Exercise 1.12.

The following pattern of numbers is called *Pascal's triangle*.

```
        1
      1   1
    1   2   1
  1   3   3   1
1   4   6   4   1
```

The numbers at the edge of the triangle are all 1, and each number inside the triangle is the sum of the two numbers above it.[1] Write a procedure that computes elements of Pascal's triangle by means of a recursive process.

Answer.

The distribution of elements inside the triangle implies that the problem might become easier if we denote their position in an appropriate coordinates. Figure 1 shows a way to expressing Pascal's

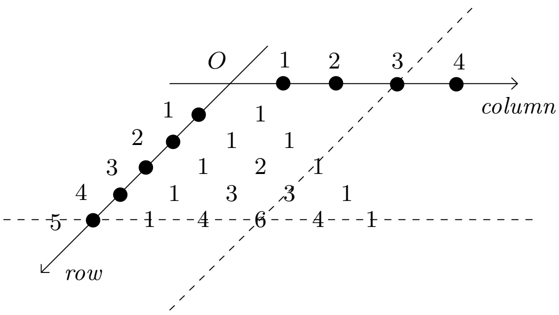

Figure 1. A way to expressing Pascal's triangle in the coordinates.

triangle in the coordinates, in which the horizonal axis denotes the column an element locates and the row of it can be labeled by an downward inclined axis. Hence, the element 6 inside the triangle above is appointed with the coordinate $(3, 5)$.

Obviously, elements which are not in Pascal's triangle are all 0, that is

- If the *column* coordinate of an element is smaller than 1, the value of it would be 0.

- If the *row* coordinate of an element is smaller than its *column* coordinate, the value of it would be 0.

To those that are the elements of the triangle, we observe that

- The edge of the triangle are those elements whose *column* coordinate are either equal to 1 or equal to its *row* coordinate.

- Any number inside the triangle with the coordinates (c, r) is the sum of the two elements above it, whose coordinates are $(c-1, r-1)$ and $(c, r-1)$ respectively.

We can easily translate this description into a recursive procedure:

```
(define (pascal-triangle row column)
  (cond ((or (< cloumn 1) (< row column)) 0)
        ((or (= column 1) (= row column)) 1)
        (else (+ (pascal-triangle (- row 1) (- column 1))
                 (pascal-triangle (- row 1) column)))))
```


1. The elements of Pascal's triangle are called the *binomial coefficients*, because the nth row consists of the coefficients of the terms in the expansion of $(x+y)^n$. This pattern for computing the coefficients appeared in Blaise Pascal's 1653 seminal work on probability theory, *Traité du triangle arithmétique*. According to Knuth (1973), the same pattern appears in the *Szu-yuen Yü-chien*("The Precious Mirror of the Four Elements"), published by the Chinese mathematician Chu Shih-chieh in 1303, in the works of the twelfth-century Persian poet and mathematician Omar Khayyam, and in the works of the twelfth-century Hindu mathematician Bháscara Áchárya.