exists a $b \in B$ such that $f(b) = c$. It follows that there exists an $a \in A$ such that $g(a) = b$ by definition of g as an onto function. Therefore, there always exists some $a \in A$ such that $(f \circ g)(a) = c$.

Thus, by definition of onto, if f and g are onto, then $f \circ g$ is onto. d(^_^)>

11. (a)

Theorem 2. *Let $f : B \to C$ and $g : A \to B$. $f \circ g$ being onto does not imply that g is onto.*

Proof. Consider the case when $A = B = C = \mathbb{R}$, $f(x) = \ln x$, and $g(x) = x^2$. It follows that $(f \circ g)(x) = \ln(x^2)$.

Claim 1. *f is onto.*

Proof. Let $y \in \mathbb{R}$. Note that $y = \ln(e^y) = f(e^y)$. Since $e^y \in \mathbb{R}$, $f(x)$ is onto by definition of onto. d(^_^)>

Claim 2. *g is not onto.*

Proof. Consider $-1 \in \mathbb{R}$. Note that $x = \sqrt{-1} \notin \mathbb{R}$. Therefore, there is no such $x \in \mathbb{R}$ such that $g(x) = -1$. Thus, g is not onto. d(^_^)>

Claim 3. *$f \circ g$ is onto.*

Proof. Let $y \in \mathbb{R}$. Note that $y = \ln\left(\left(e^{\frac{y}{2}}\right)^2\right) = f\left(e^{\frac{y}{2}}\right)$. Since $e^{\frac{y}{2}} \in \mathbb{R}$, $f(x)$ is onto by definition of onto. d(^_^)>

Notice that $f \circ g$ is onto, but g is not onto. Therefore, $f \circ g$ being onto does not imply that g is onto. d(^_^)>

(b)

Theorem 3. *Let $f : B \to C$ and $g : A \to B$. $f \circ g$ being onto implies that f is onto.*

Proof. Suppose $c \in C$. By definition of $f \circ g$ as an onto function, there exists an $a \in A$ such that $(f \circ g)(a) = f(g(a)) = c$. Note that the existence of $f(g(a))$ requires $g(a) \in B$, so $g(a) \in B$.

Let $b = g(a)$. Thus, $f(b) = c$. Therefore, if $f \circ g$ is onto then f is onto. d(^_^)>

2 Additional Lemmas with Proofs

Lemma 1. *Let $a, b \in \mathbb{Z}$. $a + b$ is even if and only if a and b have the same parity, and odd if and only if a and b have different parities. (Reproduced from homework 4).*

Proof. **Case 1.** a and b are both even.

By definition of even:

$$\begin{aligned} a &= 2k \\ b &= 2l \end{aligned} \tag{7}$$

Therefore,

$$\begin{aligned} a + b &= 2k + 2l \\ &= 2(k + l) \end{aligned} \tag{8}$$

Let $m = k + l$. Then

$$a + b = 2m \tag{9}$$

By definition of even, $a + b$ is even.

Case 2. a and b are both odd.

By definition of odd:

$$\begin{aligned} a &= 2k + 1 \\ b &= 2l + 1 \end{aligned} \tag{10}$$

Therefore,

$$\begin{aligned} a + b &= 2k + 1 + 2l + 1 \\ &= 2(k + l + 1) \end{aligned} \tag{11}$$

Let $m = k + l + 1$. Then,

$$a + b = 2m \tag{12}$$

By definition of even, a + b is even.

Case 3. a and b have different parity.

WLOG let a be even and b be odd. By definition of even:

$$a = 2k \tag{13}$$

By definition of odd:

$$b = 2l + 1 \tag{14}$$

Therefore,

$$\begin{aligned} a + b &= 2k + 2l + 1 \\ &= 2(k + l) + 1 \end{aligned} \tag{15}$$

Let $m = k + l$. Then,

$$a + b = 2m + 1 \tag{16}$$

By definition of odd, a + b is odd.

In Summary:

a parity	b parity	$a + b$ parity
even	even	even
even	odd	odd
odd	even	odd
odd	odd	even

Therefore, $a + b$ is only even when a's parity = b's parity, and only odd when a's parity $\neq$ b's parity, concluding the lemma. `d(^_^)>`

3 Scratch Work