Randomness and determinism in Diophantine approximation: small linear forms, lattice flows and some applications

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Relevance to coding and lattices:

- DA is of growing interest in electronics: lattice coding and interference alignment (the latter to be briefly described);
- Minkowski's theorems is a basis for much of DA, and a useful tool at the same time;
- Dictionary between Homogeneous dynamics (lattice orbits) and Diophantine Approximation (will be discussed where possible)

The idea of interference alignment

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Consider the system of linear equations:

$$\begin{cases} y_1 = h_{11}x_1 + \cdots + h_{1K}x_K \\ \qquad\quad \vdots \qquad\qquad \vdots \\ y_B = h_{B1}x_1 + \cdots + h_{BK}x_K \end{cases}$$

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Interpretation: h_{ij} are channel coefficients
$x_1, \ldots, x_K$ are signals transmitted from K different transmitters
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the receiver can recover all the symbols $x_1, \ldots, x_K$. This requires that

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$\Rightarrow$ The bandwidth is simply divided between the users.

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Then the system is

$$\mathbf{Y} = x_1 \mathbf{H}_1 + \underbrace{(x_2 \mathbf{H}_2 + \cdots + x_K \mathbf{H}_K)}_{\in \mathbf{Span}\{\mathbf{H}_2, \ldots, \mathbf{H}_K\}}$$

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Various assumptions:
- varying channel coefficients,
- multiple antennae at receivers
- diagonal form of the channel coefficient matrices
- ...

Rational dimensions framework

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This observation can be distinguished from another one, say

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Fact: for any $\varepsilon > 0$ for almost every collection $(h_1, \ldots, h_K)$ of real numbers there exists a constant $\gamma = \gamma(\varepsilon, h_1, \ldots, h_K) > 0$ such that

$$|h_1(x_1 - x_1') + \cdots + h_K(x_K - x_K')| \geq \frac{\gamma}{Q^{K-1+\varepsilon}}$$

Rational dimensions: single antenna, multiple data streams

Using that **Fact** a single antenna can be turned into a multiple antennae to simultaneously transmit several data streams, say $x_1, \dots, x_M$, $x_i \in \{0, \dots, Q\}$.

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$$x = \lambda(c_1 x_1 + \cdots + c_M x_M)$$

where $|c_i| = O(1)$ are precoding coefficients and λ is a scalar reflecting power constraints.

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$$Q = O(P^{\frac{1-\varepsilon}{2(M+\varepsilon)}}) \qquad \text{and} \qquad \lambda = P^{1/2} Q^{-1}$$

ensures that the separation is bigger than the standard deviation of the noise, while meeting power constraints.

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Just as with vector alignment, it is possible to align alone real numbers.

2-user X-channel (Motahari, *et al*)

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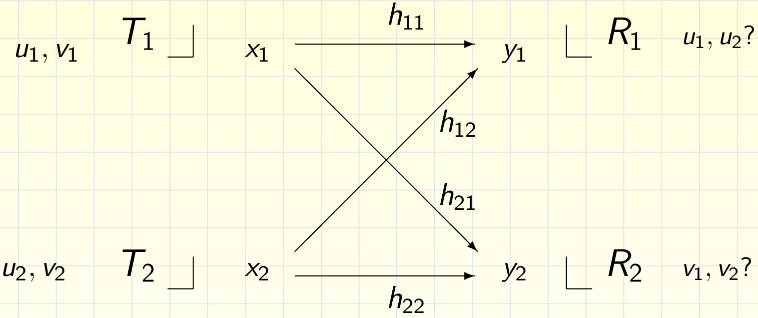

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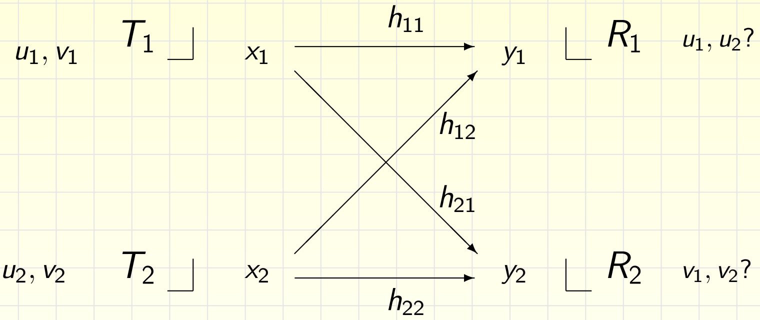

T_1 simultaneously transmits two data streams u_1 (intended for R_1) and v_1 (intended for R_2). Similarly, T_2 transmits independent two data streams u_2 (intended for R_1) and v_2 (intended for R_2). h_{ij} are the the channel coefficients. Let x_i is the signal transmitted by T_i.

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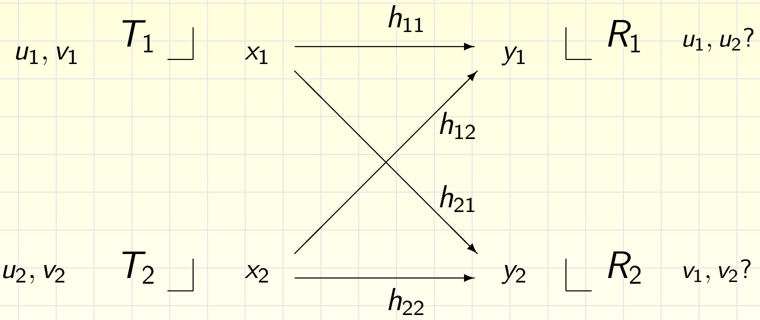

T_1 simultaneously transmits two data streams u_1 (intended for R_1) and v_1 (intended for R_2). Similarly, T_2 transmits independent two data streams u_2 (intended for R_1) and v_2 (intended for R_2). h_{ij} are the the channel coefficients. Let x_i is the signal transmitted by T_i. The received signals by R_1 and R_2 are

$$\begin{aligned} y_1 &= h_{11}x_1 + h_{12}x_2 + z_1\,, \\ y_2 &= h_{21}x_1 + h_{22}x_2 + z_2\,. \end{aligned}$$

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If $\quad x_1 = \lambda(h_{22}u_1 + h_{12}v_1) \quad \text{and} \quad x_2 = \lambda(h_{21}u_2 + h_{11}v_2)$ then

$$y_1 = \lambda\Big(h_{11}h_{22}u_1 + h_{12}h_{21}u_2 + h_{11}h_{12}(v_1+v_2)\Big) + z_1,$$
$$y_2 = \lambda\Big(h_{21}h_{22}(u_1+u_2) + h_{12}h_{21}v_1 + h_{11}h_{22}v_2\Big) + z_1$$

At R_1: v_1 and v_2 are aligned along the same real number, $h_{11}h_{12}$
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At R_1: v_1 and v_2 are aligned along the same real number, $h_{11}h_{12}$
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Again, assuming $u_i, v_i \in \{0, \dots, Q\}$ we achieve the required separation (and normalising power) in each of the equation by taking

$$Q = O(P^{\frac{1-\varepsilon}{2(3+\varepsilon)}}), \qquad \lambda = O(P^{\frac{1+\varepsilon}{3+\varepsilon}})\,.$$

DOF$=\frac{4}{3}$ almost surely.

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Dirichlet's theorem: Let $\alpha_{i,j} \in \mathbb{R}$, where $1 \le j \le n$, $1 \le i \le m$, and $Q > 1$. Then there exist $q_1, \dots, q_n, p_1, \dots, p_m \in \mathbb{Z}$ such that

$$|q_1\alpha_{i,1} + \cdots + q_n\alpha_{i,n} - p_i| < Q^{-\frac{n}{m}} \qquad (1 \le i \le m)$$

$$1 \le \max_{1 \le j \le n} |q_j| \le Q\,.$$

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Minkowski's theorem for systems of linear forms: Let $\beta_{i,j} \in \mathbb{R}$, where $1 \le i, j \le k$, and let $C_1, \dots, C_k > 0$. If

$$|\det(\beta_{i,j})_{1 \le i,j \le k}| \le \prod_{i=1}^{k} C_i, \tag{1}$$

then there exist a non-zero integer point $\mathbf{x} = (x_1, \dots, x_k)$ such that

$$\begin{cases} |x_1\beta_{i,1} + \cdots + x_k\beta_{i,k}| < C_i\,, & (1 \le i \le k-1) \\ |x_1\beta_{k,1} + \cdots + x_n\beta_{k,k}| \le C_k\,. \end{cases} \tag{2}$$

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Proof: uses Minkowski's theorem for convex bodies.

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Minkowski can be used beyond the real case. Example:

Dirichlet's theorem for $\mathbb{C}$ (one dimensional): For any $z \in \mathbb{C}$ and any $Q > 1$ there exist $p, q \in \mathbb{Z}[i] = \{a + bi : a, b \in \mathbb{Z}\}$ ($i = \sqrt{-1}$) such that

$$|qz - p| < \frac{4}{\pi Q}, \qquad 1 \le |q| \le Q\,.$$

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Badly approximable systems/matrices: Let $\alpha_{i,j} \in \mathbb{R}$, where $1 \le j \le n$, $1 \le i \le m$, and $Q > 1$. Then $A = (\alpha_{i,j})_{i,j}$ is badly approximable if there exist $c > 0$ such that for all $Q > 1$ the only integer solution $(q_1, \dots, q_n, p_1, \dots, p_m)$ to the system

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is zero.

Minkowski's theorem: If B is a convex body in $\mathbb{R}^n$ symmetric about $\mathbf{0}$ and $\operatorname{vol} B > 2^n$ then B contains a non-zero integer point.

Minkowski can be used beyond the real case. Example:

Dirichlet's theorem for $\mathbb{C}$ (one dimensional): For any $z \in \mathbb{C}$ and any $Q > 1$ there exist $p, q \in \mathbb{Z}[i] = \{a + bi : a, b \in \mathbb{Z}\}$ ($i = \sqrt{-1}$) such that

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Littlewood's conjecture (LC), (1930): For any $\alpha_1, \alpha_2 \in \mathbb{R}$ and any $\varepsilon > 0$ there exists $q \in \mathbb{Z}_{\neq 0}$, $p_1, p_2 \in \mathbb{Z}$ such that

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Admissible lattices: examples

Admissible lattices: $\Lambda \in \mathcal{L}_n := \mathrm{GL}(n,\mathbb{R})/\mathrm{SL}(n,\mathbb{Z})$ is admissible if

$$\mathrm{Nm}(\Lambda) := \inf\{|\lambda_1 \cdots \lambda_n| : (\lambda_1, \dots, \lambda_n) \in \Lambda_{\neq \mathbf{0}}\} > 0\,.$$

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$$\begin{aligned} \mathrm{Nm}(\Lambda) &= \inf\Big\{ \prod_{1\leq i\leq n} |g(\alpha_i)| : g \in \mathbb{Z}[x]_{\neq 0},\ \deg g \leq n-1 \Big\} \\ &= \inf\Big\{ |\mathrm{Resultant}(f,g)| : g \in \mathbb{Z}[x]_{\neq 0},\ \deg g \leq n-1 \Big\} \geq 1\,. \end{aligned}$$

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The proof does not extend to weighted badly approximable points.

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are best approximations and satisfy

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Fact: An irrational $x = [a_0; a_1, a_2 \dots]$ is in **Bad** $\Leftrightarrow\ \exists\, M\ \forall\, i \in \mathbb{N}\ a_i \leq M$

More on dimension 1

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The constant $1/(\sqrt{5})$ *is best possible.*

Recall:

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- **Folklore Conjecture:** Cubic irrationals are not in **Bad**
- However, for any irrational algebraic x and any $\varepsilon > 0$ there exists a constant $c(x, \varepsilon) > 0$ such that

  (Roth, 1955) $$\left|x - \frac{p}{q}\right| \geq \frac{c(x,\varepsilon)}{q^{2+\varepsilon}}.$$

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Remark. Dirichlet Improvable points/matrices can be introduced in higher dimensions, but their characterisation is not that simple. It is known that almost every matrix is not Dirichlet Improvable.

More on dimension 1: the Three Distance Theorem

To understand the gaps between $q\alpha - p$, where $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ is fixed and $p, q \in \mathbb{Z}$, $0 \le q \le Q$ it is enough to describe the distribution of

$$\{\alpha\}, \{2\alpha\}, \dots, \{Q\alpha\} \tag{4}$$

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The Three Distance Theorem: For any $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and any integer $Q \ge 1$ the points (4) partition $[0, 1]$ into $Q + 1$ intervals which lengths take at most 3 different values δ_A, δ_B and δ_C with $\delta_C = \delta_A + \delta_B$.

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The length of the intervals, the number of intervals of each type and even the order in which the intervals of various type emerge can be determined using continued fractions!!!

Theorem: Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $[a_0; a_1, a_2, \dots]$ be the continued fraction expansion of α and $p_k/q_k = [a_0; a_1, \dots, a_k]$ and $D_k = q_k\alpha - p_k$ ($k \geq 0$). Then for any $Q \in \mathbb{N}$ there exists a unique integer $k \geq 0$ such that

$$q_k + q_{k-1} \leq Q < q_{k+1} + q_k \qquad (5)$$

and unique integers r and s satisfying

$$Q = rq_k + q_{k-1} + s, \qquad 1 \leq r \leq a_{k+1} \qquad \text{and} \qquad 0 \leq s \leq q_k - 1 \quad (6)$$

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$$\begin{aligned} N_A &= Q + 1 - q_k && \text{are of length } \delta_A = |D_k|, \\ N_B &= s + 1 && \text{are of length } \delta_B = |D_{k+1}| + (a_{k+1} - r)|D_k|, \\ N_C &= q_k - s - 1 && \text{are of length } \delta_C = \delta_A + \delta_B. \end{aligned}$$

More on dimension 1: Metric viewpoint

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Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ and let $W(\psi)$ be the set of $x \in [0,1]$ such that

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Khintchine's Theorem (1924) *If ψ is monotonic, then*

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More on dimension 1: Metric viewpoint

Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ and let $W(\psi)$ be the set of $x \in [0,1]$ such that

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Example. Take $\psi(q) = q^{-1}(\log q)^{-1-\varepsilon}$ with $\varepsilon > 0$. The Khintchine sum converges. Hence for x picked at random (7) has only a finite number of solution. Also means that for almost every x there exists a constant $c(x,\psi) > 0$ such that

$$|qx - p| \geq \frac{c(x,\psi)}{(q\log q)^{1+\varepsilon}}$$

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$$|q_1\alpha_{i,1} + \cdots + q_n\alpha_{i,n} - p_i| < \psi(Q) \qquad (1 \le i \le m)$$
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holds for infinitely many $(q_1, \ldots, q_n, p_1, \ldots, p_m) \in \mathbb{Z}^{n+m}$.

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Theorem (Khintchine-Groshev /1924-1938/): Suppose that ψ is monotonic. Then

$$\mathbf{Prob}(X \in W(n,m;\psi)) = \begin{cases} 0, & \text{if } \sum_{q=1}^{\infty} q^{n-1}\psi(q)^m < \infty\,, \\ 1, & \text{if } \sum_{q=1}^{\infty} q^{n-1}\psi(q)^m = \infty\,. \end{cases}$$

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The Khintchine-Groshev theorem for non-degenerate manifolds (BBKM, 1998-2002): Suppose that ψ is monotonic and $\mathcal{M}\subset\mathbb{R}^n$ is non-degenerate. Then

$$\mathbf{Prob}(X\in W(n,1;\psi)|X\in\mathcal{M})=\begin{cases}0, & \text{if } \sum_{q=1}^{\infty} q^{n-1}\psi(q)<\infty\,,\\ 1, & \text{if } \sum_{q=1}^{\infty} q^{n-1}\psi(q)=\infty\,.\end{cases}$$

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$$\Lambda_X = \begin{pmatrix} 1 & x_1 & \dots & x_n \\ 0 & 1 & \dots & 0 \\ 0 & & \ddots & \\ 0 & \dots & 0 & 1 \end{pmatrix} \mathbb{Z}^{n+1}$$

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Fact: X is VWA if and only if there exists $\varepsilon >$ such that

$$\delta(g_t\Lambda_X) \le e^{-\varepsilon t} \qquad \text{for arbitrarily large } t.$$