Thm 18 $\Rightarrow$

$$\begin{pmatrix} B \\ S_0 \end{pmatrix} = \begin{pmatrix} B(1+r) & B(1+r) \\ S_1(\omega_1) & S_1(\omega_2) \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

$$\Rightarrow \begin{cases} \psi_1 = \dfrac{1}{1+r} \dfrac{(1+r)S_0 - S_1(\omega_2)}{S_1(\omega_1) - S_1(\omega_2)} \\ \psi_2 = \dfrac{1}{1+r} \cdot \dfrac{S_1(\omega_1) - (1+r)S_0}{S_1(\omega_1) - S_1(\omega_2)} \end{cases}$$

Thus, $\psi_1 >> 0$, $\psi_2 >> 0$

$$\Rightarrow S_1(\omega_2) < (1+r)S_0 < S_1(\omega_1)$$

Thus, (b, D) is arbitrage-free

$$\Rightarrow \boxed{\frac{S_1(\omega_2)}{1+r} < S_0 < \frac{S_1(\omega_1)}{1+r}}$$

1时刻股票价格折到现在来看
既有可能亏钱，也有可能赚钱

3.5 Martingale Measures

Remark 20: $b = D\psi$

$$\Rightarrow \begin{pmatrix} B_0 \\ S_0^1 \\ \vdots \\ S_0^N \end{pmatrix} = \begin{pmatrix} B_1(\omega_1) & B_1(\omega_2) & \cdots & B_1(\omega_K) \\ S_1^1(\omega_1) & S_1^1(\omega_2) & \cdots & S_1^1(\omega_K) \\ \vdots & \vdots & & \vdots \\ S_1^N(\omega_1) & S_1^N(\omega_2) & \cdots & S_1^N(\omega_K) \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_K \end{pmatrix}$$

$$\Rightarrow \begin{cases} B_0 = B_1(w_1)\psi_1 + B_1(w_2)\psi_2 + \cdots + B_1(w_K)\psi_K & (*) \\ S_0^i = S_1^i(w_1)\psi_1 + S_1^i(w_2)\psi_2 + \cdots + S_1^i(w_K)\psi_K \end{cases}$$

If the interest rate of the bond $= r =$ constant, i.e. $B_0 = B$, $B_1 = (1+r)B$

$$(*) = 1 = (1+r)(\psi_1 + \cdots + \psi_K)$$

$$\psi_1 + \cdots + \psi_K = \frac{1}{1+r}$$

Thus, for $1 \le i \le N$

$$S_0^i = \frac{S_1^i(w_1)}{1+r} \cdot \frac{\psi_1}{\psi_1 + \cdots + \psi_K} + \frac{S_1^i(w_2)}{1+r} \cdot \frac{\psi_2}{\psi_1 + \cdots + \psi_K} + \cdots + \frac{S_1^i(w_K)}{1+r} \cdot \frac{\psi_K}{\psi_1 + \cdots + \psi_K}$$

Remark 21: Define

$$Q(\{w_j\}) = \frac{\psi_j}{\psi_1 + \cdots + \psi_K} \quad (**) \quad ☆$$

Then Q is a probability measure.

Definition 22: The probability Q defined by $(**)$ is called a risk-neutral (probability) measure.

Remark 23: In general, Q in $(**)$ is not unique (b=Dψ中ψ不一定是唯一解)

Theorem 24*: In an arbitrage-free market model $(b, D) \trianglelefteq (\bar{S}_0, \bar{S}_1)$ there is a risk-neutral measure Q s.t.

☆☆ 非常重要 $$S_0^i = \frac{1}{1+r} E_Q[S_1^i] \quad \forall\, 1 \le i \le N.$$

where E_Q means the expectation w.r.t. the probability measure Q

Example 25: As in Example 19

$$b=\begin{pmatrix} B \\ S_0 \end{pmatrix} \quad P=\begin{pmatrix} B(1+r) & B(1+r) \\ S_1(w_1) & S_1(w_2) \end{pmatrix}$$

$$\Rightarrow Q(\{w_1\}) = \frac{\psi_1}{\psi_1+\psi_2} = \frac{(1+r)S_0 - S_1(w_2)}{S_1(w_1)-S_1(w_2)}$$

$$Q(\{w_2\}) = \frac{\psi_1}{\psi_1+\psi_2} = \frac{S_1(w_1)-(1+r)S_0}{S_1(w_1)-S_1(w_2)}$$

and $E_Q(S_1) = Q(\{w_1\})S_1(w_1) + Q(\{w_2\})S_1(w_2)$

$$= \frac{(1+r)S_0 S_1(w_1) - \cancel{S_1(w_1)S_1(w_2)} + \cancel{S_1(w_1)S_1(w_2)} - (1+r)S_0 S_1(w_2)}{S_1(w_1)-S_1(w_2)}$$

$$= (1+r)\cdot\frac{S_0(S_1(w_1)-S_1(w_2))}{S_1(w_1)-S_1(w_2)} = (1+r)S_0$$

$$\Rightarrow S_0 = \frac{1}{1+r}E_Q(S_1) = E_Q\left(\frac{S_1}{1+r}\right) \qquad (***)$$

Remark 26*: Let $X_0^i = S_0^i$

$X_1^i = \frac{S_1^i}{1+r}$ discounted stock price

Then $(***) \Rightarrow X_0^i = E_Q(X_1^i) = E_Q(X_1^i \mid \mathcal{F}_0)$ $\quad \mathcal{F}_0 = \{\phi, \Omega\}$

$\Rightarrow (X_k^i, \mathcal{F}_k)_{k=0,1}$ is a martingale.

Hence, the risk-neutral measure $\mathbb{Q}$ is called a martingale measure.

Remark 27*: (1) (X_k^i) = martingale w.r.t. $\mathbb{Q}$

$\Rightarrow (h^i \cdot X_k^i)$ = martingale w.r.t. $\mathbb{Q}$

h^i = constant

$\Rightarrow (\bar{h} \cdot \bar{X}_k)_{k=0,1}$ = martingale w.r.t. $\mathbb{Q}$

(2) The r.v. $Y^i = \frac{S_1^i}{1+r} - S_0^i$ is called the discounted net gain

Thus $E_{\mathbb{Q}}[Y^i] = 0, \ \forall 1 \le i \le N$

Remark 18 可改写为 $S_0^i = E_{\mathbb{Q}}\left(\frac{S_1^i}{1+r}\right) \ \forall 1 \le i \le N$

(b, D) = arbitrage free $\Longleftrightarrow \exists \psi \in \mathbb{R}^k, \psi \gg 0$, s.t. $b = D\psi$

Definition 28*: (1) Two probability measures

☆常用!!

$\mathbb{P}$ and $\mathbb{Q}$ are called equivalent, denoted by $\mathbb{P} \sim \mathbb{Q}$ if

$\mathbb{P}(A) = 0 \Longleftrightarrow \mathbb{Q}(A) = 0 \quad \forall A \in \mathcal{F}$

(2) An equivalent risk-neutral measure is called a pricing measure

or equivalent martingale measure (EMM).

Example 29 : (1) $\Omega = \{1,2,3,4\}$

(Discrete)

$\mathcal{F}$ = the collection of all subsets of Ω

$P_1(\{1\}) = \frac{1}{2}$, $P_1(\{2\}) = \frac{1}{4}$, $P_1(\{3\}) = \frac{1}{6}$, $P_1(\{4\}) = \frac{1}{12}$

$P_2(\{1\}) = \frac{1}{5}$, $P_2(\{2\}) = \frac{1}{5}$, $P_2(\{3\}) = \frac{1}{5}$, $P_2(\{4\}) = \frac{2}{5}$

$P_3(\{1\}) = \frac{1}{4}$, $P_3(\{2\}) = \frac{1}{2}$, $P_3(\{3\}) = 0$, $P_3(\{4\}) = \frac{1}{4}$

Then $P_1 \sim P_2$, $P_1 \not\sim P_3$, $P_2 \not\sim P_3$ ← 看等于0的集合是否一致

$\{\emptyset\}$ $\{\emptyset\}$ $\{\emptyset\}$ $\{\emptyset, \{3\}\}$ $\{\emptyset\}$ $\{\emptyset, \{3\}\}$

(continuous) (2) $(\Omega, \mathcal{F}, \mathbb{P})$, X = r.v. with $X \geq \frac{1}{2}$, $E[X] = 1$

Define: $Q(A) = \int_A X \, d\mathbb{P}$ ← 保证Q是一个概率测度

Then (i) Q is a probability

(ii) $\mathbb{P} \sim Q$

思路: $\left.\begin{array}{l} \mathbb{P}(A) = 0 \Longrightarrow Q(A) = 0 \\ Q(A) = 0 \Longrightarrow \mathbb{P}(A) = 0 \end{array}\right\} \Rightarrow \mathbb{P} \sim Q$

$Q(A) = \int_A X \, d\mathbb{P} \geq \int_A \frac{1}{2} \, d\mathbb{P} = \frac{1}{2}\mathbb{P}(A) \geq 0$

if $Q(A) = 0$, then $0 \geq \frac{1}{2}\mathbb{P}(A) \geq 0 \Rightarrow \mathbb{P}(A) = 0$

if $P(A)=0$

$$Q(A)=\mathbb{E}[XI_A] \underset{XI_A=0\ P\text{-a.s.}}{=\!=\!=} 0$$

Theorem 30* (Fundamental Theorem of Asset Pricing)

(再次改写 Remark 18)

对于 Remark 18: arbitrage-free $\Longleftrightarrow \exists\, \psi\in\mathbb{R}^k,\ \psi\gg 0$, s.t. $b=D\psi$

其中 $k\to$ 大时很难获得 ψ

$\downarrow$

A market model is arbitrage-free

$\Longleftrightarrow \mathcal{P}=\{Q: Q$ is a neutral-risk measure with $P\sim Q\}\neq\emptyset$

proof: "$\Longleftarrow$" Suppose that $Q\in\mathcal{P}$

Let $h\in\mathbb{R}^{N+1}$ with

$h\cdot S_1\geq 0$ P-a.s. and $\mathbb{E}_Q[h\cdot S_1]>0$

已知: 若为 arbitrage

则 (i) $h\cdot S_1\leq 0$

(ii) $h\cdot S_1\geq 0$ P-a.s. $\left.\begin{matrix}\\ \\ \end{matrix}\right\}\ \mathbb{E}(h\cdot S_1)>0$

$P(h\cdot S_1>0)>0$

$\boxed{\begin{matrix}(i)\Rightarrow(ii)\\ (ii)\Rightarrow(i)\end{matrix}}$

Since Q is a martingale measure,

$$h\cdot S_0=h\cdot\mathbb{E}_Q\left[\frac{S_1}{1+r}\right]=\mathbb{E}_Q\left[\frac{h\cdot S_1}{1+r}\right]>0$$ 不满足 (i)

$\Rightarrow$ This market model is arbitrage-free

由 ψ 不唯一 $\Rightarrow$ $\mathbb{Q}$ 不唯一

$\Downarrow$ $\mathbb{Q}\sim\mathbb{P}$

$\mathbb{P}$ 不唯一

那么 $\mathbb{P}$ 唯一时，是什么样，有哪些条件？

" $\Longrightarrow$ " See Föllmer – Schied P.7.

Example 3.1: 1 bond, 1 stock

$\Omega=\{\omega_1,\omega_2,\cdots,\omega_K\}$

$$b=\begin{pmatrix}1\\S_0\end{pmatrix}\quad D=\begin{pmatrix}1+r & \cdots & 1+r\\ S_1(\omega_1) & \cdots & S_1(\omega_K)\end{pmatrix}\quad r>0$$

If this financial market is arbitrage-free

$\exists\,\psi\in\mathbb{R}^K,\ \psi\gg 0$, s.t.

$$\begin{pmatrix}1\\S_0\end{pmatrix}=b=D\psi=\begin{pmatrix}1+r & \cdots & 1+r\\ S_1(\omega_1) & \cdots & S_1(\omega_K)\end{pmatrix}\begin{pmatrix}\psi_1\\ \vdots\\ \psi_K\end{pmatrix}$$

$$\Rightarrow\begin{cases}\psi_1+\psi_2+\cdots+\psi_K=\dfrac{1}{1+r}\\ S_1(\omega_1)\psi_1+S_1(\omega_2)\psi_2+\cdots+S_1(\omega_K)\psi_K=S_0\end{cases}$$

$\begin{cases}K\text{个未知数}\\ 2\text{个方程}\end{cases}$

$\Rightarrow$ $K=2$ 时有唯一解

$K>2$ 时有无穷多解

How many strictly positive solutions does this system of equations

(i) $K=2$: assume $S_1(\omega_1)>S_1(\omega_2)$

the equivalent risk-neutral measure is unique and

$$\mathbb{Q}(\omega_1)=\frac{(1+r)S_0-S_1(\omega_2)}{S_1(\omega_1)-S_1(\omega_2)}$$

$$\mathbb{Q}(\omega_2)=\frac{S_1(\omega_1)-(1+r)S_0}{S_1(\omega_1)-S_1(\omega_2)}$$

(ii) $k > 2$ = the equivalent risk-neutral measure is no more unique.

In fact, there are infinite many. 无穷多个解

不重要。 ×

当市场上有无穷多个股票时, Theorem 30 是错的! $N \to +\infty$

Example 32:

$\Omega = \mathbb{N} = \{1, 2, 3, \cdots\}$

$\mathbb{P}(\{i\}) > 0, \forall i$

$B_0 = B_1 = 1$ (i.e. interest rate $= 0$)

For $i = 1, 2, 3, \cdots$ let $S_0^i = 1$

$$S_1^i(\omega) = \begin{cases} 0, & \text{if } \omega = i \\ 2, & \text{if } \omega = i+1 \\ 1, & \text{o.w.} \end{cases}$$

(1) Claim: This model is arbitrage-free

Suppose that $\bar{h} = (h^0, h^1, h^2, \cdots)^T$ is a portfolio such that

$$\bar{h} \cdot \bar{S}_1(\omega) \geq 0, \ \forall \omega \in \Omega.$$

and

$$\bar{h} \cdot \bar{S}_0 \leq 0$$

consider $\omega = 1$

$$\bar{S}_1(1) = (1, S_1^1(1), S_1^2(1), \cdots, S_1^n(1), \cdots)$$

假设(i),(ii)前半部分都成立,若是 arbitrage-free, 则 $\bar{h}$ 解出为0.

$$=(1,\ 0,\ 1,\ 1,\cdots,\ 1,\cdots)$$

$$0 \le \bar{h}\cdot \bar{S}_1(1) = h^0 B_1 + h^1 S_1^1(1) + h^2 S_1^2(1) + \cdots$$

$$= h^0 + 0 + h^2 + h^3 + \cdots + h^n + \cdots$$

$$= h^0 + h^2 + h^3 + \cdots$$

$$= (1,1,1,\cdots,1)^T (h^0, h^1, h^2, \cdots)^T - h^1$$

$$= \bar{h}\cdot \bar{S}_0 - h^1 \overset{\bar{h}\cdot\bar{S}_0 \le 0}{\le} -h^1$$

$$h^1 \le 0$$

For $\omega = i > 1$

$$0 \le \bar{h}\cdot \bar{S}_1(i) = h^0 B_1(i) + h^1 S_1^1(i) + h^2 S_1^2(i) + \cdots$$

$$= \sum_{k=1}^{\infty} h^k + h^{i-1} - h^i$$

$$= \bar{h}\cdot \bar{S}_0 + h^{i-1} - h^i \le h^{i-1} - h^i$$

$$\Rightarrow 0 \ge h^1 \ge h^2 \ge h^3 \ge \cdots$$

$\bar{h}\cdot\bar{S}_0 \le 0$ and $\bar{h}\cdot\bar{S}_1 \ge 0$

$\Rightarrow h^i = 0,\ \forall i = 1, 2, \cdots$

$\Rightarrow$ There is no arbitrage opportunity.

(2) Claim: There is no equivalent martingale measure.

Suppose there is an equivalent martingale measure $\mathbb{Q}$

$$\Rightarrow \bar{S}_0 = \mathbb{E}_{\mathbb{Q}}[S_i] \quad \forall i$$

$$1 = \sum_{\substack{k=1 \\ k\neq i,\, k\neq i+1}}^{\infty} \mathbb{Q}(\{k\}) + 2\,\mathbb{Q}(\{i+1\})$$

$$1 = \sum_{k=1}^{\infty} \mathbb{Q}(\{k\}) + \mathbb{Q}(\{i+1\}) - \mathbb{Q}(\{i\})$$

$$1 = 1 + \mathbb{Q}(\{i+1\}) - \mathbb{Q}(\{i\}) \quad \forall i \geq 1$$

$$\Rightarrow \mathbb{Q}(\{i+1\}) = \mathbb{Q}(\{i+1\}) \quad \forall i \geq 1$$

$\mathbb{Q}$每个都相同，无穷多个$\mathbb{Q}$相加不可能为1

故假设错误，不存在这样的$\mathbb{Q}$.

综上：在股票个数为无穷的情况下，对于arbitrage-free的情况，我们也找不到对应的$\mathbb{Q}$.

或满足$\bar{h}\cdot\bar{S}_0 \leq 0$，$\bar{h}\cdot\bar{S}_1 \geq 0$后，依然是no arbitrage opportunity的.

Theorem 33. (Law of One Price)

Suppose that the market model in arbitrage-free and suppose that

$$\bar{h}\cdot\bar{S}_1 = \bar{k}\cdot\bar{S}_1$$

for two different portfolios $\bar{h}$ and $\bar{k}$, then

$$\bar{h} \cdot \bar{S}_0 = \bar{k} \cdot \bar{S}_0$$

Proof: $\bar{h} \cdot \bar{S}_1 = \bar{k} \cdot \bar{S}_1$ $\mathbb{P}$-a.s.

$\Rightarrow (\bar{h} - \bar{k}) \cdot \bar{S}_1 = 0$ $\mathbb{P}$-a.s.

$\overset{\mathbb{P} \sim \mathbb{Q}}{\Longrightarrow} (\bar{h} - \bar{k}) \cdot \bar{S}_1 = 0$ $\mathbb{Q}$-a.s.

$\Longrightarrow \mathbb{E}_{\mathbb{Q}}[(\bar{h} - \bar{k}) \cdot \bar{S}_1] = 0$ martingale measure 性质 $S_0^i = \mathbb{E}_{\mathbb{Q}}\left[\frac{S_1^i}{1+r}\right]$

$\Longrightarrow (1+r)(\bar{h} - \bar{k}) \cdot \bar{S}_0 = 0$

$\Longrightarrow \bar{h} \cdot \bar{S}_0 = \bar{k} \cdot \bar{S}_0$

Remark 34: If $V \in \{\bar{h} \cdot \bar{S}_1 : \bar{h} \in \mathbb{R}^{N+1}\}$, then we can define the price of V as

$$\pi(V) = \bar{h} \cdot \bar{S}_0 \quad \text{if } V = \bar{h} \cdot \bar{S}_1$$

whenever the market model is arbitrage-free

By Theorem 24. $\pi(V) = \mathbb{E}_{\mathbb{Q}}\left[\frac{V}{1+r}\right]$