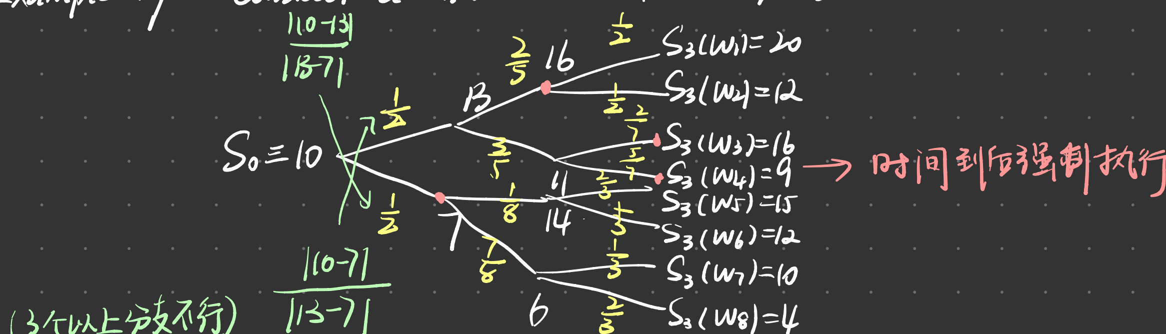
5.3 Arbitrage-free prices

Notation 16: For fixed exercise strategy τ, we denote the set of arbitrage-free prices by

$$\Pi(H_\tau) = \{\mathbb{E}^*[H_\tau] : \mathbb{P}^* \in \mathcal{P},\ \mathbb{E}^*[H_\tau] < \infty\}$$

Example 17: Consider a market model with $r=0$.



The equivalent martingale measure Q is given by

$Q(\{\omega_1\}) = \frac{1}{2} \times \frac{2}{5} \times \frac{1}{2} = \frac{1}{10}$ $\qquad$ $Q(\{\omega_5\}) = \frac{1}{24}$

$Q(\{\omega_2\}) = \frac{1}{2} \times \frac{2}{5} \times \frac{1}{2} = \frac{1}{10}$ $\qquad$ $Q(\{\omega_6\}) = \frac{1}{48}$

$Q(\{\omega_3\}) = \frac{1}{2} \times \frac{3}{5} \times \frac{2}{7} = \frac{3}{35}$ $\qquad$ $Q(\{\omega_7\}) = \frac{7}{48}$

$Q(\{\omega_4\}) = \frac{3}{14}$ $\qquad$ $Q(\{\omega_8\}) = \frac{7}{24}$

Let $\tau = \inf\{t : S_t \geq 14 \text{ or } S_t \leq 8\} \wedge 3$

Then $\tau(w_1) = \tau(w_2) = 2$

$\tau(w_3) = \tau(w_4) = 3$

$\tau(w_5) = \cdots = \tau(w_8) = 1$

Consider $H_t = (S_t - 12)^+$

$$H_\tau(w_1) = H_{\tau(w_1)}(w_1) = H_2(w_1) = (16-12)^+ = 4$$

$$H_\tau(w_2) = H_2(w_2) = (16-12)^+ = 4$$

$$H_\tau(w_3) = H_3(w_3) = (16-12)^+ = 4$$

$$H_\tau(w_4) = H_4(w_4) = (9-12)^+ = 0$$

$$H_\tau(5) = H_1(w_5) = (7-12)^+ = 0 = \cdots = H_\tau(8)$$

$$\mathbb{E}^Q[H_\tau] = \sum_i H_\tau(w_i) \cdot Q(\{w_i\}) = 4 \times \frac{1}{10} + 4 \times \frac{1}{10} + 4 \times \frac{3}{35} = \frac{8}{7}$$

当w太多时，该方法不适用！

$$\Rightarrow \Pi(H_\tau) = \{\tfrac{8}{7}\}$$

Definition 18: A real number π is called an arbitrage-free price of a discounted American claim H if the following two conditions are satisfied:

(i) The price π is not too high in the sense that there exists some stopping time (不是 exercise strategy，即不存在内幕消息) $\tau \in \mathcal{T} := \{\tau : \text{stopping time with } \tau \le T\}$ and $\pi' \in \Pi(H_\tau)$ such that $\pi \le \pi'$

(ii) The price π is not too low in the sense that there exists no $\tau' \in \mathcal{T}$ such that $\pi < \pi'$ for all $\pi' \in \Pi(H_\tau)$

总存在一个$\tau' \in \mathcal{T}$，使它所有 $\pi' \in \Pi(H_\tau)$都有 $\pi' \le \pi$

Notation 19: The set of all arbitrage-free prices of H is denoted by $\Pi(H)$ and we define

$$\pi_{\inf}(H) = \inf \Pi(H)$$

and

$$\pi_{\sup}(H) = \sup \Pi(H)$$

Recall: European contingent claim

$\Pi(H)$: one point, open interval, ϕ

$$\pi_{\inf}(H) = \inf_{P^* \in \mathcal{P}} E^*[H] \qquad \pi_{\sup}(H) = \sup_{P^* \in \mathcal{P}} E^*[H]$$

Remark 20: Any arbitrage-free price π for H must be of the form $\pi = \mathbb{E}^*[H_\tau]$ for some $\mathbb{P}^* \in \mathcal{P}$ and $\tau \in \mathcal{T}$. This implies $\sup_{\tau\in\mathcal{T}} \inf_{\mathbb{P}^*\in\mathcal{P}} \mathbb{E}^*[H_\tau] \leq \pi \leq \sup_{\tau\in\mathcal{T}} \sup_{\mathbb{P}^*\in\mathcal{P}} \mathbb{E}^*[H_\tau]$ for all $\pi \in \Pi(H)$

In particular, if the market model is complete, i.e. $\mathcal{P} = \{\mathbb{P}^*\}$ ($\#(\mathcal{P})=1$), then $\pi = \mathbb{E}^*[H_\tau]$, $\tau\in\mathcal{T}$

去掉空集的情况

Theorem 21: If $\mathbb{E}^*|H_t| < \infty$ for all t and for all $\mathbb{P}^* \in \mathcal{P}$. The set of all arbitrage-free prices for H is an interval with endpoint

(一点也可看作一个区间)

$$\pi_{\inf}(H) = \inf_{\mathbb{P}^*\in\mathcal{P}} \sup_{\tau\in\mathcal{T}} \mathbb{E}^*[H_\tau] = \sup_{\tau\in\mathcal{T}} \inf_{\mathbb{P}^*\in\mathcal{P}} \mathbb{E}^*[H_\tau]$$

and

$$\pi_{\sup}(H) = \sup_{\mathbb{P}^*\in\mathcal{P}} \sup_{\tau\in\mathcal{T}} \mathbb{E}^*[H_\tau] = \sup_{\tau\in\mathcal{T}} \sup_{\mathbb{P}^*\in\mathcal{P}} \mathbb{E}^*[H_\tau]$$

Moreover, $\Pi(H)$ either consists of one single point or does not contain its upper endpoint $\pi_{\sup}(H)$.

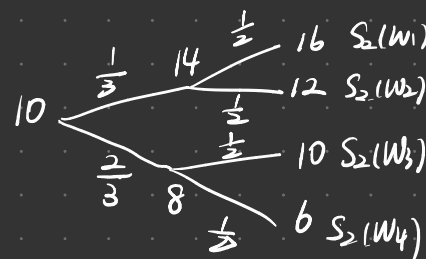
(i) $\Pi(H) = \{\pi_{\sup}(H)\} = \{\pi_{\inf}(H)\} \rightarrow$ 一点

(ii) $\Pi(H) = (\pi_{\inf}(H), \pi_{\sup}(H)) \rightarrow$ 开区间

(iii) $\Pi(H) = [\pi_{\inf}(H), \pi_{\sup}(H)) \rightarrow$ 左开右闭区间

Example 22: Consider a market model:

with $r=0$, $T=2$

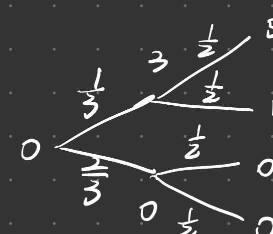


This is a complete market model with equivalent martingale measure.

$$Q(\{w_1\}) = Q(\{w_2\}) = \frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$$

$$Q(\{w_3\}) = Q(\{w_4\}) = \frac{2}{3} \times \frac{1}{2} = \frac{1}{3}$$

Suppose $H_t = (S_t - 11)^+$, then payoff



$\mathcal{T}$ = the collection of stopping times $= \{\tau_1, \tau_2, \tau_3, \tau_4, \tau_5\}$

$\tau_1 \equiv 0$

$\tau_2 \equiv 1$

怎么看 $\tau_3 \equiv 2$

$\rightarrow$ $\tau_4(w_1) = \tau_4(w_2) = 1$, $\tau_4(w_3) = \tau_4(w_4) = 2$

$\tau_5(w_1) = \tau_5(w_2) = 2$, $\tau_5(w_3) = \tau_5(w_4) = 1$

Stopping time 定义

$\{\tau = t\} \in \mathcal{F}_t$

$\{\tau = 0\} \in \mathcal{F}_0 = \{\phi, \Omega\}$

(1) $\{\tau = 0\} = \Omega \Rightarrow \tau \equiv 0 \Rightarrow \tau_1$

(2) $\{\tau = 0\} = \phi$

$\{\tau = 1\} \in \mathcal{F}_1 = \sigma(\{w_1, w_2\}, \{w_3, w_4\}) = \{\phi, \{w_1, w_2\}, \{w_3, w_4\}, \Omega\}$

(i) $\{\tau = 1\} = \Omega \Rightarrow \tau \equiv 1 \Rightarrow \tau_2$

(ii) $\{\tau = 1\} = \phi \Rightarrow \tau \equiv 2 \Rightarrow \tau_3$ $\leftarrow$ τ在0,1,2中，若$\tau \neq 0$，$\tau \neq 1$，则必有$\tau = 2$

(iii) $\left.\begin{array}{l} \{\tau = 1\} = \{w_1, w_2\} \Rightarrow \tau(\{w_1\}) = \tau(\{w_2\}) = 1 \\ \{\tau = 2\} = \{w_3, w_4\} \Rightarrow \tau(\{w_3\}) = \tau(\{w_4\}) = 2 \end{array}\right\} \Rightarrow \tau_4$

(iv) $\left.\begin{array}{l} \{\tau = 1\} = \{w_3, w_4\} \Rightarrow \tau(\{w_3\}) = \tau(\{w_4\}) = 1 \\ \{\tau = 2\} = \{w_1, w_2\} \Rightarrow \tau(\{w_1\}) = \tau(\{w_2\}) = 2 \end{array}\right\} \Rightarrow \tau_5$

$$\mathbb{E}_Q[H_{\tau_1}] = \mathbb{E}_Q[H_0] = 0$$

$$\mathbb{E}_Q[H_{\tau_2}] = \mathbb{E}_Q[H_1] = 3\times\frac{1}{3} = 1 \quad = \sum_{i=1}^{4} Q(\{w_i\})\, H_1(w_i) = \frac{1}{6}\times 3\times 2$$

$$\mathbb{E}_Q[H_{\tau_3}] = \mathbb{E}_Q[H_2] = 5\times\frac{1}{6} + 1\times\frac{1}{6} = 6\times\frac{1}{6} = 1$$

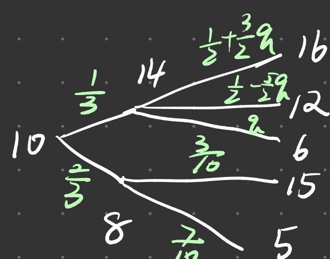
$$\mathbb{E}_Q[H_{\tau_4}] = 3\times\frac{1}{3} + 0 = 1$$

$$\mathbb{E}_Q[H_{\tau_5}] = 5\times\frac{1}{6} + 1\times\frac{1}{6} = 6\times\frac{1}{6} = 1$$

$\Longrightarrow$ arbitrage-free price $= \sup_{\tau\in\mathcal{T}} \mathbb{E}_Q[H_\tau] = 1$

$$\pi(H) = 1$$

Example 23: Consider a market model with $r=0$, $T=2$



$$\begin{pmatrix} 1 \\ 14 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 16 & 12 & 6 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}$$

$$\begin{cases} 14 = 16 q_1 + 12 q_2 + 6 q_3 \\ 1 = q_1 + q_2 + q_3 \end{cases} \Rightarrow \begin{cases} 14 - 6 q_3 = 16 q_1 + 12 q_2 \\ 1 - q_3 = q_1 + q_2 \end{cases}$$

$$\Rightarrow \begin{cases} 14 - 6 q_3 = 16 q_1 + 12 q_2 \\ 12 - 12 q_3 = 12 q_1 + 12 q_2 \end{cases} \Rightarrow \begin{cases} q_1 = \frac{1}{2} + \frac{3}{2} q_3 \\ q_2 = \frac{1}{2} - \frac{5}{2} q_3 \\ q_3 = q_3 \end{cases} \quad q_3 \in (0, \tfrac{1}{5})$$

$$\begin{pmatrix}1\\10\end{pmatrix} = \begin{pmatrix}1 & 1\\14 & 8\end{pmatrix}\begin{pmatrix}\psi_1\\\psi_2\end{pmatrix}$$

$$\begin{cases}\psi_1+\psi_2=1\\ 14\psi_1+8\psi_2=10\end{cases} \Rightarrow \begin{cases}\psi_1=\frac{1}{3}\\ \psi_2=\frac{2}{3}\end{cases}$$

Then the equivalent martingale measures are of the form

$$Q(\{w_1\}) = \frac{1}{6}+\frac{1}{2}q$$

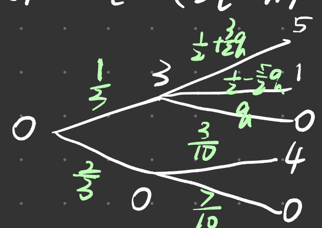
$$Q(\{w_2\}) = \frac{1}{6}-\frac{5}{6}q$$

$$Q(\{w_3\}) = \frac{1}{3}q$$

$$Q(\{w_4\}) = \frac{2}{3}\times\frac{3}{10} = \frac{1}{5}$$

$$Q(\{w_5\}) = \frac{2}{3}\times\frac{7}{10} = \frac{7}{15} \qquad \text{for } q\in(0,\tfrac{1}{5})$$

Consider $H_t = (S_t-11)^+$



Stopping time $\{\tau=t\}\in\mathcal{F}_t$

$\{\tau=0\}\in\{\Omega,\phi\}$

(1) $\{\tau=0\}=\Omega \Rightarrow \tau_1\equiv 0$

(2) $\{\tau=0\}=\phi$

$\{\tau=1\}\in\sigma(\{w_1,w_2,w_3\},\{w_4,w_5\})$

$= (\Omega, \phi, \{w_1, w_2, w_3\}, \{w_4, w_5\})$

(i) $\{\tau=1\} = \phi \Rightarrow \{\tau=2\} = \Omega \Rightarrow \tau_2 \equiv 2$

(ii) $\{\tau=1\} = \Omega \Rightarrow \tau_3 \equiv 1$

(iii) $\{\tau=1\} = \{w_1, w_2, w_3\}$

$\Rightarrow \{\tau=2\} = \{w_4, w_5\}$

则 $\tau_4(w_1) = \tau_4(w_2) = \tau_4(w_3) = 1$

$\tau_4(w_4) = \tau_4(w_5) = 2$

(iv) $\{\tau=1\} = \{w_4, w_5\}$

$\Rightarrow \{\tau=2\} = \{w_1, w_2, w_3\}$

则 $\tau_5(w_1) = \tau_5(w_2) = \tau_5(w_3) = 2$

$\tau_5(w_4) = \tau_5(w_5) = 1$

$$\mathbb{E}_Q[H_{\tau_1}] = \mathbb{E}[H_0] = \boxed{0}_{\min}$$

$$\mathbb{E}_Q[H_{\tau_2}] = \mathbb{E}[H_2] = \left(\tfrac{1}{6}+\tfrac{1}{2}q\right)\times 5 + \left(\tfrac{1}{6}-\tfrac{5}{6}q\right)\times 1 + \tfrac{1}{5}\times 4 = \tfrac{5}{3}q + \tfrac{9}{5}$$

$$\inf_q \mathbb{E}_q[H_{\tau_2}] = \boxed{\tfrac{9}{5}}_{\inf}, \quad \sup_q \mathbb{E}_q[H_{\tau_2}] = \tfrac{32}{15}$$

$$\mathbb{E}_Q[H_{\tau_3}] = \mathbb{E}[H_1] = \tfrac{1}{3}\times 3 + \tfrac{2}{3}\times 0$$

$$= 3\times\left(\tfrac{1}{6}+\tfrac{1}{2}q\right) + 3\times\left(\tfrac{1}{6}-\tfrac{5}{6}q\right) + 3\times\tfrac{1}{3}q$$

$$= \boxed{1}_{\min}$$

$$\mathbb{E}_Q[H_{\tau_4}] = \tfrac{1}{3}\times 3 + 4\times\tfrac{1}{5} = \boxed{\tfrac{9}{5}}_{\min}$$

$$\mathbb{E}_Q[H_{\tau_5}] = 5\times\left(\tfrac{1}{6}+\tfrac{1}{2}q\right) + 1\times\left(\tfrac{1}{6}-\tfrac{5}{6}q\right) + 0 = \tfrac{5}{3}q + 1$$

$$\inf_q \mathbb{E}_q[H_{\tau_5}] = \boxed{1}_{\inf}, \quad \sup_q \mathbb{E}_q[H_{\tau_5}] = \tfrac{1}{5}\times\tfrac{5}{3}+1 = \tfrac{4}{3}$$

Thus, we have

$$\Pi_{\inf}(H) = \sup_{\tau\in\mathcal{T}} \inf_{q\in\mathcal{D}} \mathbb{E}_q[H_\tau] = \sup\left\{0, \overset{\text{open}}{\tfrac{9}{5}}, 1, \overset{\text{close}}{\tfrac{9}{5}}, 1\right\} = \overset{\text{close}}{\tfrac{9}{5}}$$

$$\Pi_{\sup}(H) = \sup_{\tau\in\mathcal{T}} \sup_{q\in\mathcal{D}} \mathbb{E}_q[H_\tau] = \sup\left\{0, \overset{\text{open}}{\tfrac{32}{15}}, 1, \tfrac{9}{5}, \overset{\text{open}}{\tfrac{4}{3}}\right\} = \tfrac{32}{15}$$

由于 $q\in(0,\frac{1}{5})$

那么 inf 右端点均无法取到

"inf 取不到, min 可取到"

实际上对 q 求范围都是开区间!

$$\pi(H) \in [\tfrac{9}{5}, \tfrac{32}{15})$$

Example 24: As in Example 22. Let

$$\Omega^0 = \{w_1, w_2, w_3, w_4\}$$

$$\mathcal{F}^0 = \sigma(\{w_1\}, \{w_2\}, \{w_3\}, \{w_4\})$$

$$\mathbb{P}^0(\{w_i\}) > 0 \quad \forall\, i = 1, 2, 3, 4$$

$$\mathcal{F}_0^0 = \{\emptyset, \Omega\}$$

$$\mathcal{F}_1^0 = \sigma(\{w_1, w_2\}, \{w_3, w_4\})$$

$$\mathcal{F}_2^0 = \mathcal{F}^0$$

This market model $(S, \mathbb{F}^0)$ is complete on $(\Omega^0, \mathcal{F}^0, \mathbb{P}^0)$

Enlarge this model by adding two external states w^+ and w^-. Let

$$\Omega = \Omega^0 \times \{w^+, w^-\}$$

$$= \{(w_1, w^+), (w_1, w^-), \cdots, (w_4, w^+), (w_4, w^-)\}$$

$$\mathbb{P}(\{(w_i, w^+)\}) = \mathbb{P}(\{(w_i, w^-)\}) = \tfrac{1}{2}\mathbb{P}^0(\{w_i\}) \text{ for } i = 1, 2, 3, 4$$

$$\mathcal{F}_0 = \{\emptyset, \Omega\}$$

$$\mathcal{F}_1 = \sigma(\{(w_1, w^+), (w_2, w^+)\}, \{(w_1, w^-), (w_2, w^+)\}, \{(w_3, w^+), (w_4, w^+)\}, \{(w_3, w^-), (w_4, w^-)\})$$

$$\mathcal{F}_2 = \sigma(\{(w_i, w^+)\}, \{(w_i, w^-)\}, i = 1, 2, 3, 4)$$

Then this market model is incomplete.

Then the collection of all equivalent martingale measures $\mathcal{P}$ consists of all probability measure $\mathbb{P}^*_q$ with

$$\left.\begin{aligned} \mathbb{P}^*_q(\{(w_1, w^+)\}) &= \mathbb{P}^*_q(\{(w_2, w^+)\}) = \tfrac{1}{6} q \\ \mathbb{P}^*_q(\{(w_3, w^+)\}) &= \mathbb{P}^*_q(\{w_4, w^+)\}) = \tfrac{1}{3} q \end{aligned}\right) \quad \mathbb{P}(\Omega^\circ \times \{w^+\}) = q$$

$$0 < q < 1$$

$$\left.\begin{aligned} \mathbb{P}^*_q(\{(w_1, w^-)\}) &= \mathbb{P}^*_q(\{w_2, w^-\}) = \tfrac{1}{6}(1-q) \\ \mathbb{P}^*_q(\{(w_3, w^-)\}) &= \mathbb{P}^*_q(\{w_4, w^-\}) = \tfrac{1}{3}(1-q) \end{aligned}\right) \quad \mathbb{P}(\Omega^\circ \times \{w^-\}) = 1-q$$

(1) Consider the discounted American contingent claim H

$$H_0 \equiv 0$$

$$\mathcal{H}_1 \equiv 1$$

$$\mathcal{H}_2 = \begin{cases} 2, & \text{if } \omega = (\omega_i, \omega^+) \quad i=1,2,3,4 \\ 0, & \text{if } \omega = (\omega_i, \omega^-) \quad i=1,2,3,4 \end{cases}$$

If $q \geq \frac{1}{2}$ $\tau \equiv 2$ is an optimal stopping time for P_q^*

$$\mathbb{E}_q^*[H_\tau] = 2q$$

$$\Rightarrow \inf_{\frac{1}{2} \leq q < 1} \mathbb{E}_q^*[H_\tau] = 1, \quad \sup \mathbb{E}_q^*[H_\tau] = 2$$

If $q < \frac{1}{2}$ $\tau \equiv 1$ is an optimal stopping time for P_q^*

$$\mathbb{E}_q^*[H_\tau] = 1$$

Thus $\pi_{\inf}(\mathcal{H}) = \sup_\tau \inf_q \mathbb{E}_q^*(\mathcal{H}) = \sup\{1, 1\} = 1$

$$\pi_{\sup}(\mathcal{H}) = \sup_\tau \sup_q \mathbb{E}_q^*(\mathcal{H}) = \sup\{2, 1\} = 2$$

$$\Pi(\mathcal{H}) = [1, 2)$$

(2) Consider the discounted American contingent claim $\mathcal{H}$.

$$\mathcal{H}_0 = 0$$

$$\mathcal{H}_1 = 0$$

$$\mathcal{H}_2 = \begin{cases} 2 & \text{if } w = (w_i, w^+) \quad i = 1, 2, 3, 4 \\ 0 & \text{if } w = (w_i, w^-) \quad i = 1, 2, 3, 4 \end{cases}$$

$\tau \equiv 2$ is an optimal stopping time for P_q^*

$$\Rightarrow \inf_{0<q<1} E_q^*[H_\tau] = \inf_{0<q<1} 2q = 0$$

$$\sup_{0<q<1} E_q^*[H_\tau] = \sup_{0<q<1} 2q = 1$$

$$\pi_{\inf}(\mathcal{H}) = \sup_{\tau} \{0\} = 0$$

$$\pi_{\sup}(\mathcal{H}) = \sup_{\tau} \{1\} = 2$$

$$\Pi(H) = (0, 2)$$

Definition 25: A discounted American claim is <u>attainable</u>, if there exists a stopping time $\tau \in \mathcal{T}$ and a self-financing strategy $\bar{h}$ whose value process satisfies

欧式无此条件! $\boxed{V_t(\bar{h}) \geq H_t \text{ for all } t}$ and $V_\tau(\bar{h}) = H_\tau$ $\mathbb{P}$-a.s.

The trading strategy $\bar{h}$ is called a <u>hedging strategy</u> for H.

避险策略

Remark 26: If the market model is complete, every American claim is attainable.

Theorem 27: For a discounted American claim H satisfying $\mathbb{E}^* |H_t| < \infty$ for all t and for all $\mathbb{P}^* \in \mathcal{P}$, the following conditions are equivalent:

(1) H is attainable

(2) H admits a unique arbitrage-free price $\pi(H)$ i.e. $\Pi(H) = \{\pi(H)\}$

(3) $\pi_{\sup}(H) \in \Pi(H)$