4.0 Multi-Period Model

Assumption:

(1) trading time: $0, 1, 2, \cdots, T$ ← 交易的时间不再是只有0和1两个时刻

(2) Ω: nonempty sample space ← 不用再假定 Ω 是有限的 (situation at $t=T$).

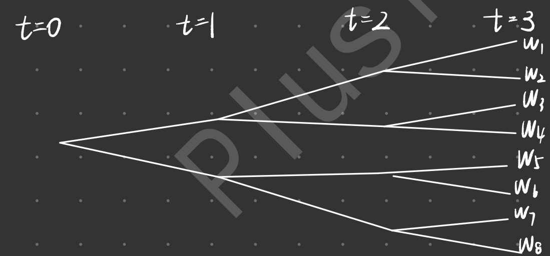
(3) information $= \mathcal{F}_t$ for $t = 0, 1, 2, \cdots, T$ ← 市场上 t 时刻的资讯

(4) The price is adapted to $(\mathcal{F}_t)$: the price of time t is $\mathcal{F}_t$-measurable for all t.

☆ 在 One-period Model 时不明显，因此常被忽略，而在 multi-period model 中，这一条很关键！
它意味着价格在 t 时刻总能被 $\mathcal{F}_t$ 的信息所确定下来

Example $t = 0, 1, 2, 3$

$\Omega = \{w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8\}$ ← 这是在 $T=3$ 时刻 (最终时刻) 的情况



$\mathcal{F}_3 = \sigma(\{w_1\}, \{w_2\}, \cdots, \{w_8\}) \rightarrow 2^8$ 个元素

$\mathcal{F}_2 = \sigma(\{w_1, w_2\}, \{w_3, w_4\}, \{w_5, w_6\}, \{w_7, w_8\}) \rightarrow 2^4$ 个元素

$\mathcal{F}_1 = \sigma(\{w_1, w_2, w_3, w_4\}, \{w_5, w_6, w_7, w_8\}) \rightarrow 2^2$ 个元素

$\mathcal{F}_0 = \sigma(\{w_1, w_2, \cdots, w_8\}) \rightarrow 2^1$ 个元素

那么当 r.v. $X \in \mathcal{F}_1$，则也就是必有 $X(w_1) = X(w_2) = X(w_3) = X(w_4)$，$X(w_5) = X(w_6) = X(w_7) = X(w_8)$

4.1 The Market Model

Definition 1: The market model is a pair $(\bar{S}, \mathbb{F})$, where $\mathbb{F} = (\mathcal{F}_t)_{t=0,1,\cdots,T}$ = filtration,

价格 信息

$\bar{S} = (\bar{S}_t)_{t=0,1,\cdots,T}$: $(\mathcal{F}_t)$-adapted stochastic process

$$\bar{S}_t = \begin{pmatrix} B_t \\ S_t^1 \\ \vdots \\ S_t^N \end{pmatrix} = \begin{pmatrix} B_t \\ S_t \end{pmatrix} \leftarrow \text{t时刻所以债券和股票的价格}$$

$$S_t = \begin{pmatrix} S_t^1 \\ \vdots \\ S_t^N \end{pmatrix}$$

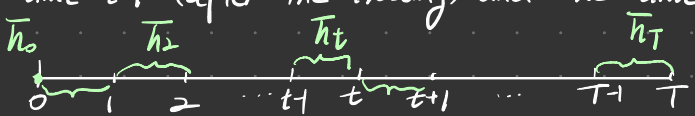
Definition 2: A stochastic process $(X_t)_{t=0,1,\cdots,T}$ is called previsible (or predictable)

if X_0 is $\mathcal{F}_0$-measurable

X_t is $\mathcal{F}_{t-1}$-measurable $\forall t = 1, 2, \cdots, T$

Definition 3: A trading strategy $(\bar{h}_t)_{t=0,\cdots,T}$, $(\bar{h}_t = (h_t^0, h_t^1, \cdots, h_t^N)^T)$ is an $\mathbb{R}^{N+1}$-valued predictable process, where h_t^i is the number of the shares of ith asset between the time $t-1$ (after the trading) and the time t (before the trading).

指在t=0时刻交易前就已经拥有的股票和 $\rightarrow$



债券张数 时间0时债券与股票的价格 → $\bar{S}_0 \uparrow \bar{h}_1 \ \bar{S}_1 \ \bar{S}_2 \ \cdots \ \bar{S}_{t-1} \ \bar{S}_t \ \bar{h}_{t+1} \ \bar{S}_{t+1} \ \cdots \ \bar{S}_{T-1} \ \bar{S}_T$

时间0交易后,时间1交易前,所拥有的股票和债券张数

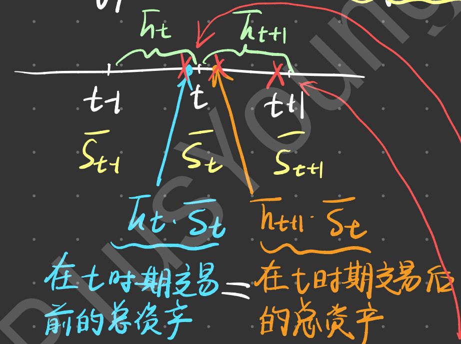
Remark 4: The total value of the portfolio $\bar{h}$ at time $t-1$ (after the trading) is

$$\bar{h}_t \cdot \bar{S}_{t-1} = h_t^0 B_{t-1} + h_t^1 S_{t-1}^1 + \cdots + h_t^N S_{t-1}^N = h_t^0 B_{t-1} + \sum_{i=1}^{N} h_t^i S_{t-1}^i$$

At time t before the trading, the value of the portfolio $\bar{h}$ has changed to

$$\bar{h}_t \cdot \bar{S}_t = h_t^0 B_t + h_t^1 S_t^1 + \cdots + h_t^N S_t^N = h_t^0 B_t + \sum_{i=1}^{N} h_t^i S_t^i$$

Definition 5: A trading strategy $\bar{h}$ is called self-financing 指无外来资金 if $\bar{h}_t \cdot \bar{S}_t = \bar{h}_{t+1} \cdot \bar{S}_t$, $\forall t$



☆☆

tip: S是市场价格,它随着t自动变化

h是交易份额,它只有在整点t会发生变动

即可进行交易(份额变动),因此在 t-1 到 t

之间 $\bar{h}$ 不会发生变化!!

Remark 6: (1) If $\bar{h}$ is self-financing, then

$$\bar{h}_{t+1} \cdot \bar{S}_{t+1} - \bar{h}_t \cdot \bar{S}_t = \bar{h}_{t+1} \cdot \bar{S}_{t+1} - \bar{h}_{t+1} \cdot \bar{S}_t + \bar{h}_{t+1} \cdot \bar{S}_t - \bar{h}_t \cdot \bar{S}_t$$

$$= (\bar{h}_{t+1} \cdot \bar{S}_{t+1} - \bar{h}_{t+1} \cdot \bar{S}_t) + \underbrace{(\bar{h}_{t+1} \cdot \bar{S}_t - \bar{h}_t \cdot \bar{S}_t)}_{=0}$$

$$= \bar{h}_{t+1} \cdot \bar{S}_{t+1} - \bar{h}_{t+1} \cdot \bar{S}_t$$

即当 $\bar{h}$ 是 self-financing 时 $\rightarrow = \bar{h}_{t+1}(\bar{S}_{t+1} - \bar{S}_t)$

t+1交易前与t交易前的价差只由股票价差来决定

$\Rightarrow$ the losses and gains resulting from the asset price fluctuation are the only source of variation of the portfolio values.

(2) If $\bar{h}$ is self-financing, the value of the portfolio $\bar{h}$ at time t (before the trading)

$$\bar{h}_t \cdot \bar{S}_t = \underbrace{\bar{h}_0 \bar{S}_0}_{\text{initial endowment}} + \underbrace{\sum_{k=1}^{t} \bar{h}_k \cdot (\bar{S}_k - \bar{S}_{k-1})}_{\text{每个阶段所赚的钱}}$$

Remark 7: Often it is assumed that B_t is the role of a (locally) riskless asset.

We may take

$B_0 = 1$ and spot rate $r_t > 0$

That is

at time $t-1$ $\qquad$ at time t

$$B_{t-1} \longrightarrow B_{t-1}(1+r_t)$$

Thus

$$B_t = \prod_{k=1}^{t} (1+r_k) \qquad (\Rightarrow B_t > 0 \ \ P\text{-a.s. for all } t)$$

Notation 8: The assumption "$B_t > 0$ for all t" allows us to use B as a "numéraire".

Define the discounted price process. ← 即转化为所有 $r_t=0$ 的情况

$$X_t^i = \frac{S_t^i}{B_t}, \quad t=0,1,2,\cdots,T, \quad i=1,2,\cdots,N$$

当 $B_0=1$ 时，B_t 就表现为折现因子（t 折现到 0 时刻的价格）

$$X_t^0 = \frac{B_t}{B_t} = 1$$ ← B_t 为无风险资产，所以任意时刻折现到 0 时刻价格都为初始时刻价格，即为 $B_0=1$

$$\overline{X}_t = (X_t^0, X_t^1, \cdots, X_t^N)^T = \begin{pmatrix} 1 \\ x_t \end{pmatrix}$$

$$x_t = (X_t^1, \cdots, X_t^N)^T$$

Definition 9: (1) The discounted value process:

$V = (V_t)_{t=0,1,\cdots,T}$ associated with a trading strategy $\overline{h}$ is given by

$$V_t(\overline{h}) = \overline{h}_t \cdot \overline{X}_t \quad \text{for } t=0,1,\cdots,T$$

(2) The gain process $G=(G_t)_{t=0,1,\cdots,T}$ associated with $\overline{h}$ is given by

$$G_0 := 0$$

$$G_t := \overline{h}_t \cdot \overline{X}_t - \overline{h}_0 \cdot \overline{X}_0 = \sum_{k=1}^{t} \overline{h}_k (\overline{X}_k - \overline{X}_{k-1}) = \sum_{k=1}^{t} h_k (x_k - x_{k-1})$$

可去掉债券部分，$X_k^0 - X_{k-1}^0 = 1-1=0$