Programming Language Semantics and Compiler Design / Sémantique des Langages de Programmation et Compilation

Maths Reminders

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Master 1 info

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Some proof techniques

Proofs by contradiction, reducto-ad-absurdum, contraposition,. . .

. . . they rely on the principles of propositional and predicate logics.

Proof by structural induction

- Proof for the basic elements, atoms, of the set.
- Proof for composite elements (created by applying) rules:
  - assume it holds for the immediate components (induction hypothesis)
  - prove the property holds for the composite element

Induction on the shape of a derivation tree

- Proof for 'one-rule' derivation trees, i.e., axioms.
- Proof for composite trees:
  - For each rule *R*, consider a composite tree where *R* is the last rule applied
  - Proof for the composite tree
    - Assume it holds for subtrees, or premises of the rule (induction hypothesis)
    - Proof for the composite tree

Outline - Maths Reminders

Proof by induction

Proof by structural induction

A notation: derivation tree

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Proof by induction

Proving a predicate $P(n)$ that depends on some parameter $n \in \mathbb{N}$.

Example (Predicate)

- $P(n) =$ The sum of the n natural numbers is $\frac{n \times (n+1)}{2}$."
- $P(n) =$ "If $q \geq 2$, we have $n \leq q^n$.
- $P(n) =$ "Every polynomial of degree n has at most n roots."

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Principle

- Prove the **base case**: prove that $P(0)$ holds (or $P(k)$ if the minimal value of the parameter is k).
- Prove the **induction step**: prove that if for some n, $P(n)$ holds, then $P(n+1)$ holds.

The principle of induction ensures that $P(n)$ holds, for any $n \geq k$.

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  - Suppose that the property holds for some $n \in \mathbb{N}$.
  - We have:

$$\begin{array}{ll} \sum_{i=0}^{n+1} i & = \sum_{i=0}^{n} i + n + 1 \\ & = \frac{n(n+1)}{2} + n + 1 \quad \text{(induction hypothesis)} \\ & = \frac{(n+2)\times(n+1)}{2} \end{array}$$

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Principle of complete induction

- Prove the **base case**: prove that $P(0)$ holds (or $P(k)$ if the minimal value of the parameter is k).
- Prove the **complete induction step**: prove that if for some n, $\forall m \leq n : P(m)$ holds, then $P(n+1)$ holds.

The principle of induction ensures that $P(n)$ holds for any $n \geq k$.

Proof by complete induction: example

The n-th fibonacci number f_n is defined as follows:

- $f_0 = 0$
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Example (The n-th Fibonacci number is even iff n is a multiple of 3.)

- Base case. We can see that it holds for $n = 0$.
- complete induction step.
  - Let us suppose that the property holds for any integer lesser than or equal to some $n \in \mathbb{N}$.
  - Consider f_{n+1} and distinguish three cases according to the rest of the division of $n+1$ by 3.
    - Case $n+1 \mod 3 = 0$. $f_{3k} = f_{3k-1} + f_{3k-2} = \text{odd} + \text{odd} = \text{even}$
    - Case $n+1 \mod 3 = 1$. $f_{3k+1} = f_{3k} + f_{3k-1} = \text{even} + \text{odd} = \text{odd}$
    - Case $n+1 \mod 3 = 2$. $f_{3k+2} = f_{3k+1} + f_{3k} = \text{odd} + \text{even} = \text{odd}$

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Proof by induction

Proof by structural induction

A notation: derivation tree

Inductive/Compositional definitions

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- $f : E \times E \times \ldots \times E \rightarrow E$ a partial function,
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A construction rule for a set states either:

- that a *basis element* belongs to the set, or
- how to produce a new element from existing elements (*production rule* given by a partial function).

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Definition (Inductive definition)

An inductive definition on E is a family of rules defining the smallest subset of E that is *closed* by these rules.

Inductive definitions: examples

Example (Natural numbers)

How can define them?

- basis element 0
- 1 rule: $x \mapsto \mathit{succ}(x)$

2 is the natural number defined as $\mathit{succ}(\mathit{succ}(0))$

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Example (Palindromes on $\{a, b\}$)

- basis elements ϵ, a, b
- 2 rules: $w \mapsto a \cdot w \cdot a$, $w \mapsto b \cdot w \cdot b$

Binary trees

Definition and examples

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A tree is a binary tree if each node has *at most two children* (possibly empty).

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The smallest set $Bt(Elt)$ s.t.:

$$Bt(Elt) = \{EmptyT\} \cup \{Node(tL, e, tR) \mid e \in Elt \land tL, tR \in Bt(Elt)\}$$

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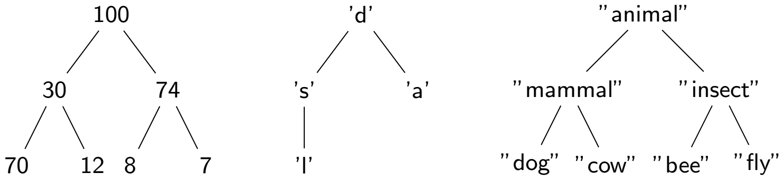
Example (Binary trees of natural numbers)

$$Bt(\mathbb{N}) = \{EmptyT\} \cup \{Node(tL, e, tR) \mid e \in \mathbb{N} \wedge tL, tR \in Bt(\mathbb{N})\}$$

Example (Binary trees)



Proof by Structural Induction

Proving that the proof holds for any element "however it is built".

Principle

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  - assume it holds for the immediate components (induction hypothesis)
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All proofs by induction are proofs by structural induction where the inductive set is $\mathbb{N}$.

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Example (Properties of size and depth of a binary tree)

Let us consider $t \in Bt(Elt)$, a binary tree:

- $\mathrm{depth}(t)$ be the depth of tree t: length of longest path from root to leaf.
- $\mathrm{size}(t)$ be the size of tree t: number of nodes + leaves.

For any type Elt and any $t \in Bt(Elt)$:

- $\mathrm{depth}(t) \leq \mathrm{size}(t)$,
- $\mathrm{size}(t) \leq 2^{\mathrm{depth}(t)-1}$.

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- basis element: 0;
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Example (Palindromes on $\{a, b\}$)

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Notation for $t = f(x_1, \ldots, x_n)$

$$\frac{x_1 \quad \ldots \quad x_n}{t}\ f$$

"t is built/obtained from $x_1, \ldots, x_n$" by applying operator f.

Example (Derivation trees)

- $2 = \mathrm{succ}(\mathrm{succ}(\mathrm{succ}(0)))$ is a natural number

$$\cfrac{\cfrac{\overline{0}}{1}}{2}\ \mathrm{succ}$$

- aba is a palindrome:

$$\frac{\overline{b}}{aba}$$

- $ababa$ is a palindrome:

$$\cfrac{\cfrac{\overline{a}}{bab}}{ababa}$$

Abstract syntax trees and derivation trees

Consider an abstract syntax tree, produced by syntactic analysis.

Derivation tree

For each node, computing information from the information of its sons.

Example (Derivation trees in type analysis)

We obtain for each node a type (or error) based on types of its sons.

Definition of derivation trees by a formal system

Generally, we have information, stored in some environment Γ. The formal system states how to *deduce* knowledge from existing knowledge, where knowledge is of the form $\Gamma \vdash \mathcal{P}$, which means $\mathcal{P}$ holds on Γ.

- a set of axiom schemes
- a set of inference rules : a rule of the form

$$\frac{\Gamma_1 \vdash \mathcal{P}_1 \quad \cdots \quad \Gamma_n \vdash \mathcal{P}_n}{\Gamma \vdash \mathcal{C}}$$

"if the hypothesis (premisses) $\mathcal{P}_i$ hold, then the conclusion $\mathcal{C}$ holds.