Section 2.7—Inverse Functions

<u>Definition of the Inverse of a Function</u>

Let f & g be two functions such that

$f(g(x)) = x$ for every x in the domain of g

and $g(f(x)) = x$ for every x in the domain of f

- The function g is the inverse of the function f and is denoted f^{-1} (read "f – inverse").
- Thus $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.
- The domain of f is equal to the range of f^{-1}, and vice versa.

<u>Example</u>—Show that each function is the inverse of the other.

$$f(x) = 4x - 7 \qquad g(x) = \frac{x+7}{4}$$

$f(g(x)) = \cancel{4}\left(\frac{x+7}{\cancel{4}}\right) - 7$

$g(f(x)) = \frac{4x - 7 + 7}{4}$

$f(g(x)) = x + 7 - 7$

$f(g(x)) = x$

$g(f(x)) = \frac{\cancel{4}x}{\cancel{4}} = x$

<u>In an inverse</u>:

- In f^{-1}, -1 is **<u>not</u>** an exponent and **<u>does not mean</u>** $f^{-1} = \frac{1}{f}$
- The two functions "undo" each other.

<u>Finding the Inverse of a Function</u>

1. Replace $f(x)$ with y in the equation.
2. Interchange x & y.
3. Solve for y.
   - If this equation is not a function, then f does not have an inverse.
   - If this equation is a function, it does have an inverse.
4. If f has an inverse, replace y in Step 3 with $f^{-1}(x)$.
5. Check this by showing that $f(f^{-1}(x)) = x$ *and* $f^{-1}(f(x))$.

Example—Find the inverse of each of the following:

a. $f(x) = 2x + 7$

$y = 2x + 7$

$x = 2y + 7$

$-7 \qquad -7$

$x - 7 = 2y$

$\frac{x-7}{2} = \frac{\cancel{2}y}{\cancel{2}}$

$y = \frac{x-7}{2}$

$\boxed{f^{-1}(x) = \frac{x-7}{2}}$

b. $f(x) = 4x^3 - 1$

$y = 4x^3 - 1$

$x = 4y^3 - 1$

$+1 \qquad +1$

$x + 1 = 4y^3$

$\frac{x+1}{4} = \frac{\cancel{4}y^3}{\cancel{4}}$

$\sqrt[3]{y^3} = \sqrt[3]{\frac{x+1}{4}}$

$y = \sqrt[3]{\frac{x+1}{4}}$

$\boxed{f^{-1}(x) = \sqrt[3]{\frac{x+1}{4}}}$

c. $f(x) = \frac{3}{x} - 1$

$y = \frac{3}{x} - 1$

$x = \frac{3}{y} - 1$

$+1 \qquad +1$

$y\,[x+1] = \left(\frac{3}{\cancel{y}}\right)\cancel{y}$

$\frac{y\cancel{(x+1)}}{\cancel{x+1}} = \frac{3}{x+1}$

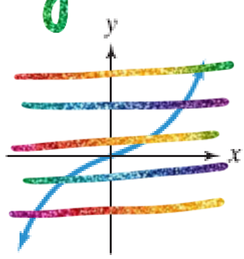
$y = \frac{3}{x+1}$

$\boxed{f^{-1}(x) = \frac{3}{x+1}}$

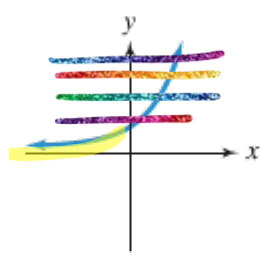
Horizontal Line Test for Inverse Function—a function f has an inverse if there is no horizontal line that intersects the graph of function f at more than one point.

Example—which of the following graphs represent functions that have inverse functions?

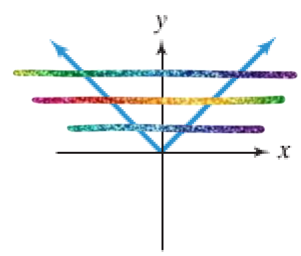
yes



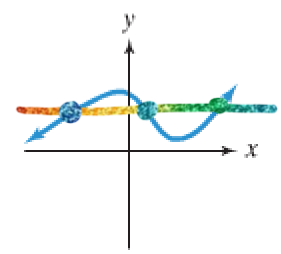
yes



no



no



One-to-One Function—a function in which no two different ordered pairs have the same second component

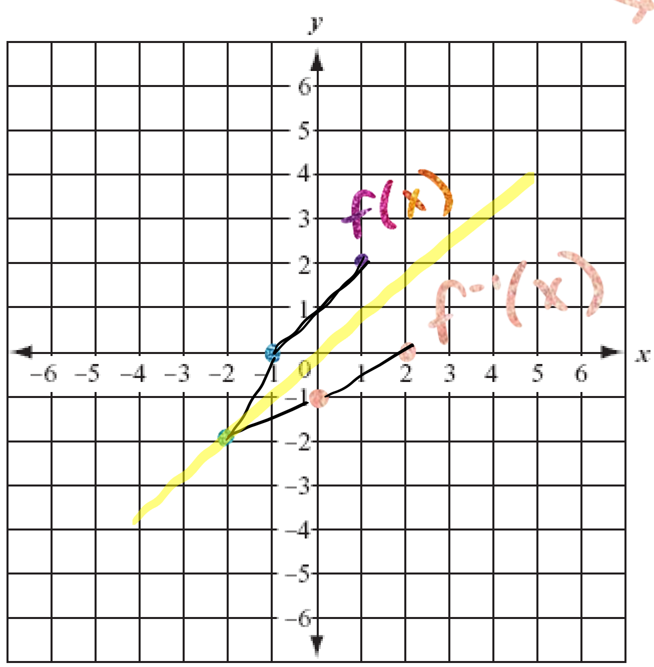
- ✓ any function that passes the horizontal line test is a one-to-one function.

- The graph of f^{-1} is a reflection of the graph of f about the line y = x.
- If the point (a, b) is on the graph of f, then (b, a) is on the graph of f^{-1}.

Example—the graph of f consists of two line segments, one from $(-2,-2)$ to $(-1,0)$ and a second from $(-1,0)$ to $(1,2)$. Graph f and use the graph to graph the inverse.

$\to (2,1)$

$(0,-1)$



Example—Find the inverse of $f(x) = x^2 + 1$, *if* $x \geq 0$. Graph f and f^{-1} in the same coordinate system

$y = x^2 + 1$

$x = y^2 + 1$

$-1 \qquad -1$

$\sqrt{x-1} = \sqrt{y^2}$

$y = \sqrt{x-1}$

$f^{-1}(x) = \sqrt{x-1}$

f(x) | f⁻¹(x)

x	y	(x,y)	
0	1	(0,1)	(1,0)
1	2	(1,2)	(2,1)
2	5	(2,5)	(5,2)