Amortized Analysis

$O(f(n))$

$op_1, op_2, op_3, \ldots, op_i, \ldots op_n$

$n \times O(f(n)) = O(n f(n))$

w	w	w	w	w	w amortized 20
10	10	10	10	60	$30 \leq 3 \cdot 20$

w	w	w	w	w	w <u>not</u> amortized 20
60	10	10	10	10	$80 \not\leq 3 \cdot 20$

Examples

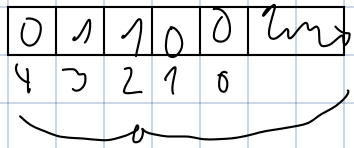
stack: init, empty, push, pop

```
multipop(S,k):
    while not empty(S) and k>0:
        pop(S)
        k-=1
```

init, push, push, pop, push, multipop(2), push, multipop(2)

$n \cdot n \in O(n^2)$

binary counter: init, inc

A: 

init, inc, inc, ...

$n \cdot k \in O(n \log n)$

Potential function methode

Potential func

$D_0 \xrightarrow{op_1} D_1 \xrightarrow{op_2} D_{i-1} \xrightarrow{op_i} D_i \rightarrow \dots\dots D_n$

$c_1 \quad c_2 \quad c_i$

$\Phi \quad \hat{c}_1 \quad \hat{c}_2$

$\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1})$

$\Delta \Phi$

$$\sum_{i=1}^{n} \hat{c}_i = \sum_{i=1}^{n} (c_i + \Phi(D_i) - \Phi(D_{i-1}))$$

$$= \sum_{i=1}^{n} (c_i + \Phi(D_1) - \Phi(D_0) + \Phi(D_2) - \Phi(D_1) + \Phi(D_3) - \Phi(D_2)$$

$$= \sum_{i=1}^{n} c_i + \underbrace{\Phi(D_n) - \Phi(D_0)}_{\text{ok if } \geq 0}$$

$\Phi(D_0) = 0$ and $\Phi(D_i) \geq 0$

Stack

Φ : number of elements on the stack

op	c	$\Delta\Phi$	$\hat{c} = c + \Delta\Phi$
push	1	1	2
pop	1	-1	0
multipop	k	-k	0

$n \cdot 2 \in O(n)$

15	1
7	1
42	1

Dynamic Tables

$\Phi: 2(n - \frac{s}{2})$

$n = s$

$s' = 2s$

$n' = n+1$

$\Phi(D_i) - \Phi(D_{i-1})$

$2(n' - \frac{s'}{2}) - 2(n - \frac{s}{2})$

$= 2n + 2 - 2n - 2n + n$

$= 2 - n$

op	c	$\Delta\Phi$	$\hat{c}$
Simple insert	1	2	3
Copy insert	$1+n+1$	$2-n$	4

allocate

Delete

$\Phi: 2(n - \frac{s}{2}), \quad n \geq \frac{s}{2}$

$\frac{s}{2} - n, \quad$ otherwise

Agree for $n = \frac{s}{2}$

op		c	$\Delta\Phi$	$\hat{c}$
insert	$n \geq \frac{s}{2}$ Simple	AS	Before	
	$n \geq \frac{s}{2}$ copy			
	$n < \frac{s}{2}$ Simple	1	-1	0
	$n < \frac{s}{2}$ copy	cannot	happen	

delete

$n \geq \frac{S}{2}$	Simple	1	-2	-1
	copy	cannot happen		
$n < \frac{S}{2}$	Simple	1	1	2
	copy	$1+(n-1)$	$-n+1$	1

$S' = S$

$n' = n-1$

$n = \frac{S}{4}$

$$\frac{S'}{2} - n' - \left(\frac{S}{2} - n\right)$$
$$= n - (n-1) - (2n - n)$$
$$= -n + 1$$