Ring-LWE: An Efficient PQC Public Key Encryption Scheme

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Outline

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Lattice Reduction

Lattices and Rings

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Basic Ring-LWE Encryption Scheme

Coding Theory

Fourier Transforms

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Lattices

Lattice Definition

A discrete additive subgroup of $\mathbb{R}^m$.

Let

$$B = \{b_1, \ldots, b_n\}$$

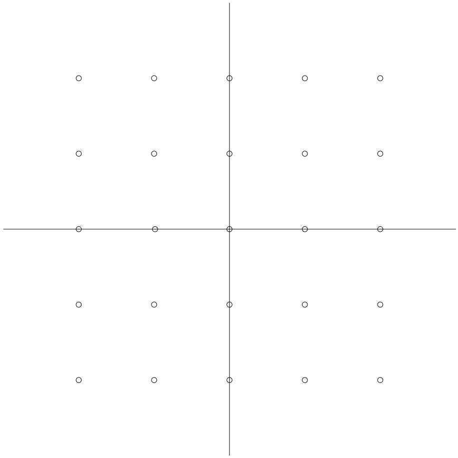
be an ordered set of vectors in $\mathbb{R}^m$. With this we generate a lattice

$$L_B = \left\{ \sum_{i=1}^{n} \lambda_i b_i | \lambda_i \in \mathbb{Z} \right\}.$$

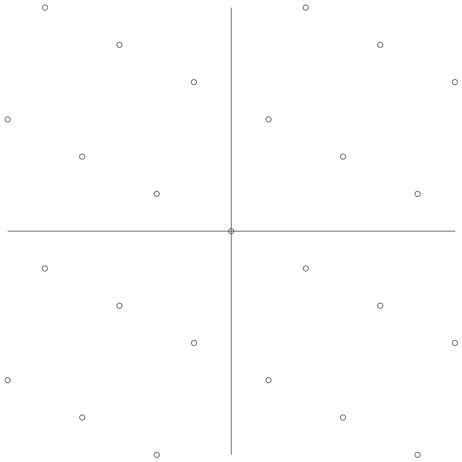
We shall also denote $(b_1, \ldots, b_n)^\top$ by B.

- B is called the basis matrix

Illustrative Example 1: $\mathbb{Z}^2$



Illustrative Example 2:



Lattice Bases

If B_1 and B_2 are bases of the same lattice if and only if

$$B_1 = UB_2$$

for some integer matrix U with determinant 1 or -1.
We define the fundamental parallelepiped associated to B as

$$P(B) = \left\{ \sum_{i=1}^{n} x_i b_i | x_i \in [0, 1) \right\}.$$

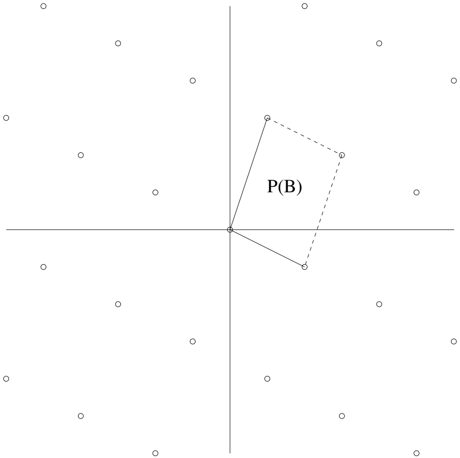
Roughly speaking

$$\{P(B) + v | v \in L_B\}$$

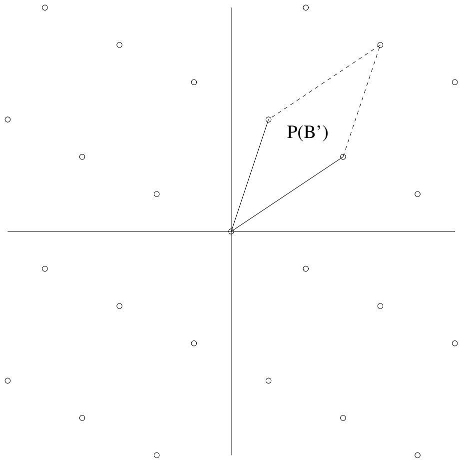
are the regions between the lattice points.

- They partition $\mathbb{R}^m$.

Fundamental Parallelepiped Illustration



Fundamental Parallelepiped Illustration 2



Lattice Determinant

We define a lattice determinant $\det(L_B)$ to be the volume of $P(B)$.

$$\det(L_B) = \mathrm{Vol}(P(B)) = \prod_i \|b_i^*\| = |\det(B^*)| \,.$$

If $m = n$, then

$$\det(L_B) = |\det(B^*)| = |\det(B)| \,.$$

More generally

$$\det(L_B) = \sqrt{|\det(B^\top B)|}.$$

The determinant is well-defined since bases differ by a factor of $\pm U$ where U is unimodular.

SVP and CVP

Given a lattice basis B.

The SVP problem is the problem of finding a vector $v \in L_B \setminus \{0\}$ such that $\|v\|_2$ is minimal.

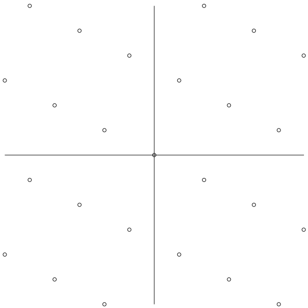
The CVP problem is the problem of given $u \notin L_B$ find a vector $v \in L_B$ such that $\|v - u\|_2$ is minimal

The are known to be hard problems.

- Suspected exponential complexity

Easy SVP Problem

If we can see the complete lattice, the SVP problem is trivial.



Easy SVP Problem

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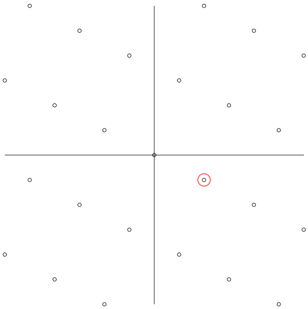
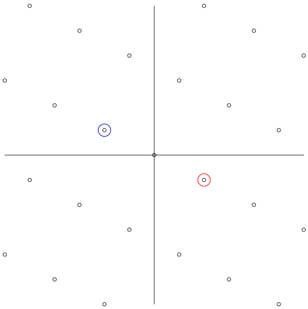


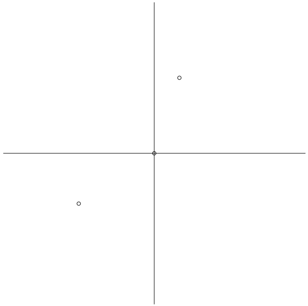
Non-Uniqueness

Note also that the shortest vector is not unique.

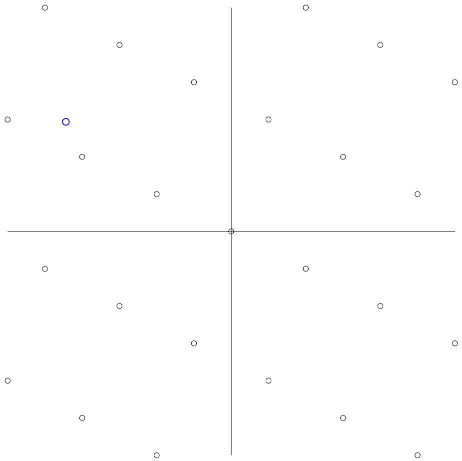


Slightly Harder SVP Problem

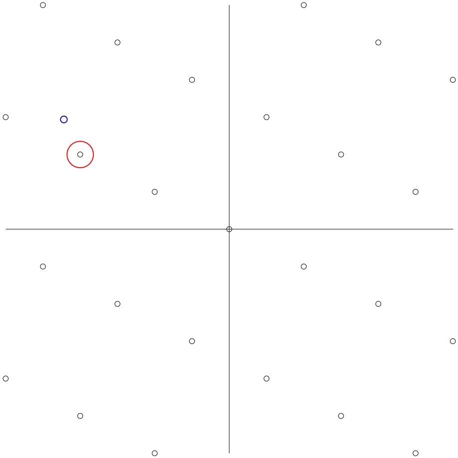
Given only a random basis, the SVP becomes harder.



CVP Illustration



CVP Illustration



Lattice Reduction

Gaussian Reduced Basis

Can often solve CVP/SVP using basis reduction

In two dimensions this is a classical algorithm

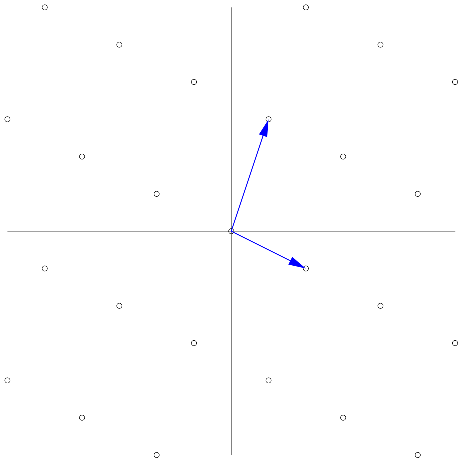
A basis $\{b_1, b_2\}$ is said to be Gaussian Reduced if

1. $|\mu_{1,2}| \leq \frac{1}{2}$
2. $\|b_1\| \leq \|b_2\|$.

where

$$\mu_{1,2} = \frac{\langle b_2, b_1 \rangle}{\langle b_1, b_1 \rangle}$$

Reduced Basis Illustration



Gaussian Reduction Algorithm

DO

1. IF $\|b_1\| > \|b_2\|$ THEN swap b_1 and b_2
2. $\mu_{1,2} = \frac{\langle b_2, b_1 \rangle}{\langle b_1, b_1 \rangle}$
3. $b_2 := b_2 - \lceil \mu_{1,2} \rfloor b_1$

WHILE $\|b_1\| > \|b_2\|$.

RETURN $(b_1, b_2)^\top$.

To obtain a Speed/Approximation Trade-off we introduce $1/2 < \delta < 1$ and replace the second condition.

1. $|\mu_{1,2}| \leq \frac{1}{2}$
2. $\delta \|b_1\| \leq \|b_2\|$.

Generalisation to *n* Dimensions

There are various ways we could attempt to generalise Gaussian Reduction to n dimensions.

Lovasz' solution, named after Lenstra, Lenstra and Lovasz, is the LLL algorithm.

It outputs a roughly orthogonal basis comprising of some short vectors.

It terminates in polynomial time in n.

An LLL Reduced Basis

1. $|\mu_{i,j}| \leq \frac{1}{2}$ for all $1 \leq j < i \leq n$.
2. $\delta \left\| b_i^* \right\|^2 \leq \left\| b_{i+1}^* + \mu_{i+1,i} b_i^* \right\|^2$ for $i = 1 \ldots n - 1$.

- $\mu_{i,j}$ are the standard Gram-Schmidt coefficients.
- We say B is *LLL reduced with respect to* δ, (or $\delta - LLL$ reduced) if both conditions are satisfied.

An LLL Reduced Basis

The first LLL condition produces an approximation to Gram-Schmidt.

- We are forced to approximate since our basis must span the same lattice.
- Makes the basis roughly orthogonal.

The second condition makes the basis vectors small, and roughly in increasing size

- So the first basis vector is an appromixation to the short-vector in the lattice

Lattices and Rings

Size Matters

A major issue in using lattice in cryptography is we seem to need to hold a lot of data.

After all a lattice basis requires storing $n \times n$ elements!

We want to be able to reduce this to n elements.

For this we use rings of polynomials

$$R = \mathbb{Z}[X]/(F(X)),$$

where $\deg(F) = n$.

Polynomial Rings

Clearly an element in a ring of polynomials will require *n* elements to define it

- One for each coefficient.

$$\mathbf{a} \longmapsto a(X).$$

This is the *vector representation* of the ring.

- We can clearly add vectors/polynomials.

Polynomial Rings

We can also, in a polynomial ring, multiply vectors/polynomials to get another polynomial of degree n

$$c = a \cdot b \pmod{F}.$$

We can think of this as vectors by looking at the *matrix representation* of the ring

$$\mathbf{c} = M_a \cdot \mathbf{b}.$$

i.e. M_a is a matrix (depending on a) which acts like multiplication by a on vectors.

Matrix Representation

So what does M_a represent?

All the vectors $\mathbf{c} = M_a \cdot \mathbf{b}$ as $\mathbf{b} \in \mathbb{Z}^n$ define a lattice

This is the lattice of the ideal of R generated by the polynomial a

Finding a good basis for the lattice generated by M_a is equivalent to finding a short generator of the ideal generated by a.

$$\begin{aligned} \langle a \rangle &= \{c(X) = a(X) \cdot b(X) \pmod{F(X)} : b(X) \in R\} \\ &\equiv \{\mathbf{c} = M_a \cdot \mathbf{b} : \mathbf{b} \in \mathbb{Z}^n\}. \end{aligned}$$

Standard and Ring LWE

Linear Algebra With Noise

LWE (Learning With Errors) is basically linear algebra with noise.

In usual linear algebra we try to solve the equation

$$\mathbf{y} = A \cdot \mathbf{x}$$

for some matrix A

In LWE we do not give you $\mathbf{y}$ but a vector with some errors in

$$\mathbf{y}' = A \cdot \mathbf{x} + \mathbf{e},$$

where $\mathbf{e}$ is "small" in some sense.

We can think of this as like a decoding problem for the linear code defined by the matrix A.

Or equivalently this is a CVP problem, as $\mathbf{e}$ is small.

Linear Algebra With Noise

However, using the above definition in crypto is a bit awkward as numbers are unbounded

- All vectors are in $\mathbb{Z}^n$
- We would like to work modulo an integer q, to bound the sizes

So we define our problem as

$$\mathbf{y} = A \cdot \mathbf{x} + \mathbf{e} \pmod{q},$$

i.e. we are given $\mathbf{y} \in \mathbb{Z}_q^n$ and are asked to find $\mathbf{x}$, given $\mathbf{e} \in \mathbb{Z}_q^n$ has "small coefficients"

- Small means when reduced into the interval $(-q/2, q/2)$.

Linear Algebra With Noise

This is actually another lattice problem, for the non square generating matrix $(A \mid q \cdot I_n)$...

$$\mathbf{y} = (A \mid q \cdot I_n) \cdot \mathbf{x} + \mathbf{e}.$$

We can also define a Ring version of LWE (called Ring-LWE)

Given a polynomial a we are asked to solve

$$y = (a \cdot x + e \pmod{F}) \pmod{q}$$

for a polynomial e with small coefficients.

The underlying ring is

$$R_q = \mathbb{Z}_q[X]/(F(X)).$$

C'Mon Feel The Noise

So how do we pick the small noise vector/polynomial?

There are various ways we *could* do this

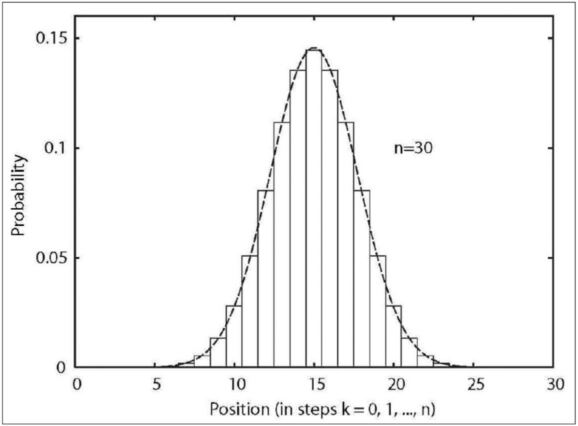
- Sample unformly from $[-b, \ldots, b]$ for a small value of b.
- Sample from $[-b, \ldots, b]$ using a non-uniform distribution.
- Sample (in the ring case) from the "complex embedding" in some way and then "pull back"

Each method has its own advantages/disadvantages

- Ease of implementation.
- Ease of analysis of resulting bounds.
- Worst-case/Average-case results (mainly for theory wonks!).

Gaussian Noise

For simplicity we will use sampling of an approximation to the discrete Gaussian...



Sampling Made Easy

Take a normal Gaussian of standard deviation σ.

The probability of being more than $6 \cdot \sigma$ away from the mean is so small we can ignore it.

So we create a table which approximates the CDF from $-6 \cdot \sigma$ upto $6 \cdot \sigma$

1. Let B be the smallest power of two which is larger than $6 \cdot \sigma$.
2. Initialize DGtable as a table indexed by 0 to $2 \cdot B - 1$.
3. For i from $-B$ to $B - 1$ do
   3.1 $x \leftarrow (x + .5)/(\sigma \cdot \sqrt{2})$.
   3.2 $v \leftarrow (1 + \mathrm{erf}(x))/2$.
   3.3 $j \leftarrow \lceil 2^{32} \cdot v \rfloor$.
   3.4 $\mathrm{DGtable}[i + B] = j$.
4. Return $[B, \mathrm{DGtable}]$.

Sampling Made Easy

To sample we then just pull a uniform random number in the range $[0, \ldots, 2^{32}]$ and do a binary search

1. $x \leftarrow \text{rand}(2^{32})$.
2. $l \leftarrow 0, u \leftarrow 2 \cdot B$.
3. While $(u - l) > 1$ do
   3.1 $m \leftarrow \lfloor (l + u)/2 \rfloor$.
   3.2 If $x > \text{DGtable}[m]$ then $l = w$, else $u = m$.
4. Return $(u - B)$.

The resulting distribution we call χ_σ.

Basic Ring-LWE Encryption Scheme

Key Generation

A public key consists of a Ring-LWE instance (a, b), a private key is the underlying secret polynomial s such that

$$b = a \cdot s + e',$$

where e' is "small"

- We pick however s also to be "small"

So key generation goes as follows:

1. $a \leftarrow R_q$.
2. $s, e' \leftarrow \chi_\sigma^n$.
3. $b = a \cdot s + e'$ in R_q.

Encryption

This works by randomizing the LWE instance given by the public key and then embedding the message $\mu \in \{0,1\}^n$

1. $v, e, d \leftarrow \chi_\sigma^n$.
2. $c_0 = b \cdot v + d + \Delta_q \cdot \mu$.
3. $c_1 = a \cdot v + e$

where

$$\Delta_q = \left\lfloor \frac{q}{2} \right\rfloor .$$

Note

$$\begin{aligned} c_0 - s \cdot c_1 &= (b \cdot v + d + \Delta_q \cdot \mu) - s \cdot (a \cdot v + e) \\ &= ((a \cdot s + e') \cdot v + d + \Delta_q \cdot \mu) - s \cdot (a \cdot v + e) \\ &= \Delta_q \cdot \mu + e' \cdot v + d - e \cdot s \\ &= \Delta_q \cdot \mu + \text{“small”}. \end{aligned}$$

Decryption

Since we have

$$c_0 - s \cdot c_1 = \Delta_q \cdot \mu + \text{“small”}.$$

we decrypt as follows:

1. $f = c_0 - s \cdot c_1$.
2. $\mu = \left\lfloor \left\lceil \frac{2}{q} \cdot f \right\rfloor \right\rfloor$.

BUT this only works if the “small” is small enough.

- We need to fix the parameters so that small is small enough

Three Problems

If we fix the parameters so that “small” is guaranteed to be **always** small enough we get huge parameters

- Like those seen in FHE schemes!

So we need to apply some coding theory to make the parameters smaller.

We do lots of operations like $a \cdot b \pmod{F}$ which are costly, $O(n^2)$.

- Would prefer $O(n)$

If we pick F and q well we can use FFT techniques

The above scheme is not CCA secure it is only CPA secure

- Need a way of transforming to get a CCA secure scheme

We now address these three problems.

Coding Theory

Coding Theory

We could just apply some complex BCH code to get a really excellent encoding scheme

- Good error correction properties.
- Complex to code correctly/well due to complex decoding algorithm.
- Open source good implementations are available, but they are polluted by the GPL.

However, these are overkill as we only need to correct very few bits in practice.

We pick a $(16, 8, 5)$ linear code.

This takes a byte and expands it into two bytes

- Allows us to correct a byte in the presence of two error bits in the 16

The $(16, 8, 5)$ Code

Generator Matrix:

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{pmatrix} \in \mathbb{F}_2^{8\times 16}$$

The $(16, 8, 5)$ Code

Parity Check Matrix:

$$H = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \in \mathbb{F}_2^{8 \times 16}$$

The $(16, 8, 5)$ Code

To encode we take a byte b and represent it as a bit vector $\mathbf{b} \in \mathbb{F}_2^8$ and then compute

$$\mathbf{c} = \mathbf{b} \cdot G.$$

To decode we compute the syndrome

$$\mathbf{s} = H \cdot \mathbf{c}.$$

We can then look up the resulting decoding value in a simple look up table of size $137 = 1 + 16 + (16 \cdot 15)/2$.

- LUT has one entry for each zero, one and two bit error.
- LUT gives the error vector which we can then add onto $\mathbf{c}$ to recover the actual message.

Coding Theory

If the per-bit error rate is p_0 then the probability we recover a byte correctly using this encoding scheme is

$$p_1 = (1 - p_0)^{16} + 16 \cdot (1 - p_0)^{15} \cdot p + \frac{16 \cdot 15}{2} \cdot (1 - p_0)^{14} \cdot p_0^2.$$

When $p_0 \approx 2^{-18}$ we obtain

$$p_1 \approx 1 - 2^{-44}.$$

Coding Theory

Given a vector $\mathbf{b}$ in $\mathbb{F}_2^n$ where n is a multiple of eight we can apply the encoding scheme $n/8$ times in parallel to obtain a vector of length $2 \cdot n$.

$$\mathbf{c} = \text{Encode}(\mathbf{b}).$$

Likewise we can decode a bit-vector vector of length $2 \cdot n$ to a bit-vector vector of length n via

$$\mathbf{b} = \text{Decode}(\mathbf{c}).$$

If $p_0 \approx 2^{-18}$ probability of transmitting 64 bytes (i.e. 512 bits) correctly is

$$p_2 = p_1^{64} \approx 1 - 2^{-38}.$$

We encode the 64 bytes as 128 bytes.

Fourier Transforms

Fourier Transforms

We pick F and q for our Ring LWE problem in a special way

1. $F = X^N + 1 = X^{2^n} + 1$
2. $q \equiv 1 \pmod{2 \cdot N}$.

This implies that $\mathbb{F}_q$ contains a $(2 \cdot N)$th root of unity.

- Let α denote a mutually agreed one.

Since $F = X^{2^n} + 1$ picking our noise via sampling Gaussian coefficients is essentially the same as sampling in the canonical embedding and pulling back.

Idea is to represent a polynomial $a(X)$ by its evaluations at the roots of unity α^i for odd $i \in [1, \ldots, 2 \cdot N]$.

Fourier Transforms

Evaluating $a(X)$ at the odd powers of α is the same as evaluating the FFT of $a(X)$.

$$\mathbf{a} \leftarrow \mathrm{FFT}(a) \in \mathbb{F}_q^N.$$

Interpolating the vector back to a polynomial is the same as evaluating the inverse FFT of **a**.

$$a \leftarrow \mathrm{FFT}^{-1}(\mathbf{a}) \in R_q.$$

Since N is a power of 2 the FFT is fast

Fourier Transforms

If we define $\oplus$ and $\otimes$ as coordinate wise addition and multiplication modulo q then we have

$$\mathbf{a} \oplus \mathbf{b} = \mathsf{FFT}(a+b),$$
$$\mathbf{a} \otimes \mathbf{b} = \mathsf{FFT}(a \cdot b).$$

This means that operations in our ring can be done in time $O(n)$ as soon as we have passed into the FFT domain.

- Mapping to/from FFT domain takes time $O(n \cdot \log n)$.

CCA Secure Scheme

CCA Parameters

We are going to need a ring size of $N = 1024$ bits

- Want to transmit keys of size 256 bits.
- Padding scheme needs 256 bits of randomness.
- Our encoding scheme doubles this to $2 \cdot (256 + 256) = 1024$.

We pick a Gaussian standard deviation of $\sigma = 3.2$

We therefore pick $q = 765953$

- Gives good security.
- Per bit error rate of $p_0 \approx 2^{-18}$.

We first define a CPA secure scheme which includes the FFT and coding theory modifications...

CPA Secure Scheme: KeyGen

1. $a \leftarrow R_q$.
2. $s, e' \leftarrow \chi_\sigma$.
3. $\mathbf{a} \leftarrow \mathsf{FFT}(a)$.
4. $\mathbf{s} \leftarrow \mathsf{FFT}(s)$.
5. $\mathbf{e}' \leftarrow \mathsf{FFT}(e')$.
6. $\mathbf{b} \leftarrow (\mathbf{a} \otimes \mathbf{s}) \oplus \mathbf{e}'$.
7. $\mathfrak{sk} \leftarrow \mathbf{s}$.
8. $\mathfrak{pk} \leftarrow (\mathbf{a}, \mathbf{b})$.
9. Return $(\mathfrak{pk}, \mathfrak{sk})$

CPA Secure Scheme: $\mathsf{Enc}_1(m, \mathfrak{pk})$

The encryption mechanism takes as input the public key $\mathfrak{pk} = (\mathbf{a}, \mathbf{b})$ and a message $m \in \{0, 1\}^b$, where $b < N/2$.

1. $\mu \leftarrow \mathsf{Encode}(m)$, treat μ as an element in R_2 (this involves applying Encode a total of $\lceil b/8 \rceil$ times).
2. $v, e, d \leftarrow \chi_\sigma$.
3. $\mathbf{v} \leftarrow \mathsf{FFT}(v)$, $\mathbf{e} \leftarrow \mathsf{FFT}(e)$.
4. $x \leftarrow d + \Delta_q \cdot \mu \bmod q$.
5. $\mathbf{x} \leftarrow \mathsf{FFT}(x)$.
6. $\mathbf{c}_0 \leftarrow (\mathbf{b} \otimes \mathbf{v}) \oplus \mathbf{x}$.
7. $\mathbf{c}_1 \leftarrow (\mathbf{a} \otimes \mathbf{v}) \oplus \mathbf{e}$.
8. Return $(\mathbf{c}_0, \mathbf{c}_1)$.

CPA Secure Scheme: $\text{Dec}_1(m, \mathfrak{pk})$

On input of a ciphertext $\mathbf{c} = (\mathbf{c}_0, \mathbf{c}_1)$ and a secret key $\mathfrak{sk} = \mathbf{s}$ the decryption is performed as follows:

1. $\mathbf{f} \leftarrow \mathbf{c}_0 \ominus (\mathbf{s} \otimes \mathbf{c}_1)$.
2. $f \leftarrow \text{FFT}^{-1}(\mathbf{f})$.
3. Convert f into centered-representation.
4. $\mu \leftarrow \left| \left\lfloor \frac{2}{q} f \right\rceil \right|$. Note μ can be considered as a string of $2 \cdot b$ bits, as $N > 2 \cdot b$ we only take the first $2 \cdot b$ bits of μ and ignore all non-zero trailing bits as "errors".
5. $m \leftarrow \text{Decode}(\mu)$.
6. Return m.

CCA From CPA

To define a CCA scheme from the above CPA scheme we use the Fujisaki-Okamoto transform:

If the original encryption scheme $(\mathrm{Enc}_1, \mathrm{Dec}_1)$ can encrypt messages of b bits in length, this this IND-CCA scheme encrypts messages of $b - 256$ bits in length.

The scheme makes use of a hash function H which outputs at least 256-bit hash values, and takes as inputs bit strings of length b.

CCA Secure Pubic Key Scheme

$\text{Enc}_2(m, \mathfrak{pk})$:

1. $s \leftarrow \{0, 1\}^{256}$.
2. $\mu \leftarrow m \| s$.
3. $r \leftarrow H(\mu)$.
4. $(\mathbf{c}_0, \mathbf{c}_1) \leftarrow \text{Enc}_1(\mu, \mathfrak{pk})$, where all randomness is generated from the seed r.
5. Return $(\mathbf{c}_0, \mathbf{c}_1)$.

$\text{Dec}_2(\mathbf{c}, \mathfrak{sk})$:

1. $\mu \leftarrow\leftarrow \text{Dec}_1(\mathbf{c}, \mathfrak{sk})$.
2. $m \| s \leftarrow \mu$, where s is 256 bits long.
3. $r \leftarrow H(\mu)$.
4. $\mathbf{c}' \leftarrow \text{Enc}_1(\mu, \mathfrak{pk})$, where all randomness is generated from the seed r.
5. If $\mathbf{c} \neq \mathbf{c}'$ then return $\perp$.
6. Return m.

CCA Secure Key Encapsulation

For key-encapsulation in post-quantum schemes we aim to transmit a key of 256 bits in length.

Thus we can easily adapt the previous IND-CCA encryption scheme to do this by selecting a message of size 256 bits at random.

- So $b - 256 = 256$.
- Which implies $b = 512$.
- So $N > 2 \cdot 512 = 1024$.
- This is why we wanted $N = 1024$ for our ring.

CCA Secure Key Encapsulation

Key Encapsulation: This takes as input a public key $\mathfrak{pk}$ and outputs an encapsulation $\mathbf{c} = (\mathbf{c}_0, \mathbf{c}_1)$ and the key $\mathfrak{k}$ it encapsulates.

1. $\mathfrak{k} \leftarrow \{0, 1\}^{256}$.
2. $(\mathbf{c}_0, \mathbf{c}_1) \leftarrow \mathsf{Enc}_2(\mathfrak{k}, \mathfrak{pk})$.
3. Return $((\mathbf{c}_0, \mathbf{c}_1), \mathfrak{k})$.

Key Decapsulation: This takes as input a secret key key $\mathfrak{sk}$ and an encapsulation $\mathbf{c} = (\mathbf{c}_0, \mathbf{c}_1)$, and outputs the key $\mathfrak{k}$ it encapsulates, or the error symbol $\perp$.

1. $\mathfrak{k} \leftarrow \mathsf{Dec}_2(\mathbf{c}, \mathfrak{sk})$.

Questions?