CSC 348 – Midterm

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Contents

1 Questions

1. $P =$ "I can assign partial credit for a question"

   $Q =$ "You leave it blank"

   $(\neg P \leftrightarrow Q) \equiv$"I cannot assign partial credit for a question if and only if you leave it blank"

   $(P \leftrightarrow \neg Q) \equiv$ "I can assign partial credit for a question if and only if you do not leave it blank"

P	Q	$\neg P \leftrightarrow Q$	$P \leftrightarrow \neg Q$
T	T	F	F
T	F	T	T
F	T	T	T
F	F	F	F

   Their truth tables are the same, so "I cannot assign partial credit for a question if and only if you leave it blank" is equivalent to "I can assign partial credit for a question if and only if you do not leave it blank".

2. Yes. $(v_5, v_6, v_2, v_5, v_3, v_2, v_1, v_3, v_4, v_0, v_1, v_6)$ is an Eulerian trail.

3. $$\begin{aligned} \text{Base Step} \quad & 1, 2, 4, 7, 8, 11, 13, 14 \in S \\ \text{Recursive Step} \quad & x \in S \to x + 15 \in S \end{aligned} \tag{1}$$

4. **Theorem 1.** *For some universe U, $A \triangle B = A \cup B \leftrightarrow A \cap B = \emptyset$*

   First, we will define the following lemmas:

Lemma 1. *Let A and B be sets in the universe U. $A \setminus B = A \cap \overline{B}$*

Proof. By definition of $A \setminus B$,

$$A \setminus B = \{x \in U | (x \in A) \wedge (x \notin B)\} \tag{2}$$

By definition of set complement,

$$A \setminus B = \left\{x \in U | (x \in A) \wedge (x \in \overline{B})\right\} \tag{3}$$

By definition of intersection,

$$A \setminus B = A \cap B \tag{4}$$

Therefore, $A \setminus B = A \cap B$. □

Lemma 2. *Let A be a set in the universe U. $A \setminus \emptyset = A$*

Proof. By definition of set difference,

$$A \setminus \emptyset = \{x \in U | (x \in A) \wedge (x \notin \emptyset)\} \tag{5}$$

Since there are no elements in $\emptyset$, x can never be an element of $\emptyset$. Therefore,

$$A \setminus \emptyset = \{x \in U | (x \in A) \wedge T\} \tag{6}$$

By definition of disjunction, this simplifies to

$$A \setminus \emptyset = \{x \in U | x \in A\} \tag{7}$$

By definition of A,

$$A \setminus \emptyset = A \tag{8}$$

Therefore, $A \setminus \emptyset = A$. □

Lemma 3. $A \triangle B = (A \cup B) \setminus (A \cap B)$

Proof. By definition of symmetric difference,

$$A \triangle B = \{x \in U | (x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A)\} \tag{9}$$

Applying the distributive property for propositions,

$$A \triangle B = \{x \in U | \left[x \in A \vee (x \in B \wedge x \notin A)\right] \wedge \left[x \notin B \vee (x \in B \wedge x \notin A)\right]\} \tag{10}$$

Applying the distributive property again,

$$A \triangle B = \{x \in U | (x \in A \vee x \in B) \wedge (x \in A \vee x \notin A) \wedge (x \notin B \vee x \in B) \wedge (x \notin B \vee x \notin A)\} \tag{11}$$

By the law of the excluded middle, we can reduce some terms from this expression.

$$A \triangle B = \{x \in U | (x \in A \vee x \in B) \wedge T \wedge T \wedge (x \notin B \vee x \notin A)\} \tag{12}$$

By definition of disjunction,

$$A \triangle B = \{x \in U | (x \in A \vee x \in B) \wedge (x \notin B \vee x \notin A)\} \tag{13}$$

Applying DeMorgan's Law,

$$A \triangle B = \{x \in U | (x \in A \vee x \in B) \wedge \neg(x \in B \wedge x \in A)\} \tag{14}$$

By definition of intersection,

$$A \triangle B = \{x \in U | x \in A \vee x \in B\} \cap \{x \in U | \neg(x \in B \wedge x \in A)\} \tag{15}$$

By definition of set complement,

$$A \triangle B = \{x \in U | x \in A \vee x \in B\} \cap \overline{\{x \in U | x \in B \wedge x \in A\}} \tag{16}$$

By definition of intersection,

$$A \triangle B = \{x \in U | x \in A \vee x \in B\} \cap \overline{A \cap B} \tag{17}$$

By definition of union,

$$A \triangle B = (A \cup B) \cap \overline{A \cap B} \tag{18}$$

By Lemma 1, this is equivalent to

$$A \triangle B = (A \cup B) \setminus (A \cap B) \tag{19}$$

Therefore, $A \triangle B = (A \cup B) \setminus (A \cap B)$. $\square$

Now, we will prove Theorem 1.

Proof. Suppose $A \cap B = \emptyset$.
By Lemma 3,

$$A \triangle B = (A \cup B) \setminus \emptyset \tag{20}$$

By Lemma 1, this is equivalent to

$$A \triangle B = (A \cup B) \setminus \emptyset \tag{21}$$

By Lemma 2,

$$A \triangle B = (A \cup B) \setminus \emptyset \tag{22}$$

Since it is given that $A \cap B = \emptyset$,

$$A \triangle B = (A \cup B) \setminus (A \cap B) \tag{23}$$

This is consistent with the result in Lemma 3.
Therefore, $A \cap B = \emptyset$ if and only if $A \triangle B = A \cup B$. $\square$

5.

Theorem 2. $(A \times B) \cap (B \times A) = \emptyset$ *if and only if* $A \triangle B = A \cup B$.

First we will define a lemma:

Lemma 4. $A \cap B = \emptyset$ *if and only if* $(A \times B) \cap (B \times A) = \emptyset$

Proof. $(\rightarrow)$ Suppose $A \cap B = \emptyset$, and seeking a contradiction, that there exists an $(a, b) \in (A \times B) \cap (B \times A)$.

By the definition of intersection,

$$(a, b) \in (A \times B) \wedge (a, b) \in (B \times A) \tag{24}$$

By definition of cartesian product,

$$(a \in A) \wedge (b \in B) \wedge (a \in B) \wedge (b \in A) \tag{25}$$

Rearranging terms,

$$(a \in A \wedge a \in B) \wedge (b \in A \wedge b \in B) \tag{26}$$

By definition of intersection,

$$(a \in (A \cap B)) \wedge (b \in (A \cap B)) \tag{27}$$

$\rightarrow\leftarrow$

This is a contradiction because $(A \cap B) = \emptyset$.

Therefore, if $A \cap B = \emptyset$ then $(A \times B) \cap (B \times A) = \emptyset$.

$(\leftarrow)$ Suppose $(A \times B) \cap (B \times A) = \emptyset$, and seeking a contradiction, that there exists an $x \in (A \cap B)$.

By definition of intersection, $x \in A$ and $x \in B$.

Thus, by definition of cartesian product,

$$(x, x) \in (A \times B) \tag{28}$$

Additionally,

$$(x, x) \in (B \times A) \tag{29}$$

Given (28) and (29) are both true, by definition of intersection,

$$(x, x) \in (A \times B) \cap (B \times A) \tag{30}$$

$\rightarrow\leftarrow$

This is a contradiction because by our assumption, $(A \times B) \cap (B \times A) = \emptyset$.

Therefore, $A \cap B = \emptyset \leftrightarrow (A \times B) \cap (B \times A) = \emptyset$. $\square$

Now, we will prove Theorem 2.

Proof. ($\leftarrow$) Suppose $A \triangle B = A \cup B$.

By Theorem 1, $A \cap B = \emptyset$.

By Lemma 4, $(A \times B) \cap (B \times A) = \emptyset$.

Therefore, $(A \times B) \cap (B \times A) = \emptyset \rightarrow A \triangle B = A \cup B$.

($\rightarrow$) Suppose $A \triangle B = A \cup B$. By a symmetric argument, $A \triangle B = A \cup B \rightarrow (A \times B) \cap (B \times A) = \emptyset$.

Therefore, $A \triangle B = A \cup B \leftrightarrow (A \times B) \cap (B \times A) = \emptyset$. □

6. $\deg_{K_n}(v) = n - 1$

7.

Theorem 3. *Let $n \in \mathbb{Z}^+$ and $K_n = (V, E)$. K_n is Eulerian if and only if n is odd.*

Proof. ($\rightarrow$) Suppose that K_n is Eulerian. Thus, all vertices have even degree. Since this is a complete graph, all vertices have the same degree. Therefore, by definition of even, for all vertices $v \in V$, for some $i \in \mathbb{N}$, $\deg_{K_n}(v) = 2i$.

By definition of a complete graph, all vertices also have degree $\deg_{K_n}(v) = n - 1$, so by the transitive law of equality,

$$\begin{aligned} 2i &= n - 1 \\ 2i + 1 &= n \end{aligned} \tag{31}$$

By definition of odd, n is odd.

($\leftarrow$) Suppose that n is odd. By definition of odd, for some $j \in \mathbb{N}$, $n = 2j + 1$. By definition of a complete graph, all vertices $v \in V$ have degree

$$\begin{aligned} \deg_{K_n}(v) &= n - 1 \\ &= (2j + 1) - 1 \quad = 2j \end{aligned} \tag{32}$$

By definition of even, all vertices $v \in V$ therefore have even degree. By definition of Eulerian, K_n is thus Eulerian.

Therefore, K_n is Eulerian $\leftrightarrow$ n is odd.

□

8.

Theorem 4. *Let $n \in Z_{\geq 3}$ and $K_n = (V, E)$. K_n has $\frac{(n-1)!}{2}$ unique cycles of length n.*

Proof. For the purposes of our proof, we will define V_n as the vertex set of K_n and E_n as the edge set of K_n for any $n \in \mathbb{N}$.

Base Case. Suppose $n = 3$. K_3 has exactly 1 cycle of length 3.

$$\frac{(n-1)!}{2} = \frac{(3-1)!}{2} = 1 \tag{33}$$

Therefore, the theorem holds for $n = 3$.

Inductive Hypothesis. Suppose $n = k$. K_k has $\frac{(k-1)!}{2}$ unique cycles of length k. Additionally, let S_k be the set of unique k-length cycles of K_k.

Induction Step. Consider the graph $K_{k+1} = (V_{k+1}, E_{k+1})$. Choose a $v \in V_{k+1}$. Note that K_{k+1} can be constructed from K_k by adding v and connecting it to all the other vertices, by definition of complete graph.

Let S_k be the set of k-length cycles in K_k. By the inductive hypothesis, $|S_k| = \frac{(k-1)!}{2}$. We will construct S_{k+1}, the set of $(k+1)$-length cycles in K_{k+1}, from the elements of S_k, using the following method:

> Choose a cycle $c \in S_k$ and an edge (a, b) that c traverses. By definition of a member of S_k, c has length k.
>
> Let d be the cycle equivalent to c, but with (a, b) removed and replaced with (a, v, b). d is an element of S_{k+1}. Note that since we removed 1 edge and added 2 new edges, the length of the new cycle is $k - 1 + 2 = k + 1$, so all d we can create using this method will have length $k + 1$.

By the inductive hypothesis, there are $\frac{(k-1)!}{2}$ unique k length cycles to choose from in S_k. Additionally, each of these cycles, having a length of k, has k edges to choose from. Therefore, there are

$$k \cdot \frac{(k-1)!}{2} = \frac{k!}{2} = \frac{((k+1)-1)!}{2} \tag{34}$$

cycles that can be created using this method.

Now, we will prove that every cycle created using a different choice of cycle in $c \in S_k$ and edge in c is unique.

Claim 1. *Suppose that the cycles $c_1, c_2 \in S_k$ generated the cycles $d_1, d_2 \in S_{k+1}$ respectively. If $c_1 \neq c_2$ then $d_1 \neq d_2$.*

> *Proof.* By way of contradiction, suppose that $d_1 = d_2 = d \in S_{k+1}$.
>
> By definition of d as an element in S_{k+1}, the following are true:
>
> (a) Recall that v is an arbitrary node not in K_k. Thus, (a, v, b) is a subpath of d.
>
> (b) (a, b) is a path of both c_1 and c_2.
>
> Let p_1, p_2 be the paths created by removing (a, b) from c_1, c_2 respectively. By definition of an element of S_{k+1}, adding (a, v, b) to either p_1 or p_2 creates c.
>
> Conversely then, removing (a, v, b) from c creates both p_1 and p_2. This means that $p_1 = p_2$.

Let $p = p_1 = p_2$. By definition of p_1, adding (a, b) to p will create c_1. By definition of p_2, adding (a, b) to p will also create c_2. Therefore, $c_1 = c_2$. $\rightarrow\leftarrow$

This is impossible because $c_1 \neq c_2$ by definition. Therefore, if $c_1 \neq c_2$ then any d_1 and d_2 created from them respectively will not be the same. $\square$

Claim 2. *Suppose c is a cycle in S_k, and $(a_1, b_1), (a_2, b_2)$ are edges in c that are replaced to generate the cycles $d_1, d_2 \in S_{k+1}$ respectively. If $(a_1, b_1) \neq (a_2, b_2)$ then $d_1 \neq d_2$.*

Proof. By way of contradiction, suppose that $d_1 = d_2 = d \in S_{k+1}$. Thus, by definition of an element in S_{k+1}, the following are true:

(a) For some $t, u \in V_k$, d contains the subpath (t, v, u).

(b) There is an edge $(t, u) \in E_k$ that c passes through, and it can be replaced with (t, v, u) to build d. Conversely, replacing (t, v, u) in d with (t, u) builds c.

Let p_1, p_2 be the paths created by removing (a_1, b_1) and (a_2, b_2) from c_1 and c_2 respectively. By our original assumption, adding (a_1, v, b_1) and (a_2, v, b_2) to p_1 and p_2 respectively will build c. Conversely, removing (a_1, v, b_1) and (a_2, v, b_2) from d will create p_1 and p_2 respectively.

However, by definition of a cycle, v only occurs once in c. Thus, there is only one unique subpath (t, v, u) in d. Therefore, $(t, v, u) = (a_1, v, b_1) = (a_2, v, b_2)$, so $a_1 = a_2$ and $b_1 = b_2$. $\rightarrow\leftarrow$

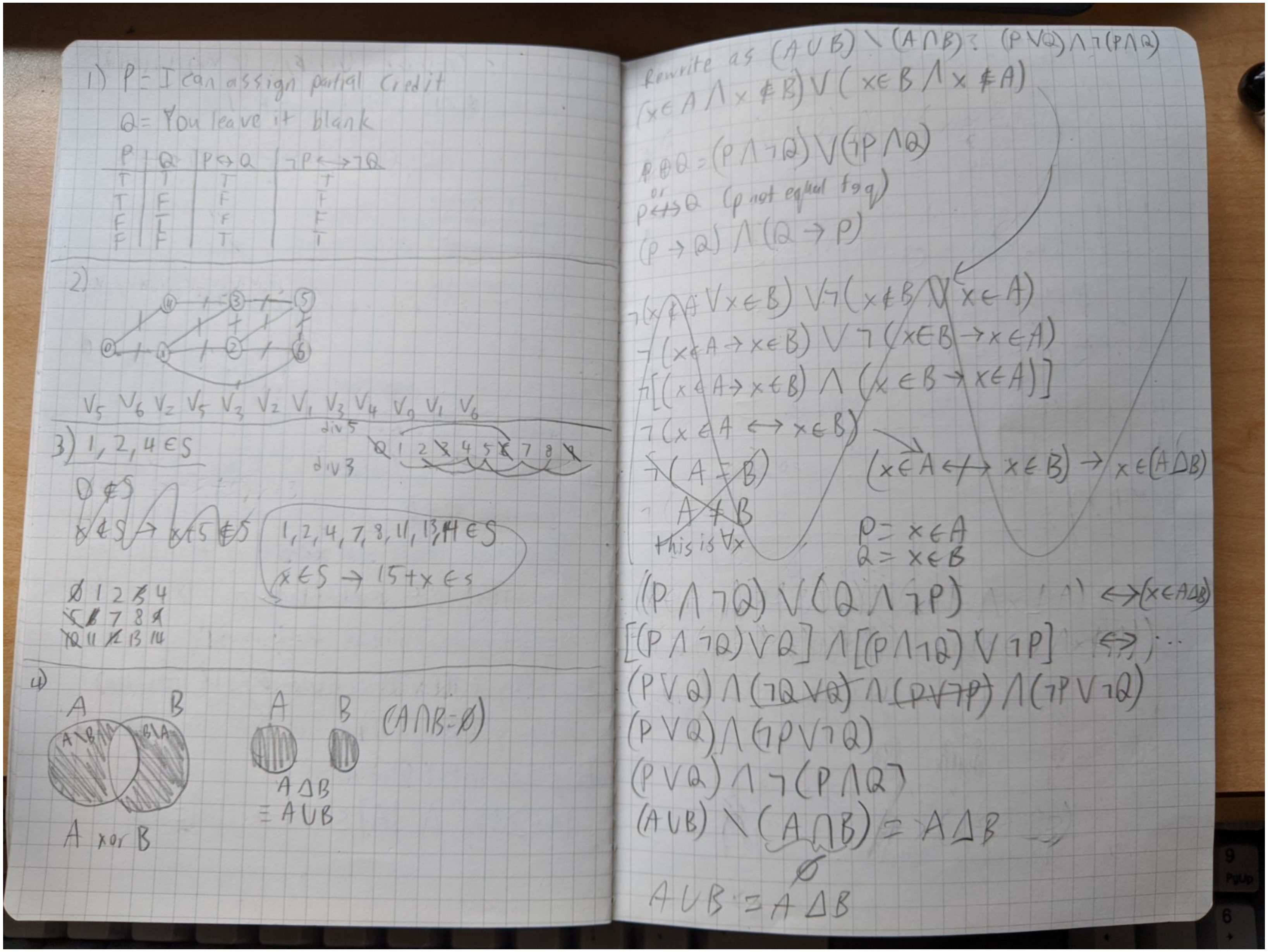
This is a contradiction because our condition is that $(a_1, b_1) \neq (a_2, b_2)$.

Therefore, if $(a_1, b_1) \neq (a_2, b_2)$ then $d_1 \neq d_2$. $\square$

By Claims 1 and 2, no cycle/pair combination will generate the same cycle. Therefore, all of our generated $(k + 1)$-length cycles are unique.

Therefore, by the principle of mathematical induction, there are $\frac{(n-1)!}{2}$ unique cycles in K_n. $\square$

2 Scratch Work



1) P = I can assign partial credit

Q = You leave it blank

P	Q	$P \leftrightarrow Q$	$\neg P \leftrightarrow \neg Q$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	T	T

2)

$V_5\ V_6\ V_2\ V_5\ V_3\ V_2\ V_1\ V_3\ V_4\ V_0\ V_1\ V_6$

3) $1, 2, 4 \in S$

div 5

div 3

$\cancel{0}\ 1\ 2\ \cancel{3}\ 4\ 5\ \cancel{6}\ 7\ 8\ \cancel{9}$

$0 \notin S$

$x \notin S \rightarrow x+5 \notin S$

$1, 2, 4, 7, 8, 11, 13, 14 \in S$

$x \in S \rightarrow 15 + x \in S$

$\cancel{0}\ 1\ 2\ \cancel{3}\ 4$

$\cancel{5}\ \cancel{6}\ 7\ 8\ \cancel{9}$

$\cancel{10}\ 11\ \cancel{12}\ 13\ 14$

4)

A B

$A \setminus B$ $B \setminus A$

A xor B

A B $(A \cap B = \emptyset)$

$A \Delta B$

$\equiv A \cup B$

Rewrite as $(A \cup B) \setminus (A \cap B)$? $(P \vee Q) \wedge \neg(P \wedge Q)$

$(x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A)$

$P \oplus Q = (P \wedge \neg Q) \vee (\neg P \wedge Q)$

or

$P \not\leftrightarrow Q$ (p not equal to q)

$(P \rightarrow Q) \wedge (Q \rightarrow P)$

$\neg(x \notin A \vee x \in B) \vee \neg(x \notin B \vee x \in A)$

$\neg(x \in A \rightarrow x \in B) \vee \neg(x \in B \rightarrow x \in A)$

$\neg[(x \in A \rightarrow x \in B) \wedge (x \in B \rightarrow x \in A)]$

$\neg(x \in A \leftrightarrow x \in B)$

$\neg(A = B)$

$A \neq B$

this is $\forall x$

$(x \in A \not\leftrightarrow x \in B) \rightarrow x \in (A \Delta B)$

$P = x \in A$

$Q = x \in B$

$(P \wedge \neg Q) \vee (Q \wedge \neg P) \leftrightarrow (x \in A \Delta B)$

$[(P \wedge \neg Q) \vee Q] \wedge [(P \wedge \neg Q) \vee \neg P] \leftrightarrow \ldots$

$(P \vee Q) \wedge \cancel{(\neg Q \vee Q)} \wedge \cancel{(P \vee \neg P)} \wedge (\neg P \vee \neg Q)$

$(P \vee Q) \wedge (\neg P \vee \neg Q)$

$(P \vee Q) \wedge \neg(P \wedge Q)$

$(A \cup B) \setminus \underbrace{(A \cap B)}_{\emptyset} = A \Delta B$

$A \cup B = A \Delta B$

5) ~~Lemma~~. $A \cap B = \emptyset \rightarrow \underbrace{(A \times B) \cap (B \times A)}_{} = \emptyset$

Suppose, seeking contradiction $\exists (a,b) \in$ ~~...~~

$[(a,b) \in (A \times B)] \wedge [(a,b) \in (B \times A)]$

$(a \in A) \wedge (b \in B) \wedge (a \in B) \wedge (b \in A)$

$(a \in A \wedge a \in B) \wedge (b \in B \wedge b \in A)$

$a \in (A \cap B) \wedge b \in (A \cap B) \rightarrow \leftarrow$

$A \cap B = \emptyset$, can't have members!

Since $A \Delta B = A \cup B$, by above thm, $A \cap B = \emptyset$.

By the lemma, $(A \times B) \cap (B \times A) = \emptyset$ $\square$

---

6) there are $n-1$ other elemenss

$\deg_{K_n}(v) = n-1$

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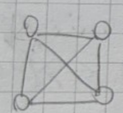
7) 

Assume n is even. $n = 2k$

Each vertex has $2k-1$ degree

$2(k-1)+1$ is odd.

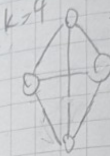
There fore not eulerian.

Assume n is odd. $n = 2k+1$

Each vertex has $2k$ degree (odd degree)

Therefore eulerian.

8)  $n=3$. $\frac{(3-1)!}{2} = 1$ (Base case)

$k=4$ 

This is excluded b/c cycle

$V_1 \; V_2 \; V_3$

3 Insertion points for V_4

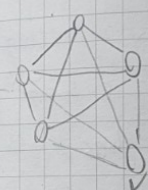
$(k-1) \cdot \text{Cycles}(k-1)$

ways to insert

$(k-1) \cdot \frac{(k-2)!}{2}$

$= \frac{(k-1)!}{2} = \frac{(4-1)!}{2}$

$= \frac{6}{2} = 3$

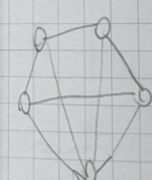
$k=5$ 

$\overbrace{V_1 \; V_2 \; V_3 \; V_4}^{k-1}$

$V_1 \; V_3 \; V_4 \; V_2$

$V_1 \; V_4 \; V_2 \; V_3$

$\left.\right\} \frac{(k-2)!}{2}$



V

V is the new vertex added

For each edge in C (a,b) (size $n-1$) there is a new cycle D that can be made by replacing that edge w/ $(a,v)(v,b)$ (size n)

It is unique from the others b/c this cycle is the only one without (a,b)

$D \in S$, the set of unique cycles for K_k

Define relation R on $(n-1)$-cycles to n-cycles (do this to prove uniqueness)

$(C, D) \in R$ if:

- For some $(v$

v_1 connected to v_a and v_b. If $a<b$ then (v_a, v_1, v_b) is a subwalk of the cycle.

Cycles in S_{k-1} have a defined direction.

Or, prove a bijective correspondence w/ $S_{k-1} \times$ (vertices of K_{k-1}) and S_k

$f(c, u): \rightarrow S_k$

$f(c, u) =$ A new cycle obtained by replacing u with u, v. u always shows up in c because c hits every element of V_{k-1}. Since c has length $k-1$, $f(c, u)$ has length k because we removed 2 edges but added 3 edges.

1-1 Suppose $f(c, u) = f(d, w) = b$.

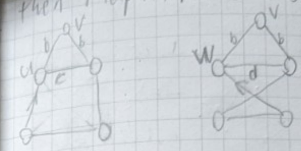
BWOC suppose $(c, u) \neq (d, w)$.

b is the result of replacing u with u, v in b

Thus, c is the result of replacing u, v with u in b

Additionally, d is the result of replacing w, v with w in b. Since v only occurs once, and cycles have a specific ordering,

then the pattern $(w, v) = (u, v)$. Additionally,



c, b   d, b

This won't work

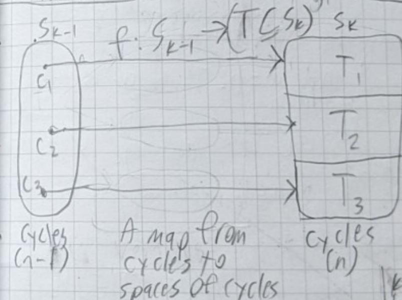
$F: S_{k-1} \rightarrow (V_{k-1} \rightarrow S_k)$

First prove that this function's outputs are functions w/ disjoint codomains

$g_i: V_{k-1} \rightarrow S_k$

Next prove that every g_i has a 1-1 mapping

$f: S_{k-1} \rightarrow (T \subseteq S_k)$



1. Prove that $T_1 \dots T_n$ are disjoint
2. Prove that T_i has $n-1$ elements
3. Thus, $\left| \bigcup_{i=1}^{k-1} f(c_i) \right| = |S_{k-1}|(n-1)$

$f: S_{k-1} \to \mathbb{P}_{k-1}(S_k)$ Power set, but every element has k-1 cardinality

V = new node

$f(c) = T$, a set defined as such:

Define a function $g: V_{k-1} \to S_k$ like so:

$g_c(u)$ = the cycle where you replace the subwalk (u) with (u, v).

Proof of 1-1:

Suppose $g_c(a) = g_c(b) = D$. $a, b \in V_{k-1}$, $D \in S_k$

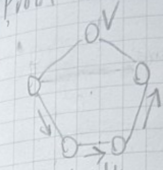
Replacing (a) in C gives D.

If we replace (b, v) in D with (b) we get C. Since v only occurs once at the subwalk (a, v), that means $(a, v) = (b, v)$ and $a = b$. Thus, g_c is 1-1.

Since V_{k-1} is a finite set, and g_c is a 1-1 function, every $u \in V_{k-1}$ gets mapped to a unique cycle. Let T = the image of V_{k-1} under g_c. Thus, $|T| = |V_{k-1}| = k-1$.

Additionally: $f(c) \neq \emptyset$ because every element has $k-1$ elements, and $k \geq 3$.

Proof that $f(c)$ has disjoint outputs



Suppose $a, b \in S_{k-1}$ and $a \neq b$ and

$f(a) = T$ $f(b) = U$

Prove $T \cap U = \emptyset$

$x \in T \to x \notin U$

Let the walk a' be x, but with v removed. Note that if you connect the ends of a', you get the cycle a.

Suppose BWOC that $x \in U$. This would mean that removing v from x results in the subwalk b', and when its ends are connected we get the cycle b. So $f(a) = f(b)$

Define a set W_{k-1}.

If $c_i \in C_{k-1}$ and you can remove the i^{th} edge to get w_i, then

For a given c, there are $k-1$ ways to do it and all results are unique.

Suppose $a \neq b$ and by way of contradiction $w_a = w_b$.

If we add the b^{th} edge of c to w_a, we get c. Therefore, if we subtract the b^{th} edge of c from c, we get w_a. We can also subtract the a^{th} edge from c to get w_a. However, this would mean that the a^{th} edge and b^{th} edge are the same, and since each edge only occurs once, $a = b$.

Actually, define as $g_c : C_{k-1} \to W_{k-1}$

Define W_n as the set of all subpaths of K_n with length $n-1$.

An $n-1$ path can be generated from $c \in C_k$ by removing an edge e.

f:

$f(w) = ((u, \ ), w + (u, v_1))$

Where $w = (v_1 \dots u)$.

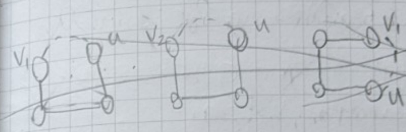
w has length $n-1$. Adding the edge (u, v_1) makes it n. Also, that makes it a cycle.

$f(w)$ is $\underline{1-1}$

Suppose $f(w_1) = f(w_2) = (u, c)$

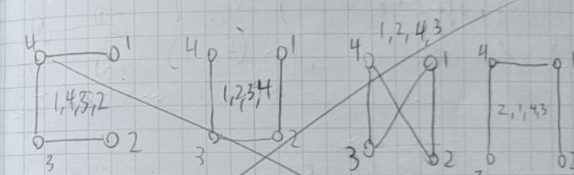
and $w_1 = (v_1 \dots u)$ and $w_2 = (v_2 \dots u)$

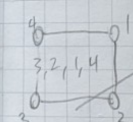
Connecting u to v_1 makes c, and connecting u to v_2 makes c. c has one occurrence



~~This would be the path $(u \dots v_1)$~~

~~new $f : W_n \to V_n \times V_n$~~





We will designate each cycle & path with a standard ordering.

Designate some arbitrary node v_1, and assign unique #'s to nodes arbitrarily.

We will assume that every cycle has 1 unique ordering.

For example: $(v_1, v_2, v_3, v_1) = (v_2, v_3, v_1, v_2)$
$= (v_1, v_3, v_2, v_1)$, etc.

Define W_n as a set of paths in K_n.
Additionally, define C_n as the set of unique n-cycles in S_n. Assume that all cycles have a single representation (i.e.

If $c \in C_n$, then define

$$W_c = \{ w \mid e \text{ is an edge of } c,\ w = \text{remove } e \text{ from } c \}$$

Since there are n edges, $|W_c| = n$

Suppose $c_1, c_2 \in C_n$ and $c_1 \neq c_2$. Proof that $W_{c_1} \cap W_{c_2} = \emptyset$

Suppose BWOC $w \in W_{c_1} \cap W_{c_2}$

$w = (v_1 \cdots v_n)$ so $c_1 = (v_1 \cdots v_n, v_1) = c_2$

But $c_1 \neq c_2$ $\rightarrow\leftarrow$

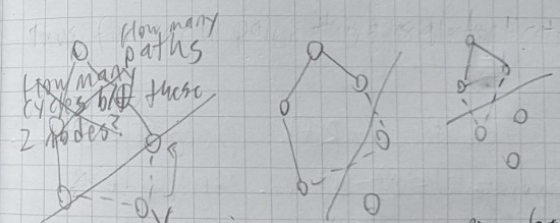
Let $W_n = \bigcup_{c \in C_n} W_c$. Since all W_c's are disjoint, $|W_{c_1}| + |W_{c_2}| = |W_{c_1} \cup W_{c_2}|$

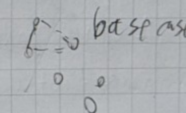
so $|W_n| = \sum_{c \in C_n} |W_c|$.

IH K_k has $\frac{(k-1)!}{2}$ unique cycles

Step suppose we add v to K_k to make it into K_{k+1}. There are k edges attached to v. There are k ways to enter v, and

Choose a $v \in K_k$. We can construct the k-length cycles in K_k by connecting the head and tail of $w \in W_{k-1}$ to v.



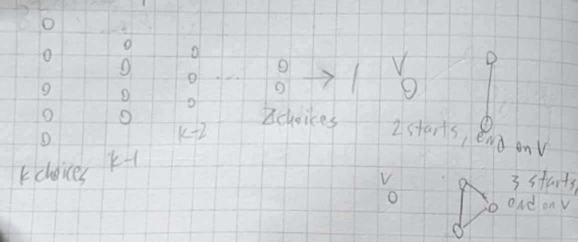


This is for EVERY PAIR.

There are $\frac{(n-1)n}{2}$ pairs.

$$\left(\frac{(n-1)n}{2} \text{ pairs}\right) \cdot \left(\frac{(n-2)!}{2} \text{ cycles/pair}\right)$$

How many paths are there b/t 2 nodes?



Let $v_1, v_2 \in V$. paths b/t $v_1, v_2 = \frac{n!}{2}$

<u>Base</u> $n = 2$. $\frac{(2-1)!}{2} = 1$. There is 1 path for a 2 node graph.

<u>IH</u> $\frac{(k-1)!}{2}$ for K_k

Step Consider $n = k+1$. pick $v_1, v_2 \in V$. Suppose we start at v_1. There are k other nodes connected to v_1 and $k-1$ nodes that aren't v_2. Consider H, the subgraph of G that only excludes v. It is isomorphic to K_{k-1}. For any given vertex $v \in H$ that isn't v_2, by the IH, there are $\frac{(k-1)!}{2}$ paths to v_2.

Since there are $(k-1)$ choices for v_1, there are $(k-1) \cdot \frac{(k-2)!}{2}$ paths from v_1 to v_2.