Theorem 29. H: discounted claim

与 one-period model 不同：

(1) If H is attainable, $\#(\Pi(H))=1$

$\mathbb{E}(H)<\infty$ 不成立（不可积）

(2) If H is not attainable, then either $\Pi(H)=\phi$ or $\Pi(H)=(\pi_{inf}, \pi_{sup})$

Definition 30. A (arbitrage-free) market model is called complete if every contingent claim is attainable. → 可以不用该条件

Theorem 31: An arbitrage-free market model is complete $\Longleftrightarrow \#(\mathcal{P})=1$

如果有该条件

5.1 Stopping Time

Consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0,1,\cdots,T}, \mathbb{P})$

$\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_T$

Definition 1: A r.v. $\tau: \Omega \longrightarrow \{0,1,2,\cdots,T\} \cup \{\infty\}$

τ is called a stopping time with respect to $\mathbb{F}=(\mathcal{F}_t)_{0\le t\le T}$ if

$$\{\tau=t\}=\{w\in\Omega : \tau(w)=t\}\in\mathcal{F}_t \quad \text{for all } t$$

Example 2: $\Omega = [0,8)$, $T=4$

$\mathcal{F} = \sigma\left([0,\tfrac{1}{2}), [\tfrac{1}{2},1), \cdots, [\tfrac{15}{2},8)\right)$

$\mathcal{F}_0 = \sigma(\phi) = \{\phi, \Omega\}$

$\mathcal{F}_1 = \sigma([0,4), [4,8))$

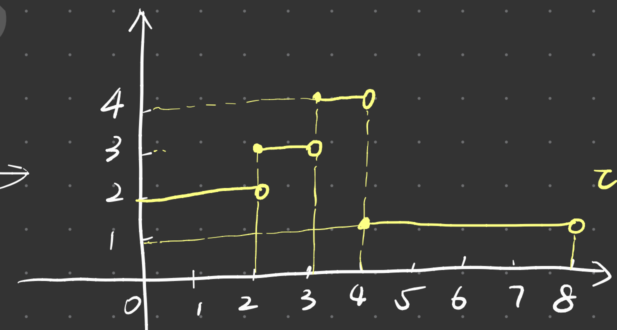
$\mathcal{F}_2 = \sigma([0,2), [2,4), [4,6), [6,8))$

$\mathcal{F}_3 = \sigma([0,1), [1,2), \cdots, [7,8))$

$\mathcal{F}_4 = \mathcal{F}$

(1) Consider a r.v. τ_1 $\longrightarrow$

check τ_1 stopping time.



$\{\tau_1 = 0\} = \phi \in \mathcal{F}_0$

$\{\tau_1 = 1\} = [4,8) \in \mathcal{F}_1$

$\{\tau_1 = 2\} = [0,2) \in \mathcal{F}_2$ $\Longrightarrow$ τ_1 is a stopping time.

$\{\tau_1 = 3\} = [2,3) \in \mathcal{F}_3$

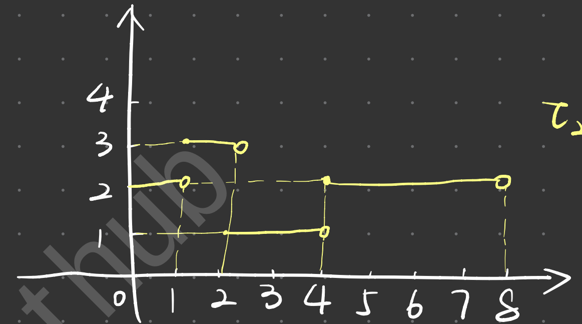
$\{\tau_1 = 4\} = [3,4) \in \mathcal{F}_4$

(2) Consider a r.v. τ_2

$\{\tau_2=0\}=\phi\in\mathcal{F}_0$

$\{\tau_2=1\}=[2,4)\notin\mathcal{F}_1$

τ_2 is <u>not</u> a stopping time.



Remark 3: (1) A r.v. $\tau:\Omega\to\{0,1,\cdots,T\}\cup\{\infty\}$ is a stopping time

$\iff \{\tau\le t\}\in\mathcal{F}_t \quad \forall t$

$\iff \{\tau> t\}\in\mathcal{F}_t \quad \forall t$ compliment

(in discrete time
$\{\tau>t\}=\{\tau\ge t+1\}$

(2) If τ and σ are stopping times, then

$\tau+\sigma \qquad \tau\vee\sigma \qquad \tau\wedge\sigma$

are stopping times. (注: 相减不是 stopping time)

proof: $\{\tau+\sigma=t\}=\bigcup_{k=0}^{t}\{\tau=k,\sigma=t-k\}$

$$=\bigcup_{k=0}^{t}\Big(\underbrace{\{\tau=k\}}_{\in\mathcal{F}_k\in\mathcal{F}_t}\cap\underbrace{\{\sigma=t-k\}}_{\in\mathcal{F}_t}\Big)\in\mathcal{F}_t$$

Definition 4: Let τ be a stopping time, then $\mathcal{F}_\tau = \{A \in \mathcal{F} : A \cap \{\tau \le t\} \in \mathcal{F}_t, \forall t \le T\}$ is called the σ-algebra of events determined prior to the stopping time τ.

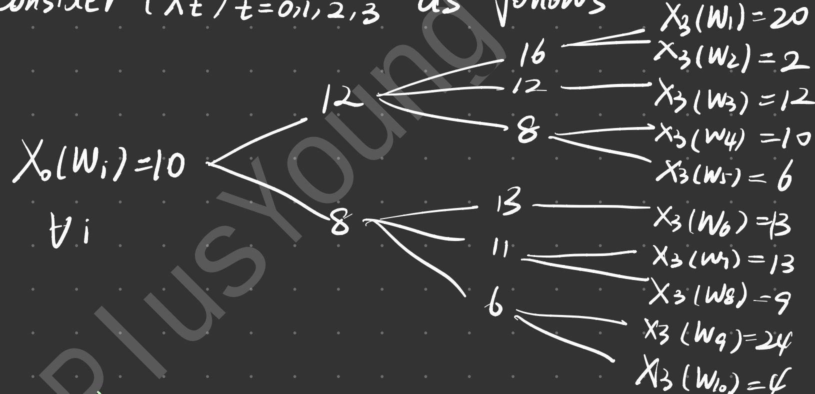
Remark 5: $\mathcal{F}_\tau$ is a σ-algebra (Exercise!)

Example 6: (1) τ_1 in Example 2.

$[3, \tfrac{7}{2}), [\tfrac{7}{2}, 4)$

$\mathcal{F}_{\tau_1} = \sigma([0,2), [2,3), \underbrace{[3,4)}_{\times}, [4,8))$

(2) Consider $(X_t)_{t=0,1,2,3}$ as follows



$\mathcal{F}_\tau^X = \sigma(\{W_1\}, \{W_2\}, \cdots, \{W_{10}\})$

$\mathcal{F}_0^X = \sigma(\{W_1, W_2, \cdots, W_{10}\})$

$\mathcal{F}_1^X = \sigma(\{W_1, \cdots, W_5\}, \{W_6, \cdots, W_{10}\})$

$\mathcal{F}_2^X = \sigma(\{w_1, w_2\}, \{w_3\}, \{w_4, w_5\}, \{w_6\}, \{w_7, w_8\}, \{w_9, w_{10}\})$

$\mathcal{F}_3^X = \mathcal{F}$

Let $\tau = \inf\{n \geq 0 : X_n \geq 18 \text{ or } X_n \leq 8\}$ ← 当股票价格超过18元或少于8元时将其卖掉.

where $\inf \phi = \infty$ ← 如果股票价格一直在8到18元之间跳动时，定义τ为无穷大

Then $\tau(w_1) = \tau(w_2) = 3$ $\quad \tau(w_3) = \infty$

$\tau(w_4) = \tau(w_5) = 2$ $\quad \tau(w_6) = \cdots = \tau(w_{10}) = 1$

τ is a stopping time.

$$\boxed{\begin{aligned} &\tau = 0 = \phi \in \mathcal{F}_0^X \\ &\tau = 1 = \sigma(\{w_6, \cdots, w_{10}\}) \in \mathcal{F}_1^X \\ &\tau = 2 = \sigma(\{w_4, w_5\}) \in \mathcal{F}_2^X \\ &\tau = 3 = \sigma(\{w_1, w_2\}) \in \mathcal{F}_3^X \end{aligned}}$$

$\mathcal{F}_\tau = \sigma(\underline{\{w_1, w_2\}}, \{w_4, w_5\}, \{w_6, \cdots, w_{10}\}, \{w_3\})$

$\hookrightarrow \{w_1\}, \{w_2\}$ 当$\tau = T$时，不能直接照搬$\{\tau = t\}$，要把里面的元素拆开（$\mathcal{F}_T$中所有元素分开）

(3) $\mathcal{F} = \sigma(\{w_1\}, \cdots, \{w_{16}\})$

$\mathcal{F}_0 = \{\phi, \Omega\}$

$\mathcal{F}_1 = \sigma(\{w_1, \cdots, w_8\}, \{w_9, \cdots, w_{16}\})$

$\mathcal{F}_2 = \sigma(\underline{\{w_1, \cdots, w_4\}, \{w_5, \cdots, w_8\}}, \{w_9, \cdots, w_{12}\}, \{w_{13}, \cdots, w_{16}\})$

$\mathcal{F}_3 = \sigma(\{w_1, w_2\}, \{w_3, w_4\}, \cdots, \{w_{15}, w_{16}\})$

$\mathcal{F}_4 = \mathcal{F}$

$\{\tau = 0\} = \phi \in \mathcal{F}_0$

$\{\tau = 1\} = \phi \in \mathcal{F}_1$

$\{\tau = 2\} = \underline{\{w_1, \cdots, w_8\}} \in \mathcal{F}_2$

$\{\tau = 3\} = \{w_9, w_{10}, w_{15}, w_{16}\} \in \mathcal{F}_3$

$\{\tau = 4\} = \{w_{11}, w_{12}\} \in \mathcal{F}_4$

$\{\tau = \infty\} = \underline{\{w_{13}, w_{14}\}}$ ← 当作 $\mathcal{F}_4$ 来看 $\in \mathcal{F}$

$\Rightarrow \tau$ is a stopping time.

$\mathcal{F}_\tau = \sigma(\{w_1, w_2, w_3, w_4\}, \{w_5, w_6, w_7, w_8\}, \{w_9 \cdots w_{16}\}, \{w_{11}, w_{12}\}, \{13\}, \{14\})$

Lemma 7: Let $\underbrace{\tau<\infty}_{X_\infty \text{无定义}}$ be a stopping time, then

$X_\tau(w) = X_{\tau w}(w)$ is $\mathcal{F}_\tau$-measurable (${\mathcal{F}}_\tau$与$\mathcal{F}_t$无一定的包含关系)

例: $\tau(w_1)=3$ 那么 $X_3(w_1) = X_{\tau(w_1)}(w_1) = 20$

$\tau(w_5)=2$ 那么 $X_2(w_5)=8$

proof claim: For any $a\in\mathbb{R}$, $\{X_\tau \le a\}\in\mathcal{F}_\tau$

若 $\underbrace{\{X_\tau\le a\}}_{A}\cap\{\tau\le k\}\in\mathcal{F}_k$, 则 $\{X_\tau\le a\}\in\mathcal{F}_\tau$

技巧 不定数τ 化成定数n

$$= \left(\bigcup_{n=0}^{T}\{X_\tau\le a, \tau=n\}\right)\cap\{\tau\le k\}$$

$$= \left(\bigcup_{n=0}^{T}\{X_n\le a\}\right)\cap\underbrace{\{\tau\le k\}}_{=\bigcup_{n=0}^{k}(\tau=n)}$$

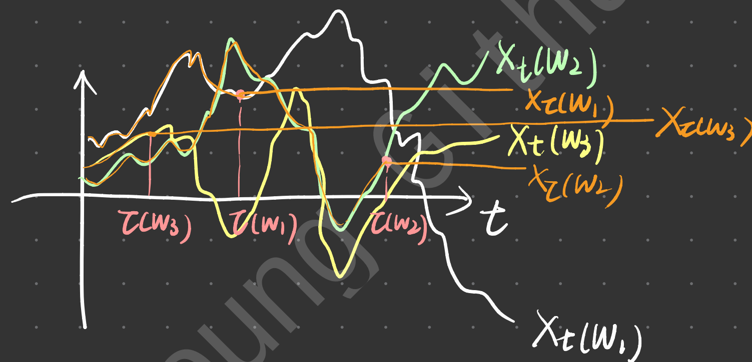
$$= \bigcup_{n=0}^{k}\left(\underbrace{\{X_n\le a\}}_{\in\mathcal{F}_n\subseteq\mathcal{F}_k}\cap\underbrace{\{\tau=n\}}_{\in\mathcal{F}_n\subseteq\mathcal{F}_k}\right)\in\mathcal{F}_k \text{ for all } k.$$

$\Longrightarrow \{X_\tau\le a\}\in\mathcal{F}_\tau$

$\Longrightarrow X_\tau$ is $\mathcal{F}_\tau$-measurable

Definition 8: For a stochastic process $(X_t)_{t=0,1,\cdots,T}$ and a stopping time τ. the stopped process.

$$X^\tau = (X^\tau_t)_{t=0,1,\cdots,T} \text{ is defined by } X^\tau_t = X_{\tau\wedge t} = \begin{cases} X_\tau, & \text{if } \tau \le t \\ X_t, & \text{if } \tau > t \end{cases}$$



Definition 9: Let M be an adapted process with $\mathbb{E}|M_t| < \infty$ for all t. The following conditions are equivalent:

$t=0,1,\cdots,T$

(1) M is a martingale.

(2) For any stopping time τ, the stopped process M^τ is a martingale.

(3) $\mathbb{E}[M_{\tau\wedge t}] = M_0$ for any stopping time τ.

Proof: (1) $\Rightarrow$ (2)

$$M^{\tau}_{t+1} - M^{\tau}_{t} = M_{(t+1)\wedge\tau} - M_{t\wedge\tau} = \begin{cases} 0, & \tau \le t \\ M_{t+1} - M_t, & \tau > t \end{cases}$$

$$= (M_{t+1} - M_t) I_{\{\tau > t\}}$$ $\leftarrow$ discrete 成立, continuous 不行

$$\mathbb{E}[M^{\tau}_{t+1} - M^{\tau}_{t} \mid \mathcal{F}_t] = \mathbb{E}[(M_{t+1} - M_t) I_{\{\tau>t\}} \mid \mathcal{F}_t]$$

$$= \underbrace{I_{\{\tau>t\}}}_{\{\tau>t\}\in\mathcal{F}_t} \mathbb{E}[M_{t+1} - M_t \mid \mathcal{F}_t] = 0$$

(2) $\Rightarrow$ (3) By definition, $M^{\tau}_0 = \mathbb{E}[M^{\tau}_t \mid \mathcal{F}_0] = M_{\tau\wedge 0} = M_0$

(3) $\Rightarrow$ (1) For $A \in \mathcal{F}_t$, consider the stopping time $\tau_1 = t I_A + T I_{A^c}$

$$\Rightarrow M_0 = \mathbb{E}[M_{T\wedge\tau_1}] = \int_\Omega M_{T\wedge\tau_1} \, d\mathbb{P} = \int_\Omega (M_{T\wedge\tau_1} I_A) \, d\mathbb{P} + \int_\Omega (M_{T\wedge\tau_1} I_{A^c}) \, d\mathbb{P}$$

$$= \int_A M_t \, d\mathbb{P} + \int_{A^c} M_T \, d\mathbb{P}$$

Moreover, consider the stopping time $\tau_2 = T$

$$\Rightarrow M_0 = \mathbb{E}[M_T] = \int_A M_T \, d\mathbb{P} + \int_{A^c} M_T \, d\mathbb{P}$$

$$\Rightarrow \int_A M_t \, d\mathbb{P} = \int_A M_T \, d\mathbb{P} \quad \forall A \in \mathcal{F}_t \quad = \int_A \mathbb{E}[M_T \mid \mathcal{F}_t] \, d\mathbb{P}$$

$\Rightarrow M_t = \mathbb{E}[M_T | \mathcal{F}_t] \ \forall t \le T$

$\Rightarrow M$ is a martingale

Corollary 10: Let U be an adapted process with $\mathbb{E}|U| < \infty, \forall t$

Then U is a supermartingale

$\Leftrightarrow$ For any stopping time τ the stopped process U^τ is a supermartingale

---

5.2 American claims

Definition 11. An American contingent claim is a security that can be exercised at time t between 0 and its expiration time T.

Definition 12: An exercise strategy for an American contingent claim C is $\mathcal{F}_T$-measurable random variable τ taking values in $\{0, 1, 2, \cdots, T\}$.

注意：这里的 τ 不一定是 stopping time

The payoff obtained by using τ is equal to $C_{\tau(\omega)} = C_{\tau(\omega)}(\omega)$ for $\omega \in \Omega$.

与欧式相比，美式的 τ 是不确定的，因为不确定走哪条 ω_i