Achieving Channel Capacity With Lattice Codes: From Fermat to Shannon

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1. Statement of Coding Problems in Information Theory
2. Lattices and Algebraic Number Theory
3. Coding for Gaussian, Fading and MIMO channels

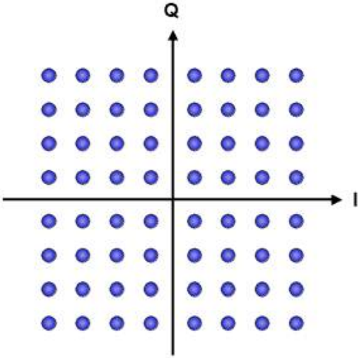
Communications in the presence of noise

- Signal vector $\mathbf{x} = [x_1, \ldots, x_n]^T$ in n-dimensional Euclidean space.
- The additive white Gaussian noise (AWGN) channel: $\mathbf{y} = \mathbf{x} + \mathbf{w}$, where signal power $P = E[\|\mathbf{x}\|^2]/n$ and noise power $= \sigma_w^2$.
- Shannon capacity (1949)

$$C = \frac{1}{2}\log(1+\rho)$$

where signal-to-noise ratio (SNR) $\rho = \frac{P}{\sigma_w^2}$.

- Shannon used random coding, but we need a concrete code to achieve the capacity.



quadrature amplitude modulation (QAM) constellation

Coding

Algebraic approach

- Hamming code
- Reed-Solomon code
- BCH code
- Algebraic-geometry code

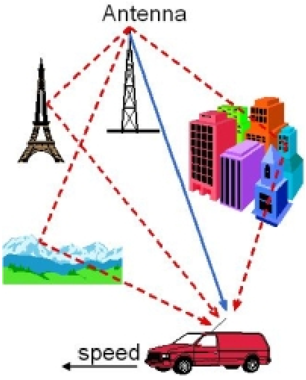
Probabilistic approach

- Convolutional code
- Turbo code
- LDPC code
- Polar code

For binary (discrete)-input channels, dream has come true with polar codes [Arikan'09] and SC-LDPC codes [Jimenez-Felstrom-Zigangirov'99].

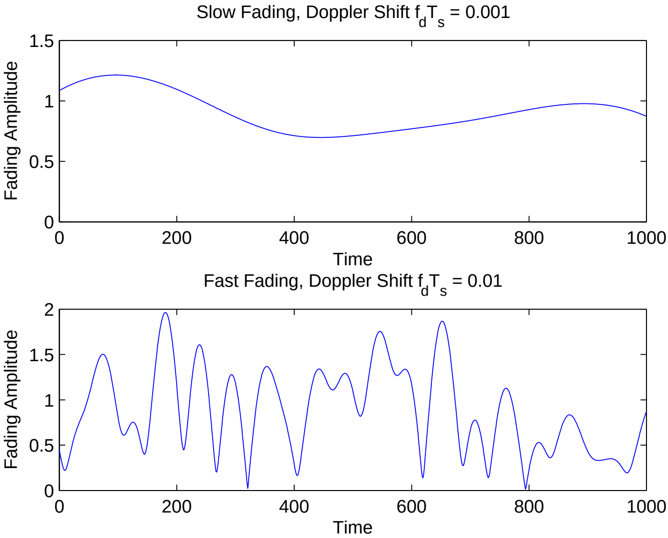
However, the question of achieving the capacity of the Gaussian channel has to be solved with lattice codes.

Multipath fading in mobile communications



- Multipath propagation in urban environment.
- Fading is multiplicative noise (large variation in signal strength)
$$y_t = h_t x_t + w_t$$
- Rayleigh fading: channel coefficient h_t is complex Gaussian.
- Time autocorrelation is modelled by a Bessel function (Jakes model) $R(\tau) = \mathrm{E}[h_t h_{t+\tau}^*] = J_0(2\pi f_d \tau)$ where $f_d = (v/c)f$ is normalized Doppler frequency shift.

Slow fading vs. fast fading



Models

- Slow fading (block fading): The fading process is nearly constant (but random) in the duration of a codeword. We need (time, frequency etc.) diversity from several independent blocks:

$$(\underbrace{h_1, h_1, \ldots, h_1}, \underbrace{h_2, h_2, \ldots, h_2}, \cdots, \underbrace{h_n, h_n, \ldots, h_n})$$

  - The length of each block is known as coherence time T.
  - Ergodicity doesn't hold due to delay constraint.
  - Capacity $C = \sum_{i=1}^{n} \log\left(1 + |h_i|^2 \rho\right)$.

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- Fast fading: The fading coefficients $\{h_t\}$ are nearly independent.
  - In reality, ergodic fading is a more accurate model.
  - Capacity $C = \mathrm{E}_H \left[\log\left(1 + |h|^2 \rho\right)\right]$.

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- These represent two extremes of stationary fading.
- Open question: to design capacity-achieving codes over fading channels (wireless systems will operate close to capacity).

Block fading channel

- Slow fading is the realistic model in delay-constrained communication systems (4G, 5G...).
- Write down the matrix form of the channel $\mathbf{Y} = \mathbf{HX} + \mathbf{W}$, where channel matrix $\mathbf{H} = \mathrm{diag}[h_1, h_2, \ldots, h_n]$.
- Set target capacity

$$C = \log \left| \mathbf{I} + \rho \mathbf{H}^{\dagger} \mathbf{H} \right| . \tag{1}$$

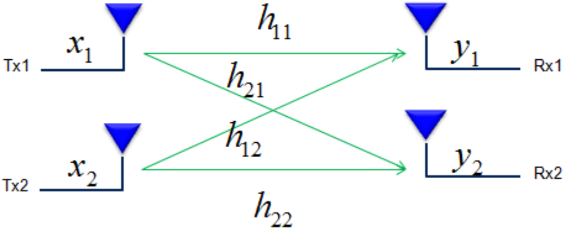
- The receiver has channel state information (CSI), while the transmitter doesn't.
- Our goal is to achieve capacity C on all channels such that (1) is true (without even knowing the distribution of $\mathbf{H}$).
- This requires a universal code on the compound channel (1), i.e., a collection of channels with the same capacity.

MIMO channel

- Capacity $\propto n$, the number of antennas.
- Channel model $\mathbf{Y} = \mathbf{HX} + \mathbf{W}$, where $\mathbf{H}$ is the $n \times n$ channel matrix, fixed (but random) in coherence time T.
- Set target capacity

$$C = \log \left| \mathbf{I} + \rho \mathbf{H}^{\dagger} \mathbf{H} \right| . \tag{2}$$

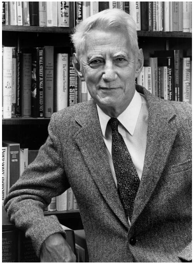
- Open question: achieve the capacity of the compound MIMO channel (2).



What are lattices?

- Lattices are regular, efficient and near-optimum arrays in the Euclidean space.
- The hexagonal lattice A_2 and FCC lattice A_3 give the densest sphere packings in dimensions 2 and 3.
- Breaking news [Viazovska (et al.)'16]: The E_8 lattice and Leech lattice are optimum for sphere packing for $n = 8$ and 24.
- Recent years see the revival of this classic area, driven by new applications, particularly coding and cryptography.

Why are lattices useful?



- Minkowski founded geometric number theory, where lattices are used to solve problems in number theory.
- Coding for the Gaussian channel is closely related to the sphere packing problem, for which lattices are near-optimum.
  - Shannon already indicated dense sphere packings to achieve channel capacity (without knowing lattices).
- Cryptographers are more interested in the hardness of lattice problems.

History of lattice coding and cryptography

- 1960s: earliest use of lattices in coding.
- 1984: lattice formulation of trellis-coded modulation.
- 1988: first book on lattice coding *Sphere Packings, Lattices and Groups*.
- 1992: ideal lattices for Rayleigh fading channels.
- 2004: capacity-achieving lattice codes for Gaussian channels.
- 2005: lattice-based space-time codes for MIMO channels.
- 1982: first use of lattices in cryptanalysis.
- 1996: first crypto scheme based on hard lattice problems.
- 2002: first book on lattice crypto *Complexity of Lattice Problems* published.
- 2005: learning with errors.
- 2006: application of ideal lattices to crypto.
- 2009: fully homomorphic encryption.
- 2012: multilinear maps.

Definition

- A lattice $\Lambda \subset \mathbb{R}^n$ is defined as

$$\Lambda = \Lambda(\mathbf{B}) = \{\mathbf{Bx} | \mathbf{x} \in \mathbb{Z}^n\}$$

  where $\mathbf{B}$ (n-by-n) is called a basis, or generator matrix. For example, the lattice $\mathbb{Z}^2$ (aka QAM) has basis $\mathbf{B} = \mathbf{I}_2$.
- May be viewed as the result of a linear transformation applied to the $\mathbb{Z}^n$ lattice (cubic lattice).
- Euclidean counterpart of a linear code (a linear code is normally defined in the Hamming space).
- The dual lattice Λ^* has basis $(\mathbf{B}^{-1})^T$. It arises in the Fourier transform of multi-dimensional signals.

Fundamental regions

- A fundamental region of a lattice is a piece that tiles the Euclidean space without any overlap or gap
- The volume of a fundamental region is called the fundamental volume

$$V(\Lambda) = \det(\Lambda) = |\det(\mathbf{B})|$$

- Fundamental parallelotope

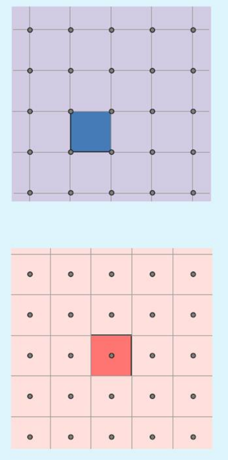
$$\mathcal{P} = \{\mathbf{y} \mid \mathbf{y} = \mathbf{B}\mathbf{x}, 0 \leq x_i < 1\}$$

- Voronoi region: the nearest-neighbor decoding region

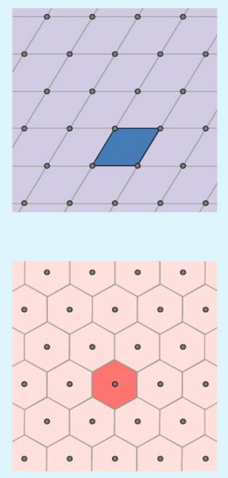
$$\mathcal{V} = \{\mathbf{y} \mid \|\mathbf{y}\| < \|\mathbf{y} - \mathbf{x}\|, \forall \mathbf{x} \in \Lambda\}$$

Examples of fundamental parallelotope and Voronoi region

Square lattice $\mathbb{Z}^2$



Hexagonal lattice A_2



An ensemble of random lattices (Loeliger ensemble)

- Consider the following family of q-ary lattices for all matrices $\mathbf{A} \in \mathbb{Z}_q^{k \times n}$,

$$\Lambda_q(\mathbf{A}) \quad = \quad \{\mathbf{y} \in \mathbb{Z}^n : \mathbf{y} = \mathbf{A}^T\mathbf{s} \mod q \text{ for } \mathbf{s} \in \mathbb{Z}^k\}$$

- These are lattices from Construction A, given the generator matrix $\mathbf{A}$ of the code.
- For a lattice vector $\mathbf{x}$,

$$\mathbf{x} \mod q = \mathbf{c} \in \mathcal{C}$$

  for a linear (n, k) code $\mathcal{C} \subset \mathbb{Z}_q^n$.
- Construction A is an important bridge between lattice theory and coding theory.

Minkowski-Hlawka Theorem

- Loeliger ensemble is discrete version of classic Minkowski-Hlawka-Siegel emsemble:

$$\mathcal{L} = \{\Lambda(\mathbf{B}) : \mathbf{B} \in \mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z})\}$$

- There exist dense lattices in Loeliger ensemble as $n, q \to \infty$ [Lolieger'97]

$$\lambda_1(\Lambda) \approx \sqrt{\frac{n}{2\pi e}} V^{1/n}(\Lambda)$$

  - Its proof is based on Shannon's random coding.
  - Rogers' proof of Minkowski-Hlawka Theorem is a special case with $k = 1$.
- Good lattices can be generated from random codes.
  - Good for coding.
  - Good for quantization.
  - Good for secrecy.
  - ...

Variations

- Similarly, Construction A may be defined with the parity-check matrix:

$$\Lambda_q^{\perp}(\mathbf{A}) \quad = \quad \{\mathbf{y} \in \mathbb{Z}^n : \mathbf{A}\mathbf{y} = \mathbf{0} \mod q\}$$

- Some special lattices/generalizations
  - Cyclic codes $\Rightarrow$ cyclic lattices
  - Negacyclic codes $\Rightarrow$ negacyclic lattices
  - Double-circulant codes: aka NTRU in crypto; quasi-cyclic codes
  - Modulo a multi-dimensional lattice (D_4, E_8), ideal $\mathfrak{q}$ of a number field...

Quest for structured lattices

Construction A

Good news: dense lattices (for coding [Loeliger'97]); hard to decode (for crypto [Regev'05]).
Bad news: hard to decode (for coding); inefficient (for both)[a].

[a] Being efficient means quasi-linear complexity; n is several hundreds to thousands in practice.

- In coding: (the study of transmitting information)
  - Gaussian channels:
    - Classic approach: dense lattices.
    - Practical approach: Constructions A, D etc. from good codes.
  - Fading channels: ideal lattices from algebraic number fields.
  - MIMO channels: division algebras over number fields.
- In crypto: (the study of hiding information)
  - Cyclic lattices, ideal lattices.
  - More efficient than general lattices.
  - New functionalities: homomorphic encryption, code obfuscation...

Algebraic number theory

- Fermat's Last Theorem (1637): When $n > 2$,

$$x^n + y^n = z^n$$

  has no nontrivial solutions $x, y, z \in \mathbb{Z}$.
- It was in the Guinness Book of World Records for "most difficult mathematical problems".
- Historically gave rise to algebraic number theory:

$$(x^p + y^p) = \prod_{i=0}^{p-1} (x + \zeta_p y)$$

- Kummer proved the theorem for all regular primes ($p \nmid h_p$ of a cyclotomic number field).
- Finally settled by Andrew Wiles in 1994, 3.5 centuries later.

Number fields

- A number field K is a finite field extension of $\mathbb{Q}$, i.e., a field which is a $\mathbb{Q}$-vector space of finite dimensions. The dimension $[K:\mathbb{Q}]$ is called the degree of K.
- Any number field can be built by adding a primitive element θ to $\mathbb{Q}$, i.e., $K=\mathbb{Q}(\theta)$ (in fact, θ is an algebraic integer).
- An algebraic integer in a number field K is an element $\alpha \in K$ which is a root of a monic irreducible polynomial with integer coefficients. Such a polynomial is called the minimum polynomial of α.
- Example: $\mathbb{Q}(\sqrt{2}) = \{x + y\sqrt{2} | x, y \in \mathbb{Q}\}$ is a number field of degree 2, i.e., a quadratic field.
- Example: If ζ_m is a primitive mth root of unity, the number field $\mathbb{Q}(\zeta_m)$ is called a cyclotomic field.

Ring of integers

- The ring of integers $\mathcal{O}_K$ of a number field K is the set of all algebraic integers of K.
- Example: $\mathbb{Z}[\sqrt{2}] = \{x + y\sqrt{2} | x, y \in \mathbb{Z}\}$ is the ring of integers of $\mathbb{Q}(\sqrt{2})$.
- Example: For the mth cyclotomic number field $\mathbb{Q}(\zeta_m)$ of degree $n = \varphi(m)$, the ring of integers is given by

$$\mathbb{Z}[\zeta_m] = \mathbb{Z} + \mathbb{Z}\zeta_m + \ldots + \mathbb{Z}\zeta_m^{n-1} \cong \mathbb{Z}[X]/\langle \Phi_m(X) \rangle.$$

- There exists an integral basis $\{\omega_i\}_{i=1}^n$ of K such that any element of $\mathcal{O}_K$ can be uniquely written as $\sum_{i=1}^n a_i \omega_i$ with $a_i \in \mathbb{Z}$ for all i.
- We can get an algebraic lattice from $\mathcal{O}_K$.

Canonical embedding

- Let θ_i for $i = 1, \ldots, n$ be the distinct roots of the minimum polynomial of θ. There are exactly n embeddings $\sigma_i : K \to \mathbb{C}$, defined by $\sigma_i(\theta) = \theta_i$, for $i = 1, \ldots, n$.
- When we apply the embedding σ_i to an arbitrary element x of K, $x = \sum_{k=1}^{n} a_k \theta^k$, $a_k \in \mathbb{Q}$, we get

$$\sigma_i(x) = \sigma_i(\sum_{k=1}^{n} a_k \theta^k) = \sum_{k=1}^{n} \sigma_i(a_k)\sigma_i(\theta)^k = \sum_{k=1}^{n} a_k \theta_i^k$$

- Let r_1 be the number of embeddings with image in $\mathbb{R}$, and $2r_2$ the number of embeddings with image in $\mathbb{C}$ so that $r_1 + 2r_2 = n$.
- Canonical (Minkowski) embedding

$$\sigma(x) = (\sigma_1(x), \ldots, \sigma_{r_1}(x), \Re(\sigma_{r_1+1}(x)), \ldots, \Im(\sigma_{r_1+r_2}(x))) \in \mathbb{R}^n.$$

From $\mathcal{O}_K$ to lattice

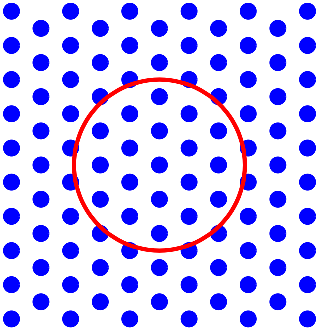
- If we take the ring of integers $\mathcal{O}_K$, we obtain a lattice with canonical embedding.
- Let $\{\omega_i\}_{i=1}^n$ be an integral basis of K. The n vectors $v_i = \sigma(\omega_i) \in \mathbb{R}^n$ form a basis of an algebraic lattice $\Lambda = \Lambda(\mathcal{O}_K) = \sigma(\mathcal{O}_K)$, whose generator matrix is given by

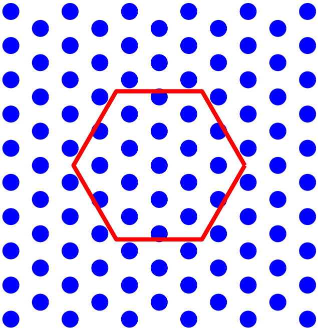
$$\mathbf{M} = \begin{pmatrix} \sigma_1(\omega_1) & \cdots & \sigma_{r_2}(\omega_1) & \Re\sigma_{r_1+1}(\omega_1) & \cdots & \Im\sigma_{r_1+r_2}(\omega_1) \\ \sigma_1(\omega_2) & \cdots & \sigma_{r_2}(\omega_2) & \Re\sigma_{r_1+1}(\omega_2) & \cdots & \Im\sigma_{r_1+r_2}(\omega_2) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \sigma_1(\omega_n) & \cdots & \sigma_{r_2}(\omega_n) & \Re\sigma_{r_1+1}(\omega_n) & \cdots & \Im\sigma_{r_1+r_2}(\omega_n) \end{pmatrix}.$$

- We can get more lattices $\Lambda' \subset \Lambda$ from ideals $\mathcal{I} \subseteq \mathcal{O}_K$, which are called ideal lattices.

Lattice coding

- A lattice code is a code constructed from a lattice in the Euclidean space. Thus, it is naturally suited to a Gaussian channel.
- Since a lattice is infinite, shaping is needed to obtain a code of given rate. The common practice is to apply a finite shaping region (cubic, spherical, or Voronoi).





AWGN-good lattices

- The issues of shaping and coding are largely separable.
- Consider (infinite) lattice coding over the AWGN channel with noise variance σ_w^2.
- For an n-dimensional lattice Λ, define the volume-to-noise ratio (VNR) by

$$\gamma_\Lambda(\sigma_w) \triangleq \frac{(V(\Lambda))^{\frac{2}{n}}}{\sigma_w^2}$$

- The error probability is given by $P_e = \mathbb{P}\{\mathsf{W}^n \notin \mathcal{V}(\Lambda)\}$.
- A sequence of lattices $\Lambda^{(n)}$ of increasing dimension n is *AWGN-good* if for a fixed VNR $\gamma_\Lambda(\sigma_w) > 2\pi e$, P_e vanishes in n [Poltyrev'94].
- This is the best possible performance, achieved only if the Voronoi region is approximately a sphere.

Constructions from number fields

- The connection between coding and dense sphere packing is well known [Conway-Sloane'93].
- Lattices from cyclotomic fields [Craig'78]: Let $p = n + 1$ be a prime. Take a principal ideal $\mathcal{I} = ((1 - \zeta_p)^m)$ in $\mathbb{Q}(\zeta_p)$, for some m. $\mathcal{I}$ yields a lattice under canonical embedding.
- Lattices from class field towers [Martinet'78]: Embedding of $\mathcal{O}_{K_i}$ in the tower; densest lattices having been constructed.

Remark

Number field constructions yield dense lattices, but have not achieved the Minkowski-Hlawka bound.

Constructions from codes

- Constructions A, B, C, D... [Leech-Sloane'71].
- Existence of AWGN-good lattices from Construction A [Loeliger'97].
- Shannon-theoretic construction [Forney et al.'00]: A lattice from Construction A is AWGN-good if the code achieves capacity of the mod-q channel.

Remark

Forney et al.'s construction yields AWGN-good lattices, but are not necessarily dense.

Achieving capacity with lattice codes

- Lattice codes achieve the $\frac{1}{2}\log(1+\rho)$ capacity of the Gaussian channel.
- This also forms the basis of the recent surge in applications to network information theory.
- Erez and Zamir's scheme [2004]
  - Voronoi shaping: the coarse lattice is good for quantization.
  - It also requires dithering.
  - The existence proof is again based on random lattices.
- Polar lattice
  - Gaussian shaping: Applying a discrete Gaussian distribution over an AWGN-good lattice [Ling-Belfiore'14].
  - Explicit construction from polar codes, $O(n\log n)$ decoding complexity [Yan-Liu-Ling-Wu'15].

Coding for fading channel

- Good Codes for the Gaussian channel usually have rather poor performance in the fading channel.
- The construction of good lattice codes for the fading channel exploits algebraic number theory [Belfiore et al. 1990s].
- A powerful tool is ideal theory for the rings of algebraic integers, leading to the construction of ideal lattice codes.
- However, capacity-achieving codes for fading channels are still unavailable[1].
- Record is a constant gap to capacity [Ordentlich-Erez'13, Luzzi-Vehkalahti'15].

[1] In the special case of i.i.d fading, capacity is acheived with polar lattices [Liu-Ling'16]

Work in 1990s

- Consider the fast (i.i.d.) Rayleigh fading channel

$$y_i = h_i x_i + w_i$$

where $(x_1, \ldots, x_n) = \mathbf{x} \in \mathbb{R}^n$ is the codeword, h_i's are i.i.d. fading coefficients of the Rayleigh distribution.

- Pairwise error probability

$$P(\mathbf{x} \to \hat{\mathbf{x}}) \leq \frac{1}{2} \prod_{i: x_i \neq \hat{x}_i} \frac{8\sigma_w^2}{(x_i - \hat{x}_i)^2} = \frac{1}{2} \frac{(8\sigma_w^2)^l}{\prod_{i: x_i \neq \hat{x}_i} (x_i - \hat{x}_i)^2}$$

if the two codewords differ in l positions.

- Design criteria
  - Maximize the diversity order $\min\{l\} = \min_{\mathbf{x} \neq \hat{\mathbf{x}}} |\{i : x_i \neq \hat{x}_i\}|$. Full diversity order n is desired.
  - Maximize the product distance: $d_{p,\min} = \min_{\mathbf{x} \neq \hat{\mathbf{x}}} \prod_{i: x_i \neq \hat{x}_i} |x_i - \hat{x}_i|$.

Ideal lattice code

- Now, suppose an ideal lattice Λ built from ideal $\mathcal{I} \subseteq \mathcal{O}_K$ is used as the coding lattice.
- By the union bound and geometric uniformity, error probability

$$P_e \leq \sum_{\mathbf{x}\in\Lambda\backslash\mathbf{0}} P(\mathbf{0} \to \mathbf{x}) = \sum_{\mathbf{x}\in\Lambda\backslash\mathbf{0}} \frac{1}{2} \frac{(8\sigma_w^2)^{l_\mathbf{x}}}{\prod_{i:x_i\neq 0} |x_i|^2}$$

  where $l_\mathbf{x}$ is the number of nonzero elements (the sum is take over a shaping region).
- Design criteria rephrased (Oggier, Viterbo'04):
  - Maximize the diversity order $\min\{l\} = \min_{\mathbf{x}\in\Lambda\backslash\mathbf{0}} |\{i : x_i \neq 0\}|$. K should be totally real to achieve full diversity n.
  - The minimum norm $\mathrm{N}_{\min} = \min_{x\neq 0, x\in\mathcal{I}} |\mathrm{N}(x)|$ should be maximized (recall algebraic norm $\mathrm{N}(x) \triangleq \prod_{i=1}^n \sigma_i(x)$).

Capacity?

- Our goal for block fading channels is to achieve capacity of compound channel (with diagonal $\mathbf{H}$)

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$$\underbrace{\mathbf{Y}}_{n \times T} = \underbrace{\mathbf{H}}_{n \times n} \underbrace{\mathbf{X}}_{n \times T} + \underbrace{\mathbf{W}}_{n \times T}$$

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- Vectorizing this equation, we obtain

$$\underbrace{\mathbf{y}}_{nT \times 1} = \underbrace{\mathcal{H}}_{nT \times nT} \underbrace{\mathbf{x}}_{nT \times 1} + \underbrace{\mathbf{w}}_{nT \times 1}$$

where $\mathcal{H} = \mathbf{I}_T \otimes \mathbf{H}$.

Fading-good lattices

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- With Gaussian shaping, the problem boils down to finding a lattice that is good for block fading.

Fading-good lattices [Campello-Ling-Belfiore'16]

We say that a sequence of lattices Λ of increasing dimension nT is universally good for the block-fading channel if for any VNR $\gamma_{(\mathbf{I}_T \otimes \mathbf{H})\Lambda}(\sigma_w) > 2\pi e$ and all (absolute) $\mathbf{H}$ s.t. $|\mathbf{H}| = D$, $P_e(\Lambda, \mathbf{H}) \to 0$ as $T \to \infty$.

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- With Gaussian shaping, the problem boils down to finding a lattice that is good for block fading.

Fading-good lattices [Campello-Ling-Belfiore'16]

We say that a sequence of lattices Λ of increasing dimension nT is universally good for the block-fading channel if for any VNR $\gamma_{(\mathbf{I}_T \otimes \mathbf{H})\Lambda}(\sigma_w) > 2\pi e$ and all (absolute) $\mathbf{H}$ s.t. $|\mathbf{H}| = D$, $P_e(\Lambda, \mathbf{H}) \to 0$ as $T \to \infty$.

- If $\mathbf{H} = \mathbf{I}$, the problem reduces to that of AWGN-good lattices.

Generalized Construction A

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Generalized Construction A [Kositwattanarerk-Ong-Oggier'13]

Let $K/\mathbb{Q}(i)$ be a relative extension of degree n.
Let $\mathfrak{p} \subset \mathcal{O}_K$ be a prime ideal above p with norm p^ℓ. Then $\mathcal{O}_K/\mathfrak{p} \simeq \mathbb{F}_{p^\ell}$.
The $\mathcal{O}_K$-lattice Λ associated to a linear code $\mathcal{C} \subset \mathbb{F}_{p^\ell}^T$ is defined as:

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- It reduces to usual Construction A: $\Lambda = \mathcal{C} + p^T$ when $K = \mathbb{Q}$.

Achieving capacity

Generalized Construction A is good for fading channels

With number fields, generalized Construction A are good for block fading [Campello-Ling-Belfiore'16].
The existence of a universal lattice can be proven by Minkowski-Hlawka, i.e., averaging over random codes $\mathcal{C}$ (with $p \to \infty$).
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- With Gaussian shaping, capacity of compound fading channels is achieved.
- However, there is a large gap between theory (given above) and state of the art [Ordentlich-Erez'13, Luzzi-Vehkalahti'15].

Universality (for $T = 1$)

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- Nevertheless, an ideal lattices $\mathcal{I}$ is "incompressible" [Luzzi-Vehkalahti'15]: the minimum Euclidean distance of faded lattice $\mathbf{H}\mathcal{I}$

$$\begin{aligned}
\min_{\mathbf{H}:|\mathbf{H}|=D} d^2_{\min}(\mathbf{H}\mathcal{I}) &= \min_{\mathbf{H}:|\mathbf{H}|=D} \min_{\mathbf{x}\in\mathcal{I},\mathbf{x}\neq\mathbf{0}} \|\mathbf{H}\mathbf{x}\|^2 \\
&= \min_{\mathbf{x}\in\mathcal{I},\mathbf{x}\neq\mathbf{0}} nD^{2/n}(x_1\cdots x_n)^{2/n} \\
&= nD^{2/n}\mathrm{N}(\mathcal{I})^{2/n}
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which follows from AM-GM inequality.

- However, $d_{\min} = 0$ for usual mod-q lattices.

Error probability of MIMO

- Recall channel model $\underbrace{\mathbf{Y}}_{n\times T} = \underbrace{\mathbf{H}}_{n\times n}\underbrace{\mathbf{X}}_{n\times T} + \underbrace{\mathbf{W}}_{n\times T}$.
- For a linear space-time code $\mathcal{S}$, consider pairwise error probability

$$P(\mathbf{0} \to \mathbf{X}) = \mathrm{E}_{\mathbf{H}}\left[Q\left(\frac{\|\mathbf{HX}\|_F}{\sqrt{2}\sigma_w}\right)\right] \quad (3)$$

- For Rayleigh fading,

$$P(\mathbf{0} \to \mathbf{X}) \leq \left|\mathbf{I} + \frac{\mathbf{XX}^\dagger}{4\sigma_w^2}\right|^{-n} \quad (4)$$

- If codeword matrix $\mathbf{X}$ has full rank, then high-SNR behavior

$$P(\mathbf{0} \to \mathbf{X}) \leq |\mathbf{XX}^\dagger|^{-n}\left(\frac{1}{4\sigma_w^2}\right)^{-n^2} \quad (5)$$

Non-vanishing determinant (NVD)

- Define
$$\Delta = \inf_{\mathbf{0} \neq \mathbf{X} \in \mathcal{S}} |\mathbf{X}\mathbf{X}^{\dagger}| \tag{6}$$
- Non-vanishing $\Delta > 0$ implies full diversity, in fact, optimum DMT (diveristy-multiplexing gains tradeoff) of the space-time code $\mathcal{S}$.
- Larger Δ gives larger coding gain.
- Many approaches were tried in 2000's, but $\Delta \to 0$ as constellation grows for most of them.
- Solution: construct a lattice code from division algebra [Sethuraman-Rajan'02] over a number field [Belfiore-Rekaya'03].

Division algebra

Definition

Let A be a ring and denote by A^* the set of invertible elements of A for multiplication. If $A^* = A\backslash\{0\}$, then A is referred to as a division algebra.

Hamilton's quaternions

Let $\{1, i, j, k\}$ be a basis for a vector space of dimension 4 over $\mathbb{R}$, which satisfy the relations $i^2 = -1$, $j^2 = -1$, $k^2 = -1$ and $k = ij = -ji$. Hamilton's quaternions are defined as the set

$$\mathbb{H} = \{x + yi + zj + wk \mid x, y, z, w \in \mathbb{R}\}.$$

The key link

- Rewrite a quaternion $q = x + yi + zj + wk$ as
$$q = (x + yi) + j(z - wi) = \alpha + j\beta$$
where $\alpha = x + yi$ and $\beta = z - wi$.
- It matrix representation
$$q \Longleftrightarrow \mathbf{X} = \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix}.$$
- Check
$$|\mathbf{X}| = |\alpha|^2 + |\beta|^2 = N(q) > 0 \quad \text{if} \quad \mathbf{X} \neq \mathbf{0}$$
The norm $N(q) = 0$ if and only if $q = 0$.
- This is the famous Alamouti code with full diversity [Alamouti'98]. It's used in DVB, WiFi and 4G.

Cyclic algebra

Definition

Let L/K be a Galois extension of degree n whose Galois group is cyclic with generator σ. Choose an element $0 \neq \gamma \in K$. A cyclic algebra $\mathcal{A} = (L/K, \sigma, \gamma)$ is defined as the direct sum

$$\mathcal{A} = L \oplus eL \oplus \cdots \oplus e^{n-1}L$$

where

$$e^n = \gamma \quad \text{and} \quad \lambda e = e\sigma(\lambda) \quad \lambda \in L.$$

The cyclic algebra may be viewed as a vector space over L, i.e., an element $x \in \mathcal{A}$ is written as

$$x = x_0 + ex_1 + \cdots + e^{n-1}x_{n-1} \quad \text{for} \quad x_i \in L.$$

The rule $\lambda e = e\sigma(\lambda)$ defines multiplication.

Matrix representation

An element $x \in \mathcal{A}$

$$x = x_0 + ex_1 + \cdots + e^{n-1}x_{n-1} \quad \text{for} \quad x_i \in L$$

can be represented by

$$\begin{pmatrix} x_0 & \gamma\sigma(x_{n-1}) & \gamma\sigma^2(x_{n-2}) & \dots & \gamma\sigma^{n-1}(x_1) \\ x_1 & \sigma(x_0) & \gamma\sigma^2(x_{n-1}) & \dots & \gamma\sigma^{n-1}(x_2) \\ \vdots & & \vdots & & \vdots \\ x_{n-2} & \sigma(x_{n-3}) & \sigma^2(x_{n-4}) & \dots & \gamma\sigma^{n-1}(x_{n-1}) \\ x_{n-1} & \sigma(x_{n-2}) & \sigma^2(x_{n-3}) & \dots & \sigma^{n-1}(x_0) \end{pmatrix}. \qquad (7)$$

Cyclic division algebra [Oggier-Belfiore-Viterbo'07]

If $0 \neq \gamma, \gamma^2, \dots, \gamma^{n-1} \in K$ are not a norm of some element of L, then $\mathcal{A} = (L/K, \sigma, \gamma)$ is a cyclic division algebra.

Golden code ($n = 2$, optional in WiMAX)

In general, a space-time code is an order of $\mathcal{A}$. Then $|\mathbf{X}|$ is reduced norm and $\Delta > 0$ naturally.
If $n = 2$, consider the cyclic division algebra [Belfiore-Rekaya-Viterbo'05]

$$\mathcal{A} = (L = \mathbb{Q}(i, \sqrt{5})/\mathbb{Q}(i), \sigma, i)$$

where $\sigma : \sqrt{5} \mapsto -\sqrt{5}$. The ring of integers $\mathcal{O}_L$ is given by

$$\mathcal{O}_L = \{a + b\theta \mid a, b \in \mathbb{Z}[i]\}$$

where $\theta = \frac{1+\sqrt{5}}{2}$. A codeword of the Golden code is of the form

$$\mathbf{X} = \begin{pmatrix} a + b\theta & c + d\theta \\ i(c + d\sigma(\theta)) & a + b\sigma(\theta) \end{pmatrix}$$

where $a, b, c, d \in \mathbb{Z}[i]$.

Construction A from division algebras

Generalized Construction A [Vehkalahti-Kositwattanarerk-Oggier'14]

Let Λ be the natural order of cyclic division algebra $\mathcal{A}$.
Take a two-sided ideal $\mathcal{J}$ of Λ and consider the quotient ring $\Lambda/\mathcal{J}$.
Define a reduction $\beta : \Lambda \to \Lambda/\mathcal{J}$.
For a linear code $\mathcal{C}$ over $\Lambda/\mathcal{J}$, $\beta^{-1}(\mathcal{C})$ is a lattice (in $\mathbb{C}^{n^2 T}$).

- $\Lambda/\mathcal{J}$ can be a matrix ring, skew polynomial ring...
- Nevertheless still possible to prove Minkowski-Hlawka using codes over rings [Campello-Ling-Belfiore'16].

NVD implies universality (for $T = 1$)

- Again, Λ with NVD $\Delta > 0$ is "incompressible" [Luzzi-Vehkalahti'15]: the minimum Euclidean distance of faded lattice $\mathbf{H}\Lambda$

$$\begin{aligned}\min_{\mathbf{H}:|\mathbf{H}|=D} d^2_{\min}(\mathbf{H}\Lambda) &= \min_{\mathbf{H}:|\mathbf{H}|=D}\ \min_{\mathbf{X}\in\Lambda,\mathbf{X}\neq\mathbf{0}} \|\mathbf{H}\mathbf{X}\|_F^2 \\ &= \min_{\mathbf{X}\in\Lambda,\mathbf{X}\neq\mathbf{0}} nD^{2/n}|\mathbf{X}|^{2/n} \\ &= nD^{2/n}\Delta^{2/n}\end{aligned}$$

which follows from Hadamard's inequality and AM-GM inequality.

Concluding remarks

- The coding problem for Gaussian channels has been solved.
- Algebraic number theory is an indispensable tool to design modern coding systems over fading/MIMO channels.
    - Achieve capacity of compound fading channels requires a combination of number theory and coding theory.
    - These can be viewed as concatenated codes (inner code is ideal/order; outer code is a code).
    - Extension to ergodic fading is possible.
- Emerging applications to multi-user networks:
    - Compute-and-forward
    - Interference alignment