CSC 348 – Homework #5

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May 13, 2020

Contents

1 Questions

1.

Theorem 1. *For all $n \in \mathbb{Z}^+$, 133 divides* $11^{n+1} + 12^{2n-1}$

Proof. Let the sequence $a_n = 11^{n+1} + 12^{2n-1}$. We will first rewrite it recursively.

Base case.

$$\begin{aligned} a_1 &= 11^{1+1} + 12^{2\cdot 1-1} \\ &= 11^2 + 12^1 \\ &= 133 \end{aligned} \tag{1}$$

Recursive step. We can derive the recursive step by transforming $a_{n+1} - a_n$ using the explicit definition of a_n.

$$\begin{aligned} a_{n+1} - a_n &= 11^{n+2} + 12^{2n+1} - \left(11^{n+1} + 12^{2n-1}\right) \\ &= 11 \cdot 11^{n+1} + 144 \cdot 12^{2n-1} - 11^{n+1} - 12^{2n-1} \\ &= (11-1) \cdot 11^{n+1} + (144-1) \cdot 12^{2n-1} \\ &= 10 \cdot 11^{n+1} + 143 \cdot 12^{2n-1} \end{aligned} \tag{2}$$

We can further rearrange the equation like so:

$$\begin{aligned} a_{n+1} &= a_n + 10 \cdot 11^{n+1} + 143 \cdot 12^{2n-1} \\ &= a_n + 10 \cdot 11^{n+1} + 10 \cdot 12^{2n-1} + 133 \cdot 12^{2n-1} \\ &= a_n + 10 \cdot \left(11^{n+1} + 12^{2n-1}\right) + 133 \cdot 12^{2n-1} \end{aligned} \tag{3}$$

We can substitute the explicit definition of a_n into (3) like so:

$$\begin{aligned} a_{n+1} &= a_n + 10a_n + 133 \cdot 12^{2n-1} \\ &= 11a_n + 133 \cdot 12^{2n-1} \end{aligned} \tag{4}$$

Thus, we have our recursive definition of a_n:

$$\begin{aligned} a_1 &= 133 \\ a_{n+1} &= 11a_n + 133 \cdot 12^{2n-1} \end{aligned} \tag{5}$$

Now, we will use this to prove $133 \mid a_n$ by induction.

Base case. Consider when $n = 1$. By (5), $a_1 = 133$. Thus, $133 \mid a_1$ is equivalent to $133 \mid 133$, which is obviously true.

Inductive hypothesis. Consider when $n = k$. Suppose that $133 \mid a_k$. This implies that there exists some $i \in \mathbb{Z}$ such that $a_k = 133i$.

Inductive step. Consider when $n = k + 1$. By (5):

$$a_{k+1} = 11a_k + 133 \cdot 12^{2k-1} \tag{6}$$

By the inductive hypothesis, we can substitute $a_k = 133i$ in and rearrange:

$$\begin{aligned} a_{k+1} &= 11 \cdot (133i) + 133 \cdot 12^{2k-1} \\ &= 133 \cdot \left(11 \cdot i + 12^{2k-1}\right) \end{aligned} \tag{7}$$

Let $j = 11 \cdot i + 12^{2k-1}$. Then, $a_{k+1} = 133j$. Therefore, $133 \mid a_{k+1}$.

Thus, by the principle of mathematical induction, $133 \mid a_n$, and $133 \mid 11^{n+1} + 12^{2n-1}$.

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2.

Theorem 2. *For all $n \in \mathbb{N}^+$ there exists $a \in \mathbb{N}^+_{odd}$ and $b \in \mathbb{N}$ such that $n = a \cdot 2^b$.*

Proof. **Base case.** Consider $n = 1$. Select $a = 1$ and $b = 0$:

$$n = 1 = a \cdot 2^b = 1 \cdot 2^0 = 1 \tag{8}$$

Thus, the theorem applies for $n = 1$.

Additionally, consider $n = 2$. Select $a = 1$ and $b = 1$:

$$n = 2 = a \cdot 2^b = 1 \cdot 2^1 = 2 \tag{9}$$

Thus, the theorem applies for $n = 2$.

Inductive hypothesis. Suppose that for all $1 \leq n < k$, there exists $a_n \in \mathbb{N}^+_{odd}$ and $b_n \in \mathbb{N}$ such that

$$n = a_n \cdot 2^{b_n} \tag{10}$$

Inductive step. Consider $n = k$.

Case 1. Consider the case when k is even. Let $i \in \mathbb{Z}$ such that $k = 2i$.

Since $i < k$, by the induction hypothesis, there exists some $2j + 1 \in \mathbb{N}^+_{odd}$ and $c \in \mathbb{N}$ such that

$$i = (2j + 1) \cdot 2^c \tag{11}$$

Multiplying both sides by 2 yields a definition for k:

$$\begin{aligned} 2i = k &= 2 \cdot (2j + 1) \cdot 2^c \\ &= (2j + 1) \cdot 2^{c+1} \end{aligned} \tag{12}$$

Suppose $a = 2j + 1$ and $b = c + 1$. Therefore,

$$k = a \cdot 2^b \tag{13}$$

Thus, for all $n \in \mathbb{N}^+_{even}$ there exists $a \in \mathbb{N}^+_{odd}$ and $b \in \mathbb{N}$ such that $n = a \cdot 2^b$.

Case 2. Consider the case when k is odd.

$$k = k \cdot 1 = k \cdot 2^0 \tag{14}$$

Let $a = k$ and $b = 0$. Thus,

$$k = a \cdot 2^b \tag{15}$$

Thus, for all $n \in \mathbb{N}^+_{odd}$ there exists $a \in \mathbb{N}^+_{odd}$ and $b \in \mathbb{N}$ such that $n = a \cdot 2^b$.

Since **Case 1** and **Case 2** are true, this covers all the cases for $n \in \mathbb{N}^+$.

Therefore, by the principle of strong mathematical induction, for all $n \in \mathbb{N}^+$ there exists $a \in \mathbb{N}^+_{odd}$ and $b \in \mathbb{N}$ such that $n = a \cdot 2^b$.

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3.

Theorem 3. *Let f_n be the nth Fibonacci number. For all $n \in \mathbb{Z}^+$, $\sum_{i=1}^{n} f_i^2 = f_n f_{n+1}$*

Proof. **Base case.** Consider $n = 1$.

$$\sum_{i=1}^{1} f_i^2 = f_1^2 = 1^2 = 1 \tag{16}$$

$$f_1 f_2 = 1 \cdot 1 = 1 \tag{17}$$

Therefore, $\sum_{i=1}^{n} f_i^2 = f_n f_{n+1}$ for $n = 1$.

Inductive hypothesis. Suppose that for some $k \in \mathbb{Z}^+$,

$$\sum_{i=1}^{k} f_i^2 = f_k f_{k+1} \tag{18}$$

Inductive step. Consider the case when $n = k + 1$.

$$\sum_{i=1}^{k+1} f_i^2 = f_{k+1}^2 + \sum_{i=1}^{k} f_i^2 \tag{19}$$

By the inductive hypothesis, this expression is equivalent to

$$f_{k+1}^2 + f_k f_{k+1} = f_{k+1}\left(f_{k+1} + f_k\right) \tag{20}$$

By definition of the Fibonacci sequence, $f_{k+1} + f_k = f_{k+2}$. Substituting into the equation:

$$f_{k+1} f_{k+2} = f_{(k+1)} f_{(k+1)+1} \tag{21}$$

Therefore, for all $n \in \mathbb{Z}^+$, $\sum_{i=1}^{n} f_i^2 = f_n f_{n+1}$.

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4.

Theorem 4. *Let f_n be the nth Fibonacci number. For all $n \in \mathbb{Z}^+$, $\sum_{i=1}^{n} f_{2i-1} = f_{2n}$.*

Proof. **Base case.** Consider $n = 1$.

$$\sum_{i=1}^{1} f_{2i-1} = f_{2\cdot 1-1} = f_1 = 1 \tag{22}$$

$$f_{2\cdot 1} = f_2 = 1 \tag{23}$$

Therefore, $n \in \mathbb{Z}^+$, $\sum_{i=1}^{n} f_{2i-1} = f_{2n}$ for $n = 1$.

Inductive hypothesis. Suppose that for some $k \in \mathbb{Z}^+$,

$$\sum_{i=1}^{k} f_{2i-1} = f_{2k} \tag{24}$$

Inductive step. Consider the case when $n = k + 1$.

$$\begin{aligned} \sum_{i=1}^{k+1} f_{2i-1} &= f_{2(k+1)-1} + \sum_{i=1}^{k} f_{2i-1} \\ &= f_{2k+1} + \sum_{i=1}^{k} f_{2i-1} \end{aligned} \tag{25}$$

By the inductive hypothesis, this expression is equivalent to

$$f_{2k+1} + f_{2k} \tag{26}$$

By definition of the Fibonacci sequence, $f_{2k+1} + f_{2k} = f_{2k+2}$. Substituting into the equation:

$$f_{2k+2} = f_{2(k+1)} \tag{27}$$

Therefore, for all $n \in \mathbb{Z}^+$, $\sum_{i=1}^{n} f_{2i-1} = f_{2n}$.

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5. For each of the following definitions of a_n, show $(a_n)_{n=0}^{3}$
   (a) $(1, -2, 4, -8)$
   (b) $(3, 3, 3, 3)$
   (c) $(8, 11, 23, 71)$
   (d) $(2, 0, 8, 0)$
6. For each of the following definitions of a_n, show $\sum_{n=0}^{3} a_n$
   (a) -5
   (b) 12
   (c) 113
   (d) 10
7. Consider the sequence $a_i = i^2$.
   (a) $(a_i)_{i=0}^{3} = (0, 1, 4, 9)$
   (b) $\sum_{i=0}^{3} a_i = (0, 1, 5, 14)$
   (c) $s = (a_i)_{i=0}^{n}$ is a $n + 1$ element sequence that looks like

   $$(0, 1, 4, 9, \ldots, (n-3)^2, (n-2)^2, (n-1)^2, n^2)$$

   Assuming s is 1-indexed, the jth element of s is $(j-1)^2$.
8. $(a_n)_{n=0}^{4} = (0, 2, 5, 33, 8589934593)$
9. (a)

   $$\begin{aligned} \text{Basis step} \quad & a_0 = 0 \\ \text{Recursive step} \quad & a_n = a_{n-1} + 2 \end{aligned} \tag{28}$$

   (b)

   $$\begin{aligned} \text{Basis step} \quad & a_0 = 1 \\ \text{Recursive step} \quad & a_n = a_{n-1} + 2 \end{aligned} \tag{29}$$

   (c)

   $$\begin{aligned} \text{Basis step} \quad & 1 \in S \\ \text{Recursive step} \quad & x \in S \to 3x \in S \end{aligned} \tag{30}$$

10. (a)

$$\begin{aligned} \text{Basis step} \quad & a_1 = 6 \\ \text{Recursive step} \quad & a_n = a_{n-1} + 6 \end{aligned} \tag{31}$$

(b)

$$\begin{aligned} \text{Basis step} \quad & a_1 = 3 \\ \text{Recursive step} \quad & a_n = a_{n-1} + 2 \end{aligned} \tag{32}$$

(c)

$$\begin{aligned} \text{Basis step} \quad & a_1 = 10 \\ \text{Recursive step} \quad & a_n = 10a_{n-1} \end{aligned} \tag{33}$$

(d)

$$\begin{aligned} \text{Basis step} \quad & a_1 = 5 \\ \text{Recursive step} \quad & a_n = a_{n-1} \end{aligned} \tag{34}$$

11. (a) Step 0: $\{(0,0)\}$
Step 1: $\{(0,0),(2,3),(3,2)\}$
Step 2: $\{(0,0),(2,3),(3,2),(4,6),(5,5),(6,4)\}$
Step 3: $\{(0,0),(2,3),(3,2),(4,6),(5,5),(6,4),(6,9),(7,8),(8,7),(9,6)\}$

(b)

$$S = \left\{(x,y) \in \mathbb{N}^2 \mid a,b \in \mathbb{N} \wedge x = 3a + 2b \wedge y = 2a + 3b\right\}$$

(c)

Theorem 5. *If* $(a,b) \in S$ *then* $5 \mid (a+b)$.

Proof. **Base case.** Suppose $(a,b) = (0,0)$. Thus, $5 \mid (0+0) \equiv 5 \mid 0$, which is true because everything divides 0.
Inductive hypothesis. Suppose $(a,b) \in S$ and $5 \mid (a+b)$.
It follows that for some $i \in \mathbb{Z}$,

$$5i = a + b \tag{35}$$

Inductive step. By the inductive hypothesis $(a,b) \in S$.
Therefore, it follows that $(a+3,b+2),(a+2,b+3) \in S$. WLOG, consider the case for $(a+3,b+2) \in S$.

$$(a+3) + (b+2) = a + b + 5$$

By the inductive hypothesis, this expression is equal to

$$5i + 5 = 5(i+1)$$

Let $j = i + 1$. Thus,

$$(a+3) + (b+2) = 5j$$

By definition of divides, $5 \mid [(a+3)+(b+2)]$.
Therefore, by the principle of mathematical induction, for all $(a,b) \in S$, $5 \mid (a+b)$.
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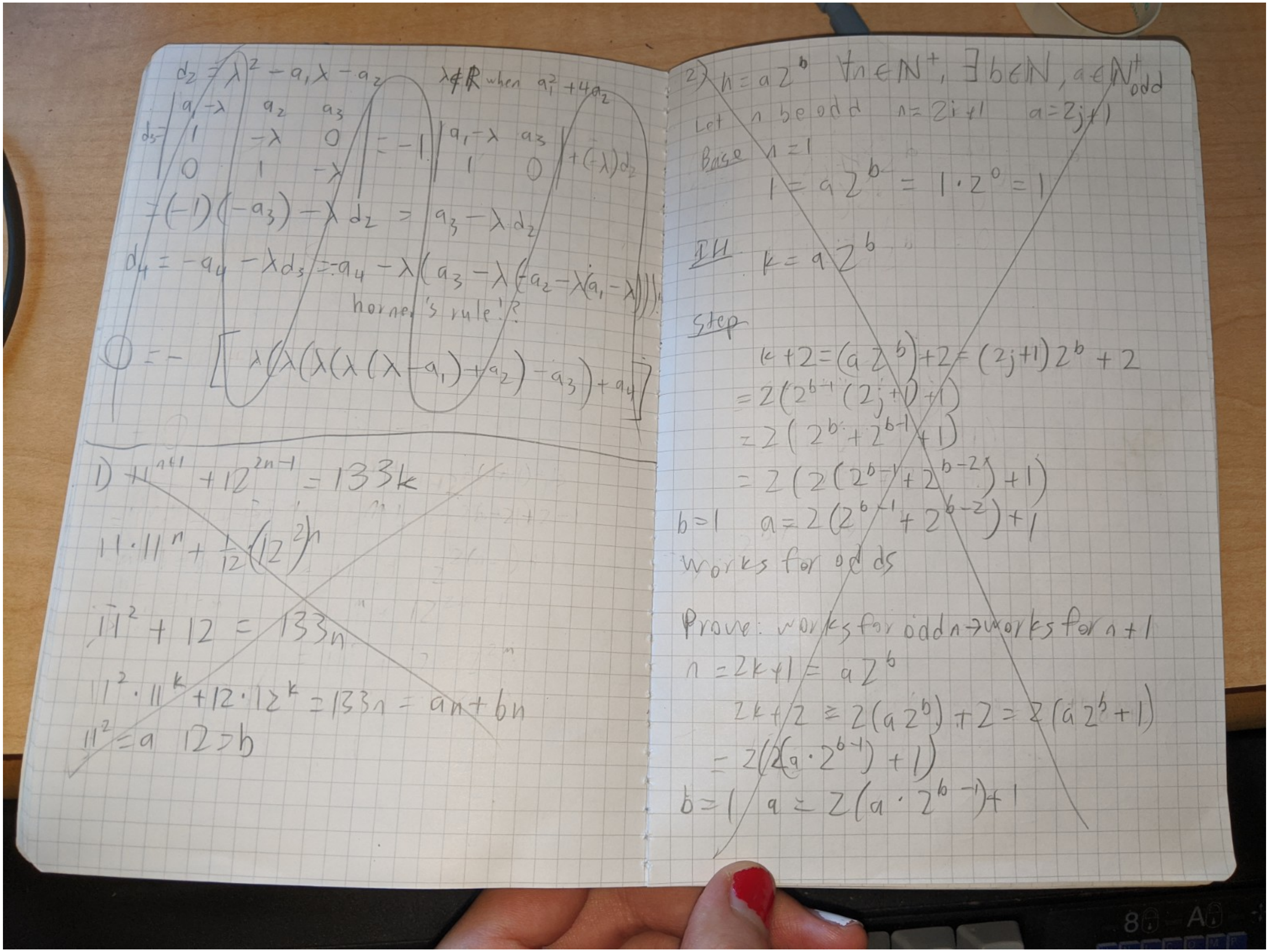
2 Scratch Work



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3) Base $n=1$ $1^2 = 1\cdot 1$

IH $n=k$ $\sum_{i=1}^{k} f_i^2 = f_k f_{k+1}$

Step

$$\sum_{i=1}^{k+1} f_i^2 = \sum_{i=1}^{k} f_i^2 + f_{k+1}^2 = f_k f_{k+1} + f_{k+1}^2$$

$$= f_{k+1}(f_k + f_{k+1}) = f_{k+1} f_{k+2}$$

4) Base $n=1$ $f_{2-1} = f_2 = 1$

IH $\sum_{i=1}^{k} f_{2i-1} = f_{2k}$

Step

$$\sum_{i=1}^{k+1} f_{2i-1} = f_{2k+1} + \sum_{i=1}^{k} f_{2i-1}$$

$$= f_{2k+1} + f_{2k} = f_{2k+2} = f_{2(k+1)} \quad \square$$

11) b) $\{(x,y)\in \mathbb{N}^2 \mid a\in \mathbb{N}, b\in \mathbb{N}, x=3a+2b, y=2a+3b\}$

11 a)

00 → 23, 32
23 → 46, 55
32 → 55, 64
46 → 69, 78
64 → 87, 96

11) c) $(0,0)\in S$

$(a,b)\in S \to (a+2, b+3)\,(a+3, b+2)\in S$

$5 \mid (a+b)$ if $(a,b)\in S$

Base $5 \mid 0$ is true

IH $(5n, 5m)\in S$

Step $(5n+2, 5m+3)\in S$

$5n+2+5m+3 = 5n+5m+5$

$= 5(n+m+1)$

1) $a_n = 121\cdot 11^{n-1}$ i $b_n = 12\cdot 144^{n-1}$ j

$a_k + b_k = 133i$

$a_{k+1} + b_{k+1}$

$121\cdot 11^{n-1} + (121\cdot 11^{n} - 121\cdot 11^{n-1})$

$(a_n + b_n)^2 = a_n^2 + b_n^2 + 2a_n b_n$

$121\cdot 11^{n-1} \cdot 12 \cdot 144^{n-1}$

$= 1452 \cdot 1584^{n-1}$

$c_{n-1} \cdot (a_n + b_n) - 2a_n b_n$

$$\overbrace{(a^n + b^n)(a+b)}^{\text{divides}} - a^n b - b^n a$$

$$= a^{n+1} + b^{n+1} \cancel{+ a^n b} + \cancel{b^n} a \cancel{- a^n b} - b^n a$$

$$121 \cdot 12^{2n-2} + 12 \cdot 11^{n-1}$$

	a	b
a^n	a^{n+1}	$a^n b$
b^n	$a b^n$	b^{n+1}

$$121 \cdot 12 \cdot 12^{2n-3} + 12 \cdot 11^2 \cdot 11^{n-3}$$

$$133 (12^{2n-3} + 11^{n-3})$$

$$(11^{n+1} + 12^{2n-1})(11^2 + 12)$$

$$11^{n+3} + 12^{2n} + 11^2 \cdot 12^{2n-1} + 12 \cdot 11^{n+1}$$

$$(11^{n+1} + 12^{2n-1})(11 + 12^2)$$

$$11^{n+2} + 12^{2n+1} + 11 \cdot 12^{2n-1} + 12^2 \cdot 11^{n+1}$$

$$11 \cdot 12^2 \cdot 12^{2n-3} + 12^2 \cdot 11^{n+1}$$

$$12^2 (11 \cdot 12^{2n-3} + 11^{n+1})$$

$$133 = 7 \cdot 19$$

$$11^2 \cdot 11^n + 12 \cdot 144^n = a_n$$

12

$$(11^{n+1} + 12^{2n-1})(11^2 + 12)$$

$$\frac{11^{n+1} + 12^{2n-1} = 121 \cdot 11^{n-1} + 12 \cdot 144^{(n-1)}}{11}$$

$$(121 \cdot 11^{n-1} + 12 \cdot 144^{n-1})(11 + 144)$$

$$\underbrace{121 \cdot 11^n + 12 \cdot 144^n}_{a_{n+1}} + 11 \cdot 12 \cdot 144^{n-1} + 12^2 \cdot 11^2 \cdot 11^{n-1}$$

$$133 = 11^2 + 12$$

$$\cancel{11 \cdot 12 (144^{n-1} + 11 \cdot 12\ 11^{n-1})}$$

$$\cancel{= 11 \cdot 12 (144^{n-1} + 12 \cdot 11^n)}$$

$$\cancel{12 (11 \cdot 144^{n-1} + 12^2}$$

$$(121 \cdot 11^{n-1} + 12 \cdot 144^{n-1})(11^2 + 144^2)$$

$$\underbrace{121 \cdot 11^{n+1} + 12 \cdot 144^{n+1}}_{a_{n+2}} + 11^2 \cdot 12 \cdot 144^{n-1}$$

$$11^2 \cdot 11^n + 12 \cdot 144^n = a_n$$

$$11^2 \cdot 11^n \cdot 144^n + 12 \cdot 144^n \cdot 11^n + 11^2 \cdot 11^{2n} + 12 \cdot 144^{2n}$$

$$= 11^n \cdot 144^n (11^2 + 12) + (a_{2n}\ ?)$$

11c) Step. $(a+3, b+2) \in S$

$$C = \begin{bmatrix} a_n \\ c_{n-1} \end{bmatrix} = \begin{bmatrix} c_{n+1} \\ a_n \end{bmatrix}$$

$a_n = 11^{n+1}$ $b_n = 12^{2n-1}$

$c_n = a_n + b_n$ $= 144$

$$\begin{bmatrix} 11 & 0 & 0 \\ 0 & 144 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_{n-1} \\ b_{n-1} \\ c_{n-1} \end{bmatrix} = \begin{bmatrix} a_n \\ b_n \\ c_n \end{bmatrix}$$

As you can see, I'm getting rather desperate.

$\vec{x}_1 = [$

$$11^{n+2} + 12^{2n+1} - 11^{n+1} - 12^{2n-1} = a_{n+1} - a_n$$

$$11(11^{n+1}) + 144(12^{2n-1}) - \dots$$

$$11^{n+1}(11-1) + 12^{2n-1}(144-1) = a_{n+1} - a_n$$

$$10 \cdot 11^{n+1} + 143 \cdot 12^{2n-1} + a_n = a_{n+1}$$

$$10 \cdot 11^{n+1} + 10 \cdot 12^{2n-1} + 133 \cdot 12^{2n-1} + a_n = a_{n+1}$$

$$133 \cdot 12^{2n-1} + 10 \cdot a_n + a_n = a_{n+1}$$

2) $k = 2j$ Suppose $j = a2^b$

$k = 2a2^b$

$k - 2 = (2j+1)2^b$

$k = (2j+1)2^b + 2 = 2(2j+1)2^{b-1} + 2$

$= 2[(2j+1)(2^{b-1}) + 1]$

potentially a fraction

$k = 2i + 1 = i + (i+1)$

$2i - 1 = (2j+1)2^c$

$2i + 1 = (2j+1)2^c + 2$

$= 2(2j+1)2^{c-1} + 2$

$= 2((2j+1)2^{c-1} + 1)$

$i = (2p+1)2^r$

$i + 1 = (2q+1)2^s$

$k = (2p+1)2^r + (2q+1)2^s$

Case 1: $r \neq s$.

WLOG $r > s$. $d = r - s$. $d + s = r$.

$k = (2p+1)2^{d+s} + (2q+1)2^s$

$= (2p+1)2^d 2^s + (2q+1)2^s$

$= 2^s[(2p+1)2^d + 2q+1]$

Case 2: $r = s$

$(i+1) - i = (2q+1)2^s - (2p+1)2^r$

$= (2q+1)2^r - (2p+1)2^r$

$= 2^r(2q - 2p)$

$= 2 \cdot 2^r (q - p)$ Always even. But 1 is odd.