3、One-Period Model

Assumption:

(1) time $t=0,1$ → 只有2个时刻 (one-period)

Trading time: $t=0$

(2) sample space $\Omega=\{\omega_1,\dots,\omega_k\}$ → 有限多种情况（对无限多种适用的定义会加(*)）

probability $P(\{\omega_k\})>0$, $\forall i=1,2,\dots,k$

at time 0: $\mathcal{F}_0=\{\phi,\Omega\}$ → 0时刻是完全无多余信息情况

at time 1: $\mathcal{F}_1$ = the collection of all possible subsets of Ω → 1时刻是全知情况

(3) 1 bond, N stocks in the financial market → 市场上只有1张债券，N张股票。

security price $\bar{S}_t=(\underbrace{B_t}_{\text{债券价格}},\underbrace{S_t^1,S_t^2,\dots,S_t^N}_{\text{N个股票t时期价格}})^T=\begin{pmatrix}B_t\\S_t\end{pmatrix}$, where $S_t=(S_t^1,S_t^2,\dots,S_t^N)$

B_t: Bond price at time t (constant 定数)

S_t^i = ith stock price at time t

S_0^i = constant 定数（第i张股票今天的价格已知）

S_1^i = r.v.: $\Omega\to\mathbb{R}^+$（第i张股票在1时刻所有的可能性）

$\bar{S}_t \in \mathbb{R}^{N+1}$

$S_t \in \mathbb{R}^N$

$$\begin{pmatrix} B_0 \\ S_0^1 \\ S_0^2 \\ \vdots \\ S_0^N \end{pmatrix} = \bar{S}_0$$

$\bar{S}_1(\omega_1) = \begin{pmatrix} B_1 \\ S_1^1(\omega_1) \\ \vdots \\ S_1^N(\omega_1) \end{pmatrix}$

$\bar{S}_1(\omega_2)$

$\vdots$

$\bar{S}_1(\omega_3)$

$\vdots$

$\bar{S}_1(\omega_K) = \begin{pmatrix} B_1 \\ S_1^1(\omega_K) \\ \vdots \\ S_1^N(\omega_K) \end{pmatrix}$

初始时刻所有价格已知

1时刻只有B_1已知

time 0

time 1

3.1 Portfolios

Definition* 1: A portfolio is a vector $\bar{h} = (h^0, h^1, \cdots, h^N)^T \in \mathbb{R}^{N+1}$

h^i is the number of shares of the ith asset → 第i种证券所拥有的数目

$\bar{h}$ 是 constant (定数) 即交易当时已确定了所购证券的数量

Remark*: The value of the portfolio $\bar{h}$ at time 0 is given by $\boxed{V_0(\bar{h})} = \underbrace{\bar{h}}_{\text{one-period model 中 } \bar{h} \text{ 无下标(时间)}} \cdot \bar{S}_0 = h^0 B_0 + h^1 S_0^1 + \cdots h^N S_0^N$ → 定数

The value of the portfolio $\bar{h}$ at time 1 is given

by $\boxed{V_1(\bar{h})} = \bar{h} \cdot \bar{S}_1 = h^0 B_1 + h^1 S_1^1 + h^2 S_1^2 + \cdots + h^N S_1^N$

↳ 随机变量(r.v.)

The profit of the portfolio $\bar{h}$ is given by

$$G(\bar{h}) = V_1(\bar{h}) - V_0(\bar{h}) = \bar{h}(\bar{S}_1 - \bar{S}_0) = \bar{h} \cdot \Delta S$$

Example 3: $\bar{S}_0 = \begin{pmatrix} 1 \\ 10 \end{pmatrix}$, $\bar{S}_1(\omega_1) = \begin{pmatrix} 1.02 \\ 12 \end{pmatrix}$, $\bar{S}_1(\omega_2) = \begin{pmatrix} 1.02 \\ 9 \end{pmatrix}$

Let $\bar{h} = \begin{pmatrix} -10 \\ 1 \end{pmatrix}$ ← 正买负卖 (正为拥有, 负为借来)

$$V_0(\bar{h}) = \bar{h} \cdot \bar{S}_0 = -10 + 10 = 0$$

$$V_1(\bar{h})(\omega_1) = \bar{h} \cdot \bar{S}_1(\omega_1) = -10.2 + 12 = 1.8$$

$$V_2(\bar{h})(\omega_2) = \bar{h} \cdot \bar{S}_1(\omega_2) = -10.2 + 9 = -1.2$$

$$\Rightarrow V_1(\bar{h}) = \begin{pmatrix} \bar{h}\cdot \bar{S}_1(w_1) \\ \bar{h}\cdot \bar{S}_1(w_2) \end{pmatrix} = \begin{pmatrix} 1.8 \\ -1.2 \end{pmatrix}$$

$$G(\bar{h}) = V_1(\bar{h}) - V_0(\bar{h}) = \begin{pmatrix} 1.8 \\ -1.2 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1.8 \\ -1.2 \end{pmatrix}$$

3.2 Derivaties Securities 衍生性金融商品

Example 4*: Forward contract (远期)

One agent agrees to sell another agent an asset at time 1 for a price K which is specified at time 0.

$$\text{payoff} = S_1^i - K$$

Example 5*: Call option (买权)

有权利但无义务去买某种东西

The owner has the right but not obligation to buy the asset at time 1 for a fixed price K, called strike price.

$$\text{payoff} = (S_1^i - K)^+ = \begin{cases} S_1^i - K > 0, & \text{if } S_1^i > K \\ 0, & \text{if } S_1^i \le K \end{cases}$$

Example 6*: Put option (卖权)

The owner has the right, but not obligation to sell the asset of time 1 for a fixed price K.

$$\text{payoff} = (K-S_1^i)^+ = \begin{cases} K-S_1^i, & S_1^i < K \\ 0, & S_1^i \geq K \end{cases}$$

Definition 7*: (1) A contingent claim is a r.v. C on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\underbrace{0 \leq C < \infty}_{\text{就是某一种信用商品的payoff}}$ $\mathbb{P}$-a.s.

(2) A contingent claim C is called a derivative of $B_1, S^1, S^2, \cdots, S^N$ if it is measurable w.r.t. $\sigma(B, S^1, \cdots, S^N)$

i.e. $C = f(B, S^1, S^2, \cdots, S^N)$ for a measurable function f on $\mathbb{R}^{N+1}$

Call / Put option 就是一种 contingent claim

Question: What is the price of a contingent claim? (如何给衍生品定价)