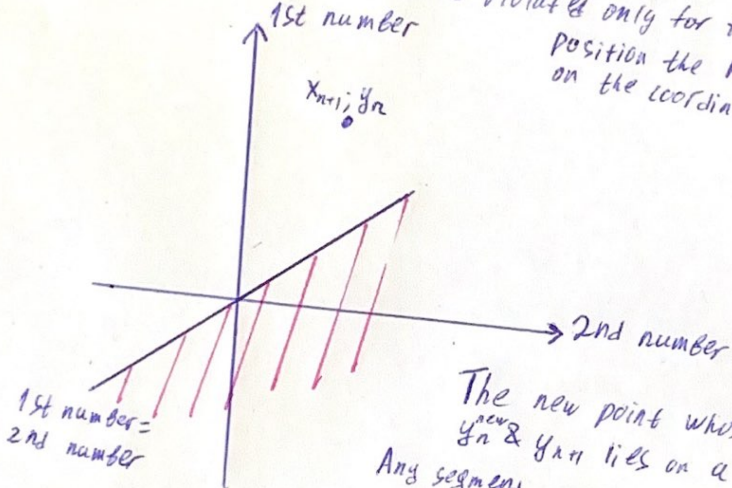
Description & proof:

- We have a vec. x, want to build a vec. y, where values in y are monotonically increasing and $d_{Eucl}(x, y)$ is minimised.
- Let's say we got some elements of y up to y_n (n can be 0). We want to append the next element.
- If x_{n+1} does not violate the monotonicity condition of the existing part of the y, then put $y_{n+1} = x_{n+1}$, additional distanse $= 0$.
- Else find the first element of y (let's say y_k) for which x_{n+1} violate the monotonicity condition. For each element ~~of~~ $y_k, -, y_n$ this condition is violated. We should change some values.

ex. for 2 values (condition is violated only for the last element of y):



Position the point with coordinates $(y_n; x_{n+1})$ on the coordinate plane.

The new point whose coordinates we use as a new y_n^{new} & y_{n+1} lies on a pink half-plane or dividing line. Any segment passing from a point (x_{n+1}, y_n) to the pink half-plane crosses the dividing line first. This means that the distanse from the $(x_{n+1}; y_n)$ to the this crossing point is less than the distanse to the point in the pink half-plane. $\Rightarrow$ the required „replacement" point lies on the line. $\Rightarrow y_n^{new} = y_{n+1}$

Now we want to find m: $\sqrt{(x_n - m)^2 + (x_{n+1} - m)^2} \to \min$. Take a derivative and equate it to 0. (derivative with respect to m):

$$\left(\sqrt{(x_n - m)^2 + (x_{n+1} - m)^2}\right)' = \frac{2m - x_n - x_{n+1}}{\sqrt{2m^2 - 2m(x_n + x_{n+1}) + x_n^2 + x_{n+1}^2}} = 0 \Rightarrow m = \frac{x_n + x_{n+1}}{2}$$

check that the denominator is not 0! $\frac{x_n^2 + 2x_n x_{n+1} + x_{n+1}^2}{2} - (\cancel{x_n^2} + 2x_n x_{n+1} + \cancel{x_{n+1}^2}) + \cancel{x_n^2} + \cancel{x_{n+1}^2} =$

$$= \frac{x_n^2 - 2x_n x_{n+1} + x_{n+1}^2}{2} = \frac{(x_n - x_{n+1})^2}{2}$$, eq. 0 only if $x_n = x_{n+1}$, where monotonicity cond. is not violated.

So, $y_n^{new} = y_{n+1} = \frac{x_n + x_{n+1}}{2}$. For cases of higher dimensions similarly.

The fact that this is the minimum can be verified by a simple substitution.