Exercise 2.13.

Show that under the assumption of small percentage tolerances there is a simple formula for the approximate percentage tolerance of the product of two intervals in terms of the tolerances of the factors. You may simplify the problem by assuming that all numbers are positive.

Answer.

Suppose that we are about to multiply two intervals $X = [c_X - t_X, c_X + t_X]$ and $Y = [c_Y - t_Y, c_Y + t_Y]$ where c_X and c_Y stand their center points and the tolerances is denoted by t_X and t_Y. We also assume that both of these intervals locate in the positive part of the number axis. With these, the percentage tolerances of X and Y can be expressed as:

$$\begin{aligned} p_X &= \frac{t_X}{c_X} \\ p_Y &= \frac{t_Y}{c_Y} \end{aligned}$$

By the definition of the multiplication of intervals, we have the product of X and Y:

$$\begin{aligned} X \times Y &= [(c_X - t_X) \times (c_Y - t_Y), (c_X + t_X) \times (c_Y + t_Y)] \\ &= [c_X \cdot c_Y - (c_X t_Y + c_Y t_X) + t_X t_Y, c_X \cdot c_Y + (c_X t_Y + c_Y t_X) + t_X t_Y] \end{aligned}$$

Notice that $t_X t_Y$ will be a wee number, hence, $X \times Y$ can be approximated by:

$$[c_X \cdot c_Y - (c_X t_Y + c_Y t_X), c_X \cdot c_Y + (c_X t_Y + c_Y t_X)]$$

This indicates the tolerance of $X \times Y$ is $c_X t_Y + c_Y t_X$. Therefore, the percentage tolerance it is:

$$\begin{aligned} p_{XY} &= \frac{c_X t_Y + c_Y t_X}{c_X \cdot c_Y} \\ &= \frac{t_X}{c_X} + \frac{t_Y}{c_Y} \\ &= p_X + p_Y \end{aligned}$$

So, it appears that the tolerance of the product of two intervals is approximately the sum of the tolerances of the factors.