Exercise 1.10. The following procedure computes a mathematical function called Ackermann's function.

```
(define (A x y)
  (cond ((= y 0) 0)
        ((= x 0) (* 2 y))
        ((= y 1) 2)
        (else (A (- x 1)
                 (A x (- y 1))))))
```

What are the values of the following expressions?

```
(A 1 10)

(A 2 4)

(A 3 3)
```

Consider the following procedures, where `A` is the procedure defined above:

```
(define (f n) (A 0 n))

(define (g n) (A 1 n))

(define (h n) (A 2 n))

(define (k n) (* 5 n n))
```

Give concise mathematical definitions for the functions computed by the procedures `f`, `g`, and `h` for positive integer values of `n`. For example, `(k n)` computes $5\,n^2$.

Answer.

Having the procedure defined above, we can easily express Ackermann's function mathematically in choices as the following:

$$A\,(x,y) = \begin{cases} 0 & y=0 \\ 2\,y & x=0 \\ 2 & y=1 \\ A\,(x-1,A\,(x,y-1)) & \text{otherwise} \end{cases}$$

Therefore, as is shown in the following, we can evaluate the expressions `(A 1 10)`, `(A 2 4)` and `(A 3 3)` step by step,

$$\begin{aligned} A\,(1,10) &= A\,(0,A\,(1,9)) \\ &= 2\times A\,(1,9) \\ &= 2\times A\,(0,A\,(1,8)) \\ &= 2\times 2\times A\,(1,8) \\ &= 2^2\times A\,(1,8) \\ &\vdots\ \ \cdots \\ &= 2^9\times A\,(1,1) \\ &= 2^9\times 2 \\ &= 2^{10} \\ &= 1024 \end{aligned}$$

$$\begin{aligned} A\,(2,4) &= A\,(1,A(2,3)) \\ &= 2^{A\,(2,3)} \\ &= 2^{A\,(1,A\,(2,2))} \\ &= 2^{2^{A\,(2,2)}} \\ &= 2^{2^{2^{A\,(2,1)}}} \\ &= 2^{2^{2^2}} \\ &= 2^{2^4} \end{aligned}$$

$$
\begin{aligned}
&= 2^{16} \\
&= 65536
\end{aligned}
$$

$$
\begin{aligned}
A\,(3{,}3) &= A\,(2{,}A\,(3{,}2)) \\
&= A\,(2{,}A(2{,}A\,(3{,}1))) \\
&= A\,(2{,}A\,(2{,}2)) \\
&= A\,(2{,}A\,(1{,}A\,(2{,}1))) \\
&= A\,(2{,}A\,(1{,}2)) \\
&= A\,(2{,}A\,(0{,}A\,(1{,}1))) \\
&= A\,(2{,}A\,(0{,}2)) \\
&= A\,(2{,}2\times 2) \\
&= A\,(2{,}4) \\
&= 2^{2^4} \\
&= 2^{16} \\
&= 65536
\end{aligned}
$$

Notice that Ackermann's function $A\,(x{,}y) = 2\,y$ whenever $x = 0$, thus, we can easily put down the definition for functions computed by procedure `f` mathematically,

$$f\,(n) = 2\,n$$

Take a glance at the process generated by evaluating $A\,(1{,}10)$ above, the evolution of this expression revealed a general pattern for functions in terms of procedure `g`,

$$g\,(n) = 2^n$$

Similarly, we can derive the mathematical expression for functions generated by procedure `h` intuitively from the evaluating progression for $A\,(2{,}4)$,

$$h\,(n) = 2^{2^n}$$