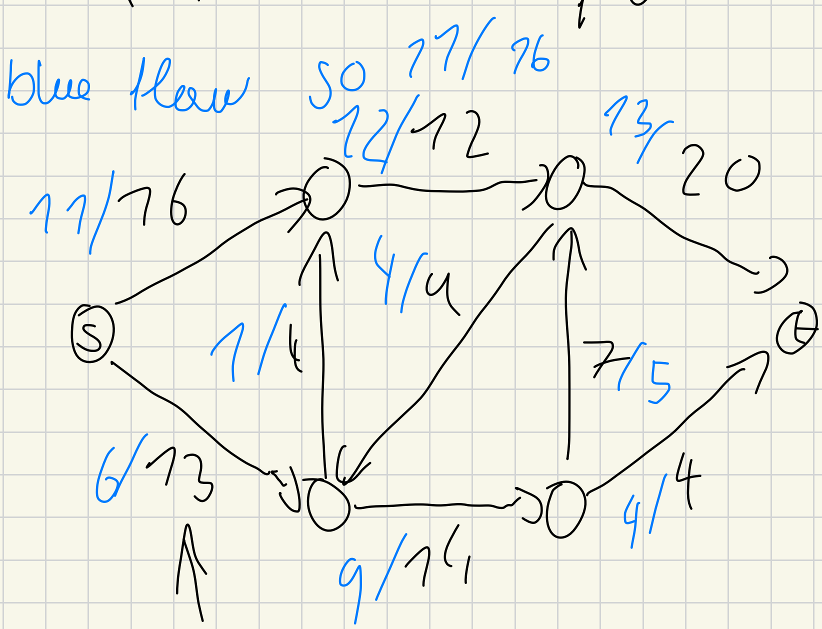
Maximum flow



flow network

Directed graph
$G=(V,E)$

$\forall (u,v) \in E \Rightarrow (v,u) \notin E$

$\forall (u,v) \in E \Rightarrow u \neq v$ (no self loops)

Source s, sink t

flow requirement:

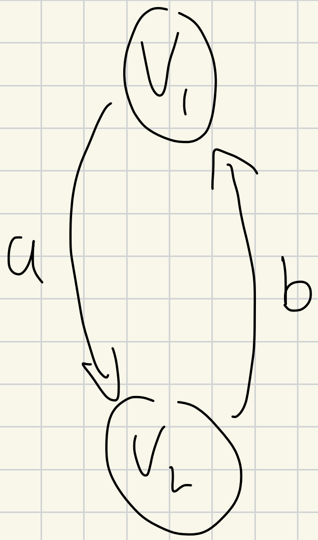
$f: V \times V \rightarrow \mathbb{R}$

- $0 \leq f(u,v) \leq c(u,v)$
- $\sum_{v \in V} f(v,u) = \sum_{v \in V} f(u,v)$, $\forall u \in V \setminus \{s,t\}$

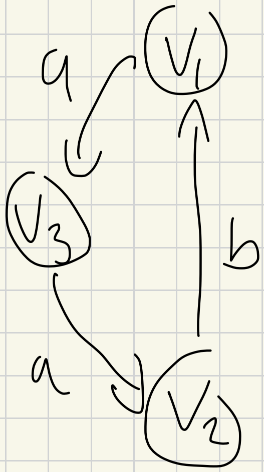
· Value $|f| = \sum_{v \in V} f(s,v) - \underbrace{\sum_{v \in V} f(v,s)}_{\text{usually } 0}$

Goal find the flow with max Value

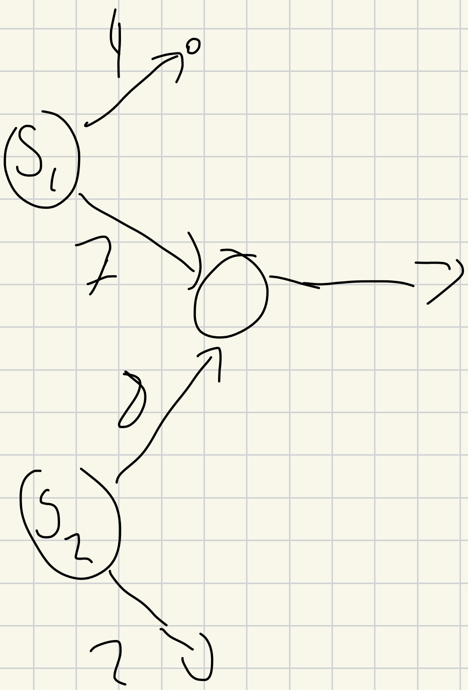
Antiparallel edges



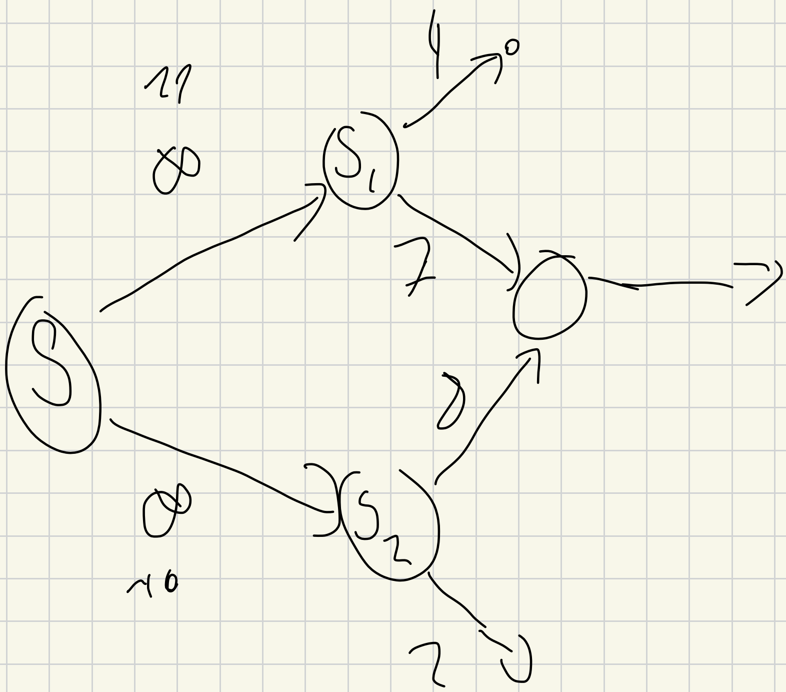
Can be trans ~~formed~~
$\to$

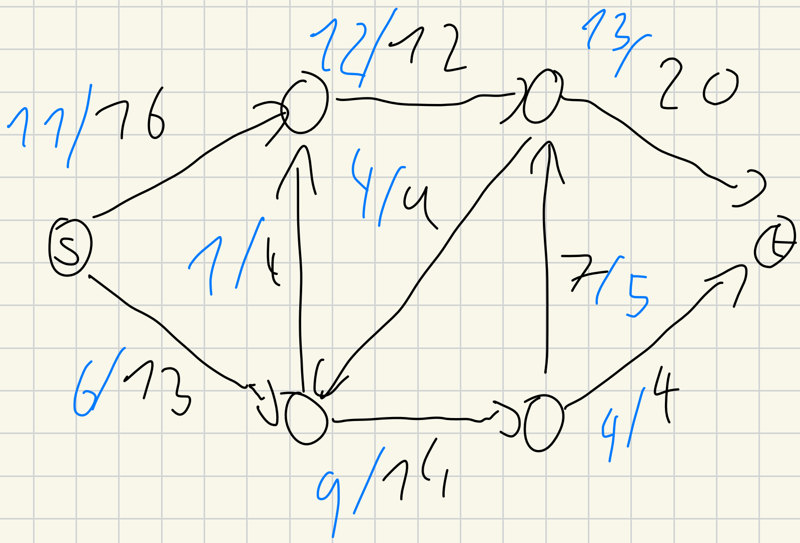


2 sources ?



make an ekstra source





Ford-Fulkerson Methode

FFM(G, s, t)

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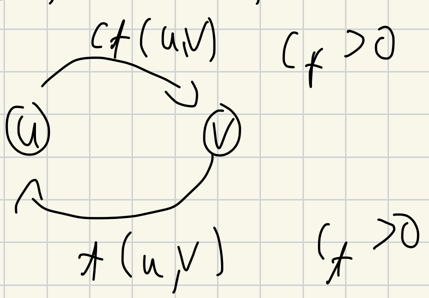
init flow f to all zero

while there exists augmenting path p in residual graph G_f augment flow f along p

G_f - residual graph (antiparallel edges allowed)

$c_f(u,v) = c_f(u,v) - f(u,v)$

$G_f = (V, E_f)$



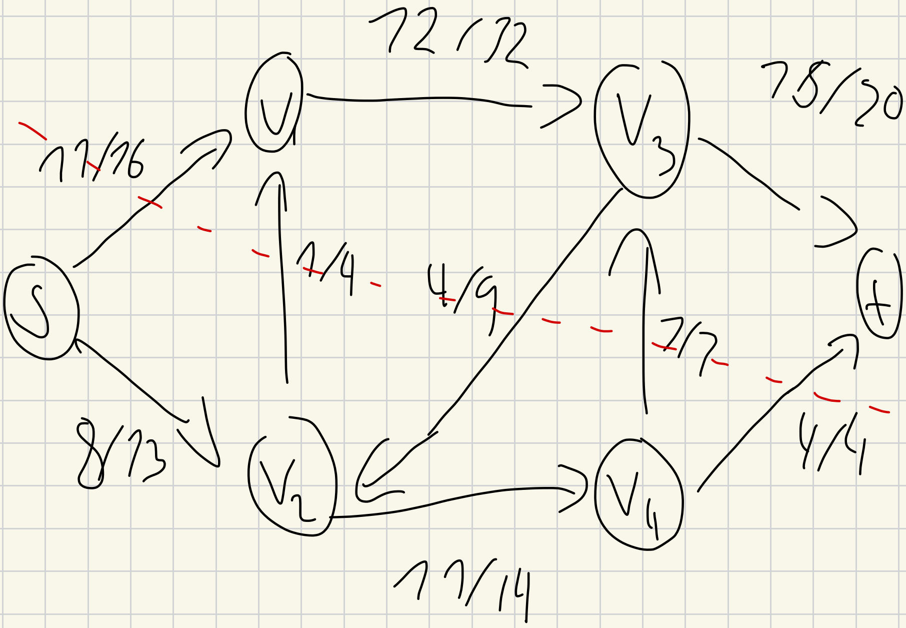
Augmentation

$$(f \uparrow f')(u,v) = \begin{cases} f(u,v) + f'(u,v) - f'(v,u), & (u,v) \in E \\ 0 \end{cases}$$

flow in G

flow in G_f

$$f'(v,u) \le c_f(v,u) = f(u,v)$$



$(11 + 1 + 4 + 7) - (4) =$

19

23.02.24

final flow / number of edges

$O(|f^*| \cdot E)$

FF(G,s,t)

for $(u,v) \in G.E$:

$(u,v).f = 0$

while $\exists p$ from s to t in G_f:

$c_f(p) = \min\{c_f(u,v) \mid (u,v) \in p\}$

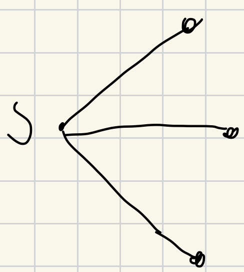
for $(u,v) \in p$:

if $(u,v) \in G.E$:

$(u,v).f \mathrel{+}= c_f(p)$

else

$(v,u).f \mathrel{-}= c_f(p)$



Edmond-Karp impl. of FF

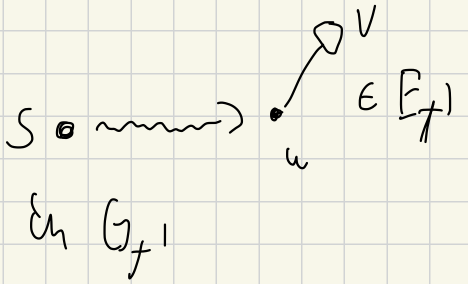
Change: choose p as a shortest path when viewing the graph as unweighted

Lemma 24.7 | $\delta_f(u,v)$ shortest path distance from u to v in G_f

$\forall v \in V \setminus \{s,t\}$: $\delta_f(s,v)$ increases monotonically with each augmentation

Proof: by contradiction. First violation from f to f' such that $\delta_{f'}(s,v) < \delta_f(s,v)$

choose v with smallest $\delta_{f'}(s,v)$



$\delta_{f'}(s,u) = \delta_{f'}(s,v) - 1$

if $(u,v) \in E_f$

$$\delta_f(s,v) \leq \delta_f(s,u) - 1 \quad \Delta\text{-}\leq$$
$$\leq \delta_{f'}(s,u) + 1 \quad \text{choice of}$$
$$= \delta_{f'}(s,v) \quad ↯$$

$\underline{\text{if } (u,v) \in E_{f'} \setminus E_f}$

(u,v) in augmenting path and shortest path

$$\delta_f(s,v) = \delta_f(s,u) - 1$$
$$\leq \delta_{f'}(s,u) - 1 = \delta_{f'}(s,v) - 2 \leq \delta_{f'}(s,v) \quad ⚡$$

Th 24.8

E-K terminates after $O(V \cdot E)$ rounds

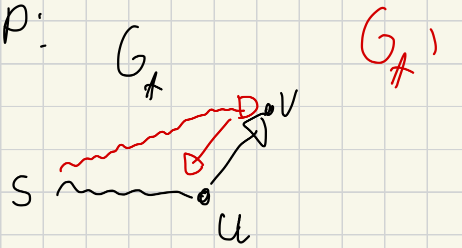
Proof:

critical if capacity $c_f(p)$

p

an edge can not become critical more than $|V|/2$ times

p:



$$\delta_f(s,v) = \delta_f(s,u) + 1$$

flow update

After $f \uparrow f_p$ (u,v) dissapears from G_f

Can only reappear if decreased

$$\delta_{f'}(s,u) = \delta_{f'}(s,v) + 1 \quad \text{, shortest path}$$
$$\geq \delta_f(s,v) + 1 \quad \text{, Lemma 24.7}$$
$$= \delta_f(s,u) \underbrace{+ 1 + 1}_{=2}$$

$$\frac{|V|-2}{2}$$

Find time $O(E)$

Running time $O(V \cdot E^2)$

Maximum Bipartite Matching