2016-2017 MID-TERM EXAM – COMPLEX ANALYSIS

1. Suppose f is a complex function in a region Ω, and both f and f^2 are harmonic. Prove that either f or $\bar{f}$ is holomorphic in Ω. (15 points)

2. Suppose $f(z) = \sum_{n\geq 0} a_n z^n, a_n \geq 0$ for all $n \geq 0$, and the radius of convergence of the series is 1. Prove that f can not be analitically continued to a larger domain containing the point 1. (15 points)

3. Suppose $I = [a, b]$ is an interval on the real axis, ϕ is a continuous function on I, and

$$f(z) = \frac{1}{2\pi i}\int_a^b \frac{\phi(t)}{t-z}dt \ \ (z \notin I).$$

Show that

$$\lim_{\epsilon\to 0}[f(x+i\epsilon) - f(x-i\epsilon)] \ \ (\epsilon > 0)$$

exists for every real x, and find it in terms of ϕ. (15 points)

4. Suppose Ω is a region, f_n is a sequence of holomorphic functions in Ω with u_n being the real part of f_n. Suppose u_n converges uniformly in compact subsets of Ω and f_n converges at least at one point $z \in \Omega$. Prove that f_n converges uniformly in compact subsets of Ω. (15 points)

5. Suppose f_n is a uniformly bounded sequence of holomorphic functions in a region Ω such that $f_n(z)$ converges for every $z \in \Omega$. Prove that f_n converges uniformly on every compact subset of Ω. (15 points)

6.
Define

$$f(z) = \frac{1}{\pi}\int_0^1 r dr \int_{-\pi}^{\pi} \frac{d\theta}{re^{i\theta} + z}.$$

Show that $f(z) = \bar{z}$ if $|z| < 1$ and that $f(z) = 1/z$ if $|z| \geq 1$. (15 points)

7. Suppose f is a C^1 complex function in $\mathbb{C}$. Let D be a Euclidean disk. Prove

$$f(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(\zeta)}{\zeta - z}d\zeta - \frac{1}{\pi}\int\int_D \frac{f_{\bar{z}}(\zeta)}{\zeta - z}d\xi d\eta, \ \ z \in D, \zeta = \xi + i\eta.$$

(10 points)