Review

Probability

Let X and Y be two continuous random variables with densities f and g, respectively, $X_1, X_2, ...$, and $Y_1, Y_2, ...$ be two sequences of continuous random variables, and a, b, c be real-valued scalars. Write down the definition or equivalent expressions of the following:

1. Expectation
   (a) $E(X)$
   (b) $E(aX + bY + c)$
   (c) $E(E(Y \mid X))$
   (d) $E(f(X)g(Y))$ if X and Y are independent
2. Variance
   (a) $\text{Var}(X)$
   (b) $\text{Var}(aX + bY + c)$
3. Covariance:
   (a) $\text{Cov}(X, Y)$
   (b) $\text{Cov}(\sum_{i=1}^{n} a_i X_i, \sum_{j=1}^{m} b_j Y_j)$
4. Correlation
   (a) $\text{Corr}(X, Y)$
   (b) Write down bounds for $\text{Corr}(X, Y)$
5. Autocorrelation between s and t:
6. Sample autocovariance of lag k:
7. Sample autocorrelation of lag k
8. Independence: X and Y are independent if and only if
9. Convergence in distribution: $X_n \xrightarrow{d} X$ if and only if
10. Convergence in probability: $X_n \xrightarrow{p} X$ if and only if

Statistics

Let θ be a parameter of interest and $\hat{\theta}$ be the corresponding estimator.

1. We say $\hat{\theta}$ is an **unbiased** estimator of θ if

2. The **mean-squared error (MSE)** of an estimator $\hat{\theta}$ with respect to θ is

3. We say $\hat{\theta}$ is a **consistent** estimator of θ if

4. A random interval $[A, B]$ is a $100(1-\alpha)\%$ **confidence interval** of θ if

5. A (sequence of) random interval(s) $[A_n, B_n]$ is a $100(1-\alpha)\%$ **asymptotic confidence interval** of θ if

Conceptual questions / Problems

1. Which one is random: sample autocorrelation or autocorrelation? Why?

2. The central limit theorem states that if $X_1, ..., X_n$ are independently identically distributed (i.i.d.) with $E(X) = \mu$ and $\mathrm{Var}(X) = \sigma^2 \in (0, \infty)$, then $\sqrt{n}(\bar{X} - \mu) \xrightarrow{d} N(0, \sigma^2)$, where $\bar{X}$ is the sample mean $n^{-1} \sum_{i=1}^{n} X_i$. Now suppose σ^2 is known.
   (a) Is $\bar{X}$ an unbiased estimator of μ?
   (b) Find $\mathrm{Var}(\bar{X})$.
   (c) Construct a 95% asymptotic confidence interval of μ.

3. Let $X_1, X_2, ...$ be a sequence of i.i.d. random variables with mean 0 and variance $\sigma^2 \in (0, \infty)$. Define $S_t = X_1 + ... + X_t$ for positive integer t.
   (a) Compute $E(S_t)$. Does it depend on t?
   (b) Compute $\mathrm{Var}(S_t)$. Does it depend on t?
   (c) Compute the autocovariance function of (S_t).
   (d) Compute the autocorrelation function of (S_t).