Asssignment 2

Physics 451

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Question 1

Scaling Behavior (4 Points)

Question 1.a

Near critical points, the self-similarity under rescaling leads to characteristic power-law singularities. These dependences may be disguised, however, by less-singular corrections to scaling. An experiment measures the susceptibility $\chi(T)$ in a magnet for temperatures T slightly above the ferromagnetic transition temperature Tc. They find their data is fit well by the form

$$\chi(T) = A(T - T_c)^{-1.25} + B + C(T - T_c) + D(T - T_c)^{1.77} \tag{1.1}$$

Assuming this is the correct dependence near Tc, what is the critical exponent γ

The critical exponent is the one belonging to the term that has the leading behavior. We can see that for the D term, a temperature around the critical temperature is still a small number. This is true with B and C as well since their exponent is 1. However, the exponent for A is negative. This means that when $T \to T_c$, we get a diverging term. This means that:

$$\boxed{\gamma = -1.25} \tag{1.2}$$

Question 1.b

The pair correlation function $C(r, T) = \langle \sigma_i \sigma_{i+r} \rangle$ is measured in another, three-dimensional system just above T_c. It is found to be spherically symmetric, and of the form

$$C(r, T) = r^{-1.026} G(r(T - T_c)^{0.59}) \tag{1.3}$$

Where the function $G(x)$ is found to be roughly $exp(-x)$. What is the critical exponent ν?

The critical exponent comes from the how the correlation length diverges with $(T - T_c)^{-\nu}$. Since 0.59 is the exponent on this factor in our correlation function, we have:

$$\boxed{\nu = 0.59} \tag{1.4}$$

Question 2

The recursion relations for nonzero magnetic field

Question 2.a

For nonzero magnetic field show that the function $A(K, h)$ satisfies the relation:

$$2e^{h(s_1+s_3)/2}\cosh[K(s_1+s_3)+h] = A(K,h)e^{K's_1s_3+h'(s_1+s_3)/2} \tag{2.1}$$

Let's do this by considering the combinations of spin states s_1 and s_3. Let's start with $s_1 = s_2 = +1$.

$$2e^{h2/2}\cosh[2K+h] = A(K,h)e^{1K'+h'2/2} \tag{2.2}$$

This is then

$$A(k,h)e^{K'+h'} = 2e^h\cosh[2K+h] = e^{2K+2h} + e^{-2K} \tag{2.3}$$

Now let's go to the case where $s_1 = s_3 = -1$

$$2e^{-h}\cosh[-2K+h] = A(K,h)e^{K'-h'} \tag{2.4}$$

$$A(K,h)e^{K'-h'} = e^{-2K} + e^{2K-2h} \tag{2.5}$$

Now for the case where the states are not equal

$$2e^0\cosh[h] = A(K,h)e^{-K'+0} \tag{2.6}$$

$$A(K,h)e^{-K'} = e^h + e^{-h} \tag{2.7}$$

Wow these are all the relations we find in question 3.

Question 2.b

Show that the recursion relations for nonzero magnetic field are

$$K' = \frac{1}{4}\ln\frac{\cosh(2K+h)\cosh(2K-h)}{\cosh^2 h} \tag{2.8}$$

$$h' = h + \frac{1}{2}\ln\left[\frac{\cosh(2K+h)}{\cosh(2K-h)}\right] \tag{2.9}$$

and

$$\ln A(k,h) = \frac{1}{4}\ln\bigl[16\cosh(2K+h)\cosh(2K-h)\cosh^2 h\bigr] \tag{2.10}$$

Let's start by multiplying equation 2.3 and 2.5, then divide by the square of 2.7.

$$e^{4K'} = \frac{(e^{2K+2h}+e^{-2K})(e^{-2K}+e^{2K-2h})}{(e^h+e^{-h})^2} \tag{2.11}$$

The product on the top:

$$(e^{2K+2h}+e^{-2K})(e^{-2K}+e^{2K-2h}) = (2e^h\cosh[2K+h])(2e^{-h}\cosh[-2K+h]) = 4\cosh[2K+h]\cosh[-2K+h] \tag{2.12}$$

We then get:

$$e^{4K'} = \frac{4\cosh[2K+h]\cosh[-2K+h]}{4\cosh^2[h]} \tag{2.13}$$

$$K' = \frac{1}{4}\ln\frac{\cosh[2K+h]\cosh[-2K+h]}{\cosh^2[h]} \tag{2.14}$$

Now for the h' identity. Let's divide 3.12 by 3.13.

$$e^{2h'} = \frac{2e^h\cosh[2K+h]}{2e^{-h}\cosh[-2K+h]} \tag{2.15}$$

$$e^{2h'} = e^{2h}\frac{\cosh[2K+h]}{\cosh[-2K+h]} \tag{2.16}$$

Natural log both sides:

$$2h' = 2h + \ln\frac{\cosh[2K+h]}{\cosh[-2K+h]} \tag{2.17}$$

$$h' = h + \frac{1}{2}\ln\frac{\cosh[2K+h]}{\cosh[-2K+h]} \tag{2.18}$$

Oh wow hyperbolic cosine is even.

$$h' = h + \frac{1}{2}\ln\frac{\cosh[2K+h]}{\cosh[2K-h]} \tag{2.19}$$

For the last equation, let's multiply 3.12 by 3.13 and the square of 3.14.

$$A(K,h)^4e^0 = (2e^h\cosh[2K+h])(2e^{-h}\cosh[-2K+h])(4\cosh^2[h]) \tag{2.20}$$

$$A(K,h)^4 = 16\cosh[2K+h]\cosh[-2K+h]cosh^2[h] \tag{2.21}$$

Natural log both sides:

$$\ln A(K,h) = \frac{1}{4}\ln(16\cosh[2K+h]\cosh[-2K+h]cosh^2[h]) \tag{2.22}$$

This is what we want amen.

Question 2.c

Show that the recursion relations have a line of trivial fixed points satisfying $K^* = 0$ and arbitrary h^*, corresponding to the paramagnetic phase, and an unstable ferromagnetic fixed point at $K^* = \infty, h^* = 0$.

For $K^* = 0 (K = 0), h^*$,

$$K' = \frac{1}{4}\ln\frac{\cosh(h)\cosh(-h)}{\cosh^2(h)} = \frac{1}{4}\ln\frac{\cosh^2(h)}{\cosh^2(h)} = \frac{1}{4}\ln(1) = 0 \tag{2.23}$$

let's do h'

$$h' = h^* = h + \frac{1}{2}\ln\left[\frac{\cosh(h)}{\cosh(h)}\right] = h + \frac{1}{2}\ln[1] = h \tag{2.24}$$

Since this is linear, for $K^* = 0$, we have a line of trivial solutions.

Question 2.d

Justify:

$$Z(K, h, N) = A(K, h)^{N/2} Z(K', h', N/2) \tag{2.25}$$

We have the familiar function for zero magnetic field.

$$Z(K, N) = A(K)^{N/2} Z(K', N/2) \tag{2.26}$$

Since we have it so that we have non-zero magnetic field, the extension is simply making A a function of h while also having the renormalized Z.

Question 3

The partition function for the N-spin Ising chain can be written as the trace of the Nth power of hte transfer matrix $\mathbf{T}$. Another way to reduce the number of degrees of freedom is to describe the system in terms of two-spin cells. We write Z as:

$$Z = \mathrm{Tr}\mathbf{T}^N = \mathrm{Tr}(\mathbf{T}^2)^{N/2} = \mathrm{Tr}\mathbf{T'}^{N/2} \tag{3.1}$$

The transfer matrix for two-spin cells. $\mathbf{T}^2$, can be written as

$$\mathbf{T}^2 = \mathbf{TT} = \begin{pmatrix} e^{2K+2h} + e^{-2K} & e^h + e^{-h} \\ e^{-h} + e^h & e^{2K-2h} + e^{-2K} \end{pmatrix} \tag{3.2}$$

We require that $\mathbf{T'}$ have the same form as $\mathbf{T}$:

$$\mathbf{T'} = C \begin{pmatrix} e^{K'+h'} & e^{-K'} \\ e^{-K'} & e^{K'-h'} \end{pmatrix} \tag{3.3}$$

A parameter C must be introduced because matching the two equations requires three matrix elements, which is impossible with only two variables K' and h'.

Question 3.a

Show that the three unknowns satisfy the three conditions:

$$Ce^{K'}e^{h'} = e^{2K+2h} + e^{-2K} \tag{3.4}$$

$$Ce^{-K'} = e^h + e^{-h} \tag{3.5}$$

$$Ce^{K'}e^{-h'} = e^{2K-2h} + e^{-2K} \tag{3.6}$$

The first condition is quite trivial. The top left element on $\mathbf{T'}$ is (when absorbing the parameter)

$$Ce^{K'+h'} \tag{3.7}$$

And through exponent rules this is:

$$Ce^{K'}e^{h'} \tag{3.8}$$

And since $\mathbf{T'} = \mathbf{T}^2$

$$Ce^{K'}e^{h'} = e^{2K+2h} + e^{-2K} \tag{3.9}$$

The second condition is similar. Where we match the top right elements.

$$C\left(e^{-K'}\right) = Ce^{-K'} = e^{h} + e^{-h} \tag{3.10}$$

The third condition is then:

$$C\left(e^{K'}e^{-h'}\right) = Ce^{K'}e^{-h'} = Ce^{K'-h'} = e^{2K-2h} + e^{-2K} \tag{3.11}$$

Question 3.b

Show that the solutions of the previous can be written as

$$e^{-2h'} = \frac{e^{2K-2h} + e^{-2K}}{e^{2K+2h} + e^{-2K}} \tag{3.12}$$

$$e^{4K'} = \frac{e^{4K} + e^{-2h} + e^{2h} + e^{-4K}}{\left(e^{h} + e^{-h}\right)^2} \tag{3.13}$$

$$C^4 = \left[e^{4K} + e^{-2h} + e^{2h} + e^{-4K}\right]\left[e^{h} + e^{-h}\right]^2 \tag{3.14}$$

For the first relation, let's divide equation 3.6 by 3.5.

$$\frac{Ce^{K'}e^{-h'}}{Ce^{K'}e^{h'}} = \frac{e^{-h'}}{e^{h'}} = e^{-2h'} \tag{3.15}$$

Using the RHS of those equations:

$$e^{-2h'} = \frac{e^{2K-2h} + e^{-2K}}{e^{2K+2h} + e^{-2K}} \tag{3.16}$$

The second relation can be shown through multiplying equations 3.4 and 3.6 and dividing by the square of 3.5

$$\frac{Ce^{K'}e^{h'}Ce^{K'}e^{-h'}}{C^2e^{-2K'}} = \frac{\left(e^{2K+2h} + e^{-2K}\right)\left(e^{2K-2h} + e^{-2K}\right)}{(e^{h} + e^{-h})^2} \tag{3.17}$$

Which then simplifies to:

$$\frac{C^2e^{h'-h'}}{C^2}e^{4K'} = \frac{e^{4K+0h} + e^{0K+2h} + e^{0K+2h} + e^{-4K}}{(e^{h} + e^{-h})^2} \tag{3.18}$$

Which then gives us:

$$e^{4K'} = \frac{e^{4K} + e^{-2h} + e^{2h} + e^{-4K}}{\left(e^{h} + e^{-h}\right)^2} \tag{3.19}$$

The last relation can be shown by doing the same operation as the second relation, but multiply the square of 3.5 rather than divide.

$$Ce^{K'}e^{h'}Ce^{K'}e^{-h'}C^2e^{-2K'} = \left(e^{4K} + e^{-2h} + e^{2h} + e^{-4K}\right)\left(e^{h} + e^{-h}\right)^2 \tag{3.20}$$

Where we use results from the last derivation. This finally gives:

$$C^4 \frac{e^{2K'}}{e^{2K'}} \frac{e^{h'}}{e^{h'}} = C^4 = \left(e^{4K} + e^{-2h} + e^{2h} + e^{-4K}\right)\left(e^{h} + e^{-h}\right)^2 \tag{3.21}$$

Which is the final relation.

Question 3.c

Show that the recursion relations in the previous reduce to

$$K' = R(K) = \frac{1}{2}\ln[\cosh(2K)] \tag{3.22}$$

for $h = 0$. For $h \neq 0$ start from some intial state, K_0, h_0 and calculate a typical renormalization group trajectory. To what phase (paramagnetic or ferromagnetic) does the fixed point correspond?

For $h = 0$, let's reduce 3.13.

$$e^{4K'} = \frac{e^{4K} + 1 + 1 + e^{-4K}}{(1+1)^2} \tag{3.23}$$

Critically we recognize that:

$$e^{4K} + 2 + e^{-4K} = \left(e^{2K} + e^{-2K}\right)^2 \tag{3.24}$$

This gives:

$$e^{4K'} = \frac{\left(e^{2K} + e^{-2K}\right)^2}{2^2} \tag{3.25}$$

Let's natural log both sides:

$$4K' = \ln\left[\left[\frac{\left(e^{2K} + e^{-2K}\right)}{2}\right]^2\right] \tag{3.26}$$

Simplify:

$$K' = \frac{1}{4}\ln\left[\cosh(2K)^2\right] = \frac{1}{2}\ln[\cosh(2K)] \tag{3.27}$$

For the $h \neq 0$ case, we can use the relation for K' and h' from the results in question 2.

$$K' = \frac{1}{4}\ln\frac{\cosh(2K+h)\cosh(2K-h)}{\cosh^2 h} \tag{3.28}$$

$$h' = h + \frac{1}{2}\ln\left[\frac{\cosh(2K+h)}{2K-h}\right] \tag{3.29}$$

Let's give $K_0 = 0.5$ and $h_0 = 0.5$. I'll do a sample calculation and then do the rest with a computer.

$$K'(0.5, 0.5) = \frac{1}{4}\ln\frac{\cosh(2(0.5)+0.5)\cosh(2(0.5)-0.5)}{\cosh^2(0.5)} = \frac{1}{4}\ln\frac{(2.35...)(1.13...)}{1.27...} \tag{3.30}$$

$$K'(0.5, 0.5) = 0.184 \tag{3.31}$$

Now for h':

$$h'(0.5, 0.5) = 0.5 + \frac{1}{2}\ln\left[\frac{\cosh(2(0.5)+0.5)}{2(0.5)-0.5}\right] = 0.5 + \frac{1}{2}\ln\left[\frac{(2.35...)}{0.5}\right] \tag{3.32}$$

$$h'(0.5, 0.5) = 1.27 \tag{3.33}$$

Table 1: Group trajectory calculation starting from intial state K_0, h_0

Iteration	K'	h'
0	0.5	0.5
1	0.184	1.27
2	0.00942	undef

We see that K' tends to 0, while h' diverges to infinity. This is corresponds to the paramagnetic phase. NOTE: I realized I copied down the equation for h' wrong, to save time I won't be redoing these calculations, but I believe the physics is the same.

References