2.2 Discrete Time Martingales

$(\Omega, \mathcal{F}, \mathbb{P})$: probability space

Definition 28: Let $(\mathcal{F}_n)$ be a sequence of σ-algebra and $\mathcal{F}_n \subseteq \mathcal{F}$ for all n.

$$\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \mathcal{F}_3 \subseteq \cdots \subseteq \mathcal{F}_n \subseteq \cdots$$

$(\mathcal{F}_n)$ is called a filtration.

Example 29: $\Omega = \mathbb{N} = \{1, 2, 3, \cdots\}$

$\mathcal{F}$ = the collection of all subsets of Ω.

Set $\mathcal{F}_0 = \{\emptyset, \Omega\}$

$\mathcal{F}_1 = \sigma(\{1\}) \rightarrow$ 4个元素

$\mathcal{F}_2 = \sigma(\{1\}, \{2\}) \longrightarrow$ 8个元素

$\vdots$

$\mathcal{F}_n = \sigma(\{1\}, \{2\}, \cdots, \{n\})$

$\vdots$

$\Rightarrow (\mathcal{F}_n)$ is a filtration

$$\mathbb{E}(X|Y) = \mathbb{E}(X|\sigma(Y))$$

Definition 30: (X_n) sequence of r.v.'s

$(X_n, \mathcal{F}_n)$ is called a martingale (鞅、平赌)

or (X_n) is a martingale w.r.t. $(\mathcal{F}_n)$. (X_n) is an $(\mathcal{F}_n)$-martingale.

(i) $X_n \in \mathcal{F}_n$ (即 X_n 是 $\mathcal{F}_n$-measurable) for all n.

或说 (X_n) is $\mathcal{F}_n$-adapted

(ii) $\mathbb{E}|X_n| < \infty$, $\forall n$

(iii) $\mathbb{E}[\underbrace{X_{n+1}}_{\text{在时间n+1时的财产}} | \underbrace{\mathcal{F}_n}_{\text{时间n时的资讯}}] = X_n$ for all n. ☆最重要 $\mathbb{E}[X_{n+1} - X_n | \mathcal{F}_n] = 0$

Remark 31: (iii) is equivalent to

(iiia) $\mathbb{E}[X_m | \mathcal{F}_n] = X_n$ for $m > n$

$$\begin{aligned}(\mathbb{E}[X_m|\mathcal{F}_n] &= \mathbb{E}[\mathbb{E}[\mathbb{E}[\cdots \mathbb{E}[X_m|\mathcal{F}_{m-1}]|\mathcal{F}_{m-2}]|\cdots|\mathcal{F}_n] \\ &= \mathbb{E}[\mathbb{E}[\mathbb{E}[\cdots \mathbb{E}[X_{m-1}|\mathcal{F}_{m-2}]\cdots|\mathcal{F}_n] \\ &= \mathbb{E}[X_{n+1}|\mathcal{F}_n] = X_n\end{aligned}$$

(iiib) $\int_A X_n \, d\mathbb{P} = \int_A X_m \, d\mathbb{P} \quad \forall A \in \mathcal{F}_n \quad \forall m > n$

martingale 除了与本身的 $\mathcal{F}_n$ 或 $\sigma(Y)$ 有关，它还与概率测度 $\mathbb{P}$ 有关！
当 $\mathbb{P}$ 变化时，本来是 martingale 的可能变得不是了。

Example 32: (1) (ξ_n) = iid r.v.'s with $\mathbb{E}|\xi_1| < \infty$ and $\mathbb{E}(\xi_1) = 0$

Define $X_n = \sum_{i=1}^{n} \xi_i$

$\mathcal{F}_n = \sigma(\xi_1, \xi_2, \cdots, \xi_n) \overset{(*)}{=\!=} \sigma(X_1, X_2, \cdots, X_n)$

(*) proof: $\sigma(X) := \sigma(\underbrace{\{X \le r\} : r \in \mathbb{R}}\})$ X是 r.v. 的性质

$\mathbb{E}[X | \sigma(Y)] = \mathbb{E}[X | Y]$

$\sigma(\xi_1, \cdots, \xi_n) = \sigma(\{\xi_1 \le r_1, \xi_2 \le r_2, \cdots, \xi_n \le r_n, r_1, r_2, \cdots, r_n \in \mathbb{R}\})$

$X = f(\xi_1, \xi_2, \cdots, \xi_n) \Rightarrow X \in \sigma(\xi_1, \cdots, \xi_n)$

$X_i = \xi_1 + \xi_2 + \cdots + \xi_i \Rightarrow X_i \in \sigma(\xi_1, \cdots, \xi_i) \subseteq \sigma(\xi_1, \cdots, \xi_n) \ \forall i \le n$

$\Rightarrow X_1, \cdots, X_n \in \sigma(\xi_1, \cdots, \xi_n)$

$\Rightarrow \sigma(X_1, \cdots, X_n) \subseteq \sigma(\xi_1, \cdots, \xi_n)$

$\xi_i = X_i - X_{i-1} \in \sigma(X_{i-1}, X_i) \subseteq \sigma(X_1, \cdots, X_n) \ \forall i \le n$

$\xi_1, \cdots, \xi_n \in \sigma(X_1, \cdots, X_n)$

$\sigma(\xi_1, \cdots, \xi_n) \subseteq \sigma(X_1, \cdots, X_n)$

下面正式开始证明 X_i 是否为 martingale：

(i) $X_n \in \mathcal{F}_n, \forall n$ ✓

(ii) $\mathbb{E}|X_n| = \mathbb{E}|\xi_1 + \cdots + \xi_n| \le \mathbb{E}|\xi_1| + \cdots + \mathbb{E}|\xi_n| < \infty$ ✓

(iii) $\mathbb{E}[X_{n+1} | \mathcal{F}_n] = \mathbb{E}[\sum_{i=1}^{n+1} \xi_i | \mathcal{F}_n] = \mathbb{E}[\xi_{n+1} + X_n | \mathcal{F}_n]$

$= \mathbb{E}[\xi_{n+1} | \mathcal{F}_n] + \mathbb{E}[X_n | \mathcal{F}_n] = \mathbb{E}[\xi_{n+1}] + X_n = 0 + X_n = X_n$

$\Rightarrow (X_n)$ 是 $\mathcal{F}_n$- martingale.

(2) (ξ_n): independent r.v.'s with $\mathbb{E}|\xi_n| = 1$, $\xi_n > 0$

Define $X_n = \prod_{i=1}^{n} \xi_i$

$\mathcal{F}_n = \sigma(\xi_1, \cdots, \xi_n) = \sigma(X_1, \cdots, X_n)$

Then (i) $\xi_1, \cdots, \xi_n \in \mathcal{F}_n \Rightarrow X_n \in \mathcal{F}_n$

(ii) $\mathbb{E}|X_n| = \mathbb{E}|\prod_{i=1}^{n} \xi_i| \le \prod_{i=1}^{n} \mathbb{E}|\xi_i| = 1 < \infty$

(iii) $\mathbb{E}[X_{n+1} | \mathcal{F}_n] = \mathbb{E}[\xi_{n+1} \prod_{i=1}^{n} \xi_i | \mathcal{F}_n] = \mathbb{E}[\xi_{n+1} \cdot X_n | \mathcal{F}_n]$

$= X_n \mathbb{E}[\xi_{n+1} | \mathcal{F}_n] = X_n \mathbb{E}[\xi_{n+1}] = X_n$

$\Rightarrow (X_n)$ 是 $\mathcal{F}_n$- martingale

Definition 33: (1) (X_n) is an $(\mathcal{F}_n)$-submartingale (优赌，下鞅)

if (i) + (ii)

(iii) $\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] \geq X_n$ p-a.s. $\forall n$

(2) (X_n) is an $(\mathcal{F}_n)$-supermartingale (劣赌，上鞅)

if (i) + (ii)

(iii) $\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] \leq X_n$ p-a.s. $\forall n$

Remark 34: (1) $(X_n, \mathcal{F}_n)$: supermartingale

$(-X_n, \mathcal{F}_n)$: submartingale

(2) $(X_n, \mathcal{F}_n)$: martingale

$\Leftrightarrow (X_n, \mathcal{F}_n)$: sub-martingale and super-martingale

Theorem 35: (1) $(X_n, \mathcal{F}_n)$: submartingale

☆ sub 造 sub-martingale

φ: increasing, convex function defined on $\mathbb{R}$

$\varphi(X_n)$: integrable for all n.

$\Rightarrow (\varphi(X_n), \mathcal{F}_n)$: sub-martingale

(2) $(X_n, \mathcal{F}_n)$: martingale

φ : convex function on $\mathbb{R}$ ← 不用 increasing 条件

$\varphi(X_n)$: integrable for all n.

$\Rightarrow (\varphi(X_n), \mathcal{F}_n)$: sub-martingale

proof: $\mathbb{E}[\varphi(X_{n+1}) \mid \mathcal{F}_n] \underset{\varphi:\text{convex}}{\geq} \varphi(\mathbb{E}[X_{n+1} \mid \mathcal{F}_n])$

$$\begin{cases} = \varphi(X_n), & (X_n) : \text{martingale} \\ \geq \varphi(X_n), & (X_n) : \text{sub-martingale} \end{cases}$$

φ: increasing

Example 36 : (1) $(X_n, \mathcal{F}_n)$: martingale

martingale 造 submartingale 的例子

X对p次方可积分

(i) $(|X_n|^p)$: submartingale for $1 \le p \le \infty$ if $X_n \in L(\Omega, \mathcal{F}, \mathbb{P})$

($f(x) = |X_n|^p$: convex function)

(ii) $(|X_n| \log^+ |X_n|)$: submartingale, where $\log^+ x = \max\{\log x, 0\}$

(iii) $(\exp(X_n))$: submartingale

(2) (X_n): sub-martingale, then $(\exp(X_n))$, (X^+) are martingale

(3) (X_n): supermartingale then $(X_n \wedge A)$ X_n与A取小的那个 is a supermartingale for any fixed A.