Exercise 2.9.

The *width* of an interval is half of the difference between its upper and lower bounds. The width is a measure of the uncertainty of the number specified by the interval. For some arithmetic operations the width of the result of combining two intervals is a function only of the widths of the argument intervals, whereas for others the width of the combination is not a function of the widths of the argument intervals. Show that the width of the sum (or difference) of two intervals is a function only of the widths of the intervals being added (or subtracted). Give examples to show that this is not true for multiplication or division.

Answer.

We start our proof by postulating two intervals $X = [l_X, h_X]$ and $Y = [l_Y, h_Y]$. Obviously, the width of X and Y are:

$$\begin{aligned} w_X &= \frac{h_X - l_X}{2} \\ w_Y &= \frac{h_Y - l_Y}{2} \end{aligned}$$

The sum of interval X and Y can be expressed directly with the definition of addition:

$$X + Y = [l_X + l_Y, h_X + h_Y]$$

With this, we can work out the width of the resulting sum of these two intervals:

$$\begin{aligned} w_{X+Y} &= \frac{(h_X + h_Y) - (l_X + l_Y)}{2} \\ &= \frac{(h_X - l_X)}{2} + \frac{(h_Y - l_Y)}{2} \\ &= w_X + w_Y \end{aligned}$$

Similar to the addition, with the definition of subtraction:

$$X - Y = [l_X - h_Y, h_X - l_Y]$$

the width of the subtraction of X and Y can also be expressed in terms of their own width:

$$\begin{aligned} w_{X-Y} &= \frac{(h_X - l_Y) - (l_X - h_Y)}{2} \\ &= \frac{(h_X - l_X)}{2} + \frac{(h_Y - l_Y)}{2} \\ &= w_X + w_Y \end{aligned}$$

Therefore, the width of the sum and difference of two intervals are both functions only of the widths of the intervals being added.

However, this rule falls down when handles the multiplication and division. For example, using the definition of multiplication and division of intervals,

$$\begin{aligned} [1,1] \times [6,8] &= [6,8] \\ \frac{[4,12]}{[2,6]} &= \left[\frac{2}{3}, 6\right] \end{aligned}$$

which falsify the pridiction given by the rule: $[7,9]$ and $[6,18]$.