Trend and Seasonality

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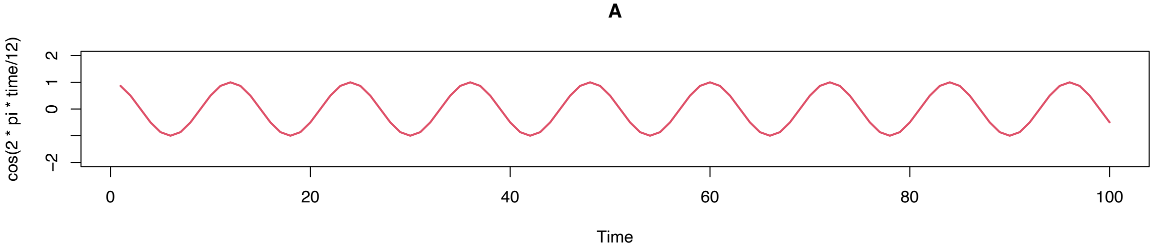
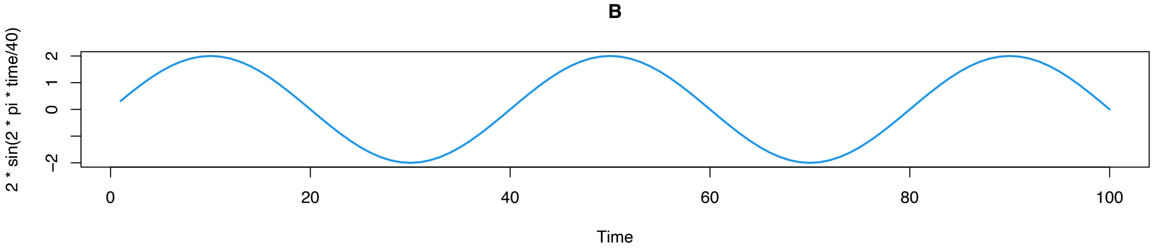
Week 4

Section 3

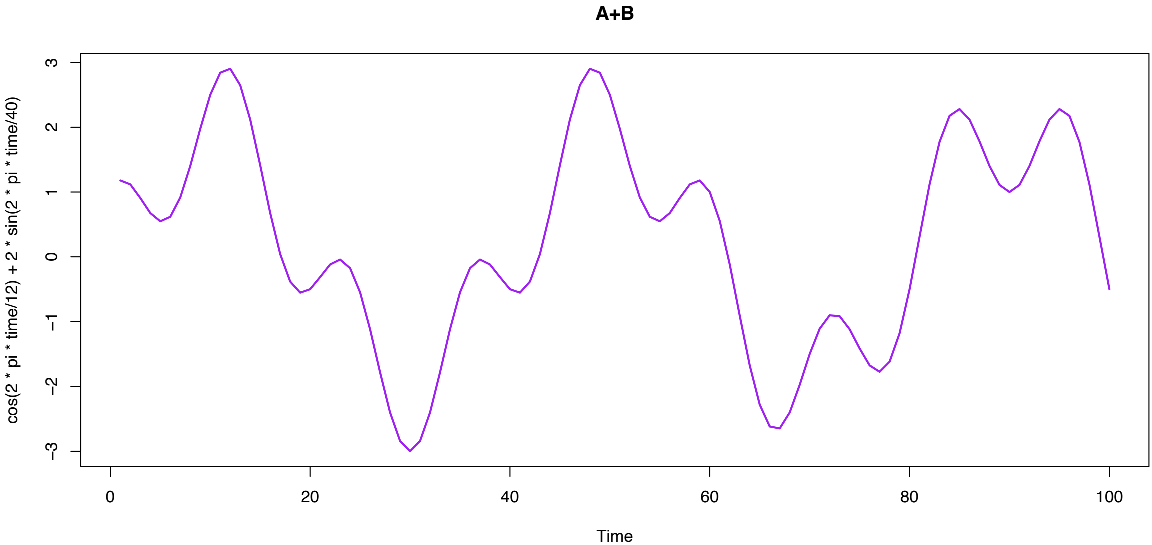
Frequency Domain

Example: Seasonality





Example: Seasonality



Recall: Parametric Seasonality Function

$$s_t = \sum_{k=1}^{K} \left(a_k \cos(2\pi tk/d) + b_k \sin(2\pi tk/d)\right)$$

- But how do we wisely choose the frequency k/d to include? In other words, do we need all k?
- What if there is no clear value of d?

Transition to Frequency domain

- This class is largely about the time domain approach: models constructed via the relationship of observations Y_t at different time points.
- Sometimes though, we will look at a time series as a composition of periodic components with different frequencies.
- This is quite natural for many time series data, which are often directly driven by periodic random events, like the purple curve in the "A+B" example a couple slides ago.

Definition: Sinusoids

We define the set of sinusoid functions as

$$\{g(t) = R\cos(2\pi ft + \Phi) \;:\; R \in R_+,\; f \in R_+,\; \Phi \in [0, 2\pi/f)\},$$

where

- R is called the *amplitude*
- f is called the *frequency*
- Φ is called the *phase*
- $1/f$ is called the *period*

Sinusoids rewritten a different way

- Estimating the phase shift Φ is nontrivial with the tools in this class, but we can rewrite the sinusoid equation to be more convenient.
- With $A = R\cos(\Phi)$ and $B = -R\sin(\Phi)$ one can rewrite sinuosoids as

$$\{g(t) = A\cos(2\pi ft) + B\sin(2\pi ft) \ : \ A, B \in R, \ f \in R_+\}.$$

- This is helpful as we can find the coefficients A and B with linear models, but that means we must find the appropriate frequencies f first. The frequency domain will help with this!

Introduction to Complex Roots via Example

- Consider this polynomial of interest: $1 - z + 0.5z^2$
- What are roots? i.e. set equal to 0 and solve for z:

$$0 = 1 - z + 0.5z^2$$

- Recall for $0 = az^2 + bz + c$, $z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- Plug in values:

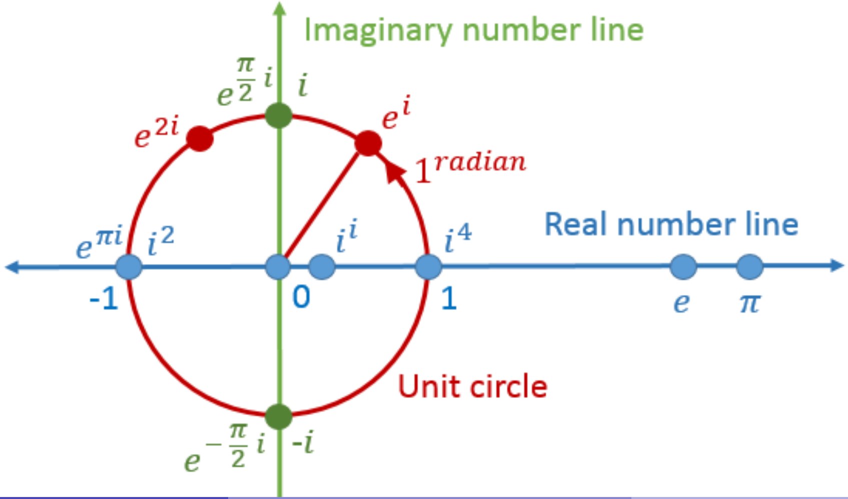
$$\frac{-(-1) \pm \sqrt{(-1)^2 - 4(0.5)(1)}}{2(0.5)} = \frac{1 \pm \sqrt{1-2}}{1} = 1 \pm \sqrt{-1}$$

- Thus the roots are $1 + i$ and $1 - i$

Brief Review of Complex Numbers

- Imaginary number: $i = \sqrt{-1}$
- Complex number: $z = a + bi$, where a,b are real valued
- $\bar{z} = a - bi$ is the complex conjugate of $z = a + bi$
- Euclidean distance: $d(a + bi) = \sqrt{a^2 + b^2}$
- We often ask if roots are within the unit circle, or $\sqrt{a^2 + b^2} \leq 1$

Complex Unit Circle



Complex Polar Coordinates

- $z = a + bi$
- $r = d(a + bi) = \sqrt{a^2 + b^2}$
- $a = r * cos(\theta)$, $b = r * sin(\theta)$
- Note Euler's equation: $e^{i\theta} = cos(\theta) + i * sin(\theta)$

$$\begin{aligned} z &= r * cos(\theta) + r * sin(\theta)i \\ &= r * e^{i\theta} \end{aligned}$$

Note

- We now define a transformation of data, which expresses the data in terms of its sinusoidal waves of different frequencies
- This will allow us to see which frequencies are prevalent in the time series

Definition: Discrete Fourier Transform

For data $x_0, \ldots, x_{n-1} \in C$ the discrete Fourier transform (DFT) is given by $b_0, \ldots, b_{n-1} \in C$, where

$$b_j = \sum_{t=0}^{n-1} x_t \exp\left(-\frac{2\pi i j t}{n}\right) \text{ for } j = 0, \ldots, n-1.$$

(In R, the DFT is calculated by the function *fft()*.)

- The frequencies j/n for $j = 0, \ldots, n-1$ as called **Fourier frequencies**.

Notes on DFT

- It always holds that $b_0 = \sum x_t$.
- When $x_0, \dots, x_{n-1} \in R$ are real numbers (in general, can be complex), then

$$b_{n-j} = \sum_t x_t \exp\left(-\frac{2\pi i(n-j)t}{n}\right)$$
$$= \sum_t x_t \exp\left(\frac{2\pi ijt}{n}\right) \exp\left(-2\pi it\right) = \bar{b}_j.$$

- For example, for $n = 11$, the DFT can be written as:

$$b_0, b_1, b_2, b_3, b_4, b_5, \bar{b}_5, \bar{b}_4, \bar{b}_3, \bar{b}_2, \bar{b}_1.$$

- For $n = 12$, it is

$$b_0, b_1, b_2, b_3, b_4, b_5, \bar{b}_6, \bar{b}_5, \bar{b}_4, \bar{b}_3, \bar{b}_2, \bar{b}_1.$$

Note that b_6 is necessarily real because $b_6 = \bar{b}_6$.

Note on DFT

- DFT $b_0, \dots, b_{n-1}$ is in one-to-one correspondence with the data $x_0, \dots, x_{n-1}$, because the original data can be uniquely recovered by its DFT, as the following theorem shows.
- $\Rightarrow$ the DFT $b_0, \dots, b_{n-1}$ and the data $x_0, \dots, x_{n-1}$ contain equivalent information.

Theorem: Inverse Fourier Transform (IDFT)

For data $x_0, \ldots, x_{n-1}$ and its DFT $b_0, \ldots, b_{n-1}$, it holds that

$$x_t = \frac{1}{n} \sum_{j=0}^{n-1} b_j \exp\left(\frac{2\pi i j t}{n}\right) \text{ for } t = 0, \ldots, n-1.$$

Proof (page 1)

- Start with the right hand side of IDFT

$$\frac{1}{n}\sum_{j=0}^{n-1} b_j \exp\left(\frac{2\pi ijt}{n}\right)$$

- Insert the DFT formula

$$= \frac{1}{n}\sum_{j=0}^{n-1}\left\{\sum_{s=0}^{n-1} x_s \exp\left(-\frac{2\pi ijs}{n}\right)\right\}\exp\left(\frac{2\pi ijt}{n}\right)$$

$$= \frac{1}{n}\sum_{s=0}^{n-1} x_s \sum_{j=0}^{n-1}\exp\left(\frac{2\pi ij(t-s)}{n}\right)$$

Proof (page 2)

- Note the inner sum equals n when $s = t$.

$$\sum_{j=0}^{n-1} \exp\left(\frac{2\pi i j(t-s)}{n}\right)$$

- Take out the j exponent

$$\sum_{j=0}^{n-1} \exp\left(\frac{2\pi i(t-s)}{n}\right)^{j}$$

- For $s \neq t$ we have that $\exp\left(\frac{2\pi i(t-s)}{n}\right) \neq 1$

Proof (page 3)

- Apply the finite geometric series formula to the inner sum

$$\begin{aligned}
&= \frac{1-\exp\left(\frac{2\pi i(t-s)}{n}\right)^n}{1-\exp\left(\frac{2\pi i(t-s)}{n}\right)} \\
&= \frac{1-\exp\left(2\pi i(t-s)\right)}{1-\exp\left(\frac{2\pi i(t-s)}{n}\right)} \\
&= \frac{1-1}{1-\exp\left(\frac{2\pi i(t-s)}{n}\right)} \\
&= 0.
\end{aligned}$$

as $\exp(ai\pi) = (-1)^a$ for integer a, and a is always even.

Aside: Sinusoids from DFT

To see why the DFT expresses the data in terms of its sinusoidal wave components, note that for $x = (x_0, \dots, x_{n-1})$ one can write

$$x = \frac{1}{n} \sum_{j=0}^{n-1} b_j \mathbf{u}_j.$$

with vectors

$$\mathbf{u}_j = (1, \exp(2\pi i j/n), \exp(2\pi i 2j/n), \dots, \exp(2\pi i (n-1)j/n))$$

for $j = 0, \dots, n-1$. That is, the sinusoid with frequency j/n evaluated at the time points $t = 0, 1, \dots, (n-1)$.

- the vectors $\mathbf{u}_j$ are an orthogonal basis: $(\mathbf{u}_l)^T \mathbf{u}_k = 0$ for $l \neq k$.

Real vs Complex

- Note that the DFT $b_0, \ldots, b_{n-1}$ of real valued data $x_0, \ldots, x_{n-1}$ can be complex valued.
- To visualize the DFT, one rather plots its absolute value.
- Note that b_0 is always just the sum of the data, which does not capture much information.
- Further because $b_{n-j} = \bar{b}_j$, it is enough to look at $|b_j|, 1 \leq j \leq n/2$.

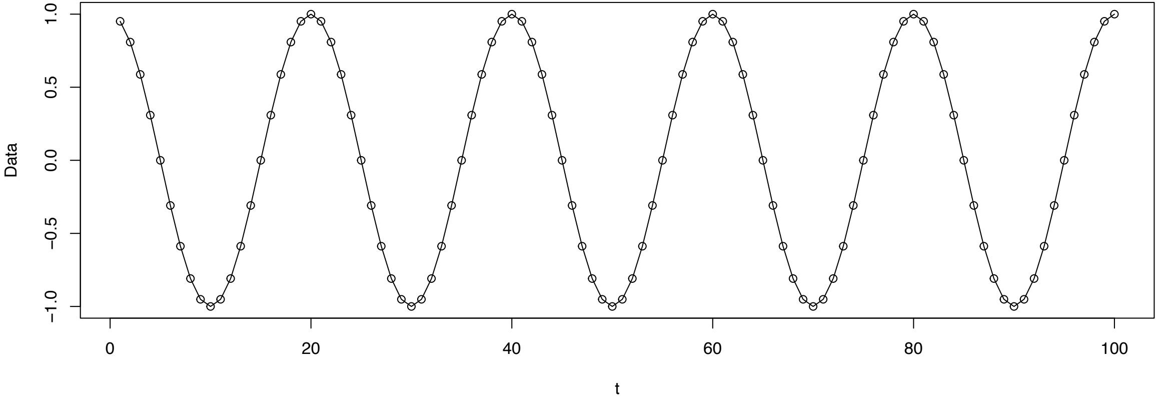
Definition: Periodogram

For real values data $x_0, \dots, x_{n-1}$ with DFT $b_0, \dots, b_{n-1}$ the **periodogram** is defined as

$$I(j/n) = \frac{|b_j|^2}{n} \quad \text{for } j = 1, \dots, \lfloor n/2 \rfloor$$

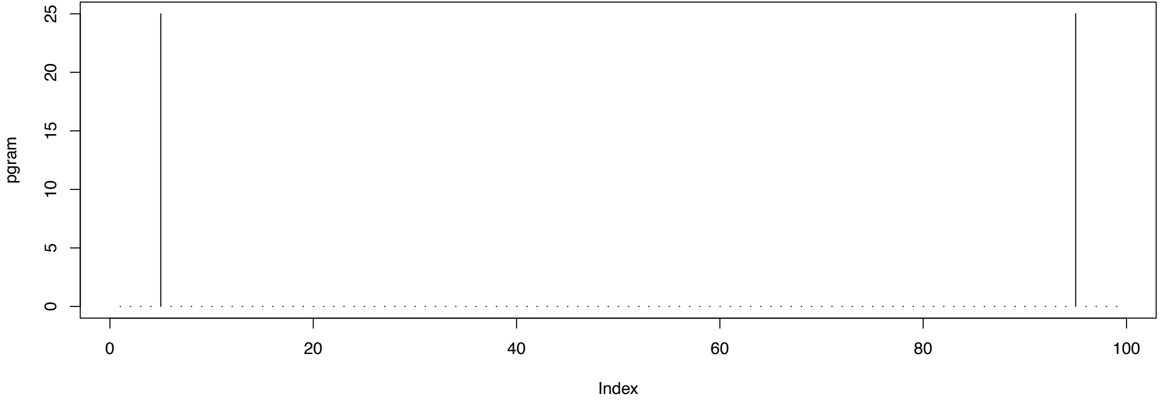
Example Data: $cos(2\pi t * 5/100)$

```
n=100; t = 1:n; cos2 = cos(2*pi*t*(5/n))
plot(t, cos2, ylab = "Data", type = "o")
```



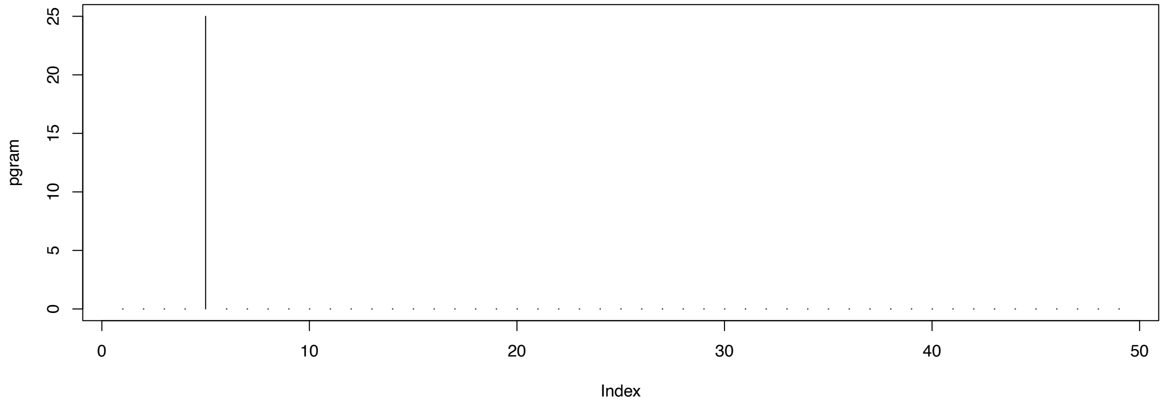
Example: $cos(2\pi t * 5/100)$

```
pgram = abs(fft(cos2)[2:100])^2/n
plot(pgram, type = "h")
```

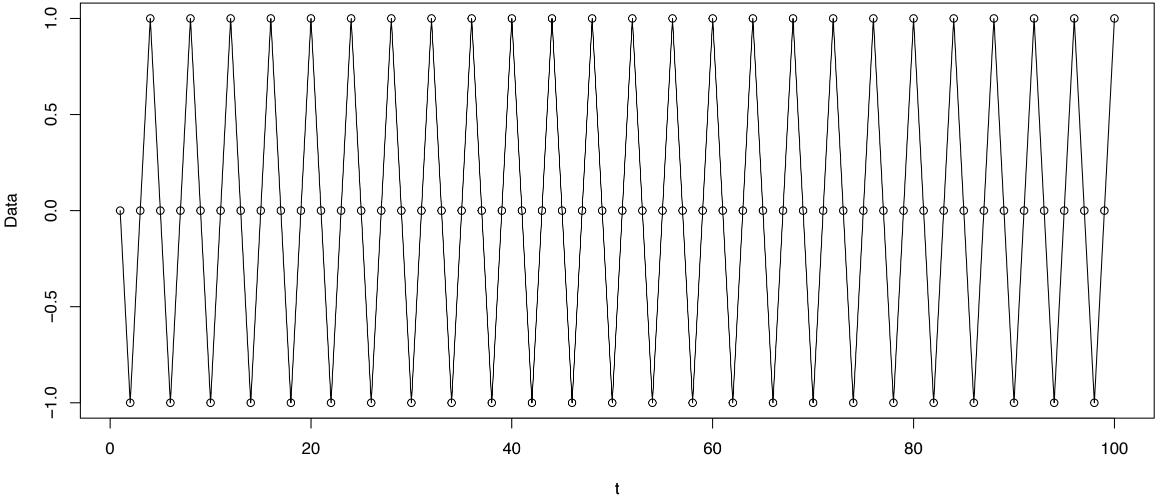


Example Periodogram: $cos(2\pi t * 5/100)$

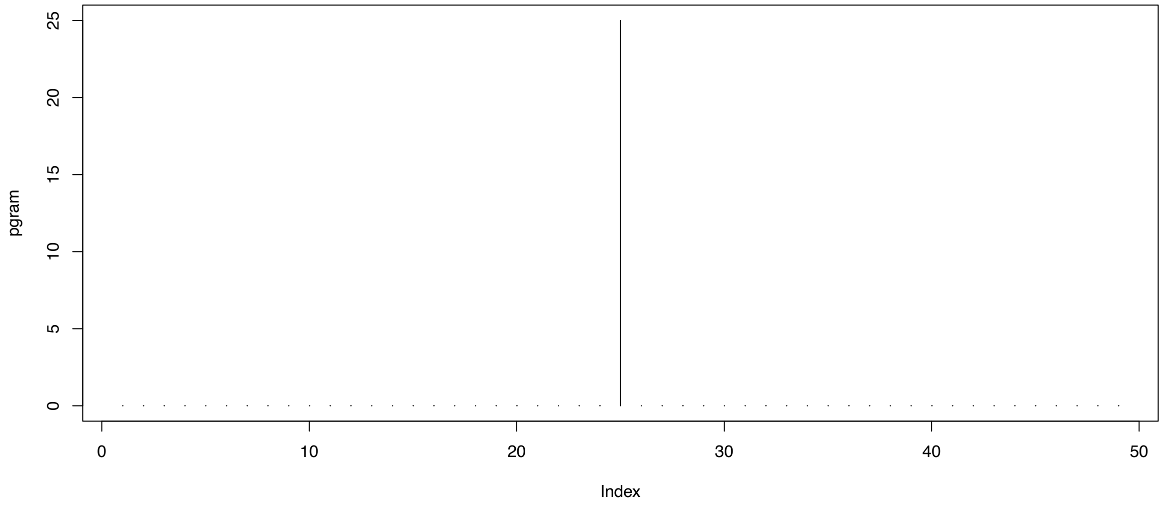
```
pgram = abs(fft(cos2)[2:50])^2/n
plot(pgram, type = "h")
```



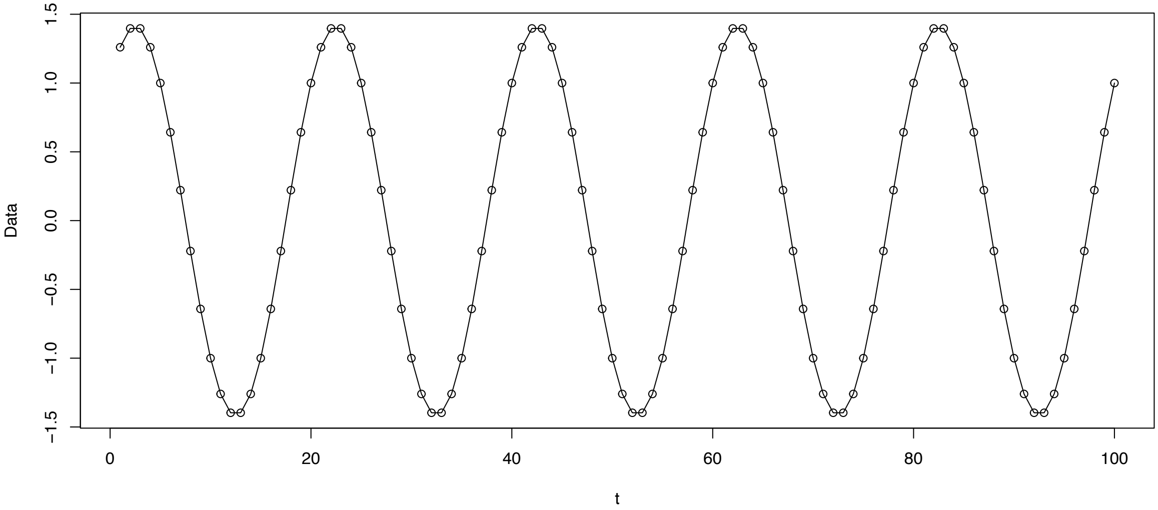
Example Data: $cos(2\pi t * 25/100)$



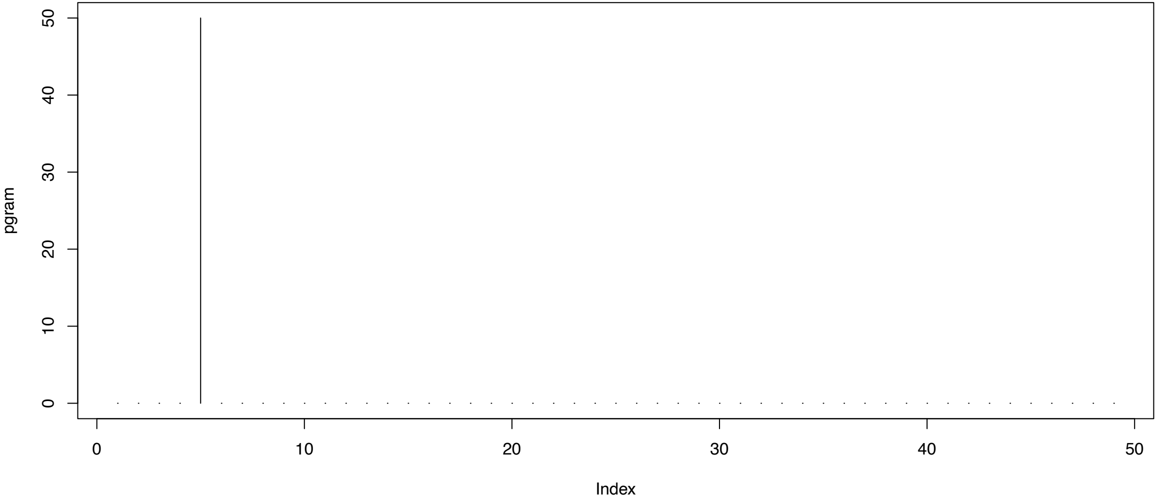
Example Periodogram: $cos(2\pi t * 25/100)$



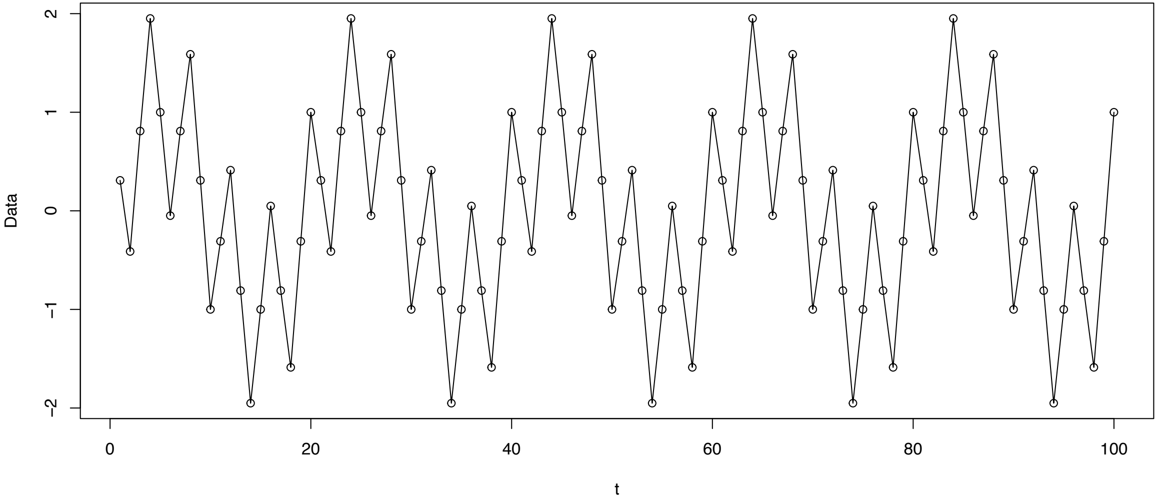
Example Data: $cos(2\pi t * 5/100) + sin(2\pi t * 5/100)$



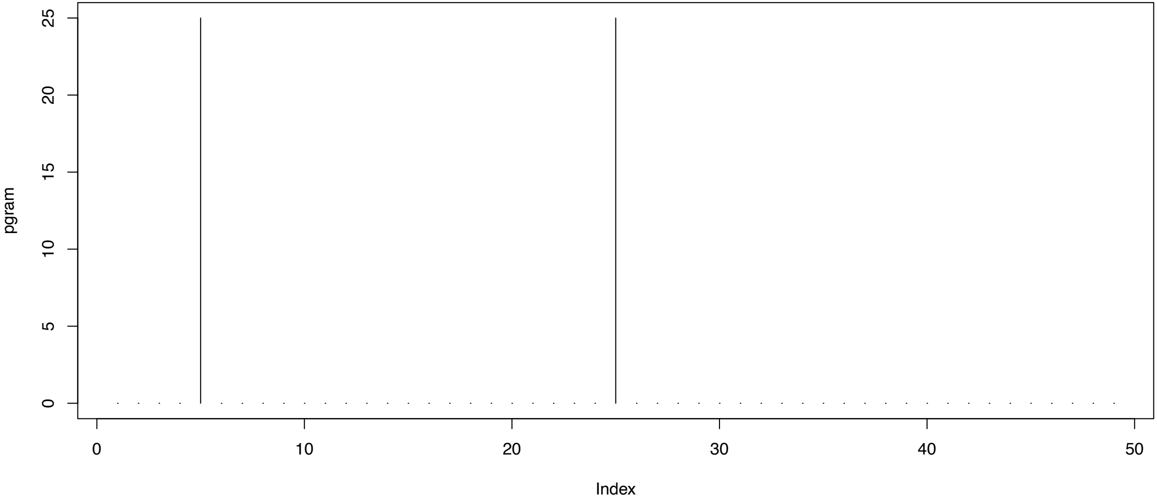
Example Periodogram: $cos(2\pi t * 5/100) + sin(2\pi t * 5/100)$



Example Data: $cos(2\pi t * 25/100) + sin(2\pi t * 5/100)$



Example Periodogram: $cos(2\pi t * 25/100) + sin(2\pi t * 5/100)$



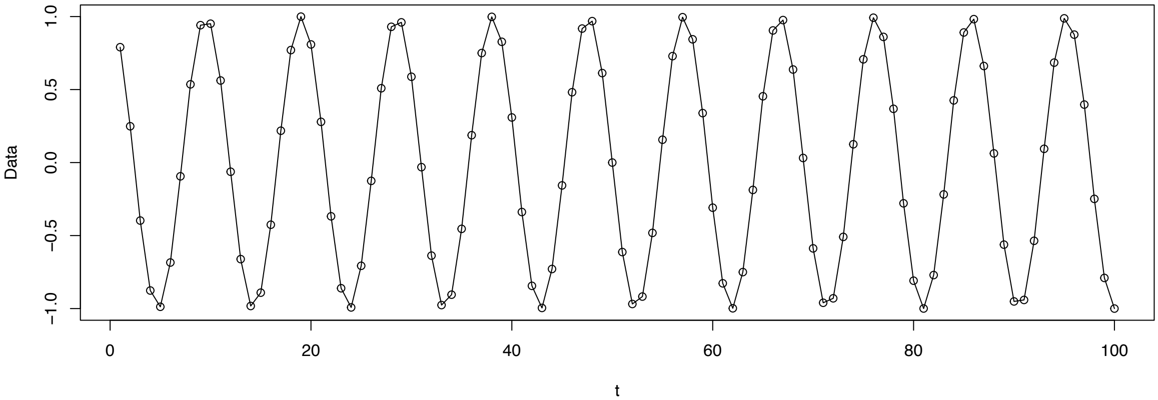
Notes on Periodogram

Recall b_j gives the jth coefficient of the data $x = (x_0, \dots, x_{n-1})$ in the basis $\mathbf{u}_0, \dots, \mathbf{u}_{n-1}$, which corresponds to the sinusoids of Fourier frequency j/n, thus:

1. If the periodogram shows a single spike for $I(j/n)$ we are sure that the data is a single sinusoid with Fourier frequency j/n.
2. If it shows two spikes, say at $I(j_1/n)$ and $I(j_2/n)$, then the data are a linear combination of two sinusoids at Fourier frequencies j_1/n and j_2/n with the strengths of these sinusoids depending on the size of the spikes.
3. Multiple spikes indicate that the data is made up of many sinusoids at Fourier frequencies.
4. Sometimes one can see multiple spikes in the DFT even when the structure of the data is not very complicated. A typical example is *leakage* due to the presence of a sinusoid at a non-Fourier frequency.

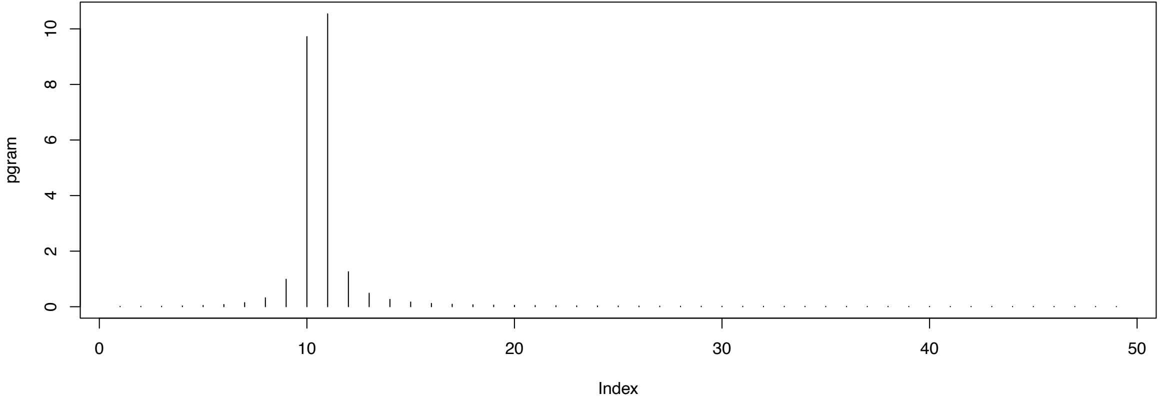
Example Data: $cos(2\pi t * 10.5/100)$

```r
t = 1:100; cos2 = cos(2*pi*t*(10.5/100))
plot(t, cos2, ylab = "Data", type = "o")
```



Example Periodogram: $cos(2\pi t * 10.5/100)$

```
pgram = abs(fft(cos2)[2:50])^2/n
plot(pgram, type = "h")
```



Theorem: Connection between periodogram and $\hat{\gamma}$

The following theorem shows an important relation between periodogram $I(j/n)$ and the sample ACVF $\hat{\gamma}(h)$ of some data $x_0, \ldots, x_{n-1}$.

For some data $x_0, \ldots, x_{n-1}$ let $\hat{\gamma}(h)$ for $h = 0, \ldots, n-1$ be its sample ACVF. Then

$$I(j/n) = \sum_{h=-(n-1)}^{n-1} \hat{\gamma}(h) \exp\left(-\frac{2\pi i j h}{n}\right) \text{ for } j = 1, \ldots, \lfloor n/2 \rfloor.$$

Proof (page 1; skipping in class)

First, by the formula for the sum of a geometric series, observe that

$$\sum_{t=0}^{n-1} \exp\left(-\frac{2\pi ijt}{n}\right) = 0 \text{ for } j = 1, \dots, \lfloor n/2 \rfloor.$$

In other words, if the data is constant i.e., $x_0 = \cdots = x_{n-1}$, then b_0 equals nx_0 and b_j equals 0 for all other j. Because of this, we can write:

$$b_j = \sum_{t=0}^{n-1} (x_t - \bar{x}) \exp\left(-\frac{2\pi ijt}{n}\right) \text{ for } j = 1, \dots, \lfloor n/2 \rfloor.$$

Proof (page 2; skipping in class)

Therefore, for $j = 1, \dots, \lfloor n/2 \rfloor$, we write

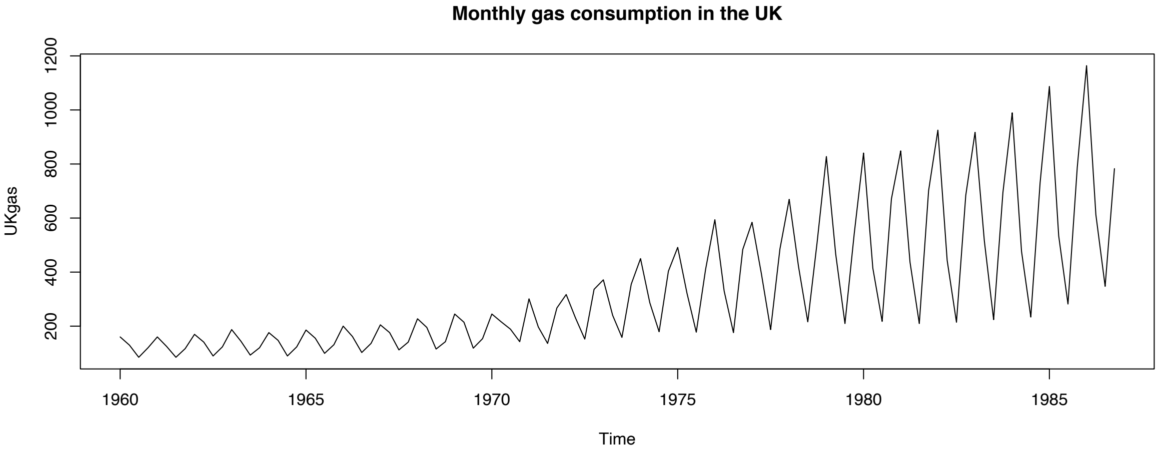
$$
\begin{aligned}
|b_j|^2 = b_j \bar{b}_j &= \sum_{t=0}^{n-1} \sum_{s=0}^{n-1} (x_t - \bar{x})(x_s - \bar{x}) \exp\left(-\frac{2\pi i j t}{n}\right) \exp\left(\frac{2\pi i j s}{n}\right) \\
&= \sum_{t=0}^{n-1} \sum_{s=0}^{n-1} (x_t - \bar{x})(x_s - \bar{x}) \exp\left(-\frac{2\pi i j (t-s)}{n}\right) \\
&= \sum_{h=-(n-1)}^{n-1} \sum_{t,s:t-s=h} (x_t - \bar{x})(x_{t-h} - \bar{x}) \exp\left(-\frac{2\pi i j h}{n}\right) \\
&= n \sum_{h=-(n-1)}^{n-1} \hat{\gamma}(h) \exp\left(-\frac{2\pi i j h}{n}\right).
\end{aligned}
$$

Section 4

Variance Stabilizing Transform

Example: UK Gas Consumption

```
plot(UKgas, main = "Monthly gas consumption in the UK")
```



Our Model

- The trend model, seasonal model, and trend+seasonal model have additive stationary process X_t.
- One implicitly assumes that the observations Y_t have a constant variance, called **homoscedasticity**
- Now suppose that the variability of the time series data set appears to be non-constant, which is **heteroscedasticity**

Heteroscedasticity

- Then, one can often transform the data with some function f and consider observations $f(Y_t)$ to obtain (approximate) homoscedasticity. This is denoted as a **Variance Stabilizing Transform**.
- To motivate the "VST", consider the situation where the variability of the data Y_t changes over time with its mean $E(Y_t) = \mu_t$.
- Specifically,
$$Var(Y_t) = g(\mu_t) \text{ for some function } g.$$
- In our model $\mu_t = m_t + s_t$

Variance Stabilizing Transform

Variance Stabilizing Transform: transform the data with some function f and consider observations $f(Y_t)$ to obtain (approximate) homoscedasticity.

Consider a first order Taylor approximation of $f(Y_t)$ around the mean μ_t

$$f(Y_t) \approx f(\mu_t) + f'(\mu_t)(Y_t - \mu_t),$$

such that

$$Var(f(Y_t)) \approx (f'(\mu_t))^2 Var(Y_t) = (f'(\mu_t))^2 g(\mu_t).$$

If we chose f such that the function $(f'(\cdot))^2 g(\cdot)$ is constant, then the variance of $f(Y_t)$ will be approximately constant over time and $f(Y_t)$ approximately homoscedastic.

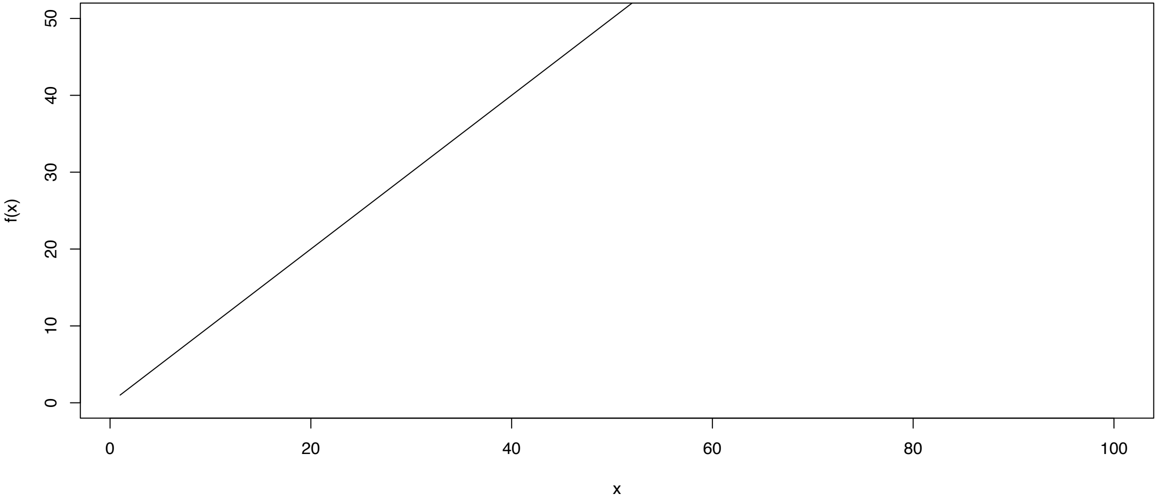
Examples

- When the variance increases *linear* with $Var(Y_t) = C\mu_t$, then for $f(x) = \sqrt{x}$ we find that $Var(\sqrt{Y_t}) \approx C/4$.
- (For example, count data are often modeled via Poisson Random variables, where the variance equals the mean.)
- When the variance increases *quadratic* with $Var(Y_t) = C\mu_t^2$, then for $f(x) = \log x$ we find that $Var(\log Y_t) \approx C$.
- The above examples are both special cases of the **Box-Cox transformation** with parameter λ, which considers the function

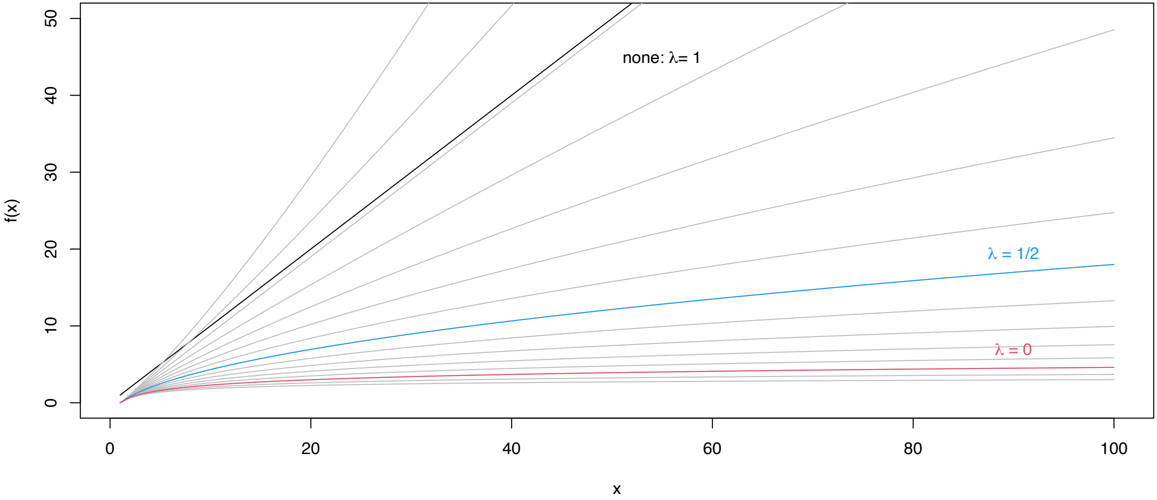
$$f(x) = f_\lambda(x) = \begin{cases} \frac{x^\lambda - 1}{\lambda} & \text{if } \lambda \neq 0 \\ \log(x) & \text{if } \lambda = 0, \end{cases}$$

where square root essentially corresponds to $\lambda = 1/2$.

No Variance Stabilizing Transform: $f(x) = x$



What the Box-Cox transformation looks like



Example: UK Gas Consumption