1.1 Probability Space

1、σ代数定义：(σ-algebra)

① $\Omega \in \mathcal{F}$

② $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$

③ $A_1, A_2, \cdots \in \mathcal{F}$，$\bigcup_{n\geq 1} A_n \in \mathcal{F}$

例子辨析：

(1) $\Omega \in \mathbb{R}$

指A中元素的个数有限

$\mathcal{F} = \{A \subseteq \mathbb{R} : \underline{\#(A)} < \infty \text{ or } \#(A^c) < \infty\}$

$\Omega \in \mathcal{F}$ 满足

若 $\#(A) < \infty$，则 $A \in \mathcal{F}$，且 $\#[(A^c)^c] < \infty$，则 $A^c \in \mathcal{F}$；

但当 $\#(A) < \infty$ 时，$\bigcup_{n\geq 1} A_n = \infty$，则 $\bigcup_{n\geq 1} A_n \notin \mathcal{F}$

因此该 $\mathcal{F}$ 不是σ代数

(2) $\Omega \in \mathbb{R}$

$\mathcal{F} = \{A \subseteq \mathbb{R} : A \text{ or } A^c \text{ is countable}\}$ is a σ-algebra (它是σ-代数)

2、可测空间定义 (mesurable space) ← 指在该空间上可以定义一个测度

Ω: non-empty (sample) space

$\mathcal{F}$: σ-algebra on Ω

$(\Omega, \mathcal{F})$ is called a measurable space

3、概率测度定义 (probability measure)

对于测度空间 $(\Omega, \mathcal{F})$, probability measure is a real-value function (实值函数) $P: \mathcal{F} \to \mathbb{R}$ (把 σ-代数映到 $\mathbb{R}$ 上 [0,1])

性质:

实变中对测度的性质/定义 ←

(1) $P(E) \geq 0$, $\forall E \in \mathcal{F}$

(2) countable additivity

} 即度量长度必为正, 度量长度可加.

E_n is the countable collection of disjoin sets in $\mathcal{F}$ ← $\mathcal{F}$ 中可数个不相交的集合

$$P\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} P(E_n)$$

概率测度独有的性质 ← (3) $P(\Omega) = 1$ → 所有事件加总概率为 1

概率测度一些常用性质汇总:

(1) $P(E) \leq 1$, $\forall E \in \mathcal{F}$

(2) $P(\emptyset) = 0$

(3) $P(E^c) = 1 - P(E)$

(4) $P(E \cup F) = P(E) + P(F) - P(E \cap F)$

(5) $E \subseteq F \Rightarrow P(E) \leq P(F)$

(6) (E_n): collection of subset of $\mathcal{F}$, $P\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} P(E_n)$

disjoin 时取 "="

If (E_n) satisfies $E_1 \subseteq E_2 \subseteq E_3 \subseteq \cdots \subseteq E_n \subseteq \cdots$

then, $P(E_n)$ converges to $P(\bigcup_{n=1}^{\infty} E_n)$. i.e.

$$P(\bigcup_{n=1}^{\infty} E_n) = \lim_{n\to\infty} P(E_n)$$

If (E_n) satisfies $E_1 \supseteq E_2 \supseteq E_3 \cdots \supseteq E_n \supseteq \cdots$

then, $P(E_n)$ converges to $P(\bigcap_{n=1}^{\infty} E_n)$. i.e.

$$P(\bigcap_{n=1}^{\infty} E_n) = \lim_{n\to\infty} P(E_n)$$

4. 概率空间定义 (probability space)

$(\Omega, \mathcal{F}, \mathbb{P})$ 合起来就称为一个概率空间 $\rightarrow$ 可测空间+已定义的测度 $\mathbb{P}$

5、概率空间典型情况：

$\Omega = \{\omega_1, \omega_2, \cdots\}$: countable set

$\mathcal{F}$ = the collection of all subsets of Ω

(p_n) : sequence of real numbers statisfies : $p_n \geq 0, \forall n$ and $\sum_{n=1}^{\infty} p_n = 1$

$$\begin{array}{cccc} \omega_1, & \omega_2, & \cdots \omega_n, & \cdots \\ \downarrow & \downarrow & \downarrow & \\ p_1 & p_2 & p_n & \cdots \end{array}$$

Define a set function $\mathbb{P} : \mathcal{F} \to \mathbb{R}$ by $P(\{\omega_n\}) = p_n$, $P(E) = \sum_{\omega_n \in E} P(\{\omega_n\}) = \sum_{\omega_n \in E} p_n$

Then $\mathbb{P}$ is a probability measure.

$(\Omega, \mathcal{F}, \mathbb{P})$ is called a discrete probability space.

Ω is called a discrete sample space

另一个好用的例子：

引入：

Question：

Ω = given

$\mathcal{C}$ = a collection of some subsets of Ω → $\mathcal{C}$不一定是σ-algebra

是否存在 another collection, say $\mathcal{G}$, of subsets of Ω such that

(i) $\mathcal{C} \subseteq \mathcal{G}$

(ii) $\mathcal{G}$是σ-algebra

Answer：Yes！$\mathcal{G}$ = the collection of all subsets of Ω

且有包住$\mathcal{C}$的最小σ-algebra：$\bigcap\limits_{\substack{\mathcal{C} \subseteq \mathcal{H} \\ \mathcal{H} = \sigma\text{-algebra}}} \mathcal{H}$

意义：有时候由Ω生成的$\mathcal{C}$并非是可测的（不是σ代数）那么我们可以找到一个包含$\mathcal{C}$的最小可测集合（最小σ代数）来测量$\mathcal{C}$的元素

注意：σ-algebra的交集还是σ-algebra.

证明：$\mathcal{F}_1$是σ-algebra，则有$\Omega_1 \in \mathcal{F}_1$，$A \in \mathcal{F}_1$，$A^c \in \mathcal{F}_1$，$\bigcup\limits_{n \geq 1} A_n \in \mathcal{F}_1$

$\mathcal{F}_2$是σ-algebra，则有$\Omega_2 \in \mathcal{F}_2$，$B \in \mathcal{F}_2$，$B^c \in \mathcal{F}_2$，$\bigcup\limits_{n \geq 1} B_n \in \mathcal{F}_2$

令$\mathcal{F} = \mathcal{F}_1 \cap \mathcal{F}_2$，对$\Omega_1 \cap \Omega_2 \in \mathcal{F}_1$，$\Omega_1 \cap \Omega_2 \in \mathcal{F}_2 \Rightarrow \Omega_1 \cap \Omega_2 \in \mathcal{F}$

对$\forall C \in \mathcal{F}$，则$C \in \mathcal{F}_1$，$C \in \mathcal{F}_2$. 那么$C^c \in \mathcal{F}_1$，$C^c \in \mathcal{F}_2 \Rightarrow C^c \in \mathcal{F}$

对$\forall C_n \in \mathcal{F}$，有$C_n \in \mathcal{F}_1$，$C_n \in \mathcal{F}_2$，则$\bigcup\limits_{n \geq 1} C_n \in \mathcal{F}_1$，$\bigcup\limits_{n \geq 1} C_n \in \mathcal{F}_2 \Rightarrow \bigcup\limits_{n \geq 1} C_n \in \mathcal{F}$

Notation：If $\mathcal{G}$ is the smallest σ-algebra continaing $\mathcal{C}$, then $\mathcal{G} = \sigma(\mathcal{C})$, we say that $\mathcal{G}$ is a σ-algebra generated by $\mathcal{C}$. （$\mathcal{G}$是$\mathcal{C}$生成的σ-代数）

例： $\Omega=\{1,2,3,4\}$， $\mathcal{C}=\{\{1,2\},\{4\}\}$

求 $\sigma(\mathcal{C})$

不能适用所有 → 方法1：

$\mathcal{C}$中包含Ω所有元素并disjoin

$\sigma(\mathcal{C})=\sigma\{\{1,2\},\{4\}\}=\sigma\{\{1,2\},\{3\},\{4\}\}$ ← 再直接求all subsets

$=\{\phi,\{1,2\},\{3\},\{4\},\{1,2,3\},\{1,2,4\},\{3,4\},\Omega\}$

易遗漏 → 方法2： ← 该方法需要检验是否个数为 2^n !!（因为易漏）

$\sigma(\mathcal{C})=\sigma\{\{1,2\},\{4\}\}=\sigma\{\underbrace{\phi,\Omega,\{1,2\},\{4\}}_{\text{先直接写出全部已知}},\underbrace{\{3,4\},\{1,2,3\},\{1,2,4\},\{3\}}_{\text{再通过“并”“补”原则添}}\}$

准确但较慢 → 方法3：

例题：已知Ω={1,2,3}，求由C={ {1,2},{1,3} }生成的σ代数。

答案：{Φ,{1},{2},{3},{1,2},{1,3},{2,3},Ω}

算法：将集合C的元素（切记集合C的元素为集合）及其与Ω的补集依次作为行及列的表头，然后在行列相交的单元格内填入对应行列的∩及∪运算结果。若有之前未出现的结果，则依次将它们新添为行及列的表头后重复上述运算。直至行列相交的单元格内的∩及∪运算没有新的结果出现。最后，Φ，Ω，及表头的内容共同构成由集合C生成的σ代数。

注意：这个算法计算量大，但是能够保证不漏算。下图中不同颜色区域，表示计算的批次。下图中的阴影单元格表示表格对角线的位置，其中的内容也是行列相交的单元格内的∩及∪运算结果，不过在图中省略了。

↗只写上半部分即可

	{1,2}	{3}	{1,3}	{2}	{1}	{2,3}
{1,2}		Φ,Ω	{1},Ω	{2},{1,2}	{1},{1,2}	{2},Ω
{3}	Φ,Ω		{3},{1,3}	Φ,{2,3}	Φ,{1,3}	{3},{2,3}
{1,3}	{1},Ω	{3},{1,3}		Φ,Ω	{1},{1,3}	{3},Ω
{2}	{2},{1,2}	Φ,{2,3}	Φ,Ω		Φ,{1,2}	{2},{2,3}
{1}	{1},{1,2}	Φ,{1,3}	{1},{1,3}	Φ,{1,2}		Φ,Ω
{2,3}	{2},Ω	{3},{2,3}	{3},Ω	{2},{2,3}	Φ,Ω	

Borel集定义： $\Omega=\mathbb{R}$， $\mathcal{C}=$ the collection of all open intervals (a,b).

then the sets in $\mathcal{B}=\sigma(\mathcal{C})$ are called Borel sets

$\mathcal{B}$不是$\mathbb{R}$中所有subsets组成的集合！即原先长度的概念没法测量$\mathbb{R}$上所有集合.

目前能在$\mathbb{R}$上想到的集合都是Borel sets.（实变中会构造）

回到前面找好用的概率空间的例子：

$\Omega = [0,1]$

$\mathcal{B}_1$ = the collection of all Borel sets of $[0,1]$

$= \mathcal{B} \cap [0,1] := \{ A \cap [0,1] : A \in \mathcal{B} \}$ ← $\mathcal{B}$中每一个元素都与$[0,1]$取交集

For $(a,b) \in \mathcal{B}_1$, define $m((a,b)) = b-a$ ← m指长度

Then we can define a probability measure $m: \mathcal{B}_1 \to \mathbb{R}$

m is called the Lebesgue measure.

该测度空间记作：$([0,1], \mathcal{B}_1, m)$

1.2 Random Variables

引入：本科阶段对随机变量的定义是作用在事件上的实值函数.但问题在于，本科中所说的事件集合$\mathcal{F}$是Ω所有子集的集合.但实际上，$\mathcal{F}$只需要是一个σ代数，并不一定要包含Ω的所有子集合，故我们对随机变量就有了新的定义方式.

在第二章涉及期望时，我们都是针对一个较小的σ-代数$\mathcal{F}$来讨论的，因此这里有必要引入对random variable更精确的定义.

随机变量的定义：A function $X: \Omega \to \mathbb{R}$ satisfies

$$\{\omega \in \Omega : X(\omega) \in B\} \in \mathcal{F}, \ \forall B \in \mathcal{B}$$

$\mathbb{R}$上的 Borel set

当$\mathcal{F}$是Ω所有子集组成的集合时，这个式子会自动满足

如果$\mathcal{F}$不为Ω的全子集组成的集合时，例如$\mathcal{F} = \sigma(\{1,2\},\{3\},\{4\})$ 此时 1|1 2|2 是无法分开量的

理解：收集满足 $X(\omega) \in$ Borel set 的 ω，其组成的集合$\in \mathcal{F}$

then, X is called a random variable (r.v.)

此时，X 对于$\mathcal{F}$来讲是 measurable

Notation: $\{X \in B\} := \{\omega \in \Omega : X(\omega) \in B\}$

例子：$\Omega = [0,1]$，$\mathcal{F} = \mathcal{B}_1$

(1) $X_1(\omega) = \omega$ for $\omega \in \Omega$, for $B \in \mathcal{B}$ (不是$\mathcal{B}_1$)

$\{X_1 \in B\} = \{\omega \in [0,1] : X_1(\omega) \in B\} = \{\omega \in [0,1] : \omega \in B\} = [0,1] \cap B \in \mathcal{B}_1$

所以 X_1 是 r.v.

(2) $X_2(\omega) = \omega^2$. For $B \in \mathcal{B}$

$\{X_2 \in B\} = \{\omega \in [0,1] : X_2(\omega) \in B\} = \{\omega \in [0,1] : \omega^2 \in B\} \overset{?}{\in} \mathcal{B}_1$

随机变量的另一种（简单）定义：

以下 statement 等价：

(1) X 是 $(\Omega, \mathcal{F})$ 的随机变量

(2) $\{X \le r\} \in \mathcal{F}, \ \forall r \in \mathbb{R}$

(3) $\{X < r\} \in \mathcal{F}, \ \forall r \in \mathbb{R}$

(4) $\{X \ge r\} \in \mathcal{F}, \ \forall r \in \mathbb{R}$

(5) $\{X > r\} \in \mathcal{F}, \ \forall r \in \mathbb{R}$

用上述等价条件继续判断 $X_2(\omega)=\omega^2$ 是否为 r.v.

① 当 $r<0$ 时，$\{X \le r\} = \{\omega \in [0,1] : X(\omega) \le r\} = \{\omega \in [0,1] : \omega^2 \le r\} = \phi \in \mathcal{B}_1$

② 当 $0 \le r \le 1$ 时，$\{X \le r\} = \{\omega \in [0,1] : X(\omega) \le r\} = \{\omega \in [0,1] : \omega^2 \le r\} = \{0 \le \omega \le \sqrt{r}\} \in \mathcal{B}_1$

③ 当 $r>1$ 时，$\{X \le r\} = \{\omega \in [0,1] : X(\omega) \le r\} = \{\omega \in [0,1] : \omega^2 \le r\} = \Omega \in \mathcal{B}_1$

$\Rightarrow$ X_2 是 r.v.

经验：当 $\mathcal{F} = \mathcal{B}$ 或 $\mathcal{B}_1$ 时，X 往往都是 r.v.

例：$\Omega = \{1,2,3,4\}$，$\mathcal{F} = \sigma(\{1,2\},\{3\},\{4\}) = \{\phi, \Omega, \{1,2\}, \{3\}, \{4\}, \{3,4\}, \{1,2,4\}, \{1,2,3\}\}$

(i) $X_1(1)=2$，$X_1(2)=3$，$X_1(3)=4$，$X_1(4)=5$

解：方法一：(定义)

$\{X_1 \in \mathcal{F}\} = \{\omega \in \Omega : X_1(\omega) \in B\}$

B与2,3,4,5是否有交集，不清楚

方法2：(等价结论)

令 $r=2\in\mathbb{R}$

$\{X\le 2\}=\{w\in\Omega: X(w)\le 2\}=\{1\}\notin\mathcal{F}$

故 X_1 不是 r.v.

(ii) $X_2(1)=X_2(2)=2$, $X_2(3)=10$, $X_2(4)=-500$

解：当 $r<-500$ 时

$\{X>r\}=\{1,2,3,4\}\in\mathcal{F}$

当 $-500\le r<2$ 时

$\{X>r\}=\{1,2,3\}\in\mathcal{F}$

当 $2\le r<10$ 时

$\{X>r\}=\{3\}\in\mathcal{F}$

当 $r\ge 10$ 时

$\{X>r\}=\phi\in\mathcal{F}$

故 X_2 是 r.v.

经验：当 Ω 是有限多个元素时，若 $\mathcal{F}$ 中只有 $\{1,2\}$ 没有 $\{1\}$，$\{2\}$，那么 $X(1)\ne X(2)$ 话，X 必不可能是 r.v.

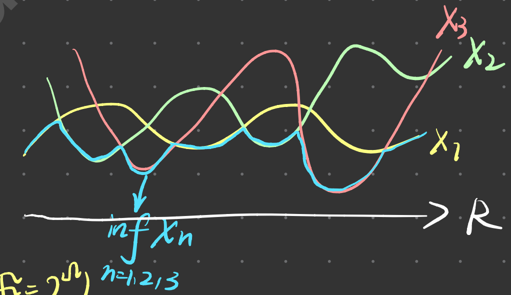
定理：(1) If X is a r.v., $f: \mathbb{R} \to \mathbb{R}$ is a Borel measurable function, then $f(X)$ is also a r.v.

(2) If X, Y are r.v.'s, $f: \mathbb{R}^2 \to \mathbb{R}$ is a Borel measurable function, then $f(X, Y)$ is also a r.v.

(3) $(X_n)_{n=1}$ is a sequence of r.v.'s, then

$$\inf_n X_n \quad \sup_n X_n \quad \liminf_{n\to\infty} X_n \quad \limsup_{n\to\infty} X_n$$

are all r.v.'s.



(4) If $(\Omega, \mathcal{F}, \mathbb{P})$ = discrete probability space (其中$\mathcal{F} = 2^{\Omega}$)

Then every real-valued function on Ω is a r.v.

(5) If $(\Omega, \mathcal{F}, \mathbb{P}) = ([0,1], \mathcal{B}_1, m)$

Then r.v. = Borel measurable function on $([0,1], \mathcal{B}_1)$

1.3 Expectation 期望

1. 示性函数 (Indicator Function)

$$I_A = \begin{cases} 1, & \text{if } \omega\in A \\ 0, & \text{if } \omega\notin A \end{cases}$$

2. 定理: I_A is a r.v. on $(\Omega, \mathcal{F}) \iff A\in\mathcal{F}$

必成立

$$\{\omega\in A: I_A(\omega)\in B, B\in\mathcal{B}\} = \{\omega\in A: \underbrace{\{0,1\}\in B}, B\in\mathcal{B}\} = \{\omega\in A\}\in\mathcal{F}$$

即 $A\in\mathcal{F}$

3. 定理: Let $A_i\in\mathcal{F}$ for all i and $\boxed{X=\sum_{i=1}^{\infty} b_i I_{A_i}}$ (called simple r.v.),

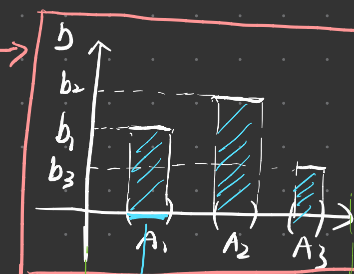
此处I_{A_i}并非一定disjoin

we define the expectation of X by $E(X)=\sum_{i=1}^{\infty} b_i\mathbb{P}(A_i)$

$$= \frac{\sum 面积}{总长度} \approx P(\Omega)=1$$

$$= \sum 面积$$

$$= \sum \underset{高}{b_i} \cdot \underset{底}{\mathbb{P}(A_i)}$$



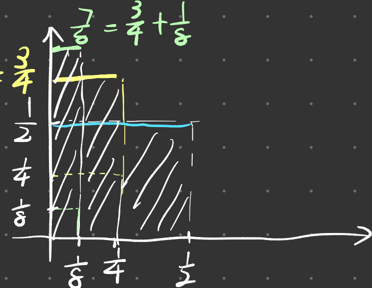
→ 总长度(概率)为1 ←

长度为$P(A_i)$

$\mathbb{P}$是测度

Notation: 若$A_i\in\mathcal{F}$, 则说A_i是$\mathcal{F}$-measurable

例: $(\Omega, \mathcal{F}, \mathbb{P}) = ([0,1], \mathcal{B}_1, m)$, $X = \sum_{i=1}^{\infty} \frac{1}{2^i} I_{[0,2^{-i}]}$



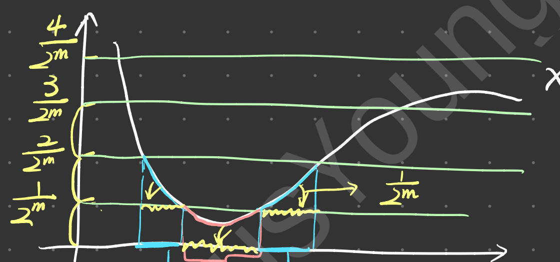
重叠的区间bi
要叠加上去!

$$E(X) = \sum_{i=1}^{\infty} \frac{1}{2^i} m([0, 2^{-i}]) = \sum_{i=1}^{\infty} \frac{1}{2^i} \cdot 2^{-i} = \sum_{i=1}^{\infty} \frac{1}{4^i}$$

$$= \frac{\frac{1}{4}}{1 - \frac{1}{4}} = \frac{1}{3}$$

Suppose X is a positive r.v., Since X is measurable, the set Λ_{mn}:

$$\Lambda_{mn} := \{ w : \frac{n}{2^m} \le X(w) \le \frac{n+1}{2^m} \}$$



$\Lambda_{m0} = \{ w : 0 \le X(w) \le \frac{1}{2^m} \}$

$\Lambda_{m1} = \{ w : \frac{1}{2^m} \le X(w) \le \frac{2}{2^m} \}$

Let $X_m = \sum_{n=0}^{\infty} \boxed{\frac{n}{2^m}} \Lambda_{mn}$

→ 高取Λ_{mn}对应y的最低处

Then for all $\omega \in \Omega$, $X_m(\omega)\uparrow$ and $X(\omega) = \lim_{m\to\infty} X_m(\omega)$ (递增级数必有极限)

X_m随m增大而增大(即y轴分得更细,整体值会上浮

随着m增大,X被不断细分,最终 $\frac{n}{2^m}$ 会趋于Λ_{mn}对应的y值.

推论:

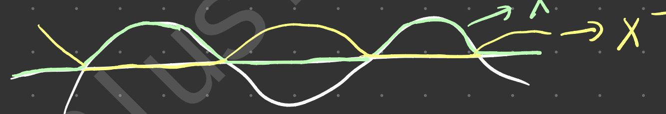
(1) If $E(X_m)=+\infty$ for some m, define $E(X)=+\infty$.

(2) If $E(X_m)<\infty$ for all m, define $E(X)=\lim_{m\to\infty}E(X_m)=\lim_{m\to\infty}\sum_{n=0}^{\infty}\frac{n}{2^m}\mathbb{P}\left(\frac{n}{2^m}\le X\le\frac{n+1}{2^m}\right)$

该积分被称为 Lebesgue Integral. ← 对y轴进行化分

4、定义: X (general) r.v.

$X = X^+ - X^-$ (where $X^+=\max\{X,0\}$, $X^-=\max\{-X,0\}$)



(1) Unless $E(X^+)$ and $E(X^-)$ are $+\infty$, we define $E(X)=E(X^+)-E(X^-)$

(2) If $E|X| = E(X^+)-E(X^-)<\infty$, X has a finite expectation. 有限期望值

We denote by $E(X)=\int_\Omega X\,d\mathbb{P}=\int_\Omega X(\omega)\cdot\mathbb{P}(d\omega)$

(3) For $A \in \mathcal{F}$, define $\int_A X \, d\mathbb{P} = E(\underline{X I_A})$ called integral of X with respect to $\mathbb{P}$ over A.

让不是A的地方都为0

(4) X is integrable w.r.t. $\mathbb{P}$ over A if the above integral exists and is finite.

Remarks:

(1) If X has a cdf (cumulative distribution function) 分布函数 $F(x) = \mathbb{P}(X \le x)$

then $E(X) = \int_{-\infty}^{+\infty} x \, dF(x)$

$$\Uparrow$$

理解:

$$\begin{aligned} E(X) &= \lim_{m\to\infty} \sum_{n=1}^{\infty} \frac{n}{2^m} \mathbb{P}\left(\frac{n}{2^m} < X < \frac{n+1}{2^m}\right) \\ &= \lim_{m\to\infty} \sum_{n=1}^{\infty} \frac{n}{2^m} \left[\mathbb{P}\left(X < \frac{n+1}{2^m}\right) - \mathbb{P}\left(X < \frac{n}{2^m}\right)\right] \\ &= \lim_{m\to\infty} \sum_{n=1}^{\infty} \frac{n}{2^m} \left[\mathbb{P}\left(X \le \frac{n+1}{2^m}\right) - \mathbb{P}\left(X \le \frac{n}{2^m}\right)\right] \\ &= \lim_{m\to\infty} \sum_{n=1}^{\infty} \frac{n}{2^m} \left[F\left(\frac{n+1}{2^m}\right) - F\left(\frac{n}{2^m}\right)\right] \\ &= \int_{-\infty}^{+\infty} x \, dF(x) \end{aligned}$$

Moreover, if g is a Borel measurable function on $\mathbb{R}$.

$$E[g(X)] = \int_{\Omega} g(X) \, d\mathbb{P} = \int_{-\infty}^{+\infty} g(x) \, dF(x)$$

(2) If X has a pdf (probability density function) f w.r.t. $\mathbb{P}$

then $E(X) = \int_{-\infty}^{+\infty} x \cdot f \, dx$

and $E[g(X)] = \int_{-\infty}^{+\infty} g(x) f(x) \, dx$

(3) If X has a mass function p w.r.t. $\mathbb{P}$ then

$$E(X) = \sum_{n=1}^{\infty} x_n \, p(x_n)$$

and

$$E[g(X)] = \sum_{n=1}^{\infty} g(X) \, p(g(X))$$

例：(1) $\Omega = \{1, 2, 3, 4\}$

$\mathcal{F} = \sigma(\{1\}, \{2\}, \{3\}, \{4\})$

$\mathbb{P}(\{1\}) = \frac{1}{2}$, $\mathbb{P}(\{2\}) = \frac{1}{4}$, $\mathbb{P}(\{3\}) = \frac{1}{6}$, $\mathbb{P}(\{4\}) = \frac{1}{12}$

Let $X = 5 I_{\{1\}} + 2 I_{\{2\}} - 4 I_{\{3,4\}}$

求 $E(X)$

解：$X = \{5, 2, -4\}$

$$E(X) = \sum_{n=1}^{3} x_n \cdot p(x_n) = 5 \times p(\{1\}) + 2 \times p(\{2\}) - 4 \times p(\{3, 4\})$$

(2) Suppose X is normally distributed on $(\Omega, \mathcal{F}, \mathbb{P})$ with mean 0 and variance 1. i.e. X has a pdf:

$$f(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$$

Then $E(X) = \int_{-\infty}^{+\infty} x \cdot f(x)\,dx = 0$

奇函数

$$E(X^3) = \int_{-\infty}^{+\infty} x^3 f(x)\,dx = \int_{-\infty}^{+\infty} \underbrace{x^3 \cdot \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)}\,dx = 0$$

$$E(e^X) = \int_{-\infty}^{+\infty} e^x f(x)\,dx = \int_{-\infty}^{+\infty} e^x \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}+x}\,dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}(x-1)^2} \cdot e^{\frac{1}{2}}\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \cdot e^{\frac{1}{2}} \int_{-\infty}^{+\infty} e^{-\left(\frac{1}{\sqrt{2}}y\right)^2}\,d\left(\frac{1}{\sqrt{2}}y\right) \cdot \sqrt{2}$$

$$= e^{\frac{1}{2}}$$

5、性质：

(1) Absolutely integrability

$$\int_A X\,d\mathbb{P} < \infty \iff \int_A |X|\,d\mathbb{P} < \infty$$

(2) Linearity

$$\int_A (aX + bY)\,d\mathbb{P} = \int_A aX\,d\mathbb{P} + \int_A bY\,d\mathbb{P}$$

(3) Additivity over sets

If (A_n) is disjoint then

$$\int_{\bigcup_n A_n} X \, d\mathbb{P} = \sum_n \int_{A_n} X \, d\mathbb{P}$$

(4) Positivity

If $X \geq 0$ P-a.e. on A, then $\int_A X \, d\mathbb{P} \geq 0$

(5) Monotonicity

If $X_1 \leq X \leq X_2$ $\mathbb{P}$-a.e. on A

$$\int_A X_1 \, d\mathbb{P} \leq \int_A X \, d\mathbb{P} \leq \int_A X_2 \, d\mathbb{P}$$

(6) Modulus inequality

$$\left| \int_A X \, d\mathbb{P} \right| \leq \int_A |X| \, dP$$

(7) $\int_A \lim_{n\to\infty} X_n \, d\mathbb{P} \overset{?}{=} \lim_{n\to\infty} \int_A X_n \, d\mathbb{P}$

检验方法 ① Dominated Convergence

If $\lim_{n\to\infty} X_n = X$, $\mathbb{P}$-a.e. on A, and $|X_n| \leq Y$ $\mathbb{P}$-a.e. for all n and $\int_A Y \, d\mathbb{P} < \infty$

则 $\lim_{n\to\infty}\int_A X_n \, d\mathbb{P} = \int_A \lim_{n\to\infty} X_n \, d\mathbb{P}$

② Monotone Convergence

If $X_n \geq 0$ (c) and $X_n \uparrow X$ $\mathbb{P}$-a.e. on A,

$$\lim_{n\to\infty}\int_A X_n \, d\mathbb{P} = \int_A \lim_{n\to\infty} X_n \, d\mathbb{P}$$

补充：Fatou's Lemma

If $X_n \geq 0$ $\mathbb{P}$-a.e. on A then

$$\int_A \liminf_{n\to\infty} X_n \, d\mathbb{P} \leq \liminf_{n\to\infty} \int_A X_n \, d\mathbb{P}$$

Jensen's inequality

突向原点

If φ is a convex function, X, $\varphi(X)$ = integral

$$E[\varphi(X)] \geq \varphi[E(X)]$$