Using the ARMA ACVF

We are going to analyse the ACF and ACVF of an ARMA process, comparing theoretical calculations against numerical computations. But first, let us familiarize ourselves with coding up ARMA and ACFs. The following R code generates 300 data points from a particular stationary ARMA process:

```
set.seed(345)
Xt = arima.sim(model=list(ar = .7,ma=-.3),n = 300)
```

1. Use this code in R to generate the time series of interest. The `set.seed()` function will make sure that we all get the same series.

2. Plot the time series. Is there anything worth noting?

3. Plot the ACF. Is this white noise? If not, we will need something other than the dashed blue lines to judge this ACF plot.

ACVF of a Causal Stationary ARMA Process

In this problem, we will show that the ACVF of a causal stationary ARMA process is always absolutely summable, which in turn implies that the ACVF converges to zero as the lag goes to infinity. Let X_t be a causal stationary ARMA process:

$$X_t = \sum_{j=0}^{\infty} \psi_j W_{t-j}, \quad \sum_{j=0}^{\infty} |\psi_j| < \infty, \quad \psi_0 = 1, \tag{1}$$

where (w_t) is a zero-mean white noise sequence with variance σ^2.

For this problem, assume that interchanging infinite summations is a valid operation.

4. Compute the autocovariance function $\gamma(h)$.
5. Show that $\sum_{h=0}^{\infty} |\gamma(h)| < \infty$. Conclude that $|\gamma(h)| \to 0$.

ACF of ARMA(1,1)

We have seen that it is fairly straightforward to get the ACVF of an ARMA model if it is in $MA(\infty)$ form (see Equation 1). This is an immediate result for MA models (ARMA with $p = 0$), but is also not too complicated for ARMA models with $p > 0$: we just need to "find the ψ_j's". Here we look at ARMA(1,1).

6. Derive the ACVF of the generic ARMA(1,1) model $X_t = \phi X_{t-1} + W_t + \theta W_{t-1}$ where W_t is white noise. In lecture we did this by dividing polynomials, so here I encourage you to instead solve the difference equations.

7. Using this ACVF, what is $\rho_X(h)$, the theoretical ACF of $\{X_t\}$?

8. Now with the theoretical ACF in hand, let's return to the ACF plot of X_t from section 2. Use your derived ACF $\gamma_X(h)$ to add red points to the plot marking the actual ACF values of the generative process (i.e. use $\phi = 0.7, \theta = -0.3$) up to lag 20.

9. We can use the R function `ARMAacf()` to double check this derived ACF (and get values from theoretical ACFs if we don't want to derive it ourselves). Use this function to add a second set of points in blue to this plot using `ARMAacf()`. Do they match the points from your derived ACF?

10. Logically, as the sample size n grows, the sample ACF (black bars) should approach the theoretical values in red/blue. Try this: in the `arima.sim()` function, increase the size of n and rerun your script. Do we see the sample statistics converging to their theoretical expected values?

    And if the sample autocorrelations have an expected value, should they also have variances/covariances? (Yes! We'll discuss this next week.)