The Lagrangian of the plaquette can be written as

$$\mathcal{L} = \frac{C_1 \dot{\eta}_1^2}{2} + \frac{C_2 \dot{\eta}_2^2}{2} - \left\{ -E_{J1} \cos \eta_1 - E_{J2} \cos \eta_2 + \frac{(\eta_2 - \eta_2 - \varphi_{ext})}{2L} + \frac{\zeta^2}{2L^*} \right\},$$

where

$$L \equiv L_1 + L_2;$$
$$L^* \equiv \frac{L_1 L_2}{L_1 + L_2};$$
$$\zeta \equiv \frac{L_2 \zeta_1 + L_1 \zeta_2}{L_1 + L_2}.$$

Then the plaquette Hamiltonian is then

$$\mathcal{H} = \frac{p_1^2}{2C_1} + \frac{p_2^2}{2C_2} - E_{J1} \cos \eta_1 - E_{J2} \cos \eta_2 + \frac{(\eta_2 - \eta_1 - \varphi_{ext})}{2L} + \frac{\zeta^2}{2L^*}.$$

Here $p_n \equiv -i\partial/\partial\eta_n$. This Hamiltonian depends on three coordinates $\{\eta_1, \eta_2, \zeta\}$. We can now express ζ via a variable φ, so that $\varphi = \zeta_1 + \eta_1 + \varphi_{ext} = \zeta_2 + \eta_2$. In the new coordinates, $\{\eta_1, \eta_2, \varphi\}$, after a bit tedious manipulations the Hamiltonian takes form

$$\mathcal{H} = \frac{p_1^2}{2C_1} + \frac{p_2^2}{2C_2} - E_{J1} \cos \eta_1 - E_{J2} \cos \eta_2 + \frac{(\eta_1 - \varphi - \varphi_{ext})^2}{2L_1} + \frac{(\eta_2 - \varphi)^2}{2L_2} - \frac{\varphi_{ext}^2}{2L}.$$

Let us introduce two types of potential:

(i) periodic potential $V_{per}(\eta_1, \eta_2) \equiv -E_{J1} \cos \eta_1 - E_{J2} \cos \eta_2$, and

(ii) aperiodic quadratic potential $V_q(\eta_1, \eta_2; \varphi, \varphi_{ext}) \equiv (\eta_1 - \varphi - \varphi_{ext})^2 / 2L_1 + (\eta_2 - \varphi)^2 / 2L_2 - \varphi_{ext}^2/2L$.

The Hamiltonian is a superposition of V_{per} and V_q (shifted by $\varphi + \varphi_{ext}$ in η_1-direction and shifted by φ in η_1-direction). Below is a schematic representation of a periodic potential.

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--- V_q (symmetric) — V_{per} (symmetric)

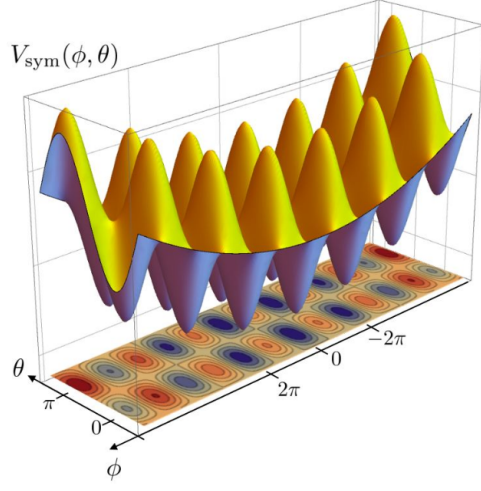--- V_q (asymmetric) — V_{per} (asymmetric)
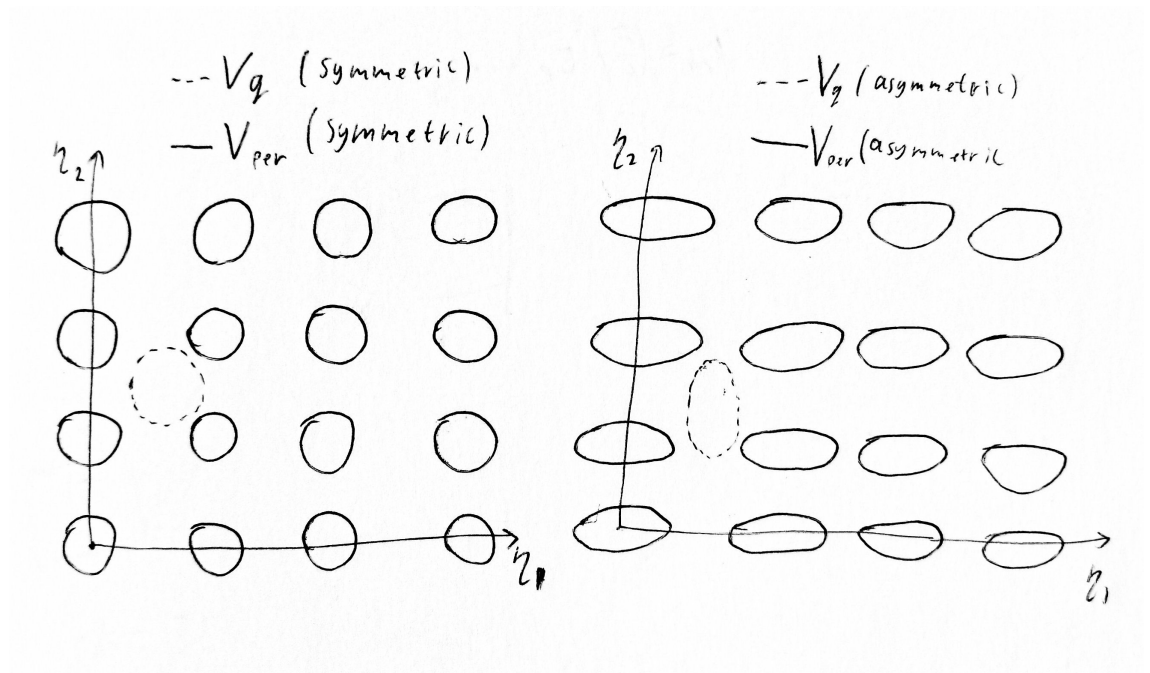
Shape of both, V_{per} and V_q changes if there is asymmetry between E_{J1} and E_{J2}; L_1 and L_2.
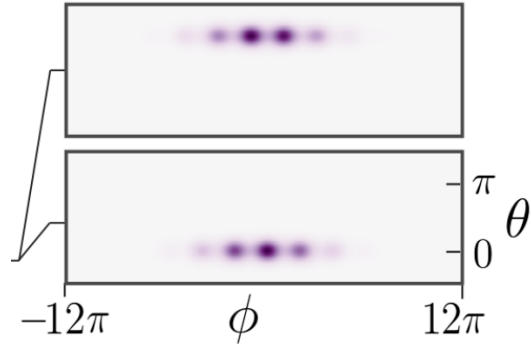
It is convenient to introduce a unit cell in a periodic potential as follows:
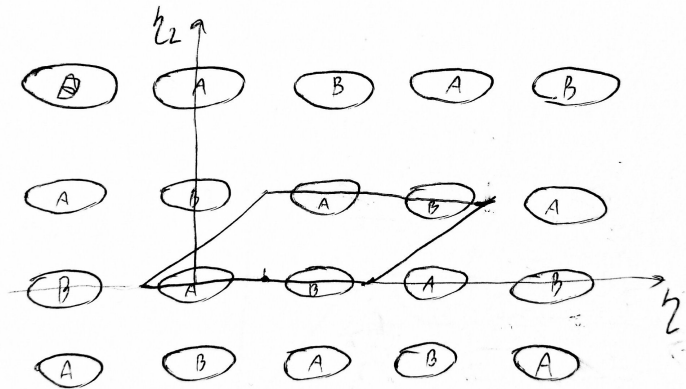
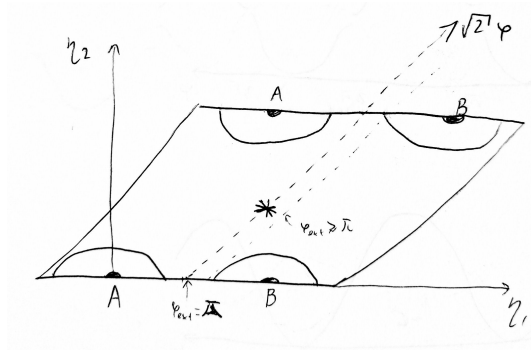

In the Northwestern university paper "Coherence properties of the 0-pi qubit", the potential (combination of periodic and quadratic ones) looks like

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and the two lowest-lying states confor to one or the other sublattice:

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Within the unit cell the center of a quadratic potential depends on φ and φ_{ext} as follows:

Here $*$-symbol represents the bottom of a quadratic potential. Looking at this figure one can construct an effective plaquette Hamiltonian

$$\mathcal{H}_p = \mathbb{A}\left(\varphi\right)\sigma^3 + \left\{\mathbb{B}\left(\varphi\right)\sigma^2 + \mathbb{C}\left(\varphi\right) + \mathbb{D}\sigma^3\right\}$$

This Hamiltonian is similar to a trivial phase of a Kitaev chain model. The σ's are the Pauli matrices acting on amplitudes of being in 'A'- or 'B'-sublattices.

Here $\mathbb{A}(\varphi) = E_p \sin\varphi$ with E_p being a constant;

Terms $\mathbb{B}, \mathbb{C}, \mathbb{D}$ are small and are caused by assymetries of a plaquette or by deviation of a flux through plaquette from $\Phi_0/2$:

$\mathbb{B} \approx t(1 + \alpha\cos\varphi)$, where t is the hopping integral between 'A'- and 'B'-sublattices. The term α appears if $E_{J1} \neq E_{J2}$ or $L_1 \neq L_2$ or $C_1 \neq C_2$.

Terms $\mathbb{C}$ and $\mathbb{D}$ appear due to deviation of the flux through a plaquette: $\mathbb{C} \propto \sin(\varphi_{ext} - \pi)\sin\varphi$; $\mathbb{D} \propto \sin(\varphi_{ext} - \pi)$.

It is also convenient to rewrite the effective Hamiltonian in a bit simpler manner

$$\mathcal{H}_p = -E_p \cos\left(\varphi + \frac{\pi}{2}\sigma^3\right) + \left\{\mathbb{B}\sigma^2 + \mathbb{C} + \mathbb{D}\sigma^3\right\}.$$

Let us see how these terms affect the ground and excited bands:
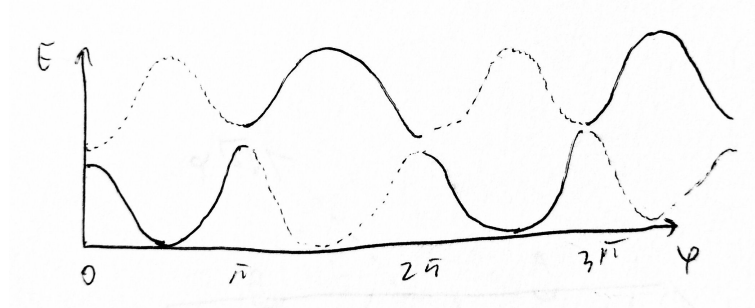
(I) Let us first consider the main term $\mathcal{H}_p = E_p \cos\left(\varphi + \frac{\pi}{2}\sigma^3\right)$. It is essentially a Josephson junction (JJ) Hamiltonian but with the phase being a hyperbolic (split-complex) number. The JJ Hamiltonian with a real variable phase looks like:
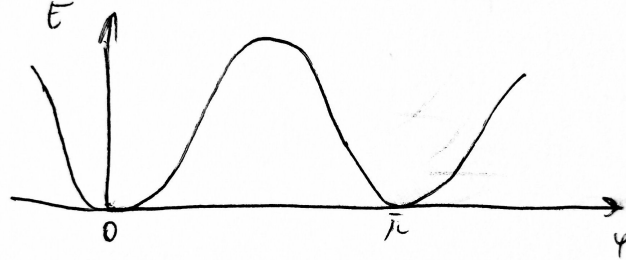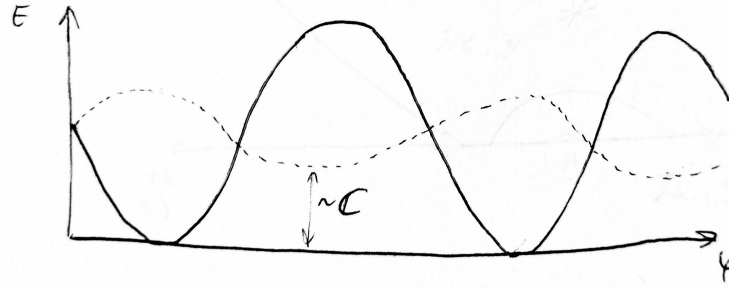

If a phase is a hyperbolic number $\hat{\varphi} = \varphi + \pi\sigma^3/2$, then the bands look like:
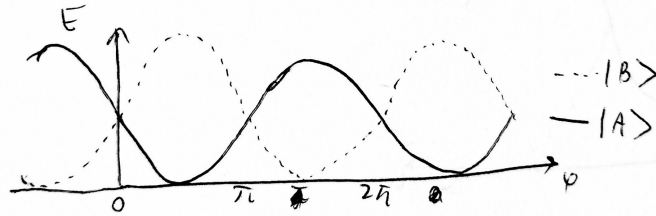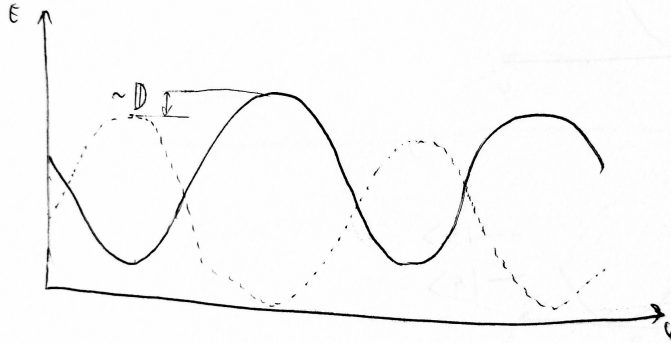

(II) Now let's introduce a $\mathbb{B}$-term, which opens a gap ($\mathcal{H}_p = -E_p \cos\left(\varphi + \frac{\pi}{2}\sigma^3\right) + \mathbb{B}\sigma^2$):

(III) The term $\mathbb{C}$ introduces an asymmetry between 0- and π-states ($\mathcal{H}_p = -E_p \cos\left(\varphi + \frac{\pi}{2}\sigma^3\right) + \mathbb{C}$):

The term $\mathbb{D}$ also introduces asymmetry between 0- and π-states, but in a different way ($\mathcal{H}_p = -E_p \cos\left(\varphi + \frac{\pi}{2}\sigma^3\right) + \mathbb{D}\sigma^3$):

For practical purposes, for example for calculating multiple plaquettes it is easy to caclulate parameters of the effective Hamiltonian $\mathcal{H}_p$ numerically and then use this effective Hamiltonian to calculate chains of plaquettes. It is supposed to be much easier than calculating Hamiltonians with many degrees of freedom and (quasi)periodic potentials.

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