Exercise 1.13.

Prove that Fib (n) is the closest integer to ${}^{\phi^n}\!/\!_{\sqrt{5}}$, where $\phi = {}^{(1+\sqrt{5})}\!/\!_2$. Hint: Let $\psi = {}^{(1-\sqrt{5})}\!/\!_2$. Use induction and the definition of the Fibonacci numbers (see section 1.2.2) to prove that Fib $(n) = {}^{(\phi^n - \psi^n)}\!/\!_{\sqrt{5}}$.

Proof.

As indicated by the problem, in order to prove that Fib (n) is the closest integer to ${}^{\phi^n}\!/\!_{\sqrt{5}}$, we had better prove

Lemma. Fib $(n) = {}^{(\phi^n - \psi^n)}\!/\!_{\sqrt{5}}$ *for all* n *greater than* 0.

first.

On the other hand, we see that the lemma Fib $(n) = {}^{(\phi^n - \psi^n)}\!/\!_{\sqrt{5}}$ can be prove by means of mathematical induction.

Base Cases:

- $(n=0)$ Fib $(n) = {}^{(\phi^n - \psi^n)}\!/\!_{\sqrt{5}}$ is true for $n=0$, because Fib $(0) = 0 = {}^{(\phi^0 - \psi^0)}\!/\!_{\sqrt{5}}$.
- $(n=1)$ Fib $(n) = {}^{(\phi^n - \psi^n)}\!/\!_{\sqrt{5}}$ also holds when $n=1$, for Fib $(1) = 1 = \frac{{}^{(1+\sqrt{5})}\!/\!_2 - {}^{(1-\sqrt{5})}\!/\!_2}{\sqrt{5}} = {}^{(\phi^1 - \psi^1)}\!/\!_{\sqrt{5}}$.

Inductive Step:

Suppose $n \geq 2$, this proposition stands true in the case $n = k$ as well as $n = k + 1$, that is, Fib $(k) = {}^{(\phi^k - \psi^k)}\!/\!_{\sqrt{5}}$ and Fib $(k+1) = {}^{(\phi^{k+1} - \psi^{k+1})}\!/\!_{\sqrt{5}}$. We must show that this proposition also holds in the case where $n = k+2$, namely, Fib $(k+2) = {}^{(\phi^{k+2} - \psi^{k+2})}\!/\!_{\sqrt{5}}$.

Using the equation

$$\phi^2 = \phi + 1 \text{ (or } \psi^2 = \psi + 1 \text{ identically)}$$

we can express the value of Fib $(k+2)$ in way akin to its predicessors':

$$\begin{aligned}
\mathrm{Fib}\,(k+2) &= \mathrm{Fib}\,(k) + \mathrm{Fib}(k+1) \\
&= {}^{(\phi^k - \psi^k)}\!/\!_{\sqrt{5}} + {}^{(\phi^{k+1} - \psi^{k+1})}\!/\!_{\sqrt{5}} \\
&= \frac{\phi^k + \phi^{k+1} - (\psi^k + \psi^{k+1})}{\sqrt{5}} \\
&= \frac{\phi^k\,(1+\phi) - \psi^k\,(1+\psi)}{\sqrt{5}} \\
&= \frac{\phi^k\,\phi^2 - \psi^k\,\psi^2}{\sqrt{5}} \\
&= {}^{(\phi^{k+2} - \psi^{k+2})}\!/\!_{\sqrt{5}}
\end{aligned}$$

Here we see that Fib $(n) = {}^{(\phi^n - \psi^n)}\!/\!_{\sqrt{5}}$ also holds when $n = k+2$. This further completes the proof of the Lemma above by strong induction that Fib $(n) = {}^{(\phi^n - \psi^n)}\!/\!_{\sqrt{5}}$ holds for all $n \geq 0$.

Finally, notice that the value of ψ^n keeps oscillating between the negtive number $\phi = {}^{(1-\sqrt{5})}\!/\!_2$ and the positive integer $\phi^0 = 1$ while the sign of it changing back and forth as n varies. So, the term ψ^n inside Fib $(n) = {}^{(\phi^n - \psi^n)}\!/\!_{\sqrt{5}}$ becomes less and less significant as n going larger and larger. Therefore, Fib (n) is the closest integer to ${}^{\phi^n}\!/\!_{\sqrt{5}}$. $\square$