Exercise 3.66.

Examine the stream `(pairs integers integers)`. Can you make any general comments about the order in which the pairs are placed into the stream? For example, about how many pairs precede the pair $(1,100)$? the pair $(99,100)$? the pair $(100,100)$? (If you can make precise mathematical statements here, all the better. But feel free to give more qualitative answers if you find yourself getting bogged down.)

Answer.

We denote an arbitrary upper triangular block in `(pairs S T)` by P_n, and consider it to be composed of three parts: the pair (S_n, T_n), the rest of the pairs in its first row (denoted by R_n), and the remaining pairs (denoted by another block P_{n+1}), as figure 1 shows.

Our problem, then, is to figure out the order in which (S_n, T_m) is placed into the stream. In other words, we have to calculate the amount of pairs in total preceding to (S_n, T_m) inside block P_0. Since blocks are nested one by another, we can pick up an arbitrary block P_n and calculate the amount of pairs preceding to (S_n, T_m) inside that block. Using this result, we can eventually figure out the amount of pairs in total preceding to (S_n, T_k) inside block P_0 by iteration.

In terms of block P_n, we will have to enumerate $m-n-1$ pairs: $(S_n, T_{n+1}), (S_n, T_{n+2}), \ldots, (S_n, T_{m-1})$ in stream R_n before (S_n, T_m) is reached. Besides, according to `interleave`, which takes elements alternately from R_n and P_{n+1}, an additional $m-n-1$ pairs inside the block P_{n+1} also need to be enumerated. Together with the pair (S_n, T_n), we will have to enumerate $2\,(m-n-1)+1$ pairs in total in block P_n, denoted by A_n.

Likewise, an additional A_n pairs need to be enumerated in stream R_{n-1}, which lies above P_n. So we have to enumerate $2\,A_n+1$ pairs in P_{n-1}, the block that embraces P_n, denoted by A_{n-1}. We can follow this pattern to compute the amount of enumetation iteratively until we extend the block to P_0:

$$\begin{aligned}
A_n &= 2\,(m-n-1)+1\\
A_{n-1} &= 2\,A_n+1\\
\ldots & \quad \ldots\\
A_0 &= 2\,A_1+1
\end{aligned}$$

Now we can calculate the value of A_0 by iteration:

$$\begin{aligned}
A_0 &= 2\,A_1+1\\
&= 2\,(2\,A_2+1)+1\\
&= 2^2\,A_2+2+1\\
&= 2^2\,(2\,A_3+1)+2+1\\
&= 2^3\,A_3+2^2+2+1\\
&= \ldots\\
&= 2^n\,A_n+2^{n-1}+2^{n-2}+\ldots+2+1\\
&= 2^n\,[2\,(m-n-1)+1]+\frac{1\times(1-2^n)}{1-2}\\
&= 2^{n+1}\,(m-n)-2^{n+1}+2^n-(1-2^n)\\
&= 2^{n+1}\,(m-n)-2^{n+1}+2^n-1+2^n\\
&= 2^{n+1}\,(m-n)-2^{n+1}+2\times2^n-1\\
&= 2^{n+1}\,(m-n)-2^{n+1}+2^{n+1}-1\\
&= 2^{n+1}\,(m-n)-1
\end{aligned}$$

So, we will have to enumerate an amount of $2^{n+1}\,(m-n)-1$ pairs before (S_n, T_m) is reached. In other words, the pair (S_n, T_m) is placed under the order of $2^{n+1}\,(m-n)$ in the stream.

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$$
\begin{array}{cccccccccc}
(S_0,T_0) & (S_0,T_1) & \dots & (S_0,T_{n-1}) & (S_0,T_n) & (S_0,T_{n+1}) & (S_0,T_{n+2}) & \dots & (S_0,T_m) & \dots \\
 & (S_1,T_1) & \dots & (S_1,T_{n-1}) & (S_1,T_n) & (S_1,T_{n+1}) & (S_0,T_{n+2}) & \dots & (S_1,T_m) & \dots \\
 & & \dots & (S_2,T_{n-1}) & (S_2,T_n) & (S_2,T_{n+1}) & (S_0,T_{n+2}) & \dots & (S_2,T_m) & \dots \\
 & & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & & & (S_{n-1},T_{n-1}) & (S_{n-1},T_n) & (S_{n-1},T_{n+1}) & (S_{n-1},T_{n+2}) & \dots & (S_{n-1},T_m) & \dots \\
 & & & & (S_n,T_n) & (S_n,T_{n+1}) & (S_n,T_{n+2}) & \dots & (S_n,T_m)\ R_n & \dots \\
 & & & & & (S_{n+1},T_{n+1}) & (S_{n+1},T_{n+2}) & \dots & (S_{n+1},T_m) & \dots \\
 & & & & & & (S_{n+2},T_{n+2}) & \dots & (S_{n+2},T_m) & \dots \\
 & & & & P_n & P_{n+1} & & \ddots & \dots & \dots
\end{array}
$$

Figure 1. The geometry interpretation of stream (`pairs S T`).