Section A

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1 Section A

Markdown Link: https://hackmd.io/@PingJung/SJk1cdfUlg

Reference:
[1] Thomas F. Munro-O'Brien, Charles N. Ryan, Performance of a low power Hall effect thruster with several gaseous propellants
[2] Käthe Dannenmayer, Stéphane Mazouffre, Elementary Scaling Relations for Hall Effect Thrusters
[3] Lee et al., Scaling Approach for Sub-Kilowatt Hall-Effect Thrusters
[4] Munro-O'Brien et al., Design, manufacture, and testing of a magnetically shielded krypton Hall effect thruster

1.1 1. Where to start ? (Problem)

We set up a series of requiremens from section 0
Power : 1000 W
Isp: 1500 s
Mass flow rate: $1.0 \times 10^{-6} [kg/s]$
Thrust: 14.7150 mN

1.2 2. What is the goal? (Target)

In section A, we try to verify and derive
1.Whether these settings are feasible from the scaling law of sub-kilowatt Xe-HETs?
2.The geometry, mechanical, or electrical constraints.

By assuming the idealistic condition (all experimental parameters are assumed to be the same except mass), we can know the upper bound of our performance.
To get a more realistic prediction, these experimental parameters need to tackle in the following section.

1.3 3. How you want to do it? (Method)

1.3.1 3.0 Nomenclature

n_n atom number density $[m^{-3}]$
U_d discharge voltage [V]
Δ voltage loss [V]
θ_d beam divergence term

α propellant conversion efficiency
γ corection factors for the presence of multiply charged ions
λ_i ionization mean free path [m]
$\bar{v}_i$ mean ion flow velocity [m/s]
v_n, v_e thermal speed of electrons and atoms [m/s]
m_n propellant atom mass [kg]
$\dot{m}_n, \dot{m}_i$ propellant mass flow rate and ion mass flow rate [kg/s]
e elementry charge [C]
T thrust [N]
γ, γ' correction factors for the presence of multiply charged ions
f_{2+} fraction of the doubly charged ions
I current [A]
h channel width [m]
d mean channel diameter [m]
η_a anode efficiency
K_B Boltzmann constant

1.3.2 3.1 Analytical Equation

From [2]
$1.\dot{m}_n = n_n m_n v_n A$
$2.T = \alpha\gamma\theta_d \dot{m}_n \bar{v}_i = \alpha\gamma\theta_d \dot{m}_n \sqrt{\frac{2e}{m_n}(U_d - \Delta)}$
$3.I_d \approx \frac{e}{m_n}\gamma'\alpha\dot{m}_n = \pi e\gamma'\alpha n_n v_n hd$
$4.P_d = U_d I_d$
$5.v_n = \sqrt{\frac{8K_B T_n}{\pi m_n}}$
$6.\eta_a = \frac{T^2}{2\dot{m}_n U_d I_d}$
$7.I_{sp} = \frac{T}{\dot{m}_n g_0}$

where
$A = \pi hd$
$\gamma = \frac{1+\frac{I^{2+}}{\sqrt{2}I^{+}}}{1+\frac{I^{2+}}{2i^{+}}} = \frac{1+\frac{f_{2+}}{\sqrt{2}(1-f_{2+})}}{1+\frac{f_{2+}}{2(1-f_{2+})}}$

1.3.3 3.2 Approximated Relation

From [2]
$1.n_n \propto \frac{\dot{m}_n}{hd}$
$2.h \propto d$

1.3.4 3.3 Scaling Relation

From [3]
$1.\dot{m}_n = C_{\dot{m}} hd$
$2.T = C_m \dot{m}_n \sqrt{U_d}$
$3.P_d = C_p U_d d^2 = C_{p,2} U_d I_d$
$4.h = C_{hd} d$

1.3.5 3.4.1 Assumptions and Unknown parameters

From [2]
1.$n_n \approx 1.2 \times 10^{-19}[m^{-3}]$
2.The propellant gas has a uniform and fixed temperature all over the channel, hence a constant propellant velocity.
3.There are no multiply charged ions in the plasma ($\gamma = \gamma' = 1$)
4.A parallel monokinetic ion beam is produced, i.e., the plasma jet divergence is null $\theta_d = 1$
5.The potential energy is fully converted into kinetic energy and all ions experience the whole potential drop, of which the magnitude is $\Delta = 0$
6.Plasma-wall interactions are taken into account through heat load to the channel walls.
7.The magnetic field is uniform; solely its value at the exit plane is considered. The channel length L is therefore the length of the region with magnetic field.

My assumptions:
8. Unknown experimental parameters in 3.1 assumed to be the same between Xe and O_2 except the mass difference:
Fixed parameters: $\alpha, \gamma, \gamma', \theta_d, \Delta, T_n$(see 3.4.1.2),$n_n$
9. All relations pass through origin without intercept.

3.4.1.1 Consider only mass influence to the scaling reltaion

3.4.1.2 Mass discrepancy between Xe and O_2
1.$m_i = m_n$ ion mass=propellant mass : $Xe^+ \approx 131.293[u]$, $O_2^+ \approx 32[u]$
2.$\bar{v_i}$ mean exhaust velocity $\propto \sqrt{\frac{1}{m_n}}$
3.v_n propellant thermal velocity $v_n = \sqrt{\frac{8K_BT_n}{\pi m_n}} =\propto \sqrt{\frac{1}{m_n}}$

3.4.1.3 Modified coefficient in 3.3
$\dot{m}_n$: $C_{\dot{m}_{O_2}} = C_{\dot{m}_{Xe}} \cdot \sqrt{\frac{m_{O_2}}{m_{Xe}}}$
I_d: $C_{I_{O_2}} = C_{I_{Xe}} \sqrt{\frac{m_{Xe}}{m_{O_2}}}$ (relate to P)
P_d: $C_{P_{O_2}} = C_{P_{Xe}} \sqrt{\frac{m_{Xe}}{m_{O_2}}}$
T: $C_{T_{O_2}} = C_{T_{Xe}}$

3.4.1.4 Modified Scaling Relation
All the coefficients are from [3]
1.$\dot{m}_{n,O_2}[kg/s] = C_{\dot{m}_{Xe}} hd = 0.003 \cdot \sqrt{\frac{m_{O_2}}{m_{Xe}}} hd[m^2]$
2.$T_{O_2}[N] = C_{T,Xe}\, \dot{m}_n \sqrt{U_d} = 892.7 \cdot \dot{m}_{n,Xe} \sqrt{U_d}[kg/(s * V^{0.5})]$
3.$P_{d,O_2}[W] = C_{p,Xe} U_d d^2 = 633.0 \cdot \sqrt{\frac{m_{Xe}}{m_{O_2}}} U_d d^2\ [V \cdot m^2]$
4.$h_{O_2}[m] = C_{hd,Xe} d = C_{hd,O_2} d = 0.242 d\,[m]$

Unknown $h, d, \dot{m}_{O_2}, U_d$,
Known: P, T

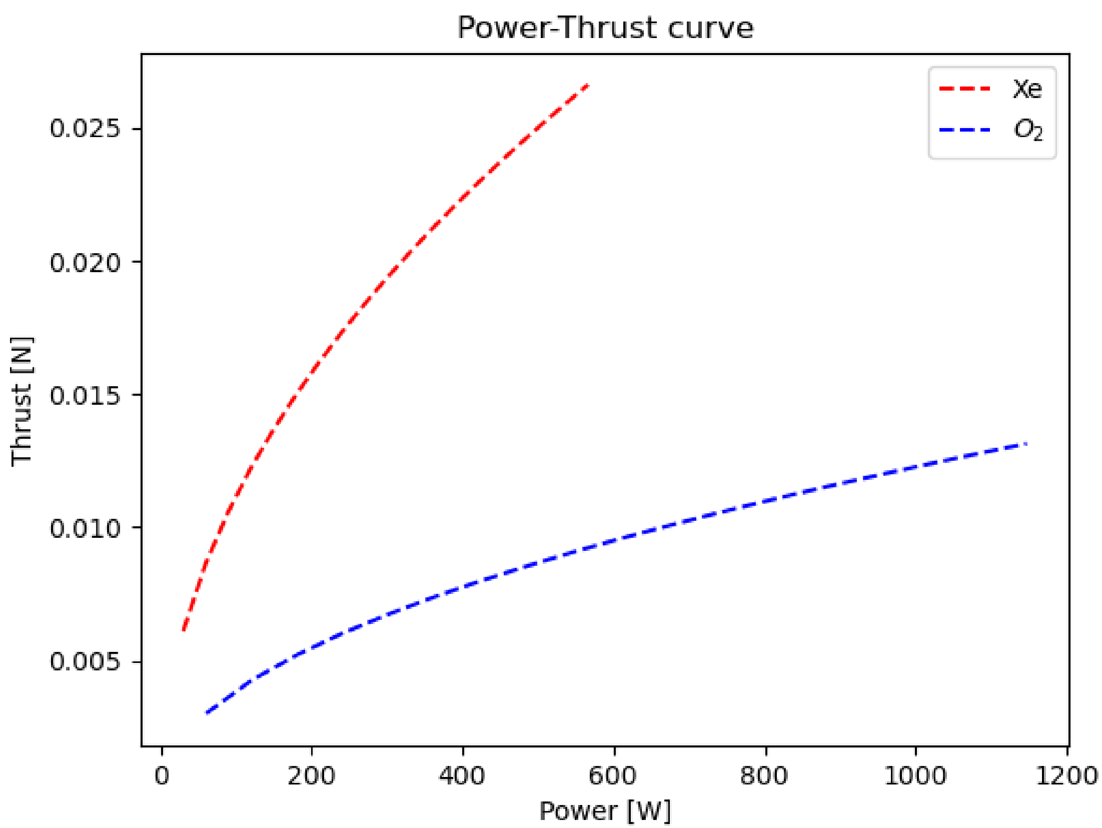
Iterative step:
1.Use P to get d
2.Use d to get h

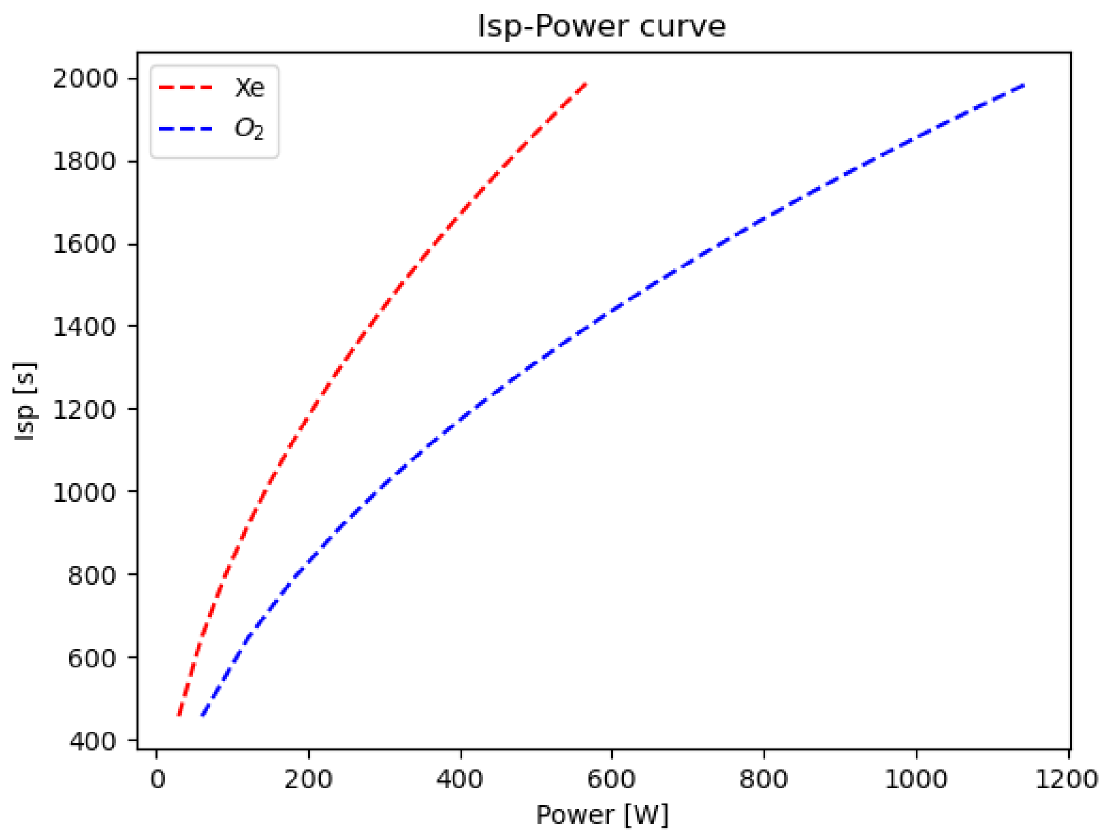
3.Use h, d to get $\dot{m}_n$
4.Use T to check U_d
5.Update U_d until it converges

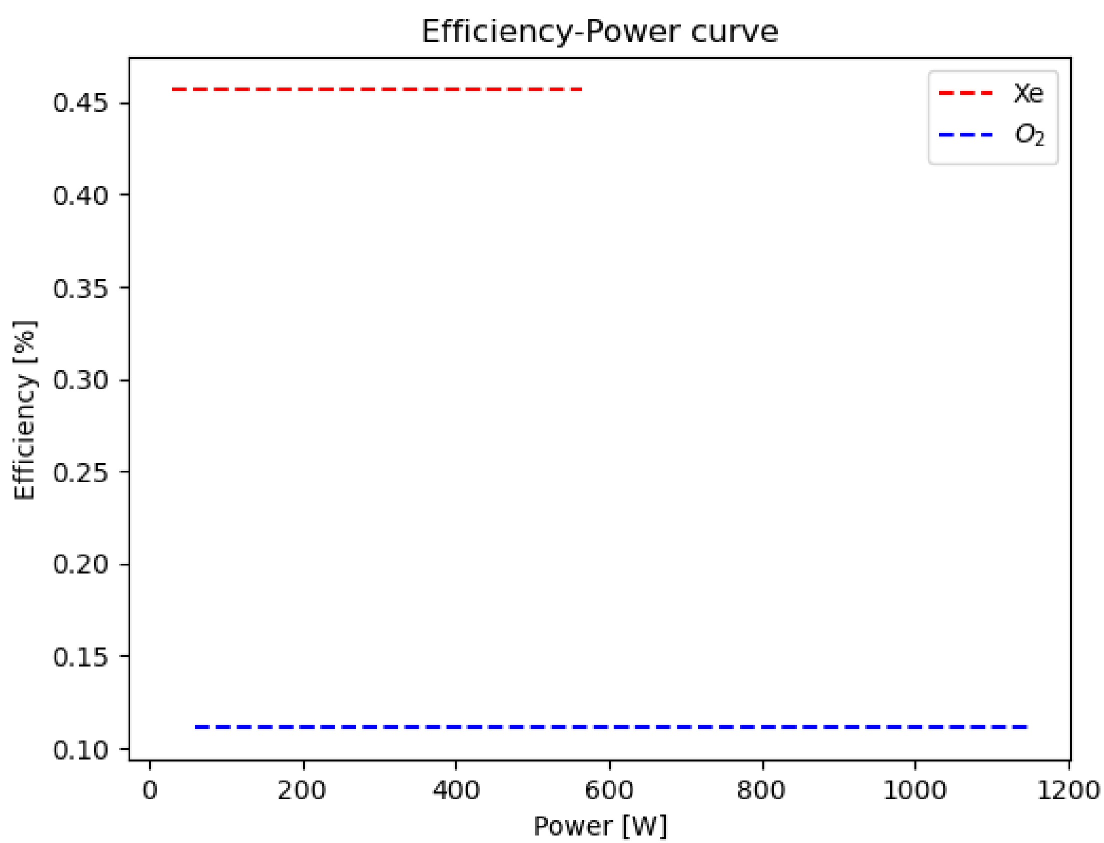
Given $P = 1000[W]$, $T = 14.7150[mN]$
Result
$d = 0.05206865912647417[m]$ (Geometry contraint)
$h = 0.012600615508606748[m]$ (Geometry contraint)
$\dot{m}_{O_2} = 0.9717255268199908[mg/s]$ (Mechanical constraint)
$U_d = 287.6638244522085[V]$ (Electrical constraint)

1.4 4. What is the result? Is it realisitic? Or why is it not feasible? (Result)

Given the range of U_d, fixed $d = 0.0434[m]$ (from [3]) and mass flow rate, we can observe the following performance




Isp-Power curve
Xe
O_2
Isp [s]
Power [W]


Efficiency-Power curve
Xe
O_2
Efficiency [%]
Power [W]

Overall, the trends relatively follow the expectation, thrust, I_{sp}, and anode efficiency are both lower compare with Xe-based HET. Nevertheless, the efficiency should not stay constant within the range of power.

1.4.1 4.1 Further improvement: release the assumption in 3.4.1 (A1)

Derive some accurate values of the following: $\alpha, \gamma, \gamma', \theta_d, \Delta, T_n$
These are the key factors since these values should eventually decrease the performance of O_2 HET in realistic condition. Once the ratio of Xe/O_2 is derived, we can adjust the slopes and makes them close to a more accurate solution.

1.4.2 4.2 Magnetic Field Estimation B and Channel Length(L) (A2)

Goal: Design magnetic field B and Channel length L

4.2.0 Nomenclature B Magnetic field [G]
λ_i ionization mean free path [m]
L channel length [m]
ν_i ionization frequency [Hz]
v_e, v_n thermal speed of the electrons and atoms[m/s]
σ_i cross section fro ionization $[m^2]$
σ_{en} electron-atom momentum exchange cross section $[m^2]$
τ_{ce} gyroperiod [s]
τ_{en} electron-atom collisional time [s]
n_n atom number density $[m^{-3}]$
T_e, T_n electron temperature, atom temperature [K]
r_{Le} Lamour radius
m_e electron mass [kg]
K_B Boltzmann constant
w microscopic electron velocity of thermal origin [m/s]

4.2.1 Criteria

1. Propellant Ionization: Melikov-Morozov criterion: ionization mean free path λ_i smaller than channel length L to ensure a sufficient ionization of the gas, atoms must stay long enough inside the channel. $\lambda_i << L$ (check Goebel p.334-335)
   with a approximation of
   $\lambda_i = \frac{v_n}{\nu_i} = \frac{v_n}{n_e <\sigma_i \, v_e>} << L$
   where
   $v_n = f(T_n)$
   $< \sigma_i \, v_e >$ Ionization reaction rate coefficient for Maxwellian electrons (check Goebel p.475) (def. from [1])
   $< \sigma_i \, v_e >= f(T_e) = \int_0^{+\infty} w\sigma_i(T_e) g(w, T_e) dw$
   $g(w, T_e)$: Maxwellian-Boltzmann electron velocity distribution function
   $g(w, T_e) = \sqrt{\frac{2}{\pi}} (\frac{m_e}{K_B T})^{\frac{3}{2}} w^2 exp[-\frac{m_e w^2}{2K_B T_e}]$
2. Electron Confinement-1: The magnetic field strength in a Hall effect thruster is such that electrons are magnetized and ions are not, or at least weakly magnetized. $r_{Le} = \frac{m_e v_e}{eB} << L$

3. Electron Confinement-2: Electron gyroperiod τ_{ce} in the magnetic barrier must be shorter than the time between two consecutive electron-atom collisions τ_{en} to warrents electrions are efficiently trapped inside the magnetic field of a HET.
$\frac{\tau_{en}}{\tau_{ce}} = \frac{eB}{m_e n_n \sigma_{en} \nu_e} >> 1$

4.2.2 Relation $v_e = f(T_e) = \sqrt{\frac{3K_B T_e}{m_e}}$ (root mean square of the total velocity)
from crit. 2 we get
$B \propto \frac{m_e v_e(T_e)}{eL} \propto \frac{1}{L}$
from crit. 3 we get
$B \propto \frac{m_e}{e} \sigma_{en} v_e(T_e) n_n$
use the relationship of $n_n \propto \frac{\dot{m}_n}{hd}$
T_e is assume to be $T_e \approx 0.2U_d$
$B_\propto \frac{\dot{m}_n \sqrt{U_d}}{hd}$
$B_\propto \frac{\sqrt{U_d}}{L}$
When the atom density is fixed the magnetic field does not depend anymore on the channel diameter d and width h.
It becomes merely a function of the channel length L.

4.2.3 Future Impovement Design magnetic field B and Channel length L using above relation.