

### 3.7 Complete Market Model

Definition 42\* : An arbitrage-free market model is complete if every contingent claim is attainable.

↓  
在 complete model 可以不用这个条件

Theorem 43\* : An arbitrage-free market is complete  $\iff \#(\mathcal{P}) = 1$  ← 这里的元素只有一个  
只当 one-period <sup>discrete time</sup> 和 multi-period model 是对的, 在 continuous model 中它不成立

proof : " $\Rightarrow$ " For  $A \in \mathcal{F}_T$ ,  $I_A$  is a contingent claim. (回顾 contingent claim 定义)

$\Rightarrow$  the arbitrage-free price is unique and  $= \mathbb{E}_Q\left[\frac{I_A}{1+r}\right] = \frac{Q(A)}{1+r}$   
corollary 39

$\Rightarrow Q$  is unique

$\Rightarrow \#(\mathcal{P}) = 1$

" $\Leftarrow$ " Suppose  $\mathcal{P} = \{Q\}$  and  $C$  is a bounded contingent claim.

$\Rightarrow \mathbb{E}_Q(C) < \infty$

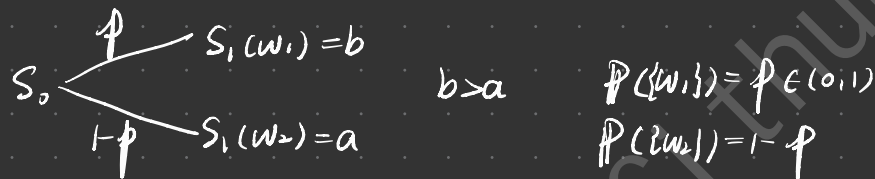
Then  $C$  has a unique arbitrage-free price  $\mathbb{E}_Q\left[\frac{C}{1+r}\right]$

$\Rightarrow$   
Cor 39  $C$  is attainable.

Example 44:  $\Omega = \{\omega_1, \omega_2\}$

$N=1 \Rightarrow$  one bond and one stock

interest rate  $r$ ,  $B_0=1$ ,  $B_1=1+r$



(i) The market model is arbitrage-free

arbitrage-free  $\Leftrightarrow S_0 \in \{ \mathbb{E}^Q \left[ \frac{S_1}{1+r} \right] = Q \sim \mathbb{P} \}$

$$= \left\{ \frac{qb + (1-q)a}{1+r} : q \in (0,1) \right\}$$

$$= \left\{ \frac{q(b-a)}{1+r} + \frac{a}{1+r} : q \in (0,1) \right\}$$

$$= \left( \frac{a}{1+r}, \frac{b}{1+r} \right)$$

(ii)  $S_0$  given in  $\left( \frac{a}{1+r}, \frac{b}{1+r} \right)$

The risk-neutral measure  $\mathbb{P}^*$  satisfies  $S_0(1+r) - \mathbb{E}^*[S_1] = p^*b + (1-p^*)a$

$\Rightarrow p^*$  is unique

由上式直接解出  $p^*$  值 (唯一),  $\mathbb{P}^*$  就对应上面的  $Q$

$$P(1w_1) = P^* = \frac{S_0(1+r) - a}{b-a} \in (0,1)$$

$$\Rightarrow \#(\Phi) = 1$$

$\Rightarrow$  The market model is complete

(iii) Given a contingent claim  $C$ , find  $h^0, h^1$  s.t.  $C = h^0 B_1 + h^1 S_1$   $\leftarrow$  attainable 另一种定义

$$\Rightarrow \begin{cases} h^0(1+r) + h^1 S_1(w_1) = C(w_1) \\ h^0(1+r) + h^1 S_1(w_2) = C(w_2) \end{cases}$$

$$\Rightarrow \begin{cases} h^0 = \frac{C(w_2)b - C(w_1)a}{(1+r)(b-a)} \\ h^1 = \frac{C(w_1) - C(w_2)}{b-a} \end{cases}$$

$\leftarrow$  即当  $C$  是 attainable 时, 可买  $h^0$  张该 Bond 和  $h^1$  张该股票来达到买  $C$  一样的结果 (复制)

总结: 若要判断  $C$  是否为 attainable

方法一: 在保证 model 是 arbitrage-free 的情况下  $\#(\Phi) = 1$   
不可少!!

方法二: 直接根据定义, 则  $C = \bar{h} \cdot \bar{S}$ , 解出  $h^0$  和  $h^1$

(iv) The arbitrage-free price  $\pi^C$  is given by:

$$\pi^C = \mathbb{E}^* \left[ \frac{C}{1+r} \right] = h^0 B_0 + h^1 S_0$$

↑  
对应方法一

$$= p^* \frac{C(w_1)}{1+r} + (1-p^*) \frac{C(w_2)}{1+r}$$

↑  
对应方法二

$$= \frac{C(w_1)b - C(w_2)a}{(1+r)(b-a)} \times 1 + \frac{C(w_1) - C(w_2)}{b-a} S_0$$

$$= \frac{C(w_1)}{1+r} \cdot \frac{S_0(1+r) - a}{b-a} + \frac{C(w_2)}{1+r} \cdot \frac{b - S_0(1+r)}{b-a}$$

In particular, if  $C = (S_1 - K)^+$  with strike price  $K \in (a, b)$

$$\pi_{(S_1 - K)^+} = \frac{b-K}{b-a} S_0 - \frac{(b-K)a}{b-a} \cdot \frac{1}{1+r}$$

$$\begin{cases} C(w_1) = b-K \\ C(w_2) = 0 \end{cases}$$