STAT 153: Practice problems for the final exam Solutions

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1 Problem 1

Short answer questions: Be brief.

(a) Define what it means for a time series to be weakly stationary.

(b) Why isn't the random walk without drift considered as a stationary time series?

(c) Define periodogram.

(d) What is the purpose of tapering the time series before constructing the periodogram?

(e) Explain the use of unit root testing.

(f) Explain briefly the idea of the parametric spectral estimation via AR processes.

Solutions:

(a) A time series (x_t) is weakly stationary if its mean $E(x_t)$ and its autocovariance function $\mathrm{Cov}(x_t, x_{t+h})$ do not depend on time.

(b) The variance of the random walk without drift increases over time. (Another acceptable reason: $\mathrm{Cov}(x_1, x_2) = \sigma_w^2 \neq 2\sigma_w^2 = \mathrm{Cov}(x_2, x_3)$)

(c) Periodogram is the squared magnitude of the discrete Fourier transform of the time series, which can be viewed as the sample spectral density:

$$I(\omega_j) = \left| \frac{1}{\sqrt{n}} \sum_{t=1}^{n} x_t e^{-2\pi i \omega_j t} \right|^2 \sum_{h=-(n-1)}^{n-1} \hat{\gamma}(h) e^{-2\pi i \omega_j h}$$

, where $\omega_j = j/n, j = 0, 1, ..., [[n/2]]$ are the fundamental frequencies.

(d) To mitigate the side lobes problem induced by the Fejér spectral window; to reduce the finite-sample bias of the periodogram.

(e) To test for nonstationarity induced by the presence of a unit root in the associated AR polynomial of a time series; to determine if differencing is necessary.

(f) Fit the best AR(p) model to the time series and use the spectral density of the fitted AR(p) as an estimate of the spectral density of the time series.

2 Problem 2

For each of the following ARMA models identify p, and q, determine whether they are causal, and determine whether they are invertible. In each case w_t is $WN(0, 1)$

(a) $x_t + 0.6x_{t-1} = w_t + 1.2w_{t-1}$

(b) $x_t - \frac{9}{4}x_{t-1} - \frac{9}{4}x_{t-2} = w_t - 3w_{t-1} + \frac{1}{9}w_{t-2} - \frac{1}{3}w_{t-3}$

(c) $x_t - x_{t-1} = w_t - \frac{1}{2}w_{t-1} - \frac{1}{2}w_{t-2}$

Solutions:

(a) The AR polynomial is $\phi(z) = 1 + 0.6z$ with root $-5/3$, which is outside of the unit circle, so the process is causal. The MA polynomial is $\theta(z) = 1 + 1.2z$ with a root $5/6$ which has absolute value less that one, so the process is not invertible.

(b) The AR polynomial is $\phi(z) = 1 - 9z/4 - 9z^2/4$ and the MA polynomial is $\theta(z) = 1 - 3z + z^2/9 - z^3/3$. We can factor out the common factor $1 - 3z$ to obtain the irredundant form $\phi(z) = 1 + 3z/4$ which has a root $-\frac{4}{3}$, and $\theta(z) = 1 + z^2/9$ with roots $\pm 3i$. Thus, this is an ARMA(1,2) process which is causal and invertible.

(c) The AR polynomial is $\phi(z) = 1 - z$, which has a root 1. The MA polynomial is $\theta(z) = 1 - z/2 - z^2/2$ which has roots -2 and 1. By factoring the common term $1 - z$ we obtain the irredundant form $\phi(z) = 1$ and $\theta(z) = 1 + z/2$. Thus this is an ARMA (0,1) process or, in other words an MA(1) process.

3 Problem 3

For an MA(1) process $x_t = w_t + \theta w_{t-1}$ show that $|\rho_x(1)| \leq 0.5$ for any θ. For which θ does $\rho_x(1)$ attain its maximum or minimum.

Solutions:

For MA(1) the ACF $\rho_x(1) = \frac{\theta}{1+\theta^2}$. Furthermore

$$\frac{d\rho_x(1)}{d\theta} = \frac{1 - \theta^2}{(1 + \theta^2)^2} = 0 \implies \theta = \pm 1$$

By the second derivative test the maximum occurs at $\theta = 1$, max $\rho_x(1) = 0.5$ and the minimum when $\theta = -1$ with min $\rho_x(1) = -0.5$. Thus $|\rho_x(1)| \leq 0.5$ for any θ.

4 Problem 4

Let x_t be a stationary time series defined by

$$x_t = 0.5x_{t-3} + w_t + 0.4w_{t-1}$$

where w_t is $\mathrm{WN}(0, 1)$.

(a) Express x_t in the form

$$x_t = \psi(B)w_t$$

where $\psi(B)$ is a rational function of the back-shift operator B. Specify the rational function ψ.

(b) Find the spectral density of x_t.

(c) Suppose that we pass the time series x_t through a linear filter, to obtain series y_t

$$y_t = \frac{1}{3}(x_{t-2} + x_{t-1} + x_t).$$

Write y_t in the form $y_t = \zeta(B)w_t$ for some rational function $\zeta(B)$ and find the spectral density of y_t.

Solutions:

(a) We can write $(1 - 0.5B^3)x_t = (1 + 0.4B)$. Thus

$$\psi(B) = \frac{1 + 0.4B}{1 - 0.5B^3}$$

(b)

$$f_{xx}(\omega) = \left| \frac{1 + 0.4e^{-2\pi i\omega}}{1 - 0.5e^{-6\pi i\omega}} \right|^2 = \frac{(1 + 0.4\cos(2\pi\omega))^2 + 0.4^2 \sin^2(2\pi\omega)}{(1 - 0.5\cos(6\pi\omega))^2 + 0.5^2 \sin^2(6\pi\omega)}.$$

(c)

$$y_t = \frac{1}{3}(1 + B + B^2)x_t = (\frac{1}{3}(1 + B + B^2)(1 + 0.4B)/(1 - 0.5B^3))w_t$$

$$\zeta(B) = \frac{1}{3} \frac{(1 + B + B^2)(1 + 0.4B)}{1 - 0.5B^3}$$

$$f_{yy}(\omega) = \frac{1}{9} \left| \frac{(1 + e^{-2\pi i\omega} + e^{-4\pi i\omega})(1 + 0.4e^{-2\pi i\omega})}{1 - 0.5e^{-6\pi i\omega}} \right|^2$$

$$= \frac{1}{9} \frac{[(1 + \cos(2\pi\omega) + \cos(4\pi\omega))^2 + (\sin(2\pi\omega) + \sin(4\pi\omega))^2][(1 + 0.4\cos(2\pi\omega))^2 + 0.4^2 \sin^2(2\pi\omega)]}{(1 - 0.5\cos(6\pi\omega))^2 + 0.5^2 \sin^2(6\pi\omega)}.$$

5 Problem 5

Let $\{x_t\}$ be a stationary time series with spectral density f_{xx}. Suppose that the time series $\{y_t\}$ is obtained by mixing a proportion α of this time series with a proportion $1 - \alpha$ of the time series delayed by k time steps $(0 \leq \alpha \leq 1)$

$$y_t = \alpha x_t + (1 - \alpha)x_{t-k}$$

Show that the spectral density of y_t is

$$f_{yy}(\omega) = (\alpha^2 + (1 - \alpha)^2 + 2\alpha(1 - \alpha)\cos(2\pi\omega k))f_{xx}(\omega).$$

Solutions:

$$\gamma_y(h) = \mathrm{Cov}(Y_t, Y_{t+h}) = \alpha^2\gamma_x(h) + \alpha(1 - \alpha)^2\gamma_x(h) + \alpha(1 - \alpha)\gamma(h - k) + \alpha(1 - \alpha)\gamma(h + k).$$

$$f_{yy}(\omega) = \sum_{h=-\infty}^{\infty} \gamma_y(h)e^{-2\pi i\omega h} = (\alpha^2 + (1 - \alpha)^2 + \alpha(1 - \alpha)(e^{-2\pi i\omega k} + e^{2\pi i\omega k}))f_{xx}(\omega)$$

$$= (\alpha^2 + (1 - \alpha)^2 + 2\alpha(1 - \alpha)\cos(2\pi\omega k))f_{xx}(\omega).$$

6 Problem 6

Consider the following model

$$x_t = w_t + 2.4w_{t-1} + 0.8w_{t-2}$$

where $\{w_t\}$ is white noise with $\sigma_w^2 = 1$.
(a) Is this process causal?
(b) Is it invertible?
(c) Calculate the autocovariance function.

Solutions:
(a) MA models are always causal.
(b) The magnitude of the roots of the MA polynomial $\theta(z) = 1 + 2.4z + 0.8z^2 = (1 + 0.4z)(1 + 2z)$ are $1/0.4$ and $1/2$. Since the latter is inside the unit circle, this MA model is not invertible.

(c) The autocovariance function is

$$\gamma(h) = \begin{cases} 1 + 2.4^2 + 0.8^2 & \text{if } h = 0 \\ 2.4(1+0.8) & \text{if } |h| = 1 \\ 0.8 & \text{if } |h| = 2 \\ 0 & \text{if } |h| \geq 3 \end{cases}. \tag{1}$$

7 Problem 7

Consider the process

$$y_t = w_t + 1.1y_{t-1} - 0.18y_{t-2}$$

where $\{w_t\}$ is white noise with $\sigma_w^2 = 1$.
(a) Is this process causal?
(b) Is it invertible?
(c) Calculate the autocovariance function. (For $j > 2$, it suffices to write down a recursive relationship.)

Solutions:

(a) The magnitude of roots of the AR polynomial $\phi(z) = 1 - 1.1z + 0.18z^2 = (1 - 0.9z)(1 - 0.2z)$ are 10/9 and 5. Since both of the roots lie outside the unit circle, the process is causal.
(b) AR models are always invertible.
(c) To find $\gamma(0), \gamma(1), \gamma(2)$, solve the following equations:

$$\begin{aligned} \gamma(0) &= 1 + (1.1^2 + 0.18^2)\gamma(0) + 2(1.1)(-0.18)\gamma(1) & (2) \\ \gamma(1) &= 1.1\gamma(0) - 0.18\gamma(1) & (3) \\ \gamma(2) &= 1.1\gamma(1) - 0.18\gamma(0) & (4) \end{aligned}$$

where the first one comes from taking variance of y_t and the latter two comes from Yule-Walker equations. Hence,

$$\gamma(h) = \begin{cases} 7.89 & \text{if } h = 0 \\ 7.35 & \text{if } |h| = 1 \\ 6.6648 & \text{if } |h| = 2 \\ 1.1\gamma(h-1) - 0.18\gamma(h-2) & \text{if } |h| \geq 3 \end{cases}. \tag{5}$$

8 Problem 8

Consider the following ARMA(p, q) processes, all of the form $\phi(B)x_t = \theta(B)w_t$, where (w_t) is $WN(0, \sigma_w^2)$. For each of the processes, determine if it is causal and/or invertible, and calculate its spectral density.
(a) $\phi(z) = 1 - 4z^2, \theta(z) = 1 - z + z^2$;
(b) $\phi(z) = 1 + \frac{3}{4}z, \theta(z) = 1 + \frac{1}{9}z^2$.

Solutions:

(a) The AR polynomial is $\phi(z) = 1 - 4z^2$, which has roots (zeros) $\pm\frac{1}{2}$ lying in the unit circle, so it is not causal. The MA polynomial is $\theta(z) = 1 - z + z^2$, which has roots (poles) $\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$, which has magnitude 1 and hence it is not invertible. The spectral density is

$$f_{xx}(\omega) = \left|\frac{1 - e^{-2\pi i\omega} + e^{-4\pi i\omega}}{1 - 4e^{-4\pi i\omega}}\right|^2 \sigma_w^2 = \frac{[1 - \cos(2\pi\omega) + \cos(4\pi\omega)]^2 + [\sin(2\pi\omega) - \sin(4\pi\omega)]^2}{[1 - 4\cos(4\pi\omega)]^2 + 16\sin^2(4\pi\omega)}\sigma_w^2.$$

(b) The AR polynomial is $\phi(z) = 1 + \frac{3z}{4}$, which has roots (zeros) $-\frac{4}{3}$ lying outside the unit circle, so it is causal. The MA polynomial is $\theta(z) = 1 + \frac{z^2}{9}$, which has roots (poles) $\pm 3i$ lying outside the unit circle, so it is invertible. The spectral density is

$$f_{xx}(\omega) = \left|\frac{1 + \frac{1}{9}e^{-4\pi i\omega}}{1 + \frac{3}{4}e^{-2\pi i\omega}}\right|^2 \sigma_w^2 = \frac{(1 + \frac{1}{9}\cos(4\pi\omega))^2 + \frac{1}{81}\sin^2(4\pi\omega)}{(1 + \frac{3}{4}\cos(2\pi\omega))^2 + \frac{9}{16}\sin^2(2\pi\omega)}\sigma_w^2.$$

9 Problem 9

Suppose that a certain time series Y_t has a quadratic trend component, a seasonal component, and a stationary component:

$$Y_t = \alpha_0 + \alpha_1 t + \alpha_2 t^2 + g(t) + X_t$$

where $\alpha_0, \alpha_1, \alpha_2$ are non-zero constants, $g(t)$ is a non-constant periodic function of t, with period 12.

(a) Show that Y_t is not stationary.
(b) Suggest linear transformations that could be applied to Y_t that would result in a stationary time series.
(c) Show that when you apply the linear transformations of part (b), the resulting time series is stationary. Express $f_{zz}(\omega)$, in terms of $\alpha_0, \alpha_1, \alpha_2$, $g(\cdot)$, and $f_{xx}(\omega)$.

Solutions:

(a) Since the function $\alpha_1 t + \alpha_2 t^2$ is not constant and not periodic, and $g(t)$ is periodic, we must have that the function $\alpha_1 t + \alpha_2 t^2 + g(t)$ is non-constant. Thus, $E(Y_t)$ depends on t, which implies Y_t is not stationary.
(b) To remove the periodic component, we could apply a seasonal difference with period 12; to remove the quadratic component, we could take second differences. Thus, we obtain the time series

$$Z_t = (1 - B)^2(1 - B^{12})Y_t$$

(c) To see that Z_t is stationary, first notice that $(1 - B^{12})$ removes the periodic component:

$$\begin{aligned}(1 - B^{12})Y_t &= \alpha_1(t - (t - 12)) + \alpha_2(t^2 - (t - 12)^2) + (g(t) - g(t - 12)) + (1 - B^{12})X_t \\ &= 12\alpha_1 + \alpha_2(24t - 144) + (1 - B^{12})X_t.\end{aligned} \tag{6}$$

Second, taking second differences removes the trend component:

$$Zt = (1 - B)^2(1 - B^{12})Y_t = (1 - B)(24\alpha_2(t - (t - 1)) + (1 - B)(1 - B^{12})X_t) = (1 - B)^2(1 - B^{12})X_t$$

Clearly, Z_t is obtained from the stationary time series X_t by taking linear combinations of time- shifted versions. Since these operations preserve stationarity, Z_t is stationary. From the expression above, the spectral density of Z_t can be written as

$$\begin{aligned}f_z(\omega) &= |1 - e^{-2\pi i\omega}|^4|1 - e^{-24\pi i\omega}|^2 f_{xx}(\omega) \\ &= [(1 - \cos(2\pi\omega))^2 + \sin^2(2\pi\omega)]^2[(1 - \cos(24\pi\omega))^2 + \sin^2(24\pi\omega)]f_{xx}(\omega).\end{aligned} \tag{7}$$

10 Problem 10

Suppose that we have 1500 observations $x_1, x_2, ..., x_{1500}$ of a time series, and we have chosen to estimate an AR(2) model,

$$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + w_t,$$

where $\{w_t\}$ is a white noise sequence with variance σ_w^2. We have computed the following sample autocovariances: $\hat{\gamma}(0) = 5.0$, $\hat{\gamma}(1) = 0.0$, $\hat{\gamma}(2) = 2.5$, $\hat{\gamma}(3) = 0.0$, $\hat{\gamma}(4) = 1.0$. Use Yule-Walker equations to estimate the coefficients $\phi(1)$, $\phi(2)$, and the noise variance σ_w^2 .

Solutions:
From Yule-Walker equations,

$$\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} \hat{\phi_1} \\ \hat{\phi_2} \end{bmatrix} = \begin{bmatrix} 0 \\ 2.5 \end{bmatrix}.$$

We have $\hat{\phi}(1) = 0$ and $\hat{\phi}(2) = 0.5$ and $\hat{\sigma_w}^2 = 5 - 1.25 = 3.75$