4.2 Arbitrage opportunity

Remark 10: By Notation 8

$$V_t = \bar{h}_t \cdot \bar{X}_t = \bar{h}_t \cdot \frac{\bar{S}_t}{B_t} = \frac{\bar{h}_t \bar{S}_t}{B_t}$$

$\Rightarrow V_t =$ the portfolio value at the end of t_{th} trading period expressed in units of the numéraire asset.

Proposition 11: For a trading stragety $\bar{h}$, the following conditions are equivalent:

(1) $\bar{h}$ is self-financing

(2) $\bar{h}_t \cdot \bar{X}_t = \bar{h}_{t+1} \cdot \bar{X}_t$ for $t = 0, 1, 2, \cdots, T-1$

(3) $V_t = V_0 + G_t$

$$= \bar{h}_0 \bar{X}_0 + \sum_{k=1}^{t} h_k (X_k - X_{k-1}) \text{ for all } t.$$

Remark 12: By Proposition 11 (2), we know that if $\bar{h}$ is self-financing,

$$\boxed{h^0_{t+1} - h^0_t = -(h_{t+1} - h_t) X_t}$$

已知 $\bar{h}_t \cdot \bar{X}_t = \bar{h}_{t+1} \cdot \bar{X}_t \Rightarrow h^0_t \cdot 1 + h_t X_t = h^0_{t+1} B_t + h_{t+1} X_t$

$$\Rightarrow (h^0_{t+1} - h^0_t) \cdot 1 + (h_{t+1} - h_t) X_t = 0$$

指买卖股票的钱全部由债券来支出/消耗

股票头寸从 $h_t \to h_{t+1}$ 这些钱从债券头寸从 $h_t^0 \to h_{t+1}^0$ 调整值价差来出/吸收

tip: 在讨论 arbitrage opportunity 时，我们会把非 self-financing 的情况去掉，因为如果有外来资金的进入，那么一定可以实现 arbitrage opportunity. 这样我们再讨论 arbitrage opportunity 的可能性就无意义了!!

Definition 13: (1) A self-financing trading strategy is called an arbitrage opportunity if its value process satisfies:

(i) $V_0 \le 0$

(ii) $V_T \ge 0$

$P(V_T > 0) > 0$

(2) A market model does not allow for arbitrage opportunity is called arbitrage-free.

Proposition 14: The market model admits an arbitrage opportunity $\Leftrightarrow$ There exists $t \in \{1, \cdots, T\}$ and a trading strategy $h \in \mathcal{F}_{t-1}$ such that

$$(*) \begin{cases} h(X_t - X_{t-1}) \ge 0 \quad P\text{-a.s.} \\ \mathbb{P}(h \cdot (X_t - X_{t-1}) > 0) > 0 \end{cases}$$

即有套利机会只需要其中某一段 $(t-1, t)$ 存在套利机会

同理，如何 muti-period model 是 arbitrage-free 的话，那每一段 one-period model 都是 arbitrage-free.

Proof: "$\Rightarrow$" Suppose that the model admits arbitrage opportunity

$\Rightarrow \exists$ arbitrage opportunity $\bar{k}=(k^0;k^T)^T$ with value function (V_t)

t是第一次发生 arbitrage opportunity 的情况 $\longrightarrow$ Let $t := \min\{s: V_s \ge 0 \ P\text{-a.s. and } P(V_s>0)>0\}$

Then $t \le T$.

By sumption we know that

either $V_{t-1}=0$ P-a.s. or $P(V_{t-1}<0)>0$

(i) $V_{t-1}=0$: Let $h=k_t$ by the definition of t

$$h\cdot(X_t - X_{t-1}) = k_t(X_t - X_{t-1}) = V_t - V_{t-1} = V_t - 0 = V_t \ge 0$$

$\Rightarrow h\cdot(X_t - X_{t-1}) \ge 0$ P-a.s.

$$P(h(X_t - X_{t-1})>0) = P(V_t>0) > 0$$

关键就是证明

$h(X_t - X_{t-1}) = V_t$

V_t性质已知

那$h(X_t - X_{t-1})$性质就可得出

(ii) $P(V_{t-1}<0)>0$: Let $h=k_t I_{\{V_{t-1}<0\}}$

Then h is $\mathcal{F}_{t-1}$-measurable and

$$h(X_t - X_{t-1}) = k_t(X_t - X_{t-1}) I_{\{V_{t-1}<0\}} = (V_t - V_{t-1}) I_{\{V_{t-1}<0\}}$$

$$\geq -V_{t+1} I_{\{V_{t+1}<0\}} > 0$$

$$\Rightarrow h(X_t - X_{t-1}) > 0$$

" $\Leftarrow$ " For t and h defined as in $(*)$

Define

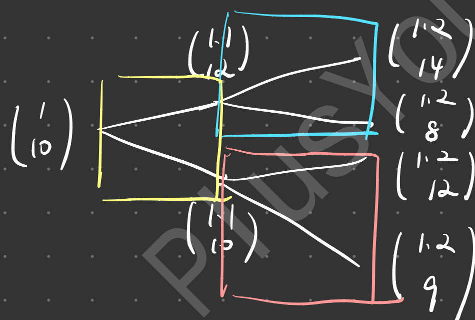
$$K_s := \begin{cases} h & \text{if } s=t \\ 0 & \text{o.w.} \end{cases}$$

Define K_s^0 by remark 12

Then $\bar{K}$ is an arbitrage opportunity.

Example 15: $\Omega = \{w_1, w_2, w_3, w_4\}$ 有一张债券一张股票

$t = 0, 1, 2$



方法: 拆成 3 个 one-period model 来看

check: $\boxed{\frac{8}{1.2} < \frac{12}{1.1} < \frac{14}{1.2}}$

$\boxed{\frac{9}{1.2} < \frac{10}{1.1} < \frac{12}{1.2}}$

$\boxed{\frac{10}{1.1} < \frac{10}{1} < \frac{12}{1.1}}$