Exercise 1.13. Prove that $\mathrm{Fib}(n)$ is the closest integer to ${}^{\phi^n}/_{\sqrt{5}}$, where $\phi = {}^{(1+\sqrt{5})}/_2$. Hint: Let $\psi = {}^{(1-\sqrt{5})}/_2$. Use induction and the definition of the Fibonacci numbers (see section 1.2.2) to prove that $\mathrm{Fib}(n) = {}^{(\phi^n-\psi^n)}/_{\sqrt{5}}$.

Proof. As is indicated in the problem, in order to prove that $\mathrm{Fib}(n)$ is the closest integer to ${}^{\phi^n}/_{\sqrt{5}}$, we had better prove

Lemma. $\mathrm{Fib}(n) = {}^{(\phi^n-\psi^n)}/_{\sqrt{5}}$ *for all* n *greater than* 0.

first.

Well, let's go! Inspired by the hint given above, the lemma $\mathrm{Fib}\ (n) = {}^{(\phi^n-\psi^n)}/_{\sqrt{5}}$ can be prove through mathematical induction.

Base Case: $(n = 0)$ Fib(0) is true because $\mathrm{Fib}\ (0) = 0 = {}^{(\phi^0-\psi^0)}/_{\sqrt{5}}$, and $(n = 1)$ Fib(1) is also true by $\mathrm{Fib}(1) = 1 = \dfrac{{}^{(1+\sqrt{5})}/_2 - {}^{(1-\sqrt{5})}/_2}{\sqrt{5}} = {}^{(\phi^1-\psi^1)}/_{\sqrt{5}}$.

Inductive Step: Suppose that $n \geq 2$ and $(n = k)$ Fib (k) and $(n = k + 1)$ Fib $(k + 1)$ holds where $\mathrm{Fib}(k) = {}^{(\phi^k-\psi^k)}/_{\sqrt{5}}$ and $\mathrm{Fib}(k+1) = {}^{(\phi^{k+1}-\psi^{k+1})}/_{\sqrt{5}}$. We must show that $(n = k+2)\,\mathrm{Fib}(k+2)$ holds, namely, that $\mathrm{Fib}\ (k + 2) = {}^{(\phi^{k+2}-\psi^{k+2})}/_{\sqrt{5}}$. Having been mentioned in the problem, we can use the defintion of the Fibonacci numbers

$$\phi^2 = \phi + 1\ (\text{or } \psi^2 = \psi + 1 \text{ identically})$$

to help us to argue:

$$\begin{aligned}
\mathrm{Fib}(k+2) &= \mathrm{Fib}(k) + \mathrm{Fib}(k+1) \\
&= {}^{(\phi^k-\psi^k)}/_{\sqrt{5}} + {}^{(\phi^{k+1}-\psi^{k+1})}/_{\sqrt{5}} \\
&= \frac{\phi^k + \phi^{k+1} - (\psi^k + \psi^{k+1})}{\sqrt{5}} \\
&= \frac{\phi^k\,(1+\phi) - \psi^k\,(1+\psi)}{\sqrt{5}} \\
&= \frac{\phi^k\,\phi^2 - \psi^k\,\psi^2}{\sqrt{5}} \\
&= {}^{(\phi^{k+2}-\psi^{k+2})}/_{\sqrt{5}}
\end{aligned}$$

Therefore, $(n = k+2)\,\mathrm{Fib}(k+2)$ holds for every $n \geq 2$, which completes the proof of Lemma by strong induction that $\mathrm{Fib}(n)$ holds for all $n \geq 0$.

In order to approach to the final conclusion, notice that the value of ψ^n keeps oscillating between the negtive real number $\phi = {}^{(1-\sqrt{5})}/_2$ and the positive integer $\phi^0 = 1$ with its sign changing alternately as n varies. So, it becomes insignificant as n going larger and larger. Therefore, $\mathrm{Fib}(n)$ is the closest integer to ${}^{\phi^n}/_{\sqrt{5}}$. $\square$