Exercise 1.13. Prove that Fib(n) is the closest integer to ${}^{\phi^n}\!/_{\sqrt{5}}$, where $\phi = {}^{(1+\sqrt{5})}\!/_2$. Hint: Let $\psi = {}^{(1-\sqrt{5})}\!/_2$. Use induction and the definition of the Fibonacci numbers (see section 1.2.2) to prove that $\mathrm{Fib}(n) = {}^{(\phi^n - \psi^n)}\!/_{\sqrt{5}}$.

Proof. As is indicated in the problem, in order to prove that Fib(n) is the closest integer to ${}^{\phi^n}\!/_{\sqrt{5}}$, we had better prove

Lemma. $\mathrm{Fib}(n) = {}^{(\phi^n - \psi^n)}\!/_{\sqrt{5}}$ *for all* n *greater than* 0.

first.

Well, let's go! Inspired by the hint given above, the lemma $\mathrm{Fib}(n) = {}^{(\phi^n - \psi^n)}\!/_{\sqrt{5}}$ can be prove through mathematical induction.

Base Case: $(n=0)$ Fib(0) is true because $\mathrm{Fib}(0) = 0 = {}^{(\phi^0 - \psi^0)}\!/_{\sqrt{5}}$, and $(n=1)$ Fib(1) is also true by $\mathrm{Fib}(1) = 1 = \dfrac{{}^{(1+\sqrt{5})}\!/_2 - {}^{(1-\sqrt{5})}\!/_2}{\sqrt{5}} = {}^{(\phi^1 - \psi^1)}\!/_{\sqrt{5}}$.

Inductive Step: Suppose that $n \geq 2$ and $(n=k)$ Fib(k) and $(n=k+1)$ Fib$(k+1)$ holds where $\mathrm{Fib}(k) = {}^{(\phi^k - \psi^k)}\!/_{\sqrt{5}}$ and $\mathrm{Fib}(k+1) = {}^{(\phi^{k+1} - \psi^{k+1})}\!/_{\sqrt{5}}$. We must show that $(n=k+2)$ Fib$(k+2)$ holds, namely, that $\mathrm{Fib}(k+2) = {}^{(\phi^{k+2} - \psi^{k+2})}\!/_{\sqrt{5}}$. Having been mentioned in the problem, we can use the defintion of the Fibonacci numbers

$$\phi^2 = \phi + 1 \text{ (or } \psi^2 = \psi + 1 \text{ identically)}$$

to help us to argue:

$$\begin{aligned}
\mathrm{Fib}(k+2) &= \mathrm{Fib}(k) + \mathrm{Fib}(k+1)\\
&= {}^{(\phi^k - \psi^k)}\!/_{\sqrt{5}} + {}^{(\phi^{k+1} - \psi^{k+1})}\!/_{\sqrt{5}}\\
&= \frac{\phi^k + \phi^{k+1} - (\psi^k + \psi^{k+1})}{\sqrt{5}}\\
&= \frac{\phi^k(1+\phi) - \psi^k(1+\psi)}{\sqrt{5}}\\
&= \frac{\phi^k\phi^2 - \psi^k\psi^2}{\sqrt{5}}\\
&= {}^{(\phi^{k+2} - \psi^{k+2})}\!/_{\sqrt{5}}
\end{aligned}$$

Therefore, $(n=k+2)$ Fib$(k+2)$ holds for every $n \geq 2$, which completes the proof of Lemma by strong induction that Fib(n) holds for all $n \geq 0$.

In order to approach to the final conclusion, notice that the value of ψ^n keeps oscillating between the negtive real number $\phi = {}^{(1-\sqrt{5})}\!/_2$ and the positive integer $\phi^0 = 1$ with its sign changing alternately as n varies. So, it becomes insignificant as n going larger and larger. Therefore, Fib(n) is the closest integer to ${}^{\phi^n}\!/_{\sqrt{5}}$. □