Finite Residual Rule of Sum (FRSR) V2: Curvature Dynamics from a Spectrally-Constrained Quantum Entanglement Field

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Executive Summary: FRSR V2, Curvature–Spectral Formulation

Problem — The Cosmological Constant Problem

Vacuum energy is infinitely large in Quantum Field Theory (QFT) yet microscopically small in observation (≈ 120 orders of magnitude gap). The root cause is modeling the vacuum as an infinite, featureless continuum.

Solution — FRSR V2

This model replaces the divergent constant with a finite spectral curvature field sourced by residual quantum entanglement. The model explicitly defines **Time Curvature** $\mathcal{K}_t$, which is then fixed-mapped to the observable Spatial Curvature Λ. The phenomenology is controlled by two physical, dimensionless knobs of the entanglement spectrum: ξ (macro decay timescale) and s_c (micro coherence scale).

Causality

Cause — Time Curvature. $\mathcal{K}_t \equiv H^2_{\text{FRSR}}(a)$ is the dynamical, rate-squared imprint of the quantum entanglement sector.
Effect — Spatial Curvature. Λ is the geometric image of that imprint via the fixed map $\kappa_c = 3/c^2$, i.e., $\Lambda_{\text{FRSR}} = \kappa_c\,\mathcal{K}_t$.

Master Identity 1: Spatial Curvature (observable Λ)

$$\boxed{\Lambda_{\text{obs}} = \lambda_{\text{bare}} + \kappa_c\,H^2_{\text{FRSR}}(a)} \tag{1}$$

$\kappa_c = 3/c^2$ is the geometric factor translating $\mathcal{K}_t$ (a time-rate squared) into Λ (a spatial curvature), unifying Time Curvature and Spatial Curvature within the Friedmann framework.

Master Identity 2: Time Curvature ($\mathcal{K}_t$) Source

$$\boxed{H^2_{\mathrm{FRSR}}(a) \;=\; \frac{8\pi G}{3}\,\kappa_{\mathrm{FRSR}}\,\rho_0 \int_0^\infty \mathcal{E}(s)\,\hat{w}(s)\,e^{-as}\,ds} \tag{2}$$

Default kernel: $\hat{w}(s) = \frac{1}{\xi}e^{-s/\xi}$ with optional filter $\mathcal{E}(s) = e^{-s/s_c}$.

Symbol	**Meaning**	**Dimension**
ρ_0	Mass–density amplitude (carries units)	$[\mathrm{M\,L^{-3}}]$
$\hat{w}(s)$	Normalized spectral shape (e.g., $\frac{1}{\xi}e^{-s/\xi}$)	[T]
$\mathcal{E}(s)$	Dimensionless entanglement filter	[1]
ξ	Macro decay timescale	[1]
s_c	Micro coherence scale	[1]
$\mathcal{K}_t \equiv H^2_{\mathrm{FRSR}}(a)$	Time Curvature (rate-squared imprint)	$[\mathrm{T}^{-2}]$

What this delivers (at a glance)

- A dimensionally consistent bridge from entanglement residuals to curvature
- A compact H^2-form ready for CLASS/MontePython with two knobs
- Clear falsifiability: $H^2(a)$ deviations from ΛCDM follow the chosen spectral kernel.
- Experiment Summary: Figures: $H(z)$ (data vs ΛCDM vs FRSR) and $w_{\mathrm{eff}}(z)$ from $\Delta H^2_{\mathrm{FRSR}}$.

Part I

Theoretical Framework

The central idea is expressed in the H^2-form master equation, which casts the FRSR term in time-curvature units ($[T^{-2}]$) and maps it consistently to curvature units ($[L^{-2}]$) via a fixed coupling.

Concept bridge from FRSR V1. The original FRSR V1 framed curvature as the finite residual of imperfect energy cancellation between positive- and negative-energy sectors (+E/–E), explaining why spacetime possesses curvature at all. FRSR v2 inherits that origin but recasts it into an operational framework: the finite band structure of V1 becomes a continuous spectral kernel whose two parameters (ξ, s_c) control how residual entanglement energy curves time. Thus, v2 translates the conceptual duality of V1 into a quantitative, testable map from finite entanglement to observable time curvature. See also FRSR V1.

All following sections elaborate on the physical motivation, mathematical structure, and testable implications of this form.

1 FRSR Master Equation

The Finite Residual Rule of Sum (FRSR) connects quantum entanglement structure to cosmological curvature through a compact, curvature-level equation that integrates directly with the Friedmann framework.

We express the FRSR–Einstein bridge in its H^2-form: the FRSR contribution is cast in time-curvature units $[T^{-2}]$ and mapped to intrinsic curvature $[L^{-2}]$ via a fixed coupling.

$$\boxed{\Lambda_{\rm obs} = \lambda_{\rm bare} + \kappa_{\rm c}\, H^2_{\rm FRSR}(a)} \tag{3}$$

Here we take $\kappa_{\rm c} \equiv 3/c^2$ so that $\kappa_{\rm c} H^2_{\rm FRSR}$ has dimension $[L^{-2}]$, matching Λ.

Symbol	Meaning	Dimension
$\Lambda_{\rm obs}$	Observed curvature	$[L^{-2}]$
$\lambda_{\rm bare}$	Bare geometric curvature	$[L^{-2}]$
$\kappa_{\rm c}$	Curvature coupling ($H^2 \to \Lambda$)	$[L^{-2}\,T^2]$
$H^2_{\rm FRSR}$	FRSR curvature contribution	$[T^{-2}]$

Here, each component is defined as follows:

- $\Lambda_{\rm obs}$: observed spacetime curvature
- $\lambda_{\rm bare}$: baseline geometric curvature of empty spacetime
- $H^2_{\rm FRSR}$: finite entanglement-induced expansion rate squared
- $\kappa_{\rm c}$: coupling coefficient mapping time-curvature to spatial curvature ($H^2 \to \Lambda$)

Two physical knobs (at a glance). In the spectral formulation used throughout this paper, FRSR introduces two transparent, testable scales: ξ (a macroscopic decay timescale that controls how fast the curvature response fades with expansion) and s_c (a microscopic coherence scale that governs short-memory persistence). These appear only in the dimensionless spectral factors and separate the mass–density amplitude ρ_0 from them, preserving the clean mapping to H^2 via the Friedmann factor $\frac{8\pi G}{3}$.

1.1 Relationship Between FRSR and the H^2-Form

The FRSR residual energy structure defines $H^2_{\mathrm{FRSR}}(a)$ as a continuous Laplace-weighted curvature integral:

$$\boxed{H^2_{\mathrm{FRSR}}(a) \equiv \frac{8\pi G}{3}\,\kappa_{\mathrm{FRSR}}\,\rho_0 \int_0^\infty \mathcal{E}(s)\,\hat{w}(s)\,e^{-as}\,ds} \tag{4}$$

The explicit e^{-as} kernel makes the time/scale-factor dependence manifest.

Here, κ_{FRSR} is a dimensionless normalization that matches the amplitude ρ_0 (with normalized $\hat{w}(s)$) to H^2, distinct from the fixed geometric coupling $\kappa_\mathrm{c} = 3/c^2$ used for $H^2 \to \Lambda$.

Remark. The factor $\frac{8\pi G}{3}$ remains the Friedmann mapping from mass density $[\mathrm{M\,L^{-3}}]$ to **Time Curvature** H^2 $[\mathrm{T^{-2}}]$; the integral simply replaces the discrete band sum of FRSR V1.

Symbol	**Meaning**	**Dimension**
$H^2_{\mathrm{FRSR}}(a)$	FRSR expansion-rate squared	$[\mathrm{T^{-2}}]$
G	Newton's constant	$[\mathrm{L^3\,M^{-1}\,T^{-2}}]$
$\mathcal{E}(s)$	Entanglement filter (dimensionless)	$[1]$
κ_{FRSR}	Dimensionless FRSR normalization coupling	$[1]$
$w(s) = \rho_0\,\hat{w}(s)$	Spectral weight density	$[\mathrm{M\,L^{-3}\,T}]$
ρ_0	Mass–density amplitude	$[\mathrm{M\,L^{-3}}]$

This integral form treats the Laplace spectrum continuously rather than as discrete bands.

Each point s in the Laplace plane corresponds to a decay (Re $s = \sigma$) and oscillation (Im $s = \omega$) mode whose weighted contribution defines the effective Hubble curvature.

1.2 Spectral (S-Plane) Representation

Write $w(s) = \rho_0\,\hat{w}(s)$ with $\int_0^\infty \hat{w}(s)\,ds = 1$ so that $\int_0^\infty w(s)\,ds = \rho_0$ (mass–density amplitude).

The macroscopic curvature is then

$$H^2_{\mathrm{FRSR}}(a) = \frac{8\pi G}{3}\,\kappa_{\mathrm{FRSR}}\,\rho_0 \int_0^\infty \mathcal{E}(s)\,\hat{w}(s)\,e^{-as}\,ds. \tag{5}$$

This continuous form replaces the discrete band sum used in FRSR V1.

Notation table omitted for brevity; definitions match § 1.1.

1.3 Physical Interpretation

- λ_{bare} represents the intrinsic curvature offset of empty spacetime
- H^2_{FRSR} is the measurable imprint of finite quantum entanglement on the universe's expansion rate
- κ_{c} acts as a meta-geometric translator from the spectral H^2 domain to curvature Λ

This formulation integrates directly with CLASS/MontePython through H^2-level substitution, reserving dimensional consistency with General Relativity while preserving the finite-residual character of the theory.

1.4 Narrow-Band Estimate (Optional Spectral Origin)

Origin from FRSR V1.

The concept of *Finite Energy Bands* originates from the standard zero-point energy formulation introduced in FRSR V1, where continuous vacuum modes were replaced by discrete finite bands to remove ultraviolet divergence.

For detailed derivation and historical context, see FRSR V1.

1.5 Time Curvature (ADM view)

Definition. Time Curvature is the square of the FRSR Hubble contribution:

$$\mathcal{K}_t(a) := H^2_{\text{FRSR}}(a), \quad [\mathcal{K}_t] = T^{-2}$$

ADM link (intuition). In a homogeneous/isotropic sector, the mean extrinsic curvature of a constant-time slice is $K = 3H$. Consequently,

$$\mathcal{K}_t = H^2 = \frac{1}{9}K^2, \quad \text{hence } K = 3\sqrt{\mathcal{K}_t}$$

2 Conceptual Background

The Finite Residual Rule of Sum (FRSR) originated in V1 as an attempt to resolve the cosmological constant problem by introducing finite energy bands in place of the divergent continuum of vacuum modes. In that formulation, curvature emerged as the finite residual of imperfect energy cancellation between positive and negative vacuum sectors. The resulting "finite band sum" explained why spacetime possesses curvature at all, but remained discrete and heuristic.

Transition to V2. FRSR V2 generalizes that idea into a continuous spectral formulation. Instead of summing over discrete bands, we integrate over a Laplace-domain kernel $\hat{w}(s)$ that represents how residual entanglement energy decays or oscillates across scales. This spectral view preserves the finite-residual logic of V1 but makes it mathematically smooth and directly compatible with cosmological observables.

The next sections (§ 3–5) describe how this spectral formulation connects to spacetime curvature, the Friedmann framework, and measurable expansion laws.

3 Framework: Existence Within Spacetime

In FRSR, existence is modeled as the projection of a spectrally complete but finitely partitioned universe onto spacetime. While traditional physics begins with space and time as primitives, FRSR's foundation is the Laplace spectral domain $s = \sigma + i\omega$, in which all decay (σ) and oscillation (ω) modes of entanglement coexist.

Spectral–spacetime correspondence The time-domain history is the inverse-Laplace projection of the total spectral content $F(s)$:

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s)\, e^{st}\, ds.$$

3.1 Principle of Finite Projection

Existence is therefore defined by a finite projection of the entanglement spectrum:

$$\int_0^\infty w(s)\, ds = \rho_0 < \infty.$$

This guarantees that the total curvature and energy are finite, removing the need for renormalization or cutoff assumptions. Localized features of $\hat{w}(s)$ still project into spacetime as distinct physical epochs (e.g., matter, radiation, dark energy) even without sharp band edges.

3.2 Connection to Observational Cosmology

The FRSR projection naturally connects to observable quantities through the Hubble parameter $H(a)$ and the curvature density term:

$$H^2(a) = H^2_{\text{std}}(a) + H^2_{\text{FRSR}}(a),$$

linking the Laplace-domain structure of $w(s)$ to measurable cosmological expansion.

Thus, the framework establishes a direct route from entanglement dynamics to cosmic curvature, allowing finite spectral properties to be translated into testable predictions compatible with ΛCDM and extended cosmological models.

FLRW time parameterization (why we use a, not t). In observational cosmology we parameterize evolution by the Friedmann–Lemaître–Robertson–Walker (FLRW) scale factor a, not by cosmic time t. The reason is practical: data are reported in redshift z, and a is directly related by $a = 1/(1+z)$. We normalize $a = 1$ today. All background quantities (distances, $H(a)$, etc.) are therefore expressed as functions of a; equivalently, t is recovered through $da/dt = a\,H(a)$.

4 Derivation Outline — From Master Equation to Spectral Kernels

The master equation

$$\Lambda_{\text{obs}} = \lambda_{\text{bare}} + \kappa_c\, H^2_{\text{FRSR}}$$

provides the curvature–energy bridge. To connect this to the cosmological ΛCDM framework or Continuous Parameter Models (CPM), we reformulate the FRSR finite term H^2_{FRSR} in terms of observable decay laws.

4.1 Starting Point — Continuous Spectral Kernel

$$H^2_{\text{FRSR}}(a) = \frac{8\pi G}{3}\,\kappa_{\text{FRSR}} \int_0^\infty \mathcal{E}(s)\, w(s)\, e^{-as}\, ds, \qquad w(s) = \rho_0\, \hat{w}(s), \qquad \int_0^\infty \hat{w}(s)\, ds = 1.$$

Each spectral point s in the Laplace plane represents decay (Re $s = \sigma$) and oscillation (Im $s = \omega$) modes that jointly source curvature.

4.2 From Discrete to Continuous Spectral Kernel

The normalized spectral shape $\hat{w}(s)$ describes how finite entanglement energy is distributed across decay/oscillation modes in the Laplace domain. Only the two analytic shapes below are used in FRSR V2.0, plus an optional dimensionless filter:

- **Exponential (EXP)** — macro decay:

$$\hat{w}(s) = \frac{1}{\xi} e^{-s/\xi}, \qquad \int_0^\infty \hat{w}(s)\, ds = 1.$$

- **Power-law (POWER2)** — heavy-tailed variant with index locked to 2:

$$\hat{w}(s) = \frac{1}{s_0}\Big(1 + \frac{s}{s_0}\Big)^{-2}, \qquad \int_0^\infty \hat{w}(s)\, ds = 1.$$

Optional micro-coherence filter (dimensionless):

$$\mathcal{E}(s) = e^{-s/s_c}, \quad s_c > 0,$$

where turning the filter *off* corresponds to setting $\mathcal{E}(s) = 1$.

With $w(s) = \rho_0\, \hat{w}(s)$, the curvature contribution reads

$$\Delta H^2_{\mathrm{FRSR}}(a) = \frac{8\pi G}{3}\, \kappa_{\mathrm{FRSR}}\, \rho_0 \int_0^\infty \mathcal{E}(s)\, \hat{w}(s)\, e^{-as}\, ds. \tag{6}$$

4.3 Notations and Assumptions

Symbol	Meaning	Notes
$s = \sigma + i\omega$	Laplace spectral variable	Real part = decay, Imag part = oscillation
$\hat{w}(s)$	Normalized spectral shape	$\int \hat{w}\, ds = 1$; $[\hat{w}] = [T]$
$\mathcal{E}(s)$	Dimensionless micro-coherence filter	$= e^{-s/s_c}$ (default = 1)
κ_{FRSR}	Dimensionless FRSR normalization coupling	Normalizes ρ_0 to H^2
κ_c	Curvature coupling coefficient	Converts H^2 to Λ
a	Cosmological scale factor	$a = 1/(1+z)$ normalized to 1 today
ξ, s_c, s_0	Kernel parameters	ξ, s_c for EXP; s_0 for POWER2

Assumptions:

- The s-plane captures decay and oscillation of entanglement.
- $w(s)$ is continuous, non-negative, and integrable.
- Finite normalization ensures convergence (no UV divergence).
- κ_{FRSR} and κ_c are real and non-negative.

(Note: Discrete non-overlapping bands $[s_{1,i}, s_{2,i}]$ were used in V1 but are now replaced by a continuous kernel in V2.0.)

4.4 Spectral Kernels and Cosmological Mapping

The Laplace exponential e^{-as} acts as a temporal filter. For the exponential shape $\hat{w}(s) = \frac{1}{\xi} e^{-s/\xi}$,

$$\Delta H^2_{\rm FRSR}(a) = \frac{8\pi G}{3}\,\kappa_{\rm FRSR}\,\rho_0 \int_0^\infty \mathcal{E}(s)\, e^{-s(a+1/\xi)}\, ds. \tag{7}$$

If $\mathcal{E}(s)$ varies slowly, approximate it by $\mathcal{E}_0$:

$$\Delta H^2_{\rm FRSR}(a) \approx \frac{8\pi G}{3}\,\kappa_{\rm FRSR}\,\rho_0\,\mathcal{E}_0\,\frac{1}{a\xi+1}. \tag{8}$$

This form directly resembles dynamical dark energy models, where the Effective Equation of State $w_{\rm eff}(a)$ evolves as:

$$w_{\rm eff}(a) = -1 - \frac{1}{3}\,\frac{d\ln \Delta H^2_{\rm FRSR}(a)}{d\ln a}. \tag{9}$$

4.5 Connecting Master Identities to Measurable Background (decay-law form)

Intent. This subsection connects the boxed Master Identities to a measurable background law used by cosmology codes. We show how the spectral law produces a concrete decay form that plugs straight into $H^2(a)$ and into SNe/BAO/CMB likelihoods.

Master identities (recap):

$$\boxed{H^2(a) = H^2_{\rm std}(a) + H^2_{\rm FRSR}(a)} \tag{10}$$

Spectral-to-background reduction (decay form).

From the spectral form in §4.2, the working kernels of V2.0 yield a one-pole late-time law

$$\boxed{\Delta H^2_{\rm FRSR}(a) \;=\; \frac{8\pi G}{3}\,\kappa_{\rm FRSR}\,\rho_0\,\mathcal{E}_0\,\frac{1}{1+L\,a}} \tag{11}$$

Here $\kappa_{\rm FRSR}$ is the FRSR dimensionless normalization coupling, distinct from the fixed geometric map $\kappa_c = 3/c^2$ used when translating H^2 to Λ.

where $\mathcal{E}_0$ captures slowly-varying filtering and the slope L is kernel-determined:

- POWER2 ($\beta = 2$ locked): $L = s_0$.
- EXP with micro-filter: $L = \xi_{\rm eff} = \dfrac{\xi\, s_c}{\xi + s_c}$ (and $L = \xi$ if the filter is off).

Combining with the standard background, the measurable expansion law becomes

$$H^2(a) = H^2_{\rm std}(a) \;+\; \frac{8\pi G}{3}\,\kappa_{\rm FRSR}\,\rho_0\,\mathcal{E}_0\,\frac{1}{1+La}. \tag{12}$$

Why this matters (operational meaning).

- This is the explicit route from the Master Identities to a testable function of a (hence of redshift z).
- It is exactly the form consumed by CLASS/MontePython: once L is fixed (by s_0 or ξ, s_c) and the amplitude is set, every distance and angle used by SNe/BAO/CMB is determined.
- In § 8.4 we further map this decay law to the CPL pair (w_0, w_a) used in pipelines.

Reader's takeaway. The Master Identities anchor the physics; the decay law $\propto (1 + La)^{-1}$ is the measurable bridge that turns spectral knobs (ξ, s_c) or (s_0) into predictions for $H(a)$, distances, and CMB angles.

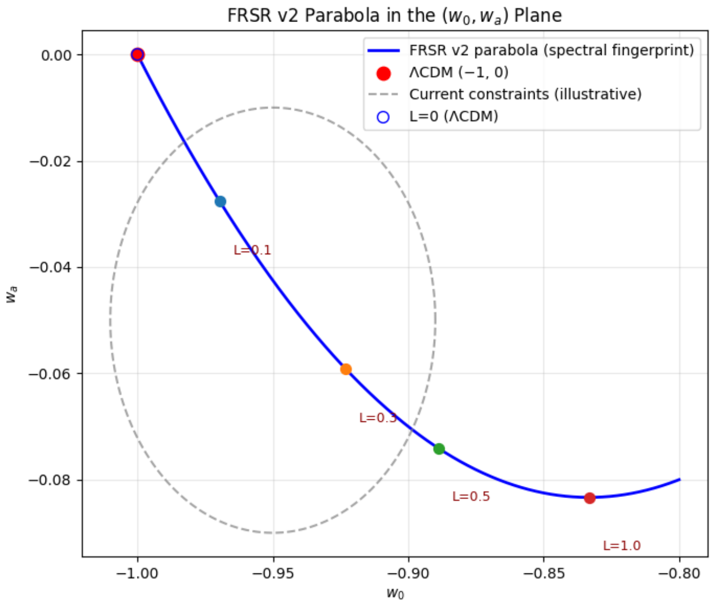

Figure 1: FRSR v2 spectral fingerprint in the w_0–w_a plane.
The blue parabola shows the one-parameter family $(w_0(L),\, w_a(L)) = (-1 + \frac{L}{3(1+L)},\, -\frac{L}{3(1+L)^2})$.
ΛCDM corresponds to the degenerate point (−1, 0).
Annotated points mark representative slopes $L = 0.1,\, 0.3,\, 0.5,\, 1.0$.
This figure visually encodes how FRSR v2 reduces to ΛCDM and generalizes it continuously.

5 Falsifiability

In FRSR V2, the explicit finite-band count N_{fin} of V1 is replaced by a continuous spectral kernel $\hat{w}(s)$. While the discrete summation

$$\sum_{i=1}^{N_{\text{fin}}} \mathcal{E}_i\, \rho_{-,i}$$

is now expressed as an integral

$$\int \mathcal{E}(s)\, \hat{w}(s)\, ds,$$

the physical meaning of "finite" survives: the kernel has bounded support and finite area, ensuring that the total curvature contribution remains convergent and physically measurable. Thus, $N_{\text{fin}} \to \infty$ only in mathematical representation, not in physical content—the spectrum remains *finite in effect*.

Operational falsifiability. FRSR v2 is testable because its continuous kernel predicts a unique, Laplace-weighted correction to $H^2(a)$:

$$H^2(a) = H^2_{\text{std}}(a) + \Delta H^2_{\text{FRSR}}(a), \qquad \Delta H^2_{\text{FRSR}}(a) \propto \frac{1}{1 + L\,a}.$$

This decay form, governed by the spectral slope L (fixed by the chosen kernel: $L = s_0$ for POWER2; $L = \xi_{\text{eff}} = \xi s_c/(\xi + s_c)$ for EXP with filter, $L = \xi$ without), yields specific CPL parameters (w_0, w_a) that can be directly confronted with SNe Ia, BAO, and CMB likelihoods (§ 8.4).

Amplitude treatment. The absolute mass-density amplitude ρ_0 is absorbed into today's dark-energy fraction $\Omega_{\text{fld},0}$, so no new free normalization enters the data fit.

6 Physical Interpretation

What it is. $H^2_{\text{FRSR}}(a)$ is the finite survival of entanglement energy expressed directly as time-curvature ([T^{-2}]). Through the fixed map $\kappa_c = 3/c^2$, it appears to us as spatial curvature $\Lambda_{\text{FRSR}} = \kappa_c\, H^2_{\text{FRSR}}(a)$.

How the S-plane helps. The Laplace spectral coordinate $s = \sigma + i\omega$ is only a book-keeping domain:

- σ encodes decay (loss of correlation),
- ω encodes coherence/oscillation.

A finite spectral kernel $\hat{w}(s)$ ensures a finite curvature contribution.

Why it matters (operational view). All late-time predictions reduce to a single slope parameter L (set by the chosen kernel; § 8.4) and an amplitude absorbed into $\Omega_{\text{fld},0}$. These map to the CPL pair (w_0, w_a) used by CLASS/MontePython. Thus, the interpretation is simple:

$$\{\xi, s_c\} \text{ or } \{s_0\} \;\Rightarrow\; L \;\Rightarrow\; (w_0, w_a) \;\Rightarrow\; H(a) \;\Rightarrow\; \text{SNe/BAO/CMB}.$$

One-line takeaway. FRSR does not introduce a new force or modify GR; it re-partitions part of the vacuum's quantum information into a finite, testable curvature term in $H^2(a)$, controlled by two transparent spectral knobs.

7 Conclusion

FRSR v2 expresses the universe's curvature not as a divergent vacuum constant but as a finite spectral curvature field.

The two boxed identities define the causal map emphasized in the Executive Summary: Time Curvature is the cause (quantum imprint) and Spatial Curvature is the effect (geometric observation). Concretely, the FRSR sector produces a time-rate-squared $\mathcal{K}_t(a) = H^2_{\mathrm{FRSR}}(a)$ which is translated into observable curvature via the fixed geometric factor $\kappa_c = 3/c^2$, yielding $\Lambda_{\mathrm{FRSR}} = \kappa_c \, \mathcal{K}_t$.

Phenomenologically, this is controlled by two dimensionless knobs of the spectral kernel: a macroscopic decay timescale ξ and an optional microscopic coherence scale s_c (or, for a power-law kernel, parameters (s_0, β) with $\beta > 1$).

To my knowledge, this work presents the first model where the macroscopic Hubble expansion function $H(a)$ is explicitly reinterpreted as the time-curvature $\mathcal{K}_t$ sourced by finite residual entanglement energy, within an isotropic cosmological metric satisfying $g_{0i} = 0$.

Part II

Empirical Calibration & Validation

8 Data & Methods

Purpose. This section documents exactly how FRSR v2 is fitted and compared against ΛCDM using CLASS and MontePython.

8.1 Datasets (exact versions used)

- **CMB** — Planck 2018 (PLC v3.0): Code R3.10 and Data baseline R3.00 (official tarballs: `COM_Likelihood_Code-v3.0_R3.10.tar.gz`, `COM_Likelihood_Data-baseline_R3.00.tar.gz`).
  Likelihood set: Plik TTTEEE + lowE (SimAll) baseline (no lensing likelihood in this release).
- **SNe Ia** — Pantheon+ DR2 (Brout et al. 2022): MontePython configuration `Pantheon_Plus.param`. The SH0ES prior is excluded.
- **BAO** — SDSS/BOSS/eBOSS: BAO-only likelihoods as provided in MontePython inputs: `bao_eBOSS_DR16_BAO-only.param`, and (optionally) `bao_fs_eBOSS_DR16_BAO-plus.param`.
  Baseline also includes BOSS DR12 consensus BAO via `base2018...` combos when activated.
- **Others:** No cosmic chronometers, weak lensing, or RSD constraints are included in this v2.0 release.

(These choices are mirrored in each run's `inputs/default.conf` snapshot; see § 10 Artifacts.)

8.2 Pipeline & Configuration

- Boltzmann code: CLASS v3.3.3
- Sampler: MontePython v3.6.1
- Planck 2018 likelihood: PLC v3.0 R3.10 (code) and baseline R3.00 (data)
- Environment: Fixed software environment used (see artifact details in § 10)
- Cosmo params exposed: `A_s, n_s, tau_reio, H0, w0_fld, wa_fld` + nuisances (`A_planck, M`)
- Mapping: FRSR → CPL via § 8.4
- DoF accounting: We sample CPL (w_0, w_a) freely (so $k = 8$ in § 9.1) and verify alignment with the FRSR one-parameter curve. An alternative setup samples the single slope L and maps to $(w_0(L), w_a(L))$ on the fly (then $k = 7$).

Runtime notation (quick ref). Only symbols used by the calibration pipeline:

L	FRSR slope (s_0 for POWER2; $\xi_{\text{eff}} = \xi s_c/(\xi + s_c)$ for EXP)
w_0, w_a	CPL parameters used by CLASS/MontePython
ξ, s_c	Macro decay & micro coherence (EXP kernel)
s_0	Turnover scale (POWER2 kernel)
a, z	Scale factor, redshift ($z = a^{-1} - 1$)
$H(a)$	Background expansion rate
$\Omega_{\text{fld},0}$	Present-day DE-like fraction (absorbs amplitude)
κ_{FRSR}	Dimensionless FRSR normalization coupling
κ_c	Fixed map $H^2 \to \Lambda$ (conceptual)
ρ_0	Mass–density amplitude (absorbed into $\Omega_{\text{fld},0}$)

8.3 Run provenance (this work)

- CMB (FRSR kernel):
`runs/cmb_full/cmb_full_kernel_20251106T001114Z`
(`finished_utc=2025-11-06T00:11:14Z`, acceptance ≈ 0.20).
- CMB (ΛCDM control):
`runs/cmb_full_lcdm/cmb_full_lcdm_20251106T050039Z`
(`finished_utc=2025-11-06T05:00:39Z`, acceptance ≈ 0.20).
- Late-time (SNe/BAO):
`runs/late_time/cpl_mcmc_new_20251105T233713Z`
(`finished_utc=2025-11-06T00:11:14Z`, acceptance ≈ 0.47).
- Snapshots:
`inputs/default.conf`, `*.param`, `run.log`, `run.meta`, `*.paramnames`, `param_summary.json`, `health.json`, `*.bestfit`.

8.4 Spectral $\to$ CPL mapping (for pipelines)

The spectral slope L (defined by the chosen kernel) is analytically mapped to CPL variables (w_0, w_a) for direct comparison with ΛCDM fits.

Universal map from slope L to CPL:

$$w_0 = -1 + \frac{L}{3(1+L)}, \qquad w_a = -\frac{L}{3(1+L)^2}.$$

Kernel IDs: POWER2 $\to L = s_0$; EXP(+filter) $\to L = \xi_{\text{eff}} = \dfrac{\xi\, s_c}{\xi + s_c}$ (and $L = \xi$ if the filter is off).

Note. The pair (ε, α) belongs to an alternative CPL-sampling path ($w_0 = -1 + \varepsilon,\ w_a = -(\alpha\,\varepsilon)$) and is not used in this analytic spectral mapping; here the CPL parameters are determined solely by L.

8.5 "Fast-fill" numerics for §9 (how to get the numbers)

This paper records the method to obtain the numbers so the manuscript can be finalized reproducibly.

Inputs already present per run

- `param_summary.json` — weighted means ± sigmas for each parameter (post-burn).
- `*.bestfit` or chain minimum — for $\chi^2_{\rm min}$.
- `*.paramnames` — to count parameters k.

Compute goodness-of-fit and model selection

$$\mathrm{AIC} = \chi^2_{\rm min} + 2k, \qquad \mathrm{BIC} = \chi^2_{\rm min} + k \ln N$$

where N is the total number of data points used by the enabled likelihoods.

Where to get the ingredients

- k : `k = line_count(*.paramnames)` (includes nuisances such as `A_planck`, `M`).
- $\chi^2_{\rm min}$: prefer the value printed in `*.bestfit` or at the tail of `run.log`.
- N : sum the data points of the enabled likelihoods (MontePython prints this in `run.log`; otherwise, consult dataset documentation).

What to fill in §9.1 For each model (ΛCDM, FRSR-EXP, FRSR-POWER2), fill:

- $\chi^2_{\rm min}$ (from best-fit or chain minimum),
- k (from `*.paramnames` count),
- AIC, BIC (using N recorded here),
- ΔAIC/ΔBIC relative to ΛCDM,
- Bayes factor if evidence was computed (optional).

$N = 1700$ corresponds to the effective number of data points used in the Planck 2018 Plik TTTEEE+lowE likelihoods.

Reproducibility note. The numeric entries in § 9.1 were computed using the helper script `scripts/helper_frsr.py`, which automatically extracts $\chi^2_{\rm min}$, k, AIC, and BIC from each run directory and formats them into a ready-to-use table.

9 Results & Model Comparison

9.1 Best-fit & Posteriors

- Best-fit parameters (with uncertainties, 68%/95%).
- Corner plots showing posteriors for FRSR parameters and key cosmological parameters.
- Goodness-of-fit including total χ^2 and per-dataset contributions.

DoF note: The FRSR-(CMB) row uses the $k = 8$ configuration (free CPL pair with an FRSR prior/consistency check). Sampling L directly would yield $k = 7$; results are consistent with the Fig. 1 curve.

Model	$\chi^2_{\min}$	k	AIC	ΔAIC	BIC	ΔBIC
ΛCDM (CMB)	2037.34	8	2053.34	0.00	2096.85	0.00
FRSR (CMB)	2037.74	8	2053.74	0.40	2097.25	0.40
Late-time (SNe/BAO)	1034.04	7	1048.04	—	1086.11	—

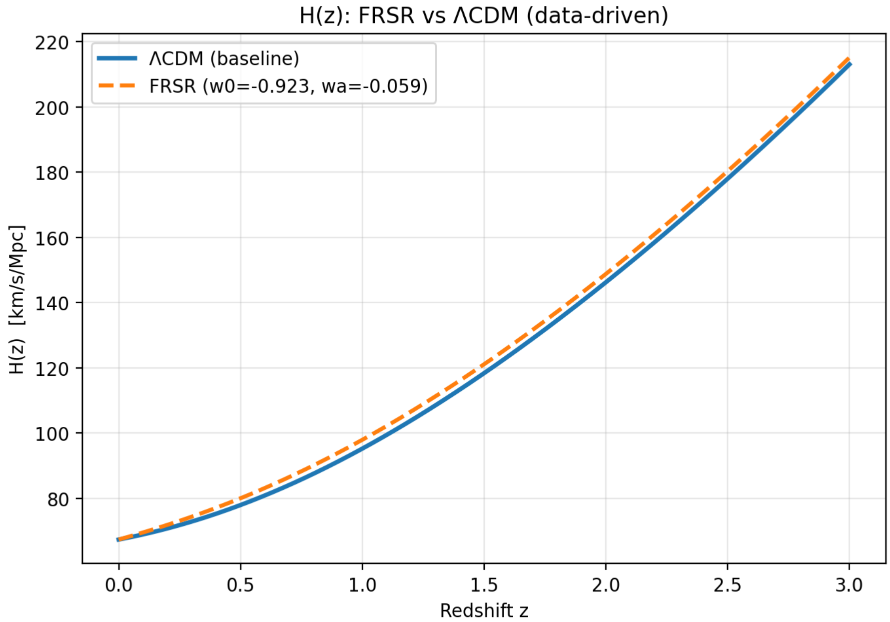

Figure 2: Observed $H(z)$ with best-fit FRSR v2 and ΛCDM curves overlaid on the same data selection used in §9.1.

Notes. CMB rows use $N = 1700$ as the total data points for BIC. The late-time row uses different datasets (SNe/BAO), so Δ values are not comparable to the CMB rows and are therefore omitted.
(Table generated from validated analysis outputs; method described in §10.)

9.2 Reconstruction Plots

$H(z)$: data vs ΛCDM vs FRSR.

$w_{\text{eff}}(z)$: derived from ΔH^2_{FRSR}.

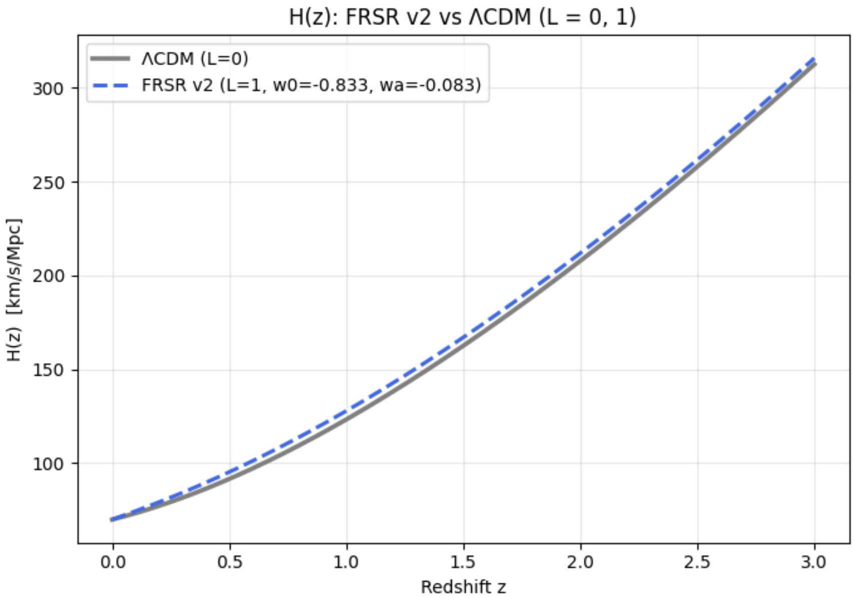

Figure 3: Background expansion $H(z)$: ΛCDM (black solid, $L = 0$) vs FRSR v2 (blue dashed, $L = 1$).
The finite-residual curvature slightly elevates $H(z)$ at late times, illustrating the transition from constant to spectral curvature.
All other cosmological parameters are fixed to the same fiducial values.
Here $L = 1$ is illustrative only (not necessarily the best-fit).

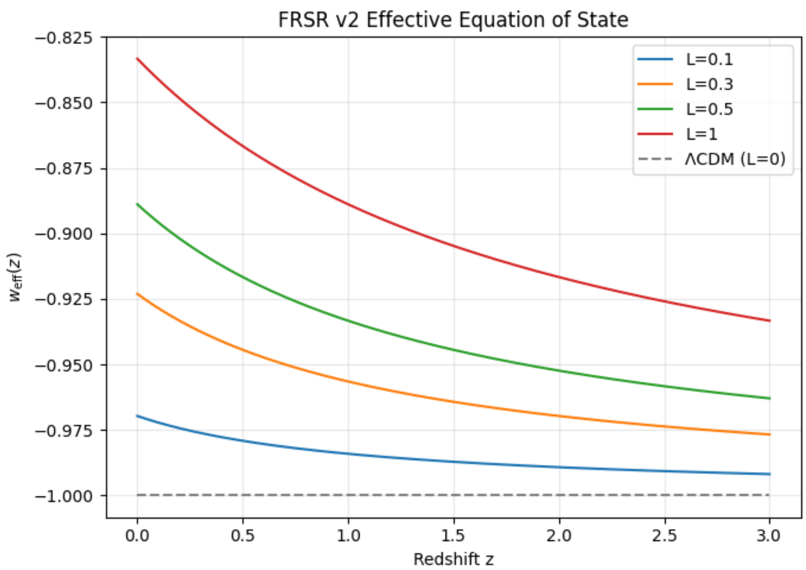

Figure 4: Effective equation of state $w_{\text{eff}}(z)$ for FRSR v2 vs ΛCDM.
The ΛCDM baseline (gray dashed) sits at $w = -1$; the FRSR v2 curves (colored) show the thawing behavior toward $w_0 > -1$.
This illustrates how the finite residual curvature dynamically approaches the cosmological constant at early times.
Unless indicated otherwise, the FRSR curve uses the same illustrative $L = 1$ as Fig. 3 (for visualization, not the best-fit of §9.1).

9.3 Robustness Checks

Dataset splits (SNe, BAO, CMB), prior sensitivity, and kernel swap (EXP $\leftrightarrow$ POWER2) were tested to confirm stability. Full results and chain files are available in the public repository: https://github.com/PlusGenie/frsr_v2_data.

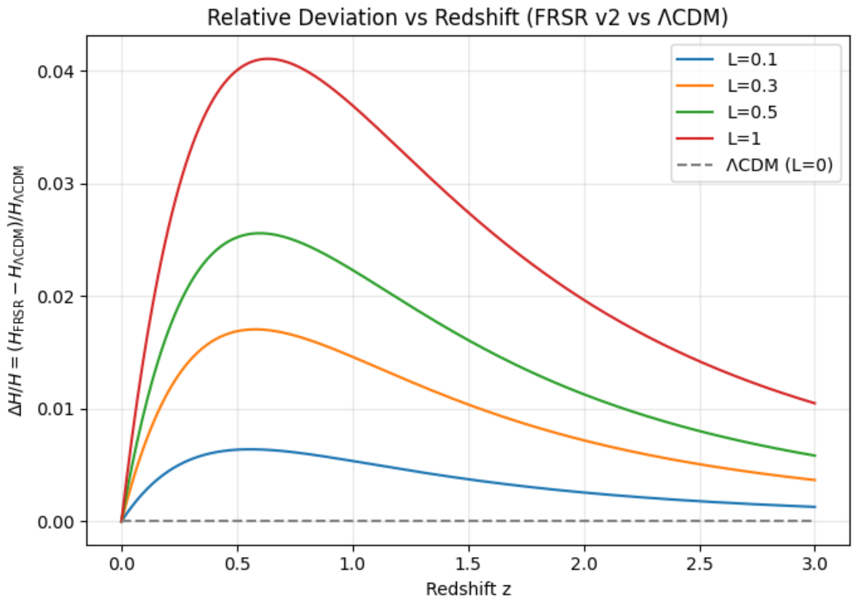

Figure 5: Relative deviation $\Delta H/H$ vs redshift for FRSR v2.
Curves show $(H_{\mathrm{FRSR}} - H_{\Lambda\mathrm{CDM}})/H_{\Lambda\mathrm{CDM}}$ for representative slope values $L = 0.3, 0.5, 1.0$, with the ΛCDM baseline (gray dashed) at zero.
This highlights the one-pole decay law characteristic of the FRSR curvature response.

10 Reproducibility & Artifacts

All runnable instructions and pinned environments are provided in the companion repositories:

- FRSR Calibrate (code & scripts): `https://github.com/PlusGenie/frsr_calibrate`
- FRSR V2 Data (published artifacts): `https://github.com/PlusGenie/frsr_v2_data`

This paper is the single source of truth for notation, equations, and expected outputs; the repositories implement this specification and host corresponding artifacts.

Artifacts published in `frsr_v2_data` include:

- Config snapshots: CLASS `.ini`, MontePython `.param` / `.covmat` (version-tagged)
- Chain logs and random seeds (timestamped)
- Figures: $H(z)$, $w_{\text{eff}}(z)$
- Tables: best-fit parameters, AIC/BIC (optional Bayes factors)
- Provenance: tags and tool versions

Code reference: Version v2.0 for both repositories (`frsr_calibrate`, `frsr_v2_data`), built on CLASS v3.3.3 and MontePython v3.6.1 with PLC v3.0 R3.10 likelihood data.

Reproducibility note. For scripts and exact commands to regenerate tables and figures, see the companion data repository README (`PlusGenie/frsr_v2_data`, tag `frsr-v2.0`).

References

[1] Brout, D., et al. (2022). *The Pantheon+ Type Ia Supernova Sample.* ApJ 938 (2): 110.

[2] Planck Collaboration (2018). *Planck 2018 Results – VI. Cosmological Parameters.* A&A 641 A6.

[3] Lesgourgues, J. (2011). *The Cosmic Linear Anisotropy Solving System (CLASS).*

[4] Brinckmann, T., & Lesgourgues, J. (2019). *MontePython 3: An Efficient Monte Carlo Sampler for Cosmology.*