3.6 Pricing

C : payoff of the contingent claim at time 1

$$(*) \begin{cases} S_0^{N+1} = \underset{\sim}{\pi(C)} = \pi^C & \text{两种写法（把 contingent claim 当作第 N+1 种股票）} \\ S_1^{N+1} = C & \leftarrow \text{已知在 } t=1 \text{ 时，价格为 } C \end{cases}$$

求 $\pi(C)$

Definition 35*: A real-number $\pi^C \geq 0$ is called an arbitrage-free price of a contingent claim C if the market model extended according to $(*)$ is arbitrage-free. The set of all arbitrage-free price for C is denoted by $\underline{\Pi(C)}$.

大写为小写 $\pi(C)$ 的集合

Theorem 36*: Suppose that $\mathcal{P} \neq \emptyset$. Then $\Pi(C) \neq \emptyset$ and

$$\Pi(C) = \{ \mathbb{E}_Q \left(\frac{C}{1+r} \right), Q \in \mathcal{P} \text{ and } \mathbb{E}_Q(C) < \infty \}$$

Proof: By Theorem 24 and Theorem 30. π^C is an arbitrage-free price for C

$\Leftrightarrow \exists \mathbb{Q} \in \mathcal{P}$ for the market model extended via (†)

$$S_0^i = \mathbb{E}_{\mathbb{Q}}\left[\frac{S_i}{1+r}\right] \quad \forall\, i = 1, 2, \cdots, N, N+1$$

Thus

$$\pi(c) \subseteq \left\{ \mathbb{E}_{\mathbb{Q}}\left[\frac{c}{1+r}\right] : \mathbb{Q} \in \mathcal{P} \text{ with } \mathbb{E}_{\mathbb{Q}}[c] < \infty \right\}$$

Conversely if $\pi^c = \mathbb{E}_{\mathbb{Q}}\left[\frac{c}{1+r}\right]$ for some $\mathbb{Q} \in \mathcal{P}$

Then $\mathbb{Q}$ is also an equivalent martingale measure for the extended market model.

$$\Rightarrow \pi(c) \supseteq \left\{ \mathbb{E}_{\mathbb{Q}}\left[\frac{c}{1+r}\right] : \mathbb{Q} \in \mathcal{P} \text{ with } \mathbb{E}_{\mathbb{Q}}[c] < \infty \right\}$$

Example 3): Consider a market model

$$b = \begin{pmatrix} 1 \\ 10 \end{pmatrix} \quad D = \begin{pmatrix} 1 & 1 & 1 \\ 9 & 11 & 12 \end{pmatrix}$$

Then $\exists\, \psi \in \mathbb{R}^3$, $\psi \gg 0$, s.t.

$$\begin{pmatrix} 1 \\ 10 \end{pmatrix} = b = D\psi = \begin{pmatrix} 1 & 1 & 1 \\ 9 & 11 & 12 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}$$

$$\begin{cases} \psi_1+\psi_2+\psi_3=1 \\ 9\psi_1+11\psi_2+12\psi_3=10 \end{cases} \Rightarrow \mathcal{P}=\{Q : Q\{\omega_i\}=a_i\}>0$$

$$\begin{cases} a_1+a_2+a_3=0 \\ 9a_1+11a_2+12a_3=10 \end{cases} \xRightarrow{Q\{\omega_1\}=a} \begin{cases} a_2+a_3=1-a \\ 11a_2+12a_3=10-9a \end{cases} \Rightarrow \begin{cases} a_2=2-3a \\ a_3=2a-1 \end{cases}$$

$$\mathcal{P}=\{Q : Q(\{\omega_1\})=a,\ Q(\{\omega_2\})=2-3a,\ Q(\{\omega_3\})=2a-1,\ \tfrac{1}{2}<a<\tfrac{2}{3}\}$$

(i) consider $C(\omega_1)=6,\ C(\omega_2)=8,\ C(\omega_3)=9$

Then $\pi^C=E_Q[C]=C(\omega_1)\cdot Q(\{\omega_1\})+C(\omega_2)\cdot Q(\{\omega_2\})+C(\omega_3)\cdot Q(\{\omega_3\})$

$=6a+8(2-3a)+9(2a-1)=7$ for all a

$\Pi(C)=\{7\}$

(ii) consider $C(\omega_1)=10,\ C(\omega_2)=8,\ C(\omega_3)=12$

Then $\pi^C=10a+8(2-3a)+12(2a-1)=4+10a$

$\Pi(C)=\{4+10a : \tfrac{1}{2}<a<\tfrac{2}{3}\}=(9,\tfrac{32}{3})$

Definition 38*: A contingent claim C is called attainable (or replicable) if

$$C = \bar{h}\cdot\bar{\xi},$$

for some $\bar{h} \in \mathbb{R}^{n+1}$. Such a portfolio strategy $\bar{h}$ is then called a replicating portfolio for C.

Corollary 39*: Suppose the market model is arbitrage-free and C is a contingent claim.

(1) C is attainable

$\iff$ it admits an unique arbitrage-free price

(2) If C is not attainable, then $\exists\, a<b$ such that

$$\Pi(C) = (a,b) \leftarrow \text{一定是个开区间}$$

Proof (1) By Theorem 33 (or Remark 34)

(2) $x_0^i = \int_\Omega x_i \, dQ_1 \quad x_0^i = \int_\Omega x_i \, dQ_2$

$$\int_\Omega x_i \, d(\lambda Q_1 + (1-\lambda)Q_2) = \lambda \int_\Omega x_i \, dQ_1 + (1-\lambda)\int_\Omega x_i \, dQ_2$$

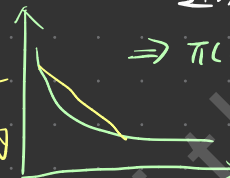
$= \lambda x_0^1 + (1-\lambda) x_0^1 = x_0^1$

所以 $\mathcal{P}$ is convex $\Rightarrow \pi(C)$ is convex

连续函数把一个凸集合转化为另一个凸集合

$\Rightarrow \pi(C)$ is an interval

凸集合指任意两点连一条线还是落在该集合内



It remains to claim that $\pi(C)$ is open.

Remark 40: If C is not attainable,

$$\Pi(C) = (\pi_{\inf}(C), \pi_{\sup}(C))$$

where $\pi_{\inf}(C) = \inf_{Q \in \mathcal{P}} E_Q\left[\frac{C}{1+r}\right]$, $\pi_{\sup}(C) = \sup_{Q \in \mathcal{P}} E_Q\left[\frac{C}{1+r}\right]$

Example 41: A financial market with one bond $B_0 = B_1 = 1$ and one stock $S_0 = \pi = 1$, $S_1 = S$

Suppose S is a Poisson distributed with parameter 1 under $\mathbb{P}$, i.e.

$$\mathbb{P}(S=k) = \frac{e^{-1}}{k!} \quad \text{for } k = 0, 1, 2$$

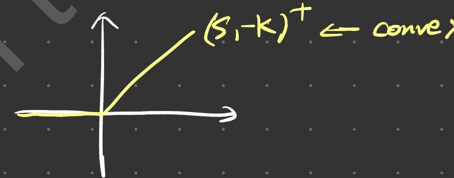
Then $\mathbb{P}$ is a risk-neutral measure with $\mathbb{E}[S_1] = 1 = S_0$

and the market model is arbitrage.

proof: $\mathbb{E}[S_1] = \mathbb{E}[S] = \sum_{k=1}^{\infty} \frac{e^{-1}}{k!} \cdot k = e^{-1} \sum_{k=0}^{\infty} \frac{1}{(k-1)!} = 1 = S_0$

Consider the contingent claim $C = (S_1 - K)^+$

For any $Q \in \mathcal{P}$



$$\mathbb{E}_Q[C] = \mathbb{E}_Q[(S_1-K)^+]$$

Jesen 不等式 $\rightarrow$ $\geq \left(\mathbb{E}_Q(S_1-K)\right)^+$

$$= (\underbrace{1}_{S_0(1+r) = S_0 = 1} - K)^+ \qquad (*)$$

Conversely, since $C \leq S_1$ $\quad (C = (S_1-K)^+)$

$$\Rightarrow \mathbb{E}_Q(C) \leq \mathbb{E}_Q[S_1] = 1 \qquad (**)$$

$(*)$和$(**)$ $\Rightarrow (1-K)^+ \leq \pi_{inf}(C) \leq \pi^C \leq \pi_{sup}(C) \leq 1$

是attainable的时候取"="

In fact we can prove that

$$\pi_{inf}(C) = (1-K)^+ \qquad [可证]$$

$$\pi_{sup}(C) = 1$$