CSC 348 – Homework #2

Astrid Augusta Yu

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1.

Theorem 1. *Let A and B be sets with some universe U. $A \setminus B = A \cap \overline{B}$.*

Proof. ($\rightarrow$) Suppose $x \in (A \setminus B)$. By definition of set difference,

$$x \in A \land x \notin B \tag{1}$$

By definition of complement:

$$x \in A \land x \in \overline{B} \tag{2}$$

By definition of intersection:

$$x \in (A \setminus \overline{B}) \tag{3}$$

Therefore, $x \in (A \setminus B) \rightarrow x \in (A \cap \overline{B})$.
By definition of subset, $(A \setminus B) \subseteq (A \cap \overline{B})$.
($\leftarrow$) Let $x \in U$. Suppose $x \in A \cap \overline{B}$.
By a symmetric argument, $x \in (A \cap \overline{B}) \rightarrow x \in (A \setminus B)$.
By definition of subset, $(A \cap \overline{B}) \subseteq (A \setminus B)$.
Because $(A \setminus B) \subseteq (A \cap \overline{B})$ and $(A \cap \overline{B}) \subseteq (A \setminus B)$, by definition of equivalent sets, $(A \setminus B) = (A \cap \overline{B})$. `d(^_^)>`

2.

Theorem 2. *Let A, B, and C be sets. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.*

Proof. ($\rightarrow$) Let $x \in A \cap (B \cup C)$. By definition of intersection:

$$(x \in A) \land [x \in (B \cup C)] \tag{4}$$

By definition of union:

$$(x \in A) \land (x \in B \lor x \in C) \tag{5}$$

Using the boolean distributive property:

$$(x \in A \land x \in B) \lor (x \in A \land x \in C) \tag{6}$$

By definition of intersection:

$$(x \in A \cap B) \lor (x \in A \cap B) \tag{7}$$

By definition of union:

$$x \in (A \cap B) \cup (A \cap B) \tag{8}$$

Therefore, $x \in [A \cap (B \cup C)] \rightarrow x \in [(A \cap B) \cup (A \cap C)]$.
By definition of subset, $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.
($\leftarrow$) Suppose $x \in (A \cap B) \cup (A \cap C)$.
By a symmetric argument,

$$x \in [(A \cap B) \cup (A \cap C)] \rightarrow x \in [A \cap (B \cup C)] \tag{9}$$

By definition of subset,

$$(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C) \tag{10}$$

Therefore, by definition of equivalent sets,

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \tag{11}$$

`d(^_^)>`

3.

Theorem 3. *Let A, B, and C be sets.* $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Proof. ($\rightarrow$) Let $x \in A \cup (B \cap C)$. By definition of union:

$$(x \in A) \lor [x \in (B \cap C)] \tag{12}$$

By definition of intersection:

$$(x \in A) \lor (x \in B \land x \in C) \tag{13}$$

Using the boolean distributive property:

$$(x \in A \lor x \in B) \land (x \in A \lor x \in C) \tag{14}$$

By definition of union:

$$(x \in A \cup B) \land (x \in A \cup B) \tag{15}$$

By definition of intersection:

$$x \in (A \cup B) \cap (A \cup B) \tag{16}$$

Therefore, $x \in [A \cup (B \cap C)] \to x \in [(A \cup B) \cap (A \cup C)]$.
By definition of subset, $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$.
($\leftarrow$) Suppose $x \in (A \cup B) \cap (A \cup C)$.
By a symmetric argument,

$$x \in [(A \cup B) \cap (A \cup C)] \to x \in [A \cup (B \cap C)] \tag{17}$$

By definition of subset,

$$(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C) \tag{18}$$

Therefore, by definition of equivalent sets,

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \tag{19}$$

d(^_^)>

4.

Theorem 4. *Let A and B be sets.* $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

Proof. ($\to$) Suppose $x \in \overline{A \cup B}$. By definition of set complement,

$$\overline{x \in (A \cup B)} \tag{20}$$

By definition of union,

$$\overline{(x \in A) \vee (x \in B)} \tag{21}$$

Using DeMorgan's Law, this is equivalent to

$$\overline{x \in A} \wedge \overline{x \in B} \tag{22}$$

By definition of intersection,

$$x \in (\overline{A} \cap \overline{B}) \tag{23}$$

Therefore, $x \in \overline{A \cup B} \to x \in \overline{A} \cap \overline{B}$.
($\leftarrow$) Suppose $x \in \overline{A} \cap \overline{B}$.
By a symmetric argument,

$$x \in \overline{A} \cap \overline{B} \to x \in \overline{A \cup B} \tag{24}$$

By definition of subset, the following are true:

$$\begin{aligned} \overline{A} \cap \overline{B} &\subseteq \overline{A \cup B} \\ \overline{A \cup B} &\subseteq \overline{A} \cap \overline{B} \end{aligned} \tag{25}$$

Therefore, by definition of set equivalence,

$$\overline{A \cup B} = \overline{A} \cap \overline{B} \tag{26}$$

`d(^_^)>`

5.

Theorem 5. *Let A and B be sets. $\overline{A \cup B} = \overline{A} \cap \overline{B}$.*

Proof. ($\rightarrow$) Suppose $x \in \overline{A \cap B}$. By definition of set complement,

$$\overline{x \in (A \cap B)} \tag{27}$$

By definition of intersection,

$$\overline{(x \in A) \wedge (x \in B)} \tag{28}$$

Using DeMorgan's Law, this is equivalent to

$$\overline{x \in A} \vee \overline{x \in B} \tag{29}$$

By definition of union,

$$x \in (\overline{A} \cup \overline{B}) \tag{30}$$

Therefore, $x \in \overline{A \cap B} \rightarrow x \in \overline{A} \cup \overline{B}$.
($\leftarrow$) Suppose $x \in \overline{A} \cup \overline{B}$.
By a symmetric argument,

$$x \in \overline{A} \cup \overline{B} \rightarrow x \in \overline{A \cap B} \tag{31}$$

By definition of subset, the following are true:

$$\begin{aligned} \overline{A} \cup \overline{B} &\subseteq \overline{A \cap B} \\ \overline{A \cap B} &\subseteq \overline{A} \cup \overline{B} \end{aligned} \tag{32}$$

Therefore, by definition of set equivalence,

$$\overline{A \cap B} = \overline{A} \cup \overline{B} \tag{33}$$

`d(^_^)>`

6.

Theorem 6. *Let $m, n, p \in \mathbb{Z}$. If $m + n$ is even and $n + p$ is even, then $m + p$ is even.*

First, let us additionally define the following lemma:

Lemma 1. *Let $a, b \in \mathbb{Z}$. $a + b$ is even if and only if a and b have the same parity, and odd if and only if a and b have different parities.*

Proof of Lemma 1. **Case 1.** a and b are both even.

By definition of even:

$$\begin{aligned} a &= 2k \\ b &= 2l \end{aligned} \tag{34}$$

Therefore,

$$\begin{aligned} a + b &= 2k + 2l \\ &= 2(k + l) \end{aligned} \tag{35}$$

Let $m = k + l$. Then

$$a + b = 2m \tag{36}$$

By definition of even, $a + b$ is even.

Case 2. a and b are both odd.

By definition of odd:

$$\begin{aligned} a &= 2k + 1 \\ b &= 2l + 1 \end{aligned} \tag{37}$$

Therefore,

$$\begin{aligned} a + b &= 2k + 1 + 2l + 1 \\ &= 2(k + l + 1) \end{aligned} \tag{38}$$

/lnoy Let $m = k + l + 1$. Then,

$$a + b = 2m \tag{39}$$

By definition of even, a + b is even.

Case 3. a and b have different parity. WLOG let a be even and b be odd.

By definition of even:

$$a = 2k \tag{40}$$

By definition of odd:

$$b = 2l + 1 \tag{41}$$

Therefore,

$$\begin{aligned} a + b &= 2k + 2l + 1 \\ &= 2(k + l) + 1 \end{aligned} \tag{42}$$

Let $m = k + l$. Then,

$$a + b = 2m + 1 \tag{43}$$

By definition of odd, a + b is odd.

In Summary:

a parity	b parity	$a + b$ parity
even	even	even
even	odd	odd
odd	even	odd
odd	odd	even

Therefore, $a + b$ is only even when a's parity = b's parity, and only odd when a's parity $\neq$ b's parity, concluding the lemma. `d(^_^)>`

Proof of Theorem 6. Let $m, n, p \in \mathbb{Z}$. Suppose $m + n$ is even and $n + p$ is even.

WLOG, let $P \in \{even, odd\}$ be the parity of m.

By Lemma 1, because $m + n$ is even and m has parity P, then n must have parity P as well.

By the same lemma, because $n + p$ is even and n has parity P, then p must have parity P as well.

Also by the same lemma, because m and p have the same parity P, $m + p$ must be even.

Therefore, if $m+n$ is even and $n+p$ is even, then $m+p$ is even. `d(^_^)>`

7.

Theorem 7.

$$\min(x, y) = \frac{x + y - |x - y|}{2} \tag{44}$$

Proof. By definition of absolute value:

$$|x - y| = \begin{cases} x - y & ifx \geq y \\ y - x & ify > x \end{cases} \tag{45}$$

In both cases, it is always equal to the positive difference, no matter if x is greater or y is greater.

WLOG suppose $x \geq y$. Thus, $|x-y| = x-y$. Substituting into the equation:

$$\begin{aligned} \min(x,y) &= \frac{x+y-(x-y)}{2} \\ &= \frac{x+y-x+y}{2} \\ &= \frac{2y}{2} \\ \min(x,y) &= y \end{aligned} \tag{46}$$

This statement is true by definition of min.

Therefore, $\min(x,y) = \frac{x+y-|x-y|}{2}$ d(^_^)>

8.

Theorem 8.

$$max(x,y) = \frac{x+y-|x-y|}{2} \tag{47}$$

Proof. WLOG suppose $x \geq y$. Therefore, $|x-y| = x-y$. Substituting into the equation:

$$\begin{aligned} \max(x,y) = x &= \frac{x+y+(x-y)}{2} \\ &= \frac{2x}{2} \\ \max(x,y) &= x \end{aligned} \tag{48}$$

This statement is true by definition of max.

Therefore, $\max(x,y) = \frac{x+y+|x-y|}{2}$ d(^_^)>

9. (a) True. $k = 9$.
   (b) True. $k = -4$.
   (c) False. $4 > 1$ so there can't be a $k \in \mathbb{Z}$ such that $4k = 1$.
   (d) False. 7 is prime, and $2 \notin \{1, 7\}$.
   (e) False. $0k = 17$ means that $0 = 17$ which means it is false for all k.
   (f) False. 17 is prime, and $3 \notin \{1, 17\}$.
   (g) True. $k = 0$.

10. (a) False
    (b) True
    (c) True
    (d) True. Let $x \in \mathbb{Z}$ and $k = 0$. Therefore $xk = 0$. By definition of divides, $\forall x(x|0)$.

(e) True. Let $y \in \mathbb{Z}$, and $k = y$. $1k = y$. By definition of divides, $\forall y(1|y)$.

(f) False. By way of contradiction, assume $\forall x \forall y(x|y)$ to be true. Let $x = 10$ and $y = 6$. Therefore, according to the statement, $(x|y) \equiv (10|6)$. However, this is false, contradicting the original premise that $\forall x \forall y(x|y)$. Therefore, the statement is false.

(g) True, for $x = 1$ as proven above.

11.

Theorem 9. *Let $a, b, c \in \mathbb{Z}$. If $a|b$ and $b|c$ then $a|c$.*

Proof. By definition of divides, the following are true:

- $ak = b$ for some $k \in \mathbb{Z}$
- $bl = c$ for some $l \in \mathbb{Z}$

Therefore, we can multiply the equations together:

$$\begin{gathered} (ak)(bl) = (b)(c) \\ aklb = cb \end{gathered} \tag{49}$$

Then, we can unmultiply[1] b from both sides:

$$akl = c \tag{50}$$

Let $m = kl$. Then, $am = c$. Therefore, by definition of divides, $a|c$.
`d(^_^)>`

[1] Because division is not closed on $\mathbb{Z}$:(