Section 4.4—Exponential and Logarithmic Equations

Exponential Equation—an equation containing a variable in an exponent

All exponential functions are one-to-one; that is no two different ordered pairs have the same second component.

Solving Exponential Equations by Expressing Each Side as a Power of the Same Base

$2^3 = 2^x$

If $b^M = b^N$, then M = N.

1. Rewrite the equation in the form $b^M = b^N$
2. Set M = N.
3. Solve for the variable.

Example—Solve

a. $5^{3x-6} = 125$ $\quad 5 \cdot 5 \cdot 5 = 5^3$

$5^{3x-6} = 5^3$

$3x - 6 = 3$

$+6 \quad +6$

$\frac{3x}{3} = \frac{9}{3}$

$\boxed{x = 3}$

b. $8^{x+2} = 4^{x-3}$

$2^{3(x+2)} = 2^{2(x-3)}$

$3(x+2) = 2(x-3)$

$3x + 6 = 2x - 6$

$-2x \quad -2x$

$x + 6 = -6$

$-6 \quad -6$

$\boxed{x = -12}$

Most exponential equations cannot be rewritten so that each side has the same base.

Using Natural Logarithms to Solve Exponential Equations

1. Isolate the exponential expression
2. Take the natural logarithm on both sides
3. Simplify using one of the following

$$\ln b^x = x \ln b \quad \text{or} \quad \ln e^x = x$$

4. Solve for the variable

Example—Solve the following

a. $5^x = 134$

$\ln 5^x = \ln 134$

$\frac{x \cancel{\ln 5}}{\cancel{\ln 5}} = \frac{\ln 134}{\ln 5}$

$x = \frac{\ln 134}{\ln 5} = \boxed{3.04}$

b. $10^x = 8000$

$\ln 10^x = \ln 8000$

$\frac{x \cancel{\ln 10}}{\cancel{\ln 10}} = \frac{\ln 8000}{\ln 10}$

$x = \frac{\ln 8000}{\ln 10} = \boxed{3.90}$

c. $7e^{2x} - 5 = 58$

$+5 \quad +5$

$\frac{7e^{2x}}{7} = \frac{63}{7}$

$e^{2x} = 9$

$\ln e^{2x} = \ln 9$

$\frac{2x}{2} = \frac{\ln 9}{2}$

$x = \frac{\ln 9}{2}$

$= \boxed{1.10}$

Logarithmic Equation—an equation containing a variable in a logarithmic expression

Using the Definition of a Logarithm to Solve Logarithmic Equations

1. Express the equation in the form $\log_b M = c$
2. Use the definition of a logarithm to rewrite the equation in exponential form:

$\log_b M = c$ means $b^c = M$

3. Solve for the variable
4. Check proposed solutions in the original equation. Include in the solution set only values for which M > 0.

Example—Solve:

a. $\log_2(x-4) = 3$

$2^3 = x - 4$

$8 = x - 4$

$+4 \quad +4$

$\boxed{12 = x}$

check:

$\log_2(12-4) = 3$

$\log_2(8) = 3$

~~4-x~~

~~4-12=-8~~

b. $\frac{4\ln(3x)}{4} = \frac{8}{4}$

$\ln_e 3x = 2$

$\frac{e^2}{3} = \frac{3x}{3}$

$x = \frac{e^2}{3} = \boxed{2.46}$

check:

$4\ln(3(2.46)) = 8$

$4\ln\left(\cancel{3}\cdot\frac{e^2}{\cancel{3}}\right) = 8$

$4\ln e^2 = 8$

$4(2) = 8$

Logarithmic expressions are defined only for logarithms of positive real numbers.

Always check proposed solutions of logarithmic equation in the original equation. Exclude values that give you the log of a negative number or a log of 0.

To rewrite $\log_b M = c$ as $b^c = M$, we have to have a single logarithm whose coefficient is one.

Example—Solve $\log x + \log(x-3) = 1$.

check:

$\log(5) + \log(5-3) = 1$

$\log 5 + \log 2 = 1$

$\log_{10} 10 = 1$ ✓

$\log_{10}[x(x-3)] = 1$

$10^1 = x(x-3)$

$10 = x^2 - 3x$

$-10 \qquad -10$

$x^2 - 3x - 10 = 0$

$\log(-2) + \log(-2-3) = 1$

$\log(-2) + \log(-5) = 1$ ✗

$(x-5)(x+2) = 0$

$x - 5 = 0 \qquad x + 2 = 0$

$+5 \quad +5 \qquad -2 \quad -2$

$\boxed{x = 5} \qquad \cancel{x = -2}$

Using the One-to-One Property of Logarithms to Solve Logarithmic Equations

1. Express the equation in the form $\log_b M = \log_b N$. (Remember: you must have a single logarithm whose coefficient is 1 on each side of the equation.)
2. Use the one-to-one property to rewrite the equation without logarithms: If $\log_b M = \log_b N$ then M = N.
3. Solve for the variable.
4. Check in the original equation. Include only the values for which M > 0 and N > 0.

Example—Solve $\ln(x-3) = \ln(7x-23) - \ln(x+1)$

check:

$\ln(4-3) = \ln(7(4)-23) - \ln(4+1)$

$\ln(1) = \ln 5 - \ln 5$

$\ln(5-3) = \ln(7 \cdot 5 - 23) - \ln(5+1)$

$\ln(2) = \ln 12 - \ln 6$

$\ln(x-3) = \ln\left(\frac{7x-23}{x+1}\right)$

$(x+1)\ x - 3 = \frac{7x-23}{\cancel{x+1}}\ \cancel{(x+1)}$

$(x+1)(x-3) = 7x - 23$

$x^2 - 3x + x - 3 = 7x - 23$

$x^2 - 2x - 3 = 7x - 23$

$-7x \qquad -7x$

$x^2 - 9x - 3 = -23$

$+23 \qquad +23$

$x^2 - 9x + 20 = 0$

$(x-4)(x-5) = 0$

$x - 4 = 0 \qquad x - 5 = 0$

$+4 \quad +4 \qquad +5 \quad +5$

$\boxed{x = 4} \qquad \boxed{x = 5}$