Homework 5

Due: May 5th, 2020

1. Prove that 133 divides $11^{n+1} + 12^{2n-1}$ for all positive integers n.
2. Prove by induction that, for all $n \in \mathbb{N}^+$, there exist some $b \in \mathbb{N}$ and $a \in \mathbb{N}^+_{\text{odd}}$ such that $n = a \cdot 2^b$. [1]
3. Let f_n be the nth Fibonacci number. Prove that, for all $n \in \mathbb{Z}^+$, $\sum_{i=1}^{n} (f_i)^2 = f_n f_{n+1}$
4. Let f_n be the nth Fibonacci number. Prove that, for all $n \in \mathbb{Z}^+$, $\sum_{i=1}^{n} f_{2i-1} = f_{2n}$
5. For each of the following definitions of a_n, show $(a_n)_{n=0}^{3}$
   (a) $a_n = (-2)^n$
   (b) $a_n = 3$
   (c) $a_n = 7 + 4^n$
   (d) $2^n + (-2)^n$
6. For each sequence in the previous question, show $\sum_{i=0}^{3} a_i$.
7. Consider the sequence $a_i = i^2$
   (a) What is $(a_i)_{i=0}^{3}$?
   (b) What is $\sum_{i=0}^{3} a_i$?
   (c) What is $(a_i)_{i=0}^{n}$?
8. Let $a_n = 2^{a_{n-1}} + 1$, where $a_0 = 0$. What is $(a_n)_{n=0}^{4}$?
9. Give recursive definitions for:
   (a) $\mathbb{N}_{even}$
   (b) $\mathbb{N}_{odd}$
   (c) $\{3^k \mid k \in \mathbb{N}\}$
10. For the following explicitly-defined sequences, give an equivalent recursive definitions for $n \in \mathbb{Z}^+$. a_1 should be the base case.
    (a) $a_n = 6n$
    (b) $a_n = 2n + 1$
    (c) $a_n = 10^n$
    (d) $a_n = 5$
11. Let S be a set of ordered pairs defined by:
    Basis step: $(0, 0) \in S$
    Recursive step: if $(a, b) \in S$, then $(a + 2, b + 3), (a + 3, b + 2) \in S$
    (a) List the elements of S after three applications of the basis step.
    (b) Find a closed-form definition for S
    (c) Show that, if $(a, b) \in S$, then $5 \mid (a + b)$

[1] Hint: use all odd n as your base case