$V_T \geq 0$ $\mathbb{Q}$-a.s. $\Rightarrow$ $V_0 = \mathbb{E}_{\mathbb{Q}}[V_T] \geq 0$ → 不存在 $V_0 < 0$ 的情况

且 $V_T > 0 \to V_0 > 0$

$V_T = 0 \to V_0 = 0$

⇒ There is no arbitrage opportunity.

⇒ This market model is arbitrage-free.

"⇒" 证明方法较多（难，略）

4.4 Arbitrage-free prices for European contingent claim

Definition 21: (1) A non-negative r.v. C on $(\Omega, \mathcal{F}, \mathbb{P})$ is called a <u>European contingent claim</u>.

(2) T is called the expiration date at maturity of C.

(3) The discounted value of C is given by $H := \frac{C}{B_T}$

H is called the discounted (European) claim, associated with C.

Assumption: $\Phi \neq \phi$

Definition 22: A contingent claim C is <u>attainable</u> (or <u>replicable</u>, <u>redundant</u>), if $\exists$ self-financing trading strategy $\bar{h}$ s.t. $C = \bar{h}_T \bar{S}_T$, $\bar{h}$ is called a <u>replicating</u> strategy for C.

Remark 23: C is attainable $\Longleftrightarrow$ the corresponding discounted claim

$$H = \bar{h}_T \cdot \bar{X}_T = V_T(\bar{h}) = V_0 + \sum_{t=1}^{T} h_t (X_t - X_{t-1}) \qquad X_T = \frac{S_T}{B_T}$$

We call $\bar{h}$ is replicating strategy for H and $H := \frac{C}{B_t}$ is attainable.

Theorem 24: (1) Any attainable discounted claim H is integral w.r.t each equivalent martingale measure, i.e. $\mathbb{E}^*[H] < \infty \quad \forall \mathbb{P}^* \in \mathcal{P}$

(2) For each $\mathbb{P}^* \in \mathcal{P}$, the value process of any replicating strategy $\bar{h}$ satisfies

$$V_t(\bar{h}) = \mathbb{E}^*[H \mid \mathcal{F}_t] \quad \mathbb{P}^*\text{-a.s. for } t = 0, 1, \cdots, T$$

(3) $(V_t)_{0 \le t \le T}$ is a nonnegative $\mathbb{P}^*$-martingale. (通过 Theorem 18 来理解)

$V_t = \frac{C}{B_t} \geq 0 \rightarrow$ nonnegative

$E(V_t^-) = 0 < \infty$

则 (V_t) 是 $\mathbb{Q}$-martingale

Definition 25: (1) A real number $\pi^H \geq 0$ is called an arbitrage-free price of a discounted claim H if there exists an adapted stochastic process X^{N+1} s.t.

(i) $X_0^{N+1} = \pi^H$

$X_t^{N+1} \geq 0$ for $t = 0, 1, \cdots, T$

$X_T^{N+1} = H$

(ii) the enlarged market model with discounted price process $\{X^0, X^1, \cdots, X^{N+1}\}$ is arbitrage-free.

(2) The set of all arbitrage-free prices of $H = \Pi(H)$

Denote

$$\pi_{inf}(H) = \inf \Pi(H) \quad \text{and} \quad \pi_{sup}(H) = \sup \Pi(H)$$

Remark 26: The (undiscounted) arbitrage-free price of the contingent claim $C := B_T H$ is given by $\pi^C := B_0 \pi^H$

one-period 这里不用注意，因为是有限的.
multi-period 则需要注意无穷的情况.

Theorem 27: $\Pi(H) = \{\mathbb{E}^*(H) : \mathbb{P}^* \in \mathcal{P}, \mathbb{E}^*[H] < \infty\}$ and in case $\Pi(H) \neq \emptyset$, we have $\pi_{inf}(H) = \inf_{\mathbb{P}^* \in \mathcal{P}} \mathbb{E}^*[H]$ $\pi_{sup}(H) = \sup_{\mathbb{P}^* \in \mathcal{P}} \Pi(H)$

由 Theorem 24 知，当 H is attainable 时，该条件一定满足

证明见 Föllmer-Schied. Chap. 5

proof: (1) By Theorem 20

π^H is arbitrage-free price $\Longleftrightarrow$ $\exists$ equivalent martingale measure $\mathbb{P}^*$ for the extended market model

$$\Rightarrow X_t^i = \mathbb{E}^*[X_T^i \mid \mathcal{F}_t] \text{ for } t = 0, 1, 2, \cdots, T, \quad i = 1, 2, \cdots, N+1$$

$$\Rightarrow X_0^{N+1} = \mathbb{E}^*[X_T^{N+1} \mid \mathcal{F}_0] \quad (\pi^H = \mathbb{E}^*[H \mid \mathcal{F}_0] = \mathbb{E}^*[H], \ \mathcal{F}_0 = \{\emptyset, \Omega\})$$

$$\Rightarrow \Pi(H) \subseteq \{\mathbb{E}^*[H] : \mathbb{P}^* \in \mathcal{P} \text{ and } \mathbb{E}^*[H] < \infty\}$$

(2) If $\pi^H = \mathbb{E}^*[H]$ for some $\mathbb{P}^* \in \mathcal{P}$, define

$$X_t^{N+1} = \mathbb{E}^*[H \mid \mathcal{F}_t] \quad t = 0,1,2,\cdots,T$$

then X^{N+1} satisfies the condition (i) in definition 25.

Moreover, $(X_t^{N+1})_{t=0,1,\cdots,T}$ is a martingale under $\mathbb{P}^*$

$\Rightarrow \pi^H$ is an arbitrage-free price

$\Rightarrow \pi^H \in \pi(H)$

$\Rightarrow \pi(H) \supseteq \{\mathbb{E}^*[H] : \mathbb{P}^* \in \mathcal{P},\ \mathbb{E}^*[H] < \infty\}$

Example 28: (1) Consider a European call option $C^{call} = (S_T - K)^+$ with strike price K and maturity T. Suppose B_t is increasing in t and $B_0 = 1$.

Then for $\mathbb{P}^* \in \mathcal{P}$.

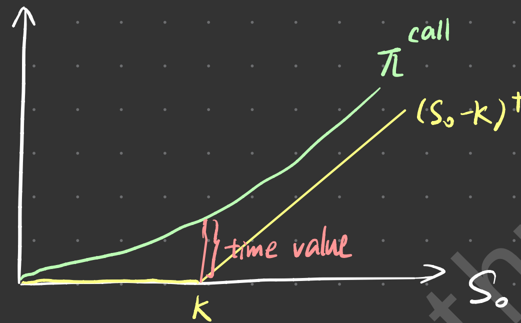
$$\pi^{call} = \mathbb{E}^*[H] = \mathbb{E}^*\left[\frac{C^{call}}{B_T}\right] = \mathbb{E}^*\left[\left(\frac{S_T}{B_T} - \frac{K}{B_T}\right)^+\right]$$

$$= \mathbb{E}^*\left[\left(X_T - \frac{K}{B_T}\right)^+\right]$$

$$\underset{\text{Jensen's ineq.}}{\geq} \left(\mathbb{E}^*\left[X_T - \frac{K}{B_T}\right]\right)^+ = \left(\underset{S_0\ (B_0=1)}{X_0} - \mathbb{E}\left[\frac{K}{B_T}\right]\right)^+$$

$$\underset{B_T \geq 1}{\geq} (S_0 - K)^+$$

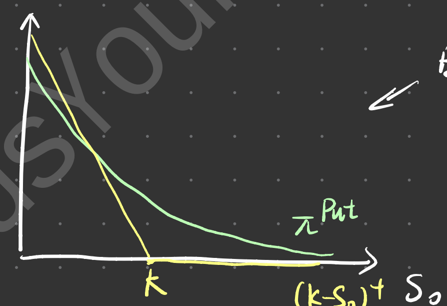
$$\Rightarrow \boxed{\pi^{Call} \geq (S_0 - K)^+} \qquad (*)$$



(2) Consider a European put option

$$C^{put} = (K - S_T)^+$$

However, (*) fails in general in this case, unless B_t = constant $\forall t$



Theorem 29. H: discounted claim

与 one-period model 不同：

(1) If H is attainable, $\#(\pi(H))=1$

$\mathbb{E}(H)<\infty$ 不成立（不可积）

(2) If H is not attainable, then either $\pi(H)=\phi$ or $\pi(H)=(\pi_{inf}, \pi_{sup})$

Definition 30. A (arbitrage-free) market model is called complete if every contingent claim is attainable. → 可以不用该条件

Theorem 31: An arbitrage-free market model is complete $\Longleftrightarrow \#(\mathcal{P})=1$

如果有该条件

5.1 Stopping Time

Consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0,1,\cdots,T}, \mathbb{P})$

$\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_T$

Definition 1: A r.v. $\tau: \Omega \longrightarrow \{0,1,2,\cdots,T\} \cup \{\infty\}$

τ is called a stopping time with respect to $\mathbb{F}=(\mathcal{F}_t)_{0\le t\le T}$ if

$$\{\tau=t\}=\{w\in\Omega : \tau(w)=t\}\in\mathcal{F}_t \quad \text{for all } t$$