Section 4.3—Properties of Logarithms

<u>Product Rule</u>—Let b, M & N be positive real numbers with b ≠ 1

$$\log_b(MN) = \log_b M + \log_b N$$

The logarithm of a product is the sum of the logarithms.

When we use the product rule to write a single logarithm as the sum of two logarithms, we say we are **expanding a logarithmic expression**.

<u>Example</u>—Use the product rule to expand each logarithmic expression.

a. $\log_6(7 \cdot 11)$

$\boxed{\log_6 7 + \log_6 11}$

b. $\log_5(100x)$

$\log_{10} 100 + \log x = \boxed{2 + \log x}$

<u>Quotient Rule</u>—Let b, M & N be positive real numbers with b ≠ 1

$$\log_b\left(\frac{M}{N}\right) = \log_b M - \log_b N$$

The logarithm of a quotient is the difference of the logarithms.

When we use the quotient rule to write a single logarithm as the difference of two logarithms, we say we are **expanding a logarithmic expression.**

<u>Example</u>—Use the quotient rule to expand each logarithmic expression.

a. $\log_8\left(\frac{23}{x}\right)$

$\boxed{\log_8 23 - \log_8 x}$

b. $\ln\left(\frac{e^5}{11}\right)$

$\ln_e e^5 - \ln 11 = \boxed{5 - \ln 11}$

<u>Power Rule</u>—Let b and M be positive real numbers with b ≠ 1 and let p be any real number

$$\log_b M^p = p \log_b M$$

The logarithm of a number with an exponent is the product of the exponent and the logarithm of that number.

When we use the power rule to "pull the exponent to the front," we say that we are **expanding a logarithmic expression**.

Example—Use the power rule to expand each logarithmic expression.

a. $\log_6 3^9$

$\boxed{9\log_6 3}$

b. $\ln \sqrt[3]{x}$

$\ln x^{1/3}$

$\boxed{\frac{1}{3}\ln x}$

c. $\log(x+4)^2$

$\boxed{2\log(x+4)}$

Sometimes it is necessary to use more than one property with logarithms.

Properties for Expanding Logarithmic Expressions

For M > 0 and N > 0:	
$\log_b(MN) = \log_b M + \log_b N$	Product Rule
$\log_b\left(\frac{M}{N}\right) = \log_b M - \log_b N$	Quotient Rule
$\log_b M^p = p\log_b M$	Power Rule

Example—Use logarithmic properties to expand each expression as much as possible.

a. $\log_b\left(x^4\sqrt[3]{y}\right)$

$\log_6 x^4 + \log_6 \sqrt[3]{y}$

$4\log_6 x + \log_6 y^{1/3}$

$\boxed{4\log_6 x + \frac{1}{3}\log_6 y}$

b. $\log_5\left(\frac{\sqrt{x}}{25y^3}\right)$

$\log_5 x^{1/2} - \log_5 25y^3$

$\frac{1}{2}\log_5 x - (\log_5 25 + \log_5 y^3)$

$\boxed{\frac{1}{2}\log_5 x - 2 - 3\log_5 y}$

To condense a logarithmic expression, we write the sum or difference of two or more logarithmic expressions as a single logarithmic expression.

Properties for Condensing Logarithmic Expressions

For M > 0 and N > 0	
$\log_b M + \log_b N = \log_b(MN)$	Product Rule
$\log_b M - \log_b N = \log_b\left(\frac{M}{N}\right)$	Quotient Rule
$p\log_b M = \log_b M^p$	Power Rule

Example—Write as a single logarithm.

a. $\log 25 + \log 4$

$\log(25 \cdot 4)$

$\boxed{\log_{10} 100 = 2}$

b. $\log(7x+6) - \log(x)$

$\boxed{\log\left(\frac{7x+6}{x}\right)}$

Coefficients of logarithms must be 1 before we can condense them using the product and quotient rules.

Example—Write as a single equation:

a. $2\ln x + \ln(x+1)$

$\ln x^2 + \ln(x+1)$

$\boxed{\ln\left[x^2(x+1)\right]}$

b. $2\ln x + \frac{1}{3}\ln(x+5)$

$\ln x^3 + \ln(x+5)^{1/3}$

$\ln x^3 + \ln \sqrt[3]{x+5}$

$\boxed{\ln\left[x^3 \sqrt[3]{x+5}\right]}$

c. $2\log(x-3) - \log x$

$\log(x-3)^2 - \log x$

$\boxed{\log\left(\frac{(x-3)^2}{x}\right)}$

d. $\frac{1}{4}\log_b x - 2\log_b 5 - 10\log_b y$

$\log_b x^{1/4} - \log_b 5^2 - \log_b y^{10}$

$\log_b \sqrt[4]{x} - \log_b 25 - \log_b y^{10}$

$\log_b \sqrt[4]{x} - (\log_b 25 + \log_b y^{10})$

$\log_b \sqrt[4]{x} - \log_b 25y^{10}$

$\boxed{\log_b\left(\frac{\sqrt[4]{x}}{25y^{10}}\right)}$

We know that calculators give the values for common logarithms (base 10) and natural logarithms (base e). To find a log with any other base, we can use the following property.

Change of Base Property—For any logarithmic bases a and b, and any positive number M

$$\log_b M = \frac{\log_a M}{\log_a b}$$

The logarithm of M with base b is equal to the logarithm of M with any new base divided by the logarithm of b with that new base.

We generally use this to write a logarithm in terms of quantities that can be evaluated with a calculator, like base 10 and base e.

Change-of-Base Property: Introducing Common and Natural Logarithms

Introducing Common Logarithms	Introducing Natural Logarithms
$\log_b M = \frac{\log_{10} M}{\log_{10} b} = \frac{\log M}{\log b}$	$\log_b M = \frac{\log_e M}{\log_e b} = \frac{\ln M}{\ln b}$

Example—Use common and natural logarithms to evaluate $\log_7 2506$.

$$\frac{\log 2506}{\log 7} = \frac{3.399}{.845} = \boxed{4.02}$$

$$\frac{\ln 2506}{\ln 7} = \frac{7.826}{1.946} = \boxed{4.02}$$