2.3. Martingale transform and Doob decomposition

2.3.1 Martingale transform

Definition 37: If $M_n \in \mathcal{F}_{n-1}$, $\forall n$, we say that (M_n) is an $(\mathcal{F}_n)$-predictable process.

Definition 38: $X = (X_n, \mathcal{F}_n)$: martingale ← 股票价格

$M = (M_n, \mathcal{F}_n)$: predictable ← 投资策略

Then the stochastic process

$$(M\cdot X)_n = M_0X_0 + \sum_{i=1}^{n} M_i\,(\underbrace{X_i - X_{i-1}}_{\text{股票价格的变化的量}})$$

is called the martingale transform of X by M.

Theorem 39. (Martingale Transform Theorem)

If M_n is bounded (有界) $\forall n$, the martingale transform $((M\cdot X)_n)$ is an $(\mathcal{F}_n)$-martingale.

proof: Claim: $E[(M\cdot X)_{n+1} \mid \mathcal{F}_n] = (M\cdot X)_n$

$$E[(M\cdot X)_{n+1} - (M\cdot X)_n \mid \mathcal{F}_n] = E[M_{n+1}(X_{n+1} - X_n) \mid \mathcal{F}_n]$$

$$= M_{n+1}\, E[X_{n+1} - X_n \mid \mathcal{F}_n] = M_{n+1}\,(E[X_{n+1} \mid \mathcal{F}_n] - E[X_n \mid \mathcal{F}_n])$$

$$\overset{M\text{: predict}}{=} M_{n+1}(X_n - X_n) = 0 \longleftarrow (X_n)\text{是Martingale}$$

☆☆ 如果 M_n 不是 bonded，例如我们采用 double strategy，即每次输则投入之前赌注的两倍（此时 M_n 个无界），那无论如何在公平赌博的情况下，一定会赚到一块钱。金融里面不允许也不存在 double strategy，因为世界上的财富是有限的，假设它前30次都输了，那第31次赌博它要投入 2^{30} 的赌注！！这是一个极大的数字！因此 M_n bounded 是一个在金融上的合理假设。

Definition 40: A sequence of r.v.'s $(Z_n)_{n\geq 1}$ is called an increasing process if

(i) $Z_1 = 0$ 或 1, $Z_n \leq Z_{n+1}$ $\forall n \geq 1$

(ii) $E[Z_n] < \infty$

Example 41: X: positive integrable r.v.'s

$Z_n = (n-1)X$

then (Z_n) is an increasing process.

Theorem 42: (Doob decomposition)

Any submartingale $(X_n, \mathcal{F}_n)$ can be written as

$$X_n = Y_n + Z_n$$

where $(Y_n, \mathcal{F}_n)$ is a martingale, $(Z_n, \mathcal{F}_n)$ is increasing predictable process

Proof: Let $Y_1 = X_1 \quad (Z_1=0)$

$$Y_n = X_1 + \sum_{i=2}^{n} \left(X_i - E[X_i \mid \mathcal{F}_{i-1}] \right) \quad \text{for } n \geq 2$$

$$Z_n = X_n - Y_n$$

Claim: (1) (Y_n): martingale

$$E[Y_{n+1} \mid \mathcal{F}_n] = E[X_1 \mid \mathcal{F}_n] + E\left[\sum_{i=2}^{n+1} \left(X_i - E[X_i \mid \mathcal{F}_{i-1}] \right) \Big| \mathcal{F}_n \right]$$

$$= X_1 + \sum_{i=2}^{n+1} \left(E[X_i \mid \mathcal{F}_n] - E[X_i \mid \mathcal{F}_{i-1}] \right)$$

$$= X_1 + \sum_{i=2}^{n} \left(X_i - E[X_i \mid \mathcal{F}_{i-1}] \right) + E[X_{n+1} \mid \mathcal{F}_n] - E[X_{n+1} \mid \mathcal{F}_n]$$

$$= X_1 + \sum_{i=2}^{n} \left(X_i - E[X_i \mid \mathcal{F}_{i-1}] \right)$$

$$= Y_n$$

(2) (Z_n) is increasing

(3) (Z_n) is predictable i.e. $E[Z_{n+1} \mid \mathcal{F}_n] = Z_{n+1}$

Remark 43: (1) The decomposition is unique. (Doob分解唯一)

(2) An increasing predicable process Z is called a compensator of the martingale X.

3、One-Period Model

Assumption:

(1) time $t=0,1$ → 只有2个时刻 (one-period)

Trading time: $t=0$

(2) sample space $\Omega=\{\omega_1,\dots,\omega_k\}$ → 有限多种情况 (对无限多种适用的定义会加(*))

probability $P(\{\omega_k\})>0$, $\forall\, i=1,2,\dots,k$

at time 0: $\mathcal{F}_0=\{\phi,\Omega\}$ → 0时刻是完全无多余信息情况

at time 1: $\mathcal{F}_1$ = the collection of all possible subsets of Ω → 1时刻是全知情况

(3) 1 bond, N stocks in the financial market → 市场上只有1张债券, N张股票。

security price $\bar{S}_t=(B_t, S_t^1, S_t^2, \dots, S_t^N)^T=\begin{pmatrix} B_t \\ S_t \end{pmatrix}$, where $S_t=(S_t^1, S_t^2, \dots, S_t^N)$

债券价格 N个股票t时期价格

B_t: Bond price at time t (constant 定数)

S_t^i = ith stock price at time t

S_0^i = constant 定数 (第i张股票今天的价格已知)

S_1^i = r.v.: $\Omega\to\mathbb{R}^+$ (第i张股票在1时刻所有的可能性)

$\bar{S}_t \in \mathbb{R}^{N+1}$

$S_t \in \mathbb{R}^{N}$

$$\begin{pmatrix} B_0 \\ S_0^1 \\ S_0^2 \\ \vdots \\ S_0^N \end{pmatrix} = \bar{S}_0$$

初始时刻所有价格已知

time 0

$$\bar{S}_1(\omega_1) = \begin{pmatrix} B_1 \\ S_1^1(\omega_1) \\ \vdots \\ S_1^N(\omega_1) \end{pmatrix}$$

$\bar{S}_1(\omega_2)$

$\vdots$

$\bar{S}_1(\omega_3)$

$\vdots$

$$\bar{S}_1(\omega_K) = \begin{pmatrix} B_1 \\ S_1^1(\omega_K) \\ \vdots \\ S_1^N(\omega_K) \end{pmatrix}$$

1时刻只有 B_1 已知

time 1