4.3 Martingale Measures

Definition 16: (1) A probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ is called a martingale measure if the discounted price process X is a (N-dimensional) $\mathbb{Q}$-martingale, i.e.

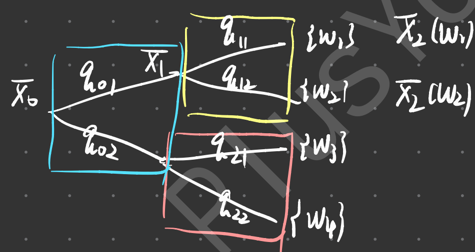
$$\mathbb{E}_{\mathbb{Q}}[X_t^i] < \infty$$

$$\mathbb{E}_{\mathbb{Q}}[X_t^i \mid \mathcal{F}_s] = X_s^i \quad \text{for } 0 \le s \le t \le T,\ i = 1, 2, \cdots, N$$

(2) A martingale measure $\mathbb{P}^*$ is called an equivalent martingale measure if $\mathbb{P}^* \sim \mathbb{P}$.

(3) Denote $\mathcal{P}$ = the collection of all equivalent martingale measures.

Example 17: As in Example 15



用 one-period model

$$\bar{X}_1 = q_1 \bar{X}_2(w_1) + q_2 \bar{X}_2(w_2)$$

解出 q_1, q_2

$$\downarrow$$

$$\mathbb{Q}\{w_1\} = q_{01} \cdot q_{11}$$

$$\mathbb{Q}\{w_2\} = q_{01} \cdot q_{12}$$

$$\mathbb{Q}\{w_3\} = q_{02} \cdot q_{21}$$

$$\mathbb{Q}\{w_4\} = q_{02} \cdot q_{22}$$

Theorem 18: The following conditions are equivalent

(1) Q is an equivalent martingale measure

(2) If $\bar{h}=(h^0,h^T)^T$ is self-financing and $\bar{h}$ is bounded, then the value process V of $\bar{h}$ is a Q-martingale.

(3) If $\bar{h}$ is self-financing and its value process V satisfies $E_Q[V_T^-]<\infty$, then V is a Q-martingale.

(4) If $\bar{h}$ is self-financing and its value process V satisfies $V_T\geq 0$ Q-a.s. then $E_Q[V_T]=V_0$

proof: (1) ⇒ (2) V = value process of $\bar{h}=(h^0,h^T)^T$ and $|h_k|\leq C$ $\forall k$. Then

证明 V_t 是 martingale

需: (1) $V_t\in\mathcal{F}_t$ (由构造易知其成立)

(2) V_t 可积分

(3) $E[V_t|\mathcal{F}_s]=V_s$

$$|V_t| = \left|V_0+\sum_{k=1}^{t} h_k\cdot(X_k-X_{k-1})\right|$$

$$\leq |V_0| + C\sum_{k=1}^{t}(|X_k|+|X_{k-1}|)$$ 可积

$\Rightarrow$ $E_Q|V_t|<\infty$ for all t

$E_Q|X_t|<\infty$

Moreover for $0\leq t\leq T-1$

$$E_Q[V_{t+1}|\mathcal{F}_t] = E_Q[V_t+h_{t+1}(X_{t+1}-X_t)|\mathcal{F}_t]$$
$$= V_t$$

$\Rightarrow (V_t)$ is a Q-martingale

The rest proof: c.f. Follow-Schied Chap 5.

Example 19: Consider a gambling (double strategy)

$$(1) \begin{cases} \xrightarrow{\frac{1}{2}} H\ (2) \\ \xrightarrow{\frac{1}{2}} T\ (0) \end{cases}$$

trading strategy: $h_0^0 = -1$ ← t=0时向bank借一元

strategy: $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ $\xrightarrow{\text{win}}$ $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$: stop!

$\xrightarrow{\text{loss}}$ $\begin{pmatrix} -3 \\ 2 \end{pmatrix}$ $\xrightarrow{\text{win}}$ $\begin{pmatrix} -3 \\ 4 \end{pmatrix}$: stop!

$\begin{pmatrix} -1 \\ 0 \end{pmatrix}$ 后再向 bank 借2元

$\xrightarrow{\text{loss}}$ $\begin{pmatrix} -7 \\ 4 \end{pmatrix}$ $\xrightarrow{\text{win}}$ $\begin{pmatrix} -7 \\ 8 \end{pmatrix}$ = stop!

$\begin{pmatrix} -3 \\ 0 \end{pmatrix}$ 后再向 bank 借4元

$\xrightarrow{\text{loss}}$ $\begin{pmatrix} -15 \\ 8 \end{pmatrix}$ ...

一定有 arbitrage!

h 不 bounded

V^- 为 ∞

Theorem 20 (Fundamental Theorem of Asset Pricing)

The market model is arbitrage-free $\Longleftrightarrow \mathcal{P} \neq \phi$ ← 连续型只有一边是对的 (有EMM 则无NA)

proof: "$\Leftarrow$" Suppose there exists an equivalent martingale measure.

By Thm 18. the value process of any self-financing strategy with

$V_T \geq 0$ Q-a.s. $\Rightarrow$ $V_0 = E_Q[V_T] \geq 0$ $\rightarrow$ 不存在 $V_0 < 0$ 的情况

且 $V_T > 0 \rightarrow V_0 > 0$

$V_T = 0 \rightarrow V_0 = 0$

$\Rightarrow$ There is no arbitrage opportunity.

$\Rightarrow$ This market model is arbitrage-free.

" $\Rightarrow$ " 证明方法较多（难，略）

4.4 Arbitrage-free prices for European contingent claim

Definition 21: (1) A non-negative r.v. C on $(\Omega, \mathcal{F}, \mathbb{P})$ is called a European contingent claim.

(2) T is called the expiration date at maturity of C.

(3) The discounted value of C is given by $H := \frac{C}{B_T}$

H is called the discounted (European) claim, associated with C.

Assumption: $\Phi \neq \phi$

Definition 22: A contingent claim C is attainable (or replicable, redundant), if $\exists$ self-financing trading strategy $\bar{h}$ s.t. $C = \bar{h}_T \bar{S}_T$, $\bar{h}$ is called a replicating strategy for C.

Remark 23: C is attainable $\Longleftrightarrow$ the corresponding discounted claim

$$H = \bar{h}_T \cdot \bar{X}_T = V_T(\bar{h}) = V_0 + \sum_{t=1}^{T} h_t (X_t - X_{t-1}) \qquad X_T = \frac{S_T}{B_T}$$