Anton Buguev BS19-RO-01
Date of birth: 21.02.2001
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1 Task 1

1.1 Problem

Let us define the following binary operation $(\_|\_)$ in $\boldsymbol{C}^2$ for any two vectors $v = (v_1, v_2)$ and $w = (w_1, w_2)$. Let $(v|w) = (-1)^{day} month \cdot v_1 w_1^* + year \cdot v_2 w_2^*$. Is this operation a dot-product? If yes then prove, otherwise explain what dot-product axioms do hold and what do fail.

1.2 Solution

First, let us substitute numbers into our defined operations and we get:

$$(v|w) = (-1)^{21} \cdot 2 \cdot v_1 w_1^* + 2001 \cdot v_2 w_2^* =$$
$$= -2 \cdot v_1 w_1^* + 2001 \cdot v_2 w_2^*$$

In order to check if this operation is dot-product we need to consider axioms of inner product (or dot-product). These properties for all vectors $u, v, w \in \boldsymbol{C}$ (complex vector space) and all scalars $a, b \in F$ (field of numbers) are:

- Conjugate symmetry: $(u|v) = (v|u)^*$, where * is a conjugate operation on scalars.
- Linearity in the first argument: $(au + bv|w) = a(u|w) + b(v|w)$.
- Positive-definiteness: $(u|u) > 0$ assuming $u \neq 0$ and $(0|0) = 0$.

Source: Lecture notes, Topic 1, Slide 32, Link

Let us proceed and consider these properties separately:

- Conjugate symmetry:

  First, we write $(w|v)^*$ by definition of our operation:

  $$(w|v)^* = (-2 \cdot w_1 v_1^* + 2001 \cdot w_2 v_2^*)^*$$

  Second, due to the properties of conjugates we can write:

  $$(w|v)^* = (-2 \cdot w_1 v_1^*)^* + (2001 \cdot w_2 v_2^*)^* =$$
  $$= -2 \cdot w_1^* (v_1^*)^* + 2001 \cdot w_2^* (v_2^*)^* =$$
  $$= -2 \cdot w_1^* v_1 + 2001 \cdot w_2^* v_2$$

  Applying commutative law for multiplication we finally get:

  $$(w|v)^* = -2 \cdot v_1 w_1^* + 2001 \cdot v_2 w_2^*$$

  Now we can see that

  $(v|w) = (w|v)^*$, thus **Conjugate symmetry axiom holds.**

- Linearity in the first argument:

  Let us take scalars a and $b \in F$ and vector $u \in \boldsymbol{C}$ and consider expression $(av + bw|u)$:

  $$(av + bw|u) = -2(av_1 + bw_1)u_1^* + 2001(av_2 + bw_2)u_2^*$$

Using distributivity of multiplication over addition we can write

$$(av + bw|u) = a(-2v_1u_1^* + 2001v_2u_2^*) + b(-2w_1u_1^* + 2001w_2u_2^*)$$

We can notice that first term is in fact $a(v|u)$ and second is $b(w|u)$, therefore we can write

$(av + bw|u) = a(v|u) + b(w|u)$, thus **Linearity in the first argument axiom holds**

- Positive-definiteness:
  First, let us write expression for $(v|v)$:
  $$(v|v) = -2v_1v_1^* + 2001v_2v_2^*$$
  Since v_1 and v_2 are complex numbers, therefore we write them in form
  $$v_1 = x_1 + iy_1,\ v_2 = x_2 + iy_2$$
  where $x_1, y_1, x_2, y_2 \in \mathbb{R}$ and i - imaginary unit, which means
  $$v_kv_k^* = (x_k + iy_k)(x_k - iy_k) = x_k^2 + y_k^2 = |v_k|^2$$
  where $|v_k|$ argument of complex number v_k, $k \in \mathbb{N}$.

  Using this property we can write:
  $$(v|v) = -2|v_1|^2 + 2001|v_2|^2$$
  As we can see, the expression above is not always positive-definite because it contains negative number, which means it can be negative for some values of v_1 and v_2. For example, if $v_1 = 100$ and $v_2 = 1$ we get

  $(v|v) = -20000 + 2001 = -17999 < 0$, thus **Positive-definiteness axiom fails**

1.3 Conclusion

One of of dot-product axioms fails, hence we can say that our defined operation is **not a dot-product**.

2 Task 3

2.1 Problem

Starting with the definitions, compute cross-correlation, convolution, and circular convolution of the signals a and b.

2.2 Solution

In our case signals a and b are:
$$a = (\mathbf{2}\ 1\ 0\ 2),\ \ b = (\mathbf{2}\ 0\ 0\ 1)$$

- Cross-correlation:
  By definition: The deterministic cross-correlation of two signals x and y is another sequence $c_{x,y} = (.., c_{-2}, c_{-1}, \boldsymbol{c_0}, c_1, c_2, ..)$ such that $c_n = \sum_{k=-\infty}^{k=\infty} x_k, y_{k-n}^*$.
  Source: Lecture notes, Topic 3, Slide 10, Link

  First, let us adapt our sequences. In order to compute circular convolution we need to extend sequences to infinite sequences with *finite support*, therefore let us add zeros in the beginning and the end of sequences a and b:

  $$a = (...\ 0\ 0\ \mathbf{2}\ 1\ 0\ 2\ 0\ 0\ ...)$$

  $$b = (...\ 0\ 0\ \mathbf{2}\ 0\ 0\ 1\ 0\ 0\ ...)$$

Since elements a_i and b_i where $i < 0$ and $i > 3$ are zeros, therefore product between these elements and any other element will always lead to 0. Hence, during calculations let us consider only such elements a_i and b_i where $0 \leq i \leq 3$. And since sequences do not contain complex numbers, conjugate operation can be omitted.

Now let us compute cross-correlation:

- $c_{-3} = \sum_{k=-\infty}^{k=\infty} a_k, b_{k+1} = \boldsymbol{a_0} \cdot b_3 = \mathbf{2} \cdot 1 = 2$
- $c_{-2} = \sum_{k=-\infty}^{k=\infty} a_k, b_{k+2} = \boldsymbol{a_0} \cdot b_2 + a_1 \cdot b_3 = \mathbf{2} \cdot 0 + 1 \cdot 1 = 1$
- $c_{-1} = \sum_{k=-\infty}^{k=\infty} a_k, b_{k+1} = \boldsymbol{a_0} \cdot b_1 + a_1 \cdot b_2 + a_2 \cdot b_3 = \mathbf{2} \cdot 0 + 1 \cdot 0 + 0 \cdot 1 = 0$
- $c_0 = \sum_{k=-\infty}^{k=\infty} a_k, b_k = \boldsymbol{a_0} \cdot \boldsymbol{b_0} + a_1 \cdot b_1 + a_2 \cdot b_2 + a_3 \cdot b_3 = \mathbf{2} \cdot \mathbf{2} + 1 \cdot 0 + 0 \cdot 0 + 2 \cdot 1 = \mathbf{6}$
- $c_1 = \sum_{k=-\infty}^{k=\infty} a_k, b_{k-1} = a_1 \cdot \boldsymbol{b_0} + a_2 \cdot b_1 + a_3 \cdot b_2 = 1 \cdot \mathbf{2} + 0 \cdot 0 + 2 \cdot 0 = 2$
- $c_2 = \sum_{k=-\infty}^{k=\infty} a_k, b_{k-2} = a_2 \cdot \boldsymbol{b_0} + a_3 \cdot b_1 = 0 \cdot \mathbf{2} + 2 \cdot 0 = 0$
- $c_3 = \sum_{k=-\infty}^{k=\infty} a_k, b_{k-3} = a_3 \cdot \boldsymbol{b_0} = 2 \cdot \mathbf{2} = 2$

We can notice that for $i \leq -4$ and $i \geq 4$ value of c_i will always be 0, because corresponding a_i and b_i are 0, hence cross-correlation of a and b is:

$$c_{a,b} = (\dots\ 0\ 0\ 2\ 1\ 0\ \mathbf{6}\ 2\ 0\ 4\ 0\ 0\ \dots)$$

- Convolution:

  By definition: The convolution $(h * x)$ between signals h and x is a sequence such that $(h * x)_n = \sum_{k \in \mathbb{Z}} h_{n-k} x_k = \sum_{k \in \mathbb{Z}} x_{n-k} h_k$.

  Source: Lecture notes, Topic 3, Slide 31, Link

  As well as before, let us extend sequences a and b to infinite sequences with *finite support* and fill them with zeros. Moreover, product of elements a_i and b_i where $i < -3$ and $i > 3$ by any other element also leads to 0, hence we will not consider them. Therefore, the convolution is:

  - $(a * b)_0 = \sum_{k=-\infty}^{k=\infty} a_k, b_{-k} = \boldsymbol{a_0} \cdot \boldsymbol{b_0} = \mathbf{2} \cdot \mathbf{2} = \mathbf{4}$
  - $(a * b)_1 = \sum_{k=-\infty}^{k=\infty} a_k, b_{1-k} = \boldsymbol{a_0} \cdot b_1 + a_1 \cdot \boldsymbol{b_0} = \mathbf{2} \cdot 0 + 1 \cdot \mathbf{2} = 2$
  - $(a * b)_2 = \sum_{k=-\infty}^{k=\infty} a_k, b_{2-k} = \boldsymbol{a_0} \cdot b_2 + a_1 \cdot b_1 + a_2 \cdot \boldsymbol{b_0} = \mathbf{2} \cdot 0 + 1 \cdot 0 + 0 \cdot \mathbf{2} = 0$
  - $(a * b)_3 = \sum_{k=-\infty}^{k=\infty} a_k, b_{3-k} = \boldsymbol{a_0} \cdot b_3 + a_1 \cdot b_2 + a_2 \cdot b_1 + a_3 \cdot \boldsymbol{b_0} = \mathbf{2} \cdot 1 + 1 \cdot 0 + 0 \cdot 0 + 2 \cdot \mathbf{2} = 6$
  - $(a * b)_4 = \sum_{k=-\infty}^{k=\infty} a_k, b_{4-k} = a_1 \cdot b_3 + a_2 \cdot b_2 + a_3 \cdot b_1 = 1 \cdot 1 + 0 \cdot 0 + 2 \cdot 0 = 1$
  - $(a * b)_5 = \sum_{k=-\infty}^{k=\infty} a_k, b_{5-k} = a_2 \cdot b_3 + a_3 \cdot b_2 = 0 \cdot 1 + 2 \cdot 0 = 0$
  - $(a * b)_6 = \sum_{k=-\infty}^{k=\infty} a_k, b_{6-k} = a_3 \cdot b_3 = 2 \cdot 1 = 2$

  Here we can notice that for $n \leq -1$ and $n \geq 7$ elements of convolution are 0, hence convolution of signals a and b is:

  $$(a * b) = (\dots\ 0\ 0\ \mathbf{4}\ 2\ 0\ 6\ 1\ 0\ 2\ 0\ 0\ \dots)$$

- Circular convolution:

  By definition: The circular convolution $(h^{(*)}x)$ between two finite sequences h and x of some fixed length $m > 0$ is the following sequence of the same length m:
  $(h^{(*)}x)_n = \sum_{k=0}^{k=m-1} h_k x_{(n-k)\text{mod } m} = \sum_{k=0}^{k=m-1} h_{(n-k)\text{mod } m} x_k$.

  Source: Lecture notes, Topic 4, Slide 43, Link

  Our signals have length 4, i.e. $m = 4$, therefore formula of circular convolution for signals a and b is:

  $$(a^{(*)}b)_n = \sum_{k=0}^{k=3} a_{(n-k)\text{mod}4} b_k = \sum_{k=0}^{k=3} b_{(n-k)\text{mod}4} a_k$$

Now let us compute circular convolution:

$$\begin{aligned}
(a^{(*)}b)_0 &= \boldsymbol{a_0} \cdot \boldsymbol{b_0} + a_1 \cdot b_3 + a_2 \cdot b_2 + a_3 \cdot b_1 = \mathbf{2} \cdot \mathbf{2} + 1 \cdot 1 + 0 \cdot 0 + 2 \cdot 0 = \mathbf{5} \\
(a^{(*)}b)_1 &= \boldsymbol{a_0} \cdot b_1 + a_1 \cdot \boldsymbol{b_0} + a_2 \cdot b_3 + a_3 \cdot b_2 = \mathbf{2} \cdot 0 + 1 \cdot \mathbf{2} + 0 \cdot 1 + 2 \cdot 0 = 2 \\
(a^{(*)}b)_2 &= \boldsymbol{a_0} \cdot b_2 + a_1 \cdot b_1 + a_2 \cdot \boldsymbol{b_0} + a_3 \cdot b_3 = \mathbf{2} \cdot 0 + 1 \cdot 0 + 0 \cdot \mathbf{2} + 2 \cdot 1 = 2 \\
(a^{(*)}b)_3 &= \boldsymbol{a_0} \cdot b_3 + a_1 \cdot b_2 + a_2 \cdot b_1 + a_3 \cdot \boldsymbol{b_0} = \mathbf{2} \cdot 1 + 1 \cdot 0 + 0 \cdot 0 + 2 \cdot \mathbf{2} = 6
\end{aligned}$$

Hence the result sequence is:

$$(a^{(*)}b) = (\mathbf{5}\ 2\ 2\ 6)$$

2.3 Answer

Cross-correlation:

$$c_{a,b} = (...\ 0\ 0\ 2\ 1\ 0\ \mathbf{6}\ 2\ 0\ 4\ 0\ 0\ ...)$$

Convolution:

$$(a * b) = (...\ 0\ 0\ \mathbf{4}\ 2\ 0\ 6\ 1\ 0\ 2\ 0\ 0\ ...)$$

Circular convolution:

$$(a^{(*)}b) = (\mathbf{5}\ 2\ 2\ 6)$$

3 Task 4

3.1 Problem

Consider a moving-average system (operator) that maps any input signal $x = (x_n)$ output signal $(y_n) = \left(\frac{(month)x_{n-1} + (day)x_{n+1}}{(year)}\right)$. Examine whether this system is memoryless, causal, shift-invariant, BIBO-stable, linear. If the system is linear then draw its infinitary matrix.

3.2 Solution

First, let us substitute numbers into formula for output signal and we get

$$(y_n) = \left(\frac{2x_{n-1} + 21x_{n+1}}{2001}\right)$$

Now let us consider properties of the properties of this system.

- Linear:

  By definition: A linear system enjoys additivity and scaling that together are known in engineering *as the superposition principle*: $T(ax + bu) = aT(x) + bT(u)$.

  Source: Lecture notes, Topic 3, Slide 13, Link

  Let us consider equation for $T(ax + bu)$:

  $$T(ax + bu) = \frac{2(ax_{n-1} + bu_{n-1}) + 21(ax_{n+1} + bu_{n+1})}{2001} =$$

  $$= \frac{a(2x_{n-1} + 21x_{n+1})}{2001} + \frac{b(2u_{n-1} + 21u_{n+1})}{2001} = aT(x) + bT(u)$$

  Hence we can conclude that this system is **linear**.

  The infinite matrix of this system is:

  $$\begin{pmatrix} ... & ... & ... & ... & ... \\ ... & 0 & \frac{21}{2001} & 0 & ... \\ ... & \frac{2}{2001} & 0 & \frac{21}{2001} & ... \\ ... & 0 & \frac{2}{2001} & 0 & ... \\ ... & ... & ... & ... & ... \end{pmatrix}$$

- Memoryless:

  By definition: A system T is *memoryless* if for every $k \in \mathbb{Z}$ and all input signals x and x' the following implication holds: $x_k = x'_k$ implies $(Tx)_k = (Tx')_k$.

  Source: Lecture notes, Topic 3, Slide 15, Link
  Let us write equations for $(Tx)_k$ and $(Tx')_k$:

$$(Tx)_k = \Big(\frac{2x_{k-1} + 21x_{k+1}}{2001}\Big),\ (Tx')_k = \Big(\frac{2x'_{k-1} + 21x'_{k+1}}{2001}\Big)$$

  Since formulas for $(Tx)_k$ and $(Tx')_k$ contain elements x_{k-1}, x'_{k-1}, x_{k+1} and x'_{k+1} which we do not have information about whether they are equal to each other as well, therefore we cannot say that our system is memoryless. Hence, this system is **not memoryless**.

- Causal:

  By definition: A system T is called *causal* if for every $k \in \mathbb{Z}$ and all input signals x and x' the following implication holds: $x_{(-\infty,k]} = x'_{(-\infty,k]}$ implies $(Tx)x_{(-\infty,k]} = (Tx')x_{(-\infty,k]}$.

  Source: Lecture notes, Topic 3, Slide 15, Link

  Similar to the previous case, equations for $(Tx)_k$ and $(Tx')_k$ contain elements x_{k+1} and x'_{k+1} which do not belong to $x_{(-\infty,k]}$ and $x'_{(-\infty,k]}$ respectively, therefore we cannot definitely say that these elements are equal as well. Hence we conclude that this system is **not causal**.

- Shift-invariant:

  By definition: A linear system T is *shift-invariant* (LSI thereafter) if for every input signal T (shifted x) is shifted Tx, i.t. if $\big((Tx)_{n-k}\big) = T(x_{n-k})$ for every input signal x and $k \in \mathbb{Z}$.

  Source: Lecture notes, Topic 3, Slide 16, Link

  Let us consider equations for $\big((Tx)_{n-k}\big)$ and $T(x_{n-k})$:

$$\big((Tx)_{n-k}\big) = (y)_{n-k} = \frac{2x_{n-k-1} + 21x_{n-k+1}}{2001}$$

$$T(x_{n-k}) = \frac{2x_{n-k-1} + 21x_{n-k+1}}{2001}$$

  As we can see, these equations are equal to each other, therefore $\big((Tx)_{n-k}\big) = T(x_{n-k})$, hence we can conclude that this system is **shift-invariant**

- BIBO-stable:

  By definition: A system T is called *bounded-input, bounded-output stable* (BIBO-stable) if a bounded input always produces bounded output: $Tx \in l^\infty$ fir all $x \in l^\infty$.

  Source: Lecture notes, Topic 3, Slide 18, Link

  According to the theorem from the same lecture notes the LSI system is BIBO-stable iff its impulse response is absolutely summable. The impulse response of this system is:

$$(..., 0, 0, \frac{21}{2001}, 0, \frac{2}{2001}, 0, 0, ...)$$

  Therefore the sum of the response can be computed:

$$\frac{21}{2001} + \frac{2}{2001} = \frac{23}{2001}$$

  As we can see the impulse response is summable, hence we can conclude that this system is **BIBO-stable**.

3.3 Answer

This system is linear, shift-invariant and BIBO-stable, however, it is not memoryless and not causal.

4 Task 5

4.1 Problem

Firstly, starting your answer with definition of DTFT, compute $A(\omega)$– DTFT or the signal a Then (also starting with definition of IDTFT) compute IDTFT for $A(\omega)$ and validate that in this case IDTFT is the inverse for DTFT indeed.

4.2 Solution

1. Definition: **The Discrete-Time Fourier Transform (DTFT)** maps each filter x (i.e., two-side infinite absolutely summable sequence) to the frequency response *spectrum* function $X(e^{j\omega}) = \sum_{k\in\mathbb{Z}} e^{-j\omega k}x_k$ of real argument ω.

   Source: Lecture notes, Topic 4, Slide 9, Link

   In our case we have sequence a:
$$a = (\mathbf{2}\ 1\ 0\ 2)$$

   Now let us compute DTFT:
$$A(\omega) = \sum_{k=0}^{k=3} e^{-j\omega k}a_k = 2e^{-j\omega\cdot 0} + e^{-j\omega\cdot 1} + 0e^{-j\omega\cdot 2} + 2e^{-j\omega\cdot 3}$$

   Hence we have
$$A(\omega) = 2 + e^{-j\omega} + 2e^{-3j\omega}$$

2. Definition: **The Inverse-DTFT** of 2π-periodic function $f(\omega)$ of the real argument is the following (two-side infinite) sequence $x = (\dots\ x_{-2}\ x_{-1}\ x_0\ x_1\ x_2\ \dots)$ where $x_n = \frac{1}{2\pi}\int_{-pi}^{\pi} f(\omega)e^{j\omega n}d\omega$ for all $n \in \mathbb{Z}$.

   Source: Lecture notes, Topic 4, Slide 10, Link

   Let us compute inverse-DTFT for $0 \le n \le 3$:

   - $\boldsymbol{a_0} = \frac{1}{2\pi}\int_{-\pi}^{\pi}(2 + e^{-j\omega} + 2e^{-3j\omega})e^{j\omega\cdot 0}d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi}(2 + e^{-j\omega} + 2e^{-3j\omega})d\omega = \frac{1}{2\pi}(2\omega - \frac{e^{-j\omega}}{j} - \frac{2e^{-3j\omega}}{3j})\Big|_{-\pi}^{\pi} =$
     $= \frac{1}{2\pi}\big(2(\pi + \pi) - \frac{1}{j}(e^{-j\pi} - e^{j\pi}) - \frac{2}{3j}(e^{-3j\pi} - e^{3j\pi})\big)$

     Applying Euler formula $e^{j\phi} = \cos\phi + j\sin\phi$ we get:
     $\boldsymbol{a_0} = \frac{1}{2\pi}\big(4\pi - \frac{1}{j}(\cos\pi - j\sin\pi - \cos\pi - j\sin pi) - \frac{2}{3j}(\cos 3\pi - j\sin 3\pi - \cos 3\pi - j\sin 3\pi)\big) = \frac{1}{2\pi}\cdot 4\pi = \mathbf{2}$

     Let us apply the same procedure for the rest values of n:
   - $a_1 = \frac{1}{2\pi}\int_{-\pi}^{\pi}(2 + e^{-j\omega} + 2e^{-3j\omega})e^{j\omega\cdot 1}d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi}(2e^{j\omega} + 1 + 2e^{-2j\omega})d\omega = 1$
   - $a_2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}(2 + e^{-j\omega} + 2e^{-3j\omega})e^{j\omega\cdot 2}d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi}(2e^{j\omega} + e^{j\omega} + 2e^{-j\omega})d\omega = 0$
   - $a_3 = \frac{1}{2\pi}\int_{-\pi}^{\pi}(2 + e^{-j\omega} + 2e^{-3j\omega})e^{j\omega\cdot 3}d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi}(2e^{3j\omega} + e^{2j\omega} + 2)d\omega = 2$

   Therefore result sequence of inverse-DTFT is:
$$a = (\mathbf{2}\ 1\ 0\ 2)$$

3. Comparison:

   As we can see, the result of inverse-DTFT is indeed corresponds to original sequence a. Hence, we can say that DTFT and inverse-DTFT are correct.

4.3 Conclusion

DTFT:
$$A(\omega) = 2 + e^{-j\omega} + 2e^{-3j\omega}$$

Inverse-DTFT:
$$a = (\mathbf{2}\ 1\ 0\ 2)$$

Inverse-DTFT sequence is equal to the original sequence.