3.3 Absence of arbitrage

Definition 8*: A portfolio $\bar{h} \in \mathbb{R}^{N+1}$ is called an arbitrage opportunity if

(i) $V_0(\bar{h}) = \bar{h} \cdot \bar{S}_0 \leq 0$ 在时刻0我的总资产 ≤ 0

(ii) $V_1(\bar{h}) = \bar{h} \cdot \bar{S}_1 \geq 0$ P-a.s. 在时刻1我的总资产 >0

but $\mathbb{P}(V_1(\bar{h}) > 0) > 0$ 且我总资产 >0 的概率是 >0 的

Remark 9. If $\Omega = \{\omega_1, \cdots, \omega_K\}$ 有限个, then an arbitrage opportunity $\bar{h}$ satisfies

(i) $V_0(\bar{h}) \leq 0$

(ii) $V_1(\bar{h})(\omega_i) \geq 0 \quad \forall\, i = 1, 2, \cdots, K$

$V_1(\bar{h})(\omega_j) > 0$ for some j.

Example 10 (1) $\Omega = \{\omega_1, \omega_2\}$

$$\bar{S}_0 = \begin{pmatrix} 1 \\ 10 \end{pmatrix} \quad \bar{S}_1(\omega_1) = \begin{pmatrix} 1.1 \\ 11 \end{pmatrix} \quad \bar{S}_1(\omega_2) = \begin{pmatrix} 1.1 \\ 12 \end{pmatrix}$$

Then $\bar{h} = \begin{pmatrix} -10 \\ 1 \end{pmatrix}$ is an arbitrage opportunity.

proof:

$$V_0(\bar{h}) = \bar{h} \cdot \bar{S}_0 = -10 + 10 = 0$$

$$V_1(\bar{h})(\omega_1) = \bar{h} \cdot \bar{S}_1(\omega_1) = -11 + 11 = 0$$

$$V_1(\bar{h})(\omega_2) = -10 + 12 = 2 > 0$$

(2) $\Omega=[0,1]$ $\mathcal{F}=\mathcal{B}_1$ $\mathbb{P}=m$

$\bar{S}_0=\begin{pmatrix}1\\10\end{pmatrix}$, $\bar{S}_1=\begin{pmatrix}1\\10Z\end{pmatrix}$ Z: uniformly distributed on $[0,1]$

Then $\bar{h}=\begin{pmatrix}10\\-1\end{pmatrix}$ is an arbitrage opportunity.

Proof: $V_0(\bar{h})=\bar{h}\cdot\bar{S}_0=10-10=0$

$V_1(\bar{h})=\bar{h}\cdot\bar{S}_1=10-10Z$

$\mathbb{P}(V_1(\bar{h})>0)=\mathbb{P}(10-10Z>0)=\mathbb{P}(Z<1)=1$

Assumption: Suppose the interest rate of the bond $=r$, i.e.

$B_0=B$, $B_1=B(1+r)$

Lemma 11^*: The market model admits an arbitrage opportunity

$\Longleftrightarrow$ $\exists$ vector $h\in\mathbb{R}^N$ such that

$h\cdot S_1\geq(1+r)h\cdot S_0$ P-a.s. 从股票角度来看(不包括 B_t) (γ)

and $P(\underbrace{h\cdot S_1}_{\text{投资股票赚钱概率一定}>0}>\underbrace{(1+r)h\cdot S_0}_{\text{指在}t=0\text{时，把买股票的钱全部投到债券里去}})>0$ (γ)

proof:

"$\Rightarrow$" Let $\bar{h} = \begin{pmatrix} h^0 \\ h \end{pmatrix}$ be an arbitrage opportunity

$V_0(\bar{h}) = h^0 B_0 + h \cdot S_0 \leq 0$ (1) $\longrightarrow -(1+r) h \cdot S_0 \geq (1+r) h^0 B_0$ (*)

$V_1(\bar{h}) = h^0 B_1 + h \cdot S_1 = h^0 (1+r) B_0 + h \cdot S_1 \geq 0$ P-a.s. (2)

$\mathbb{P}(V_1(\bar{h}) > 0) > 0$ (3)

$$h \cdot S_1 - (1+r) h \cdot S_0 \underset{(*)}{\geq} h \cdot S_1 + (1+r) h^0 B_0 \underset{(2)}{\geq} 0$$

Due to (3) we have

$$\mathbb{P}[h \cdot S_1 - (1+r) h \cdot S_0 > 0] \geq \mathbb{P}[h \cdot S_1 + h^0 (1+r) B_0] > 0$$

"$\Leftarrow$" Suppose h satisfies (γ)

Let $\bar{h} = \begin{pmatrix} h^0 \\ h \end{pmatrix}$ where $h^0 = -\dfrac{h \cdot S_0}{B_0}$ $\leftarrow$ 用原来买股票的钱需要借多少张债券

Then $V_0(\bar{h}) = h^0 B_0 + h \cdot S_0 = -h \cdot S_0 + h \cdot S_0 = 0$

$V_1(\bar{h}) = h^0 B_0 (1+r) + h \cdot S_1 = -h \cdot S_0 (1+r) + h \cdot S_1 \geq 0$

and $P(V_1(\bar{h}) > 0) > 0$

$\Rightarrow \bar{h}$ is an arbitrage opportunity.

Definition 12*: If there exists no arbitrage opportunity in a financial market model, we say that there is no arbitrage (arbitrage-free) in this market model.

3.4 No Arbitrage and Price System → Ω是有限个$=\{\omega_1,\omega_2,\cdots,\omega_k\}$

Definition 13: $D\in M_{(N+1)\times K}$ → $(N+1)\times K$的矩阵 is defined by

$$D=\begin{pmatrix} B_1(\omega_1) & B_1(\omega_2) & \cdots & B_1(\omega_K) \\ S_1^1(\omega_1) & S_1^1(\omega_2) & \cdots & S_1^1(\omega_K) \\ \vdots & \vdots & & \\ S_1^N(\omega_1) & S_1^N(\omega_2) & \cdots & S_1^N(\omega_K) \end{pmatrix}=(\bar{S}_1(\omega_1)\ \ \bar{S}_1(\omega_2)\ \cdots\ \bar{S}_1(\omega_K))$$

D is called the payoff matrix.

The vector $b=\bar{S}_0=(B_0, S_0^1, S_0^2, \cdots, S_0^N)^T$ is called the price vector.

$(\bar{S}_0, \bar{S}_1)\equiv(b, D)\in\mathbb{R}^{N+1}\times M_{(N+1)\times K}$ is called the market model.

Example 14: As in example 3.

$$b=\begin{pmatrix}1\\10\end{pmatrix},\quad D=\begin{pmatrix}1.02 & 1.02\\12 & 9\end{pmatrix}$$

Notation 15: $c=\begin{pmatrix}c_1\\ \vdots\\ c_n\end{pmatrix}\in\mathbb{R}^n$

$c>0$ if $c_i\geq 0,\ \forall i$, and $c_k>0$ for some k $\longrightarrow$ 包括X轴，Y轴但不包括原点

$c\gg 0$ if $c_i>0,\ \forall i$ $\rightarrow$ 第一象限，不包括X轴和Y轴

$c\geq 0$ if $c_i\geq 0,\ \forall i$ $\longrightarrow$ 包括X轴、Y轴和原点

Remark 16: A strategy $\bar{h}$ is an arbitrage opportunity if

$\bar{h}\cdot b\leq 0$ and $D^T\cdot\bar{h}>0$

Remark 17: An alternative definition of arbitrage opportunity

$\bar{h}\cdot b\leq 0$ and $D^T\bar{h}>0$

or

$\bar{h}\cdot b\leq 0$ and $D^T\bar{h}\geq 0$

☆ 重要！

Remark 18: (Fundamental Theorem of Asset Pricing)

In the market model (b, D) the following statements are equivalent:

(1) (b, D) is arbitrage-free

(2)$^{\times}$ $\exists\ \varphi \in \mathbb{R}^{k+1}$, $\varphi >> 0$ such that

$$\langle \varphi, L(\bar{h}) \rangle = 0 \quad \forall\ \bar{h} \in \mathbb{R}^{N+1}$$ → φ与$L(\bar{h})$的内积为0

where $L: \mathbb{R}^{N+1} \to \mathbb{R}^{k+1}$ is a linear transformation given by

$$L(\bar{h}) = \begin{pmatrix} -b^T \\ D^T \end{pmatrix} \bar{h}$$

☆☆ 常用!

(3) $\exists$ vector $\psi \in \mathbb{R}^k$, $\psi >> 0$ such that

$$b = D\psi$$

proof: (1) ⇒ (2): suppose (b, D) is arbitrage-free

⇒ There is no $\bar{h} \in \mathbb{R}^{N+1}$ s.t. $L(\bar{h}) = \begin{pmatrix} -\bar{h} \cdot b \\ D^T \bar{h} \end{pmatrix} > 0$

⇒ the set $\left\{ \begin{pmatrix} -\bar{h} \cdot b \\ D^T \cdot \bar{h} \end{pmatrix} : \bar{h} \in \mathbb{R}^{N+1} \right\}$ is a proper subset of $\mathbb{R}^{k+1}$ (真子集)

与$\mathbb{R}^{k+1}$不一样的$\mathbb{R}^{k+1}$的子集

由线性代数可知，子集一定过原点O.

$L(\bar{h}) > 0$不存在，则从二维来看$L(\bar{h})$要满足不在第一象限且过原点的情况

此时一定有 $\underset{\sim}{\varphi \gg 0}$ (φ在第一象限) 使得 $\varphi \cdot L(\bar{h}) = 0$

(2) $\Rightarrow$ (3): Let $\varphi \in \mathbb{R}^{K+1}$, $\varphi \gg 0$ s.t. $\langle \varphi, L(\bar{h}) \rangle = 0$

$$\text{Let } \varphi = \begin{pmatrix} \varphi_0 \\ \varphi_1 \end{pmatrix}, \quad \varphi_0 \in \mathbb{R}, \ \varphi_1 \in \mathbb{R}^K$$

$$\xRightarrow[\varphi \gg 0]{} \varphi_0 > 0, \ \varphi_1 \gg 0$$

$$0 = \langle \varphi, L(\bar{h}) \rangle = \begin{pmatrix} \varphi_0 \\ \varphi_1 \end{pmatrix} \begin{pmatrix} -\bar{h} \cdot b \\ D^T \bar{h} \end{pmatrix} = -\varphi_0 \bar{h} b + \langle \varphi_1 \cdot D^T \bar{h} \rangle$$

$$\Rightarrow \underset{\sim}{\bar{h} \cdot b} = \left\langle \frac{\varphi_1}{\varphi_0} \cdot D^T \bar{h} \right\rangle \underset{\psi = \frac{\varphi_1}{\varphi_0}}{=} \langle \psi^T \cdot D^T \bar{h} \rangle = \underset{\sim}{\bar{h} \cdot D\psi}, \ \forall \bar{h} \in \mathbb{R}^{N+1}$$

$$b = D\psi \text{ for some } \psi \gg 0$$

(3) $\Rightarrow$ (1) Since

$$\bar{h} \cdot b = \langle \psi^T \cdot D^T \bar{h} \rangle, \ \forall \bar{h} \in \mathbb{R}^{N+1}$$

$$\text{If } D^T \bar{h} > 0 \xRightarrow[\psi^T \gg 0]{} \bar{h} \cdot b > 0 \xRightarrow[\text{Remark 16}]{} (b, D) \text{ is arbitrage-free}$$

Example 19 (One-Period two states mode)

$$\Omega = \{w_1, w_2\}$$

$$b = \begin{pmatrix} B \\ S_0 \end{pmatrix} \quad D = \begin{pmatrix} B(1+r) & B(1+r) \\ S_1(w_1) & S_1(w_2) \end{pmatrix} \text{ with } S_1(w_1) > S_1(w_2)$$

Suppose that (b, D) is arbitrage-free

Thm 18 $\Rightarrow$

$$\begin{pmatrix} B \\ S_0 \end{pmatrix} = \begin{pmatrix} B(1+r) & B(1+r) \\ S_1(w_1) & S_1(w_2) \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

$$\Rightarrow \begin{cases} \psi_1 = \frac{1}{1+r} \frac{(1+r)S_0 - S_1(w_2)}{S_1(w_1) - S_1(w_2)} \\ \psi_2 = \frac{1}{1+r} \cdot \frac{S_1(w_1) - (1+r)S_0}{S_1(w_1) - S_1(w_2)} \end{cases}$$

Thus, $\psi_1 >> 0$, $\psi_2 >> 0$

$$\Rightarrow S_1(w_2) < (1+r)S_0 < S_1(w_1)$$

Thus, (b, D) is arbitrage-free

$$\Rightarrow \boxed{\frac{S_1(w_2)}{1+r} < S_0 < \frac{S_1(w_1)}{1+r}}$$

1时刻股票价格折到现在来看
既有可能亏钱，也有可能赚钱

3.5 Martingale Measures

Remark 20: $b = D\psi$

$$\Rightarrow \begin{pmatrix} B_0 \\ S_0^1 \\ \vdots \\ S_0^N \end{pmatrix} = \begin{pmatrix} B_1(w_1) & B_1(w_2) & \cdots & B_1(w_K) \\ S_1^1(w_1) & S_1^1(w_2) & \cdots & S_1^1(w_K) \\ \vdots & \vdots & & \vdots \\ S_1^N(w_1) & S_1^N(w_2) & \cdots & S_1^N(w_K) \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_K \end{pmatrix}$$