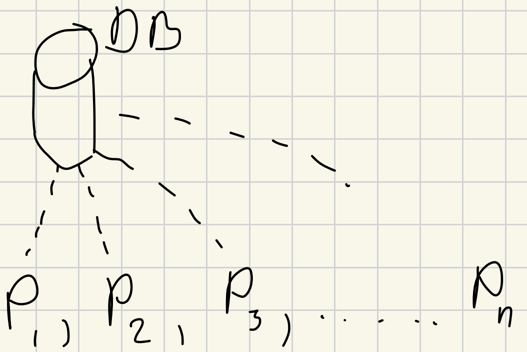
Algorithms & randomization

- average case analysis of deterministik algo.

- randomized algorithsm – algs. that flip a coin

  - correct but expected time (Las Vegas)
  - controllable time but expected correctnes (Monte Carlo)

- Simpler
- faster
- better



Access if it is the only process atempting

Attempt access with prob. p.

Event $A[i,t]$: Process P_i attempts in round t.

$Pr(A[i,t]) = p$, $Pr(\overline{A[i,t]}) = 1 - p$

Event $S[i,t] = A[i,t] \cap \bigcap_{j \neq i} \overline{A[j,t]}$

$Pr[S[i,t]] = Pr[A[i,t]] \cdot \prod_{j \neq i} Pr[\overline{A[j,t]}] =$

$p(1-p)^{n-1}$

differantiate

$1 \cdot (1-p)^{n-1} + p(n-1)(1-p)^{n-2}(-1) = 0$

$(1-p)^{n-1} = p(n-1)(1-p)^{n-2}$

$(1-p) = p(n-1)$

$p = \frac{1}{n}$

$= \frac{1}{n}\left(1 - \frac{1}{n}\right)^{n-1}$

$$\boxed{\lim_{n \to \infty} \left(1 - \frac{1}{n}\right)^{n-1} = \frac{1}{e} \quad e = 2.71}$$

$$\left(1-\frac{1}{n}\right)^n$$

$$\boxed{\frac{1}{e\cdot n} \leq P(S[i,t]) \leq \frac{1}{2\cdot n}}$$

How long before P_i gets access?

Event $F[i,t]$: P_i does not get access in $1 \dots t$

$$\Pr[F[i,t]] = P\left[\bigcap_{r=1}^{t} \overline{S[i,r]}\right]$$

$$= \prod_{r=1}^{t} \Pr\left[\overline{S[i,t]}\right]$$

$$= \left(1 - \frac{1}{n}\left(1 - \frac{1}{n}\right)^{n-1}\right)^t$$

$$\leq \left(1 - \frac{1}{en}\right)^t$$

$$t = \lceil e \cdot n \rceil$$

$$Pr[F[i,t]] = \left(1 - \frac{1}{en}\right)^{\lceil e \cdot n \rceil} \leq \left(1 - \frac{1}{en}\right)^{en}$$

$$\leq \frac{1}{e} \approx 0{,}3679$$

$$t = \lceil en \rceil \cdot (c \cdot \ln(n))$$

$$Pr[F[i,t]] = \left(1 - \frac{1}{en}\right)^{\lceil en \rceil \cdot (c \cdot \ln(n))} \leq \left(\frac{1}{e}\right)^{c \cdot \ln(n)}$$

$$= \underbrace{n^{-c}}$$

$$e^{-1 \cdot c \cdot \ln(n)} = e^{c \cdot \ln(n)} = n^{-c}$$

$$c = 3 \qquad \frac{1}{n^3}$$

Event F_t: Some process did not access in $1 \dots t$

$$F_t = \bigcup_{i=n}^{n} F[i,t)$$

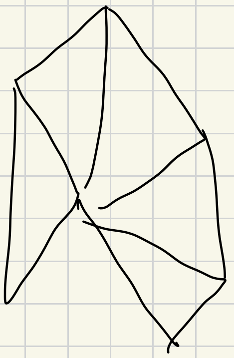
union bound

$$\Pr[F_t] \leq \sum_{i=1}^{n} \Pr[F[i,t]]$$

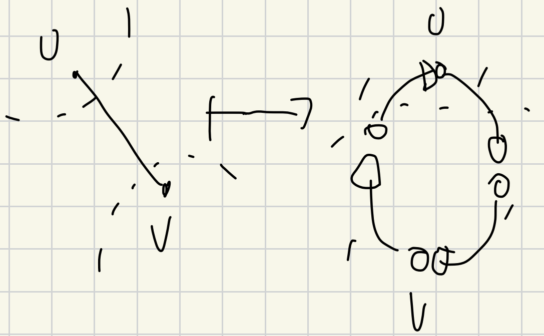
$$= n \cdot n^{-c} \qquad \text{if } c = 2 \quad n \cdot n^{-2}$$

$$= \frac{1}{n}$$

Global Minimum Cut



Use max-flow/min cut tools



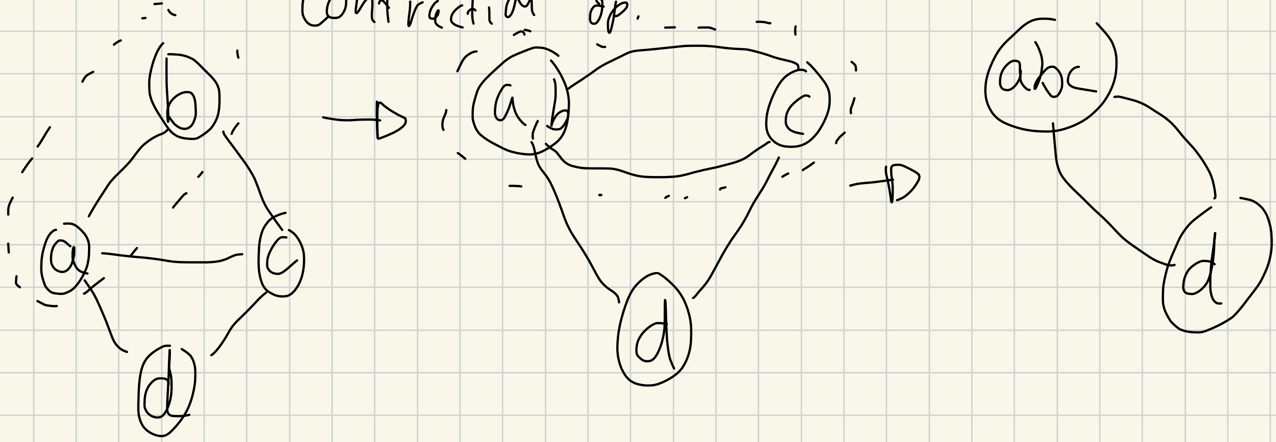
Fix S, try all t's. Return minimum cut

time: $\Theta(n) \cdot \Theta(\text{max-flow alg})$

$\Theta(V \cdot VE^2) = \Theta(V^2 \cdot E^2)$ opt $\Theta(V^2 \cdot E)$

The contraction Algorithm

contraction op.



<u>repeat</u>

if $|V| = 2$:

return $(S(v_1), S(v_2))$

select $(u,v) \in E$ uniformly at random

G = graph resulting from contracting (u,v)

into node w, $S(w) = S(u) \cup S(v)$

global min cut w. prob. $\frac{1}{\binom{n}{2}}$

Proof

F = global min cut $\quad |F| = k \quad$ So $|E| \geq \frac{k \cdot n}{2}$

Do <u>not</u> want to contract an F-edge. Happens

with prob. $\dfrac{k}{\frac{k \cdot n}{2}} = \dfrac{2}{n}$

After j iterations - all good so far

Prob. of contracting an F-edge $\leq \dfrac{k}{\frac{k \cdot (n-j)}{2}} = \dfrac{2}{(n-j)}$

Prob. of final success: E_j: no F edge contraction in iteration j

$$\Pr[E_1] \cdot \Pr[E_2 \mid E_1] \cdot \Pr[E_3 \mid E_1 \wedge E_2] \dots \Pr[E_{n-2} \mid E_1 \wedge \dots E_{n-3}]$$

$$\geq \left(1 - \frac{2}{n}\right) \cdot \left(1 - \frac{2}{n-1}\right) \cdot \left(1 - \frac{2}{n-2}\right) \cdots \left(1 - \frac{2}{3}\right) =$$

$$\frac{n-\cancel{2}}{n} \cdot \frac{n-3}{n-1} \cdot \frac{n-4}{\cancel{n-2}} \cdots \frac{\cancel{3}}{5} \cdot \frac{2}{4} \cdot \frac{1}{\cancel{3}} = \frac{2}{n(n-1)} =$$

$$\frac{1}{\frac{n(n-1)}{2}} = \frac{1}{\binom{n}{2}}$$

Contraction Algo Repetiton

Fail w. Prob $\leq 1 - \frac{1}{\binom{n}{2}}$

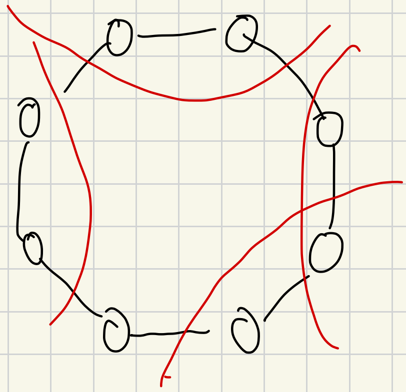
Run $\binom{n}{2}$ times. Failure $\leq \left(1 - \frac{1}{\binom{n}{2}}\right)^{\binom{n}{2}} \leq$

$\frac{1}{e}$

Run $\binom{n}{2} \ln n$ Failure $\leq \left(1 - \frac{1}{\binom{n}{2}}\right)^{\binom{n}{2} \ln n} =$

$\left(\left(1 - \frac{1}{\binom{n}{2}}\right)^{\binom{n}{2}}\right)^{\ln n} \leq \left(\frac{1}{e}\right)^{\ln n} =$

$\left(e^{\ln n}\right)^{-1} = n^{-1} = \frac{1}{n}$



$C_1, C_2, \dots, C_r$ all the global min cuts

E_i: event that Contraction algo finds C_i

$$\Pr[E_i] \geq \frac{1}{\binom{2}{n}} \qquad 1 \geq \Pr\left[\bigcup_{i=1}^{r} E_i\right] = \sum_{i=1}^{r} \Pr[E_i] \geq \frac{r}{\binom{n}{2}}$$

$$r \leq \binom{n}{2}$$

Lower bound

Waiting time for the first success

Desired event has prob p.

Expected $\underbrace{\text{waiting Time}}_{X}$ $\quad Pr[X = j] = (1-p)^{j-1} \cdot p$

$$E[X] = \sum_{j=1}^{\infty} j \cdot Pr[X = j] = \sum_{j=1}^{\infty} j \, (1-p)^{j-1} \cdot p$$

$$= \frac{p}{1-p} \sum_{j=1}^{\infty} j \, (1-p)^{j} = \frac{\cancel{p}}{\cancel{1-p}} \cdot \frac{\cancel{1-p}}{p^{\cancel{2}}} = \frac{1}{p}$$

X = # correct predictions

$X_i = 1$ if card i is predicted correct

$$E[X_i] = 0 \cdot Pr[X_i = 0] + 1 \cdot Pr[X_i = 1]$$
$$= Pr[X_i = 1] = \frac{1}{n}$$

$$E[X] = E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} \frac{1}{n} = 1$$

Choose uniformly at random between unseen cards

$$E[X_i] = Pr[X_i = 1] = \frac{1}{n-(i-1)}$$

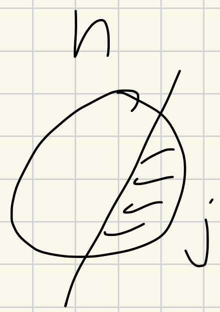
Harmonic numbers

$$E[X] = \sum_{i=1}^{n} E[X_i = 1] = \sum_{i=1}^{n} \frac{1}{n-i+1} = \sum_{i=}^{n} \frac{1}{i} \overset{\text{def}}{=} H_n$$

$$\ln(n+1) < H(n) < \ln n + 1$$

Coupon Collection

X_j = waiting time from having collected j to obtain $j+1$



Prob. of getting new: $\frac{n-j}{n}$

$$E[X_j] = \frac{1}{\frac{n-j}{n}} = \frac{n}{n-j}$$

$$E\left[\sum_{j=0}^{n-1} X_j\right] = \sum_{j=0}^{n-1} E[X_j] = \sum_{j=0}^{n-1} \frac{n}{n-j} = n\sum_{j=1}^{n} \frac{1}{j} = n H_n$$

Max 3 SAT

Propositional Variable

Formula $\underbrace{(q_1 \vee q_2 \vee \bar{q}_4)}_{C_1 \text{ Clause } 1} \wedge \underbrace{(q_3 \vee \bar{q}_2 \vee q_4)}_{C_2 \text{ Clause } 2} \wedge \underbrace{\ldots}_{C_k}$

C_i fails to be true w. prob. $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$

C_i satisfied w. prob. $1 - \frac{1}{8} = \frac{7}{8}$

$Z_i = 1$ if C_i is satisfied

$$E\left[\sum_{i=1}^{k} Z_i\right] = \sum_{i=1}^{k} E[Z_i] = \frac{7}{8} k$$

Probabilistic Method

There exists an assignment making at least $\frac{7}{8}$th off all clauses are true formula with $k \le 7$

$$k \le 7$$

$$\Downarrow$$

$$\frac{7}{8} > \frac{k-1}{k} = 1 - \frac{1}{k}$$

$$\Downarrow \quad \frac{7}{8} k > k - 1$$

must satisfy $\ge k$ clauses

Want $\geq \frac{7}{8}k$! $\quad p =$ prob of satisfying $\geq \frac{7}{8}k$

p_j = Prob. of satisfying j clauses

Expected satisfied: $\sum_{j=0}^{k} j \cdot p_j = \frac{7}{8}k$

Choose k' such that $\frac{7}{8}k - 1 \leq k' < \frac{7}{8}k$

$$\frac{7}{8}k = \sum_{j<\frac{7}{8}k} j \cdot p_j + \sum_{j\geq\frac{7}{8}k} j \cdot p_j$$

failure $(1-p)$ $\qquad p$

$$\leq \sum_{j<\frac{7}{8}k} k' \cdot p_j + \sum_{j\geq\frac{7}{8}k} k \cdot p_j = k' \sum_{j<\frac{7}{8}k} p_j + k \sum_{j\geq\frac{7}{8}k} p_j$$

$$= k'(1-p) + k \cdot p \leq k' + k \cdot p$$

$$p \geq \frac{\frac{7}{8}k - k'}{k} \geq \frac{1}{k} \cdot \frac{1}{8} = \frac{1}{8k}$$

Run in expected steps: $\frac{1}{1/8k} = 8k$