Quicksort

Random QS chooses a Random partition instead of the last element

Random Partition

RP(A, p, r

i = Random(p r)

A[r] ↔ A[i]

return P(A, P, r

Lemma 7.1

RQS runs in $O(n + X)$

X = # Comparisons

All element are distinct $= z_1 < z_2 < z_3 \dots < z_j < \dots$

$Z_{ij} = \{z_i, z_{i+1} \dots z_j$

Lemma 7.2

RQS compares z_i and z_j (once!)

$\Updownarrow$

z_i or z_j are choosen as first pivot from Z_{ij}

z_i first: compare w. all Z_{ij} incl z_j

pivot $x \notin \{z_i, z_j\}$ $\quad z_i < x < z_j$

Lemma 7.3

Prob of comp. z_i and z_j, $i<j$, is $\frac{2}{j-i+1}$

Th 7.4

Exp running time of RQS is $O(n \log n)$

Proof

X_{ij} = indicator that z_i and z_j are comp.

$$E\left[\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \right] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} E[X_{ij}]$$

$$= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} Pr[z_i \text{ and } z_j \text{ are comp.}]$$

$$= \sum^{n-1} \sum^{n} = \frac{2}{j-i+1}$$

$$i=1 \quad j=i+1 \quad j-i+1$$

$$= \sum_{i=1}^{n-1} \sum_{k=1}^{n-i} \frac{2}{k+1}$$

$H_k < \ln n + 1$

$$< \sum_{i=1}^{n-1} \sum_{k=1}^{n} \frac{2}{k}$$

$$< \sum_{i=1}^{n-1} 2 \sum_{k=1}^{n} \frac{1}{k}$$

$$= \sum_{i=1}^{n-1} 2 \cdot (\ln n + 1)$$

$$\leq 2 \cdot n \ln n + 2n$$

$$= / \in O(n \log n)$$

Selection in expected linear time

Find ith smallest element in A

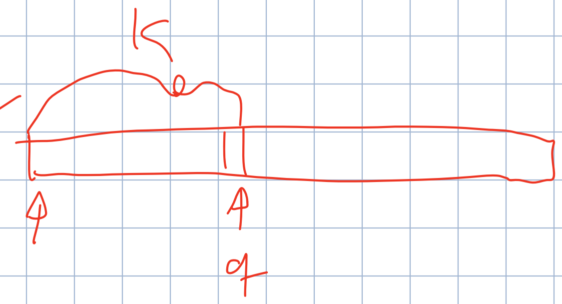
$A[p:r] \quad 1 \leq i \leq r-p+1$

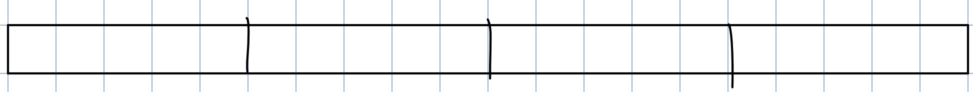
```
RS(A, p, r, i)

    if p == r
        return A[p]

    q = RP(A, p, r)
    k = q - p + 1
    if i == k:
        return A[q]
    elif i < k:
        return RS(A, p, q-1, i)
    else:  # i > k
        return RS(A, q+1, r, i-k)
```

Randomized partition





$A^{(0)}, A^{(1)}, A^{(2)} \dots A^{(j)}$ ← Active part after Partitioning

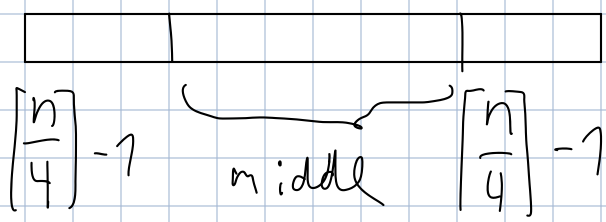
j'th partitoning helpful

if $|A^{(j)}| \le \frac{3}{4} \cdot |A^{(j-1)}|$

Lemma 9.1

A partitioning is helpful w. prob. $\frac{1}{2}$

Proof



because pivot also disapears

pivot in middle, remove $\lceil \frac{n}{4} \rceil - 1 + 1$

leaving $n - \lceil \frac{n}{4} \rceil = \lfloor \frac{3n}{4} \rfloor \leq \frac{3}{4} n$

Helpful!

Prob. of choosing a non-middle pivot:

$$\frac{2 \cdot \left(\lceil \frac{n}{4} \rceil - 1 \right)}{n} \leq \frac{2\left(\frac{n}{4} + \cancel{1 - 1} \right)}{n} = \frac{1}{2}$$

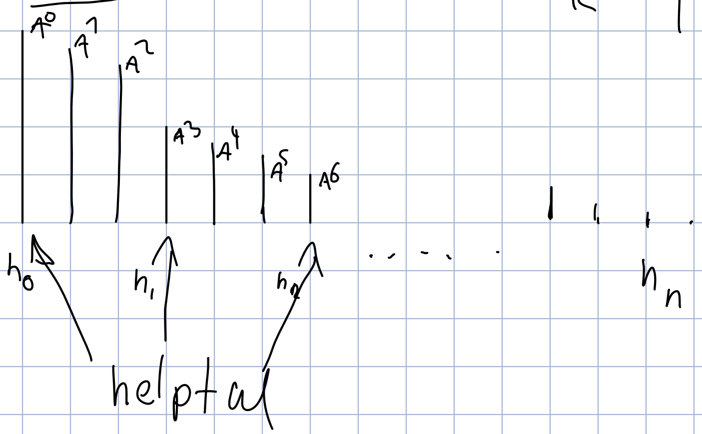
Middle pivot $\geq 1 - \frac{1}{2} = \frac{1}{2}$

Th 9.2

RS has expected running time $\Theta(n)$

proof

$n_k = |A^{h_k}|$ $n_k \le \frac{3}{4} n_{k-1}$



So, $n_k \le \left(\frac{3}{4}\right)^k \cdot n_0 = \left(\frac{3}{4}\right)^k \cdot n$

$X_k = h_{k+1} - h_k$

random variable waiting time for the $(k+1)$st helpful part, after the k'th

$$\sum_{k=0}^{m-1} \sum_{j=h_k}^{h_k+X_k-1} |A^{(j)}| \le \sum_{k=0}^{m-1} \sum_{j=h_k}^{h_k+X_k-1} n_k = \sum_{k=0}^{n-1} X_k \cdot n_k \le$$

$$\sum_{k=0}^{m-1} X_k \cdot \left(\frac{3}{4}\right)^k \cdot n = F$$

$$E[F] = \sum_{k=0}^{n-1} n \cdot \left(\frac{3}{4}\right)^k \cdot E[X_k] = 2n \sum_{k=0}^{n-1} \left(\frac{3}{4}\right)^k \le$$

$$2n \sum^{\infty} \left(\frac{3}{4}\right)^k = 2n \cdot \frac{1}{1-\frac{3}{4}} = 8n$$

$k=0$ 4