3.7 Complete Market Model

Definition 42*: An arbitrage-free market model is complete if every contingent claim is attainable.

在 complete model 可以不用这个条件

Theorem 43*: An arbitrage-free market is complete $\Longleftrightarrow \#(\mathcal{P})=1$ ← $\mathcal{P}$里的元素只有一个

(只当 one-period 和 discrete time multi-period model 是对的, 在 continuous model 中它不成立)

proof: "$\Rightarrow$" For $A \in \mathcal{F}$, I_A is a contingent claim. (回顾 contingent claim 定义)

$\underset{\text{corollary 39}}{\Longrightarrow}$ the arbitrage-free price is unique and $= \mathbb{E}_Q\left[\frac{I_A}{1+r}\right] = \frac{Q(A)}{1+r}$

$\Rightarrow Q$ is unique

$\Rightarrow \#(\mathcal{P})=1$

"$\Leftarrow$" Suppose $\mathcal{P}=\{Q\}$ and C is a bounded contingent claim.

$\Rightarrow \mathbb{E}_Q(C)<\infty$

Then C has a unique arbitrage-free price $\mathbb{E}_Q\left[\frac{C}{1+r}\right]$

$\underset{\text{Cor 39}}{\Longrightarrow}$ C is attainable.

Example 44 : $\Omega = \{\omega_1, \omega_2\}$

$N=1 \Rightarrow$ one bond and one stock

interest rate r , $B_0 = 1$, $B_1 = 1+r$

$$\begin{array}{ccl} & \overset{p}{\nearrow} & S_1(\omega_1) = b \\ S_0 & & \\ & \underset{1-p}{\searrow} & S_1(\omega_2) = a \end{array} \qquad b > a \qquad \begin{array}{l} \mathbb{P}(\{\omega_1\}) = p \in (0,1) \\ \mathbb{P}(\{\omega_2\}) = 1-p \end{array}$$

(i) The market model is arbitrage-free

arbitrage-free $\iff S_0 \in \{ \mathbb{E}_{\mathbb{Q}}[\frac{S_1}{1+r}] : \mathbb{Q} \sim \mathbb{P} \}$

$$= \left\{ \frac{q b + (1-q) a}{1+r} : q \in (0,1) \right\}$$

$$= \left\{ \frac{q(b-a)}{1+r} + \frac{a}{1+r} : q \in (0,1) \right\}$$

$$= \left(\frac{a}{1+r}, \frac{b}{1+r} \right)$$

(ii) S_0 given in $\left(\frac{a}{1+r}, \frac{b}{1+r} \right)$

The risk-neutral measure $\mathbb{P}^*$ satifies $\underline{S_0(1+r) = \mathbb{E}^*[S_1] = p^* b + (1-p^*) a}$

$\Longrightarrow p^*$ is unique $\longleftarrow$ 由上式直接解出 p^* 值 (唯一), $\mathbb{P}^*$ 就对应上面的 $\mathbb{Q}$

$\mathbb{P}(\{w_1\}) = \mathbb{P}^* = \frac{S_0(1+r)-a}{b-a} \in (0,1)$

$\Rightarrow \#(\mathbb{P}) = 1$

$\Rightarrow$ The market model is complete.

(iii) Given a contingent claim C, find h^0, h^1 s.t. $C = h^0 B_1 + h^1 S_1$ $\Leftarrow$ attainable 另一种定义

$$\Rightarrow \begin{cases} h^0(1+r) + h^1 S_1(w_1) = C(w_1) \\ h^0(1+r) + h^1 S_1(w_2) = C(w_2) \end{cases}$$

$$\Rightarrow \begin{cases} h^0 = \frac{C(w_2)b - C(w_1)a}{(1+r)(b-a)} \\ h^1 = \frac{C(w_1) - C(w_2)}{b-a} \end{cases}$$

$\leftarrow$ 即当 C 是 attainable 时，可买 h^0 张该 Bond 和 h^1 张该股票来达到买 C 一样的结果（复制）

总结：若要判断 C 是否为 attainable

方法一：在保证 model 是 arbitrage-free 的情况下 $\#(\mathbb{P}) = 1$

不可少!!

方法二：直接根据定义，则 $C = h^0 B_1 + h^1 S_1$，解出 h^0 和 h^1

(iv) The arbitrage-free price π^C is given by:

$$\pi^C = \mathbb{E}^*\left[\frac{C}{1+r}\right] = h^0 B_0 + h^1 S_0$$

对应方法一

$$= p^* \frac{c(\omega_1)}{1+r} + (1-p^*) \frac{c(\omega_2)}{1+r}$$

对应方法二

$$= \frac{c(\omega_2)b - c(\omega_1)a}{(1+r)(b-a)} \times 1 + \frac{c(\omega_1) - c(\omega_2)}{b-a} S_0$$

$$= \frac{c(\omega_1)}{1+r} \frac{S_0(1+r)-a}{b-a} + \frac{c(\omega_2)}{1+r} \cdot \frac{b-S_0(1+r)}{b-a}$$

In particular, if $c = (S_1 - K)^+$ with strike price $K \in (a,b)$

$$\pi^{(S_1-K)^+} = \frac{b-K}{b-a} S_0 - \frac{(b-K)a}{b-a} \cdot \frac{1}{1+r}$$

$$\boxed{\begin{aligned} c(\omega_1) &= b-K \\ c(\omega_2) &= 0 \end{aligned}}$$