CSC 348 – Homework #7

Astrid Augusta Yu

May 30, 2020

Contents

1 Questions

1.

Theorem 1. *$\mathbb{Z}_{\geq 10}$ is countably infinite.*

Proof. Let $f : \mathbb{Z}_{\geq 10} \to \mathbb{N}$ be defined as

$$f(x) = x - 10$$

.

Claim 1.1. *f is 1-1.*

Suppose $a, b \in \mathbb{Z}_{\geq 10}$ and $f(a) = f(b)$. Thus,

$$a - 10 = b - 10$$

$$a = b$$

Thus, f is 1-1 by definition.

Claim 1.2. *f is onto.*

Consider $y \in \mathbb{N}$. Note that $y = (y + 10) - 10 = f(y + 10)$.

Additionally, note that the lowest possible value for $y + 10$ is $0 + 10 = 10 \in \mathbb{Z}_{\geq 10}$. All values greater than that are elements of $\mathbb{Z}_{\geq 10}$ by definition of that set. Thus, $y + 10 \in \mathbb{Z}_{\geq 10}$

Therefore, f is onto by definition.

Since f is both 1-1 and onto, it is bijective. Therefore, $|\mathbb{Z}_{\geq 10}| = |\mathbb{N}|$, so $\mathbb{Z}_{\geq 10}$ is countably infinite. d(^_^)>

2.

Theorem 2. *$\mathbb{Z}^-_{odd}$ is countably infinite.*

Proof. Let $f : \mathbb{Z}^-_{odd} \to \mathbb{Z}^+_{odd}$ be defined as

$$f(x) = -x$$

Claim 2.1. *f is 1-1.*

Suppose $a, b \in \mathbb{Z}^-_{odd}$ and $f(a) = f(b)$. Thus,

$$-a = -b$$

$$a = b$$

Thus, f is 1-1 by definition.

Claim 2.2. *f is onto.*

Consider $i \in \mathbb{N}$ such that $2i+1 \in \mathbb{Z}^+_{odd}$. Note that $2i+1 = -(-(2i+1)) = f(-(2i+1))$.

Additionally, note that

$$-(2i+1) = -2i - 1 = 2(-i) + 2 - 2 - 1 = 2(-i-1) + 1$$

Suppose $j = -i - 1$. Thus,

$$-(2i+1) = 2j + 1$$

Therefore, $-(2i+1)$ is odd by definition.

Additionally, since $2i+1$ is always negative, $-(2i+1)$ is always positive. Thus, since it is positive and odd, $-(2i+1) \in \mathbb{Z}^+_{odd}$.

Therefore, f is onto by definition.

Since f is both 1-1 and onto, it is bijective. Therefore, $|\mathbb{Z}^-_{odd}| = |\mathbb{Z}^+_{odd}|$, so $\mathbb{Z}_{\geq 10}$ is countably infinite. d(^_^)>

3.

Theorem 3. *$\mathbb{Z}^-_{odd}$ is countably infinite.*

Proof. Let $f : \mathbb{Z} \to 10\mathbb{Z}$ be defined as

$$f(x) = 10x$$

Claim 3.1. *f is 1-1.*

Suppose $a, b \in \mathbb{Z}$ and $f(a) = f(b)$. Thus,

$$10a = 10b$$

$$a = b$$

Thus, f is 1-1 by definition.

Claim 3.2. *f is onto.*

Consider some number $n \in 10\mathbb{Z}$. By definition of $10\mathbb{Z}$, there exists a $m \in \mathbb{Z}$ such that $n = 10m$.

Note that $n = 10m = f(m)$. Since, by definition, $m \in \mathbb{Z}$, f is onto.

Since f is both onto and 1-1, it is bijective. Therefore, $|\mathbb{Z}| = |10\mathbb{Z}|$ and $10\mathbb{Z}$ is countably infinite. `d(^_^)>`

4.

Theorem 4. $S = \{x \in \mathbb{Z} \mid |x| < 1,000,000\}$ *is finite.*

Proof. Suppose $T = \mathbb{N}_{\leq 1,999,999}$. Note that T is finite. Define $f : S \to T$ as

$$f(x) = x + 999,999$$

Claim 4.1. *f is 1-1.*

Suppose $a, b \in \mathbb{Z}$ and $f(a) = f(b)$. Thus,

$$a + 999,999 = b + 999,999$$

$$a = b$$

Thus, f is 1-1 by definition.

Claim 4.2. *f is onto.*

Let $n \in \mathbb{N}_{\leq 1,999,999}$. Consider the value $x = n - 999,999$.

- When $n = 0$, $x = -999,999$. Note that $|x| < 1,000,000$.
- When $n = 1,999,999$, $x = 999,999$. Note that $|x| < 1,000,000$.

Thus, for all values of n, $x \in \mathbb{N}_{\leq 1,999,999}$.

Note that $n = (n - 999,999) + 999,999 = x + 999,999 = f(x)$. Thus, by definition, f is onto.

Since f is 1-1 and onto, it is bijective, so $|S| = |T|$. Since T is finite, then S is finite as well. `d(^_^)>`

5.

Theorem 5. $1, 2 \times \mathbb{N}$ *is countably infinite.*

Proof. By definition of cartesian product,

$$1, 2 \times \mathbb{N} = \{(1, x) \mid x \in \mathbb{N}\} \cup \{(2, x) \mid x \in \mathbb{N}\}$$

By Lemma 2, $|\{(1, x) \mid x \in \mathbb{N}\}| = |\mathbb{N}|$ and $|\{(2, x) \mid x \in \mathbb{N}\}| = |\mathbb{N}|$, so both are countably infinite sets. Thus, by definition of union, their union $1, 2 \times \mathbb{N}$ is also countably infinite as well. `d(^_^)>`

6. (a)

    Theorem 6. $7\mathbb{N} \cap 11\mathbb{N}$ *is countably infinite.*

    Proof. Let $S = 7\mathbb{N} \cap 11\mathbb{N}$. By definition of set S,

    $$S = 7\mathbb{N} \cap 11\mathbb{N} = \{7x \mid x \in \mathbb{N}\} \cap \{11y \mid y \in \mathbb{N}\}$$

    Thus, $n \in S$ if and only if there exists some $x, y \in \mathbb{N}$ such that $n = 7x = 11y$.
    Consider $n = 77j$ for some $j \in \mathbb{N}$. Note that $n = 7(11j) = 11(7j)$. Setting $x = 11j$ and $y = 7j$, $n = 7x = 11y$. Therefore, $n \in S$.
    Note that there are $|\mathbb{N}|$ choices for j by its definition, so there are $|\mathbb{N}|$ elements of S. Therefore, $|S| = |\mathbb{N}|$ and S is countably infinite. `d(^_^)>`

    (b)

    Theorem 7. $\mathbb{N}_{even} \cap \mathbb{N}_{odd}$ *is finite.*

    Proof. Suppose by way of contradiction that there exists some $x \in \mathbb{N}$ such that $x \in \mathbb{N}_{even}$ and $x \in \mathbb{N}_{odd}$. By definition of even and odd, $x = 2i = 2j + 1$ for some $i, j \in \mathbb{N}$. Rearranging the equation,

    $$2i - 2j = 1$$

    $$2(i - j) = 1$$

    Let $k = i - j$. Thus, $2k = 1$ and 1 is an even number. $\rightarrow\leftarrow$
    This is a contradiction because 1 is not an even number. Therefore, x does not exist, so $\mathbb{N}_{even} \cap \mathbb{N}_{odd} = \emptyset$. Thus, $\mathbb{N}_{odd} \cap \mathbb{N}_{even}$ is a finite set. `d(^_^)>`

7. (a)

    Theorem 8. $\mathbb{N} \setminus \mathbb{Z}^+$ *is finite.*

    Proof. Note that $0 \in \mathbb{N}$ but $0 \notin \mathbb{Z}^+$. Thus, $0 \in \mathbb{N} \setminus \mathbb{Z}^+$.
    By way of contradiction, suppose there exists some $x \in \mathbb{N}$ such that $x \neq 0$ and $x \notin \mathbb{Z}^+$. Since 0 is the smallest item in $\mathbb{N}$, $x \neq 0$ is equivalent to $x > 0$, so x must be positive. $\rightarrow\leftarrow$
    There is a contradiction because we assumed that $x \notin \mathbb{Z}^+$, or in other words, that x is not positive. Therefore, $\mathbb{N} \setminus \mathbb{Z}^+ = \{0\}$ and $\mathbb{N} \setminus \mathbb{Z}^+$ is finite. `d(^_^)>`

    (b)

Theorem 9. *Let $\mathbb{P}$ be the set of primes.$\mathbb{P} \setminus \mathbb{N}_{odd}$ is finite. (Note that both $\mathbb{N}_{odd}$ and $\mathbb{P}$ are countably infinite.)*

Proof. By definition of set subtraction,

$$\mathbb{P} \setminus \mathbb{N}_{odd} = \mathbb{P} \cap \mathbb{N}_{even}$$

Note that 2 is even, and prime by definition. Thus, $2 \in (\mathbb{P} \cap \mathbb{N}_{even})$

We will prove that there are no primes greater than 2. By way of contradiction, suppose there is some $x \in P$ such that $x > 2$, x is even, and x is prime. By definition of even, $x = 2j$ for some $j \in \mathbb{N}$. $\rightarrow\leftarrow$

This violates the assumption that x is prime. Thus, x cannot exist. Therefore, $\mathbb{N}_{even} \cap \mathbb{P} = \mathbb{P} \setminus \mathbb{N}_{odd} = \{2\}$ and it is a finite set. d(^_^)>

8.

Theorem 10. *If $|A| = |B|$ and $|B| = |C|$ then $|A| = |C|$.*

Proof. Since $|A| = |B|$, there exists a bijection $f : A \to B$. Since $|B| = |C|$, there exists a bijection $g : B \to C$. Thus, there exists a bijection $(g \circ f) : A \to C$. Therefore, $|A| = |C|$. d(^_^)>

9.

Theorem 11. *If A and B are sets, then $|A \cup B| = |A| + |B \setminus A|$*

Proof. Consider $S = (A \cup B) \setminus A$. By Lemma 1, $T = B \setminus A$. Note that every element in A is contained in $A \cup B$ and there are no elements of A that are not contained in $A \cup B$ by definition of union. Therefore, $(A \cup B) \setminus A = B \setminus A$ has $|A|$ less elements in it than $A \cup B$.

Written as an equation:

$$|A \cup B| - |A| = |B \setminus A|$$

Rearranging the equation produces our result.

$$|A \cup B| = |A| + |B \setminus A|$$

d(^_^)>

10.

Theorem 12. *If A and B are sets, then $|A \times B| = |A| \cdot |B|$*

Proof. By definition of cartesian product, $A \times B$ is expanded out to the following:

$$\{(a, b) \mid a \in A, b \in B\}$$

We can rewrite this as the following union:

$$S = \bigcup_{a \in A} \{(a, b) \mid b \in B\}$$

By Lemma 2, every such $\{(a, b) \mid b \in B\}$ has cardinality $|B|$. Additionally, there are $|A|$ possible $a \in A$ being iterated through. Therefore, $|S| = |A \times B| = |A| \cdot |B|$. d(^_^)>

2 Additional Lemmas with Proofs

Lemma 1. *Let A and B be sets. $(A \cup B) \setminus A = B \setminus A$.*

Proof. By definition of set subtraction,

$$(A \cup B) \setminus A = (A \cup B) \cap \overline{A} \tag{1}$$

Using the distributive property, this expression is equivalent to

$$(A \cap \overline{A}) \cup (B \cap \overline{A}) \tag{2}$$

By definition of intersection, this is reduced to

$$\emptyset \cup (B \cap \overline{A}) \tag{3}$$

By definition of union, this is reduced to

$$B \cap \overline{A} \tag{4}$$

Therefore, $(A \cup B) \setminus A = B \setminus A$. d(^_^)>

Lemma 2. *Let A and B be sets, and $a \in A$. Suppose the set $S_a = \{(a, b) \mid b \in B\}$. Then, $|S_a| = |B|$ is countably infinite.*

Proof. Let $f_a : B \to S_a$ be defined as follows:

$$f_a(b) = (a, b)$$

Claim 2.1. *f_n is 1-1.*

Suppose $x, y \in B$ such that $f_a(x) = f_a(y)$. Thus, by definition of f,

$$(a, x) = (a, y)$$

Therefore, $x = y$, and f_n is 1-1.

Claim 2.2. *f_a is onto.*

Let $(a, b) \in S$. Note that $(a, b) = f_a(b)$. Thus, there exists some $b \in B$ such that $f_a(b) = (a, b)$, so f_a is onto.

Because f_b is both 1-1 and onto, f_b is bijective. Therefore, for all $b \in B$, $|S_a| = |B|$. d(^_^)>

3 Scratch Work