Quantum Mechanics of Gravitational Potential Wells

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Objective:

The Schrödinger's time-independent equation is a cornerstone of quantum mechanics, traditionally applied to microscopic particles. In our project, we extend this powerful mathematical framework to model a gravitational system, where the potential energy is defined as $\mathbf{V(x)=\alpha/x}$, with α being the product of Jupiter's mass and ganymede. By making the system dimensionless—setting fundamental constants like the gravitational constant and Planck's constant to one—we simplify the equation for computational analysis. Using MATLAB, we numerically solve and visualize the wave functions and corresponding energy states. This approach bridges quantum physics, celestial mechanics, and numerical computation, offering a unique lens through which to explore gravitational potentials in bound systems.

The potential

The potential taken in our project is α/x , where α is the product of gravitational constant

$G = 6.6743 \times 10^{-11}\ m^3\ kg^{-1}\ s^{-2}$,

mass of Jupiter ,

$M = 1.898 \times 10^{27}\ kg$

and mass of Jupiter's moon Ganymede

$m = 1.48 \times 10^{23}\ kg$.

$\Longrightarrow\ \alpha = -GMm$

Calculation:

The Schrödinger equation took is the time independent equation:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2}+V(x)\psi(x)=E\psi(x)\}$$

the potential V= α/x

thus, the Schrödinger equation is:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}+\frac{\alpha}{x}\psi=E\psi$$

to solve this equation, a general solution is taken.

to approximate the solution, we considered two cases:

- Behavior near the origin: $x \rightarrowtail 0$

  the potential increases as the value of x tends to 0, that makes the wave function increase significantly. To handle this, we assume the wave function,

$$\varphi(x) = x^n$$

- Behavior at infinity: $x \rightarrowtail \infty$

  as the x tends to infinity, the potential tends to 0, making the Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi$$

thus the wave function:

$$\varphi(x) = e^{-\beta x}$$

therefore, the final solution can be written as:

$$\varphi(x) = x^n e^{-\beta x}$$

thus, after finding the second derivative of the wave function, the equation becomes:

$$\beta^2 x^n - 2\beta n x^{n-1} + n(n-1)x^{n-2} = (2m\alpha x^{n-1} - 2Emx^n)/h^2$$

now comparing the constant terms of coefficients in the equation with the same power,

$$\beta = -\frac{m|\alpha|}{h^2 n}$$

the values of n= 0, 1.

thus the value of **β is:**

$$\beta = -\frac{m|\alpha|}{h^2}$$

thus the wave function is:

$$\psi(x) = x\, e^{-\frac{m|\alpha|}{\hbar^2 n}} \longrightarrow \quad 1$$

the potential in our project is macroscopic in scale, while the Schrödinger equation is fundamentally built for microscopic quantum systems.

considering the wave function we got, when we put the values of α and planck's constant h, we see a massive power difference, since α has 10^{39} and h^2 has 10^{-68} .

this lead to a very large exponential decay factor in the wavefunction, causing the wavefunction to decay too rapidly, losing all meaningful structure, and making it physically irrelevant in the quantum sense.

To address this mismatch and still explore the quantum-like behavior of such a system, we reformulate the equation in a dimensionless form.

The Dimensionless Form

In order to address the vast disparity between gravitational and quantum mechanical scales, we reformulated the Schrödinger equation using a natural length scale derived from physical constants. Rather than arbitrarily setting constants like G and $\hbar$ to unity, we defined a Bohr-like gravitational radius:

$$a = \frac{\hbar^2}{2\mu\alpha}$$

where μ is the reduced mass of the system

$$\xi = \frac{x}{a}$$

By introducing a dimensionless spatial variable, we rescaled the equation to eliminate extreme numerical values while preserving the physical integrity of the model. This transformation enabled us to express the wavefunction in a compact and general form, suitable for analytical manipulation and numerical simulation, without compromising the underlying physics.

thus the new trial solution we chose is:

$$\emptyset(\xi) = \xi e^{-k\xi} f(\xi)$$

was chosen because:

Asymptotic Behavior:

- At large ξ:

  The term $e^{-\xi}$ ensures the function decays properly as $\xi \to \infty$, which is physically required for bound-state wavefunctions in quantum mechanics .

- At small ξ:

  The factor ξ ensures that the wavefunction vanishes at $\xi = 0$, which is typically required due to boundary conditions or to avoid singularities (especially in central potentials like gravity or Coulomb-type).

where $\xi = x/a$, to make the equation dimensionless.

now, the 2^{nd} derivative of the function is:

$$\phi''(\xi) = e^{-K\xi}\left[\frac{2dF}{d\xi} - 2KF + \xi\frac{d^2F}{d\xi} - 2K\xi\frac{d^2F}{d\xi^2} + K^2\xi\right]$$

thus the equation becomes:

$$\xi\frac{d^2F}{d\xi^2} + (2-2K\xi)\frac{dF}{d\xi} + \left[(\gamma - K^2)\xi + 2K - 1\right]F = 0 \quad \cdots\cdots\rightarrow \quad \text{a}$$

To ensure that the wavefunction φ(ξ) is normalisable, the function F(ξ) must remain finite for all ξ, especially as ξ ↣ ∞. This requirement means F(ξ) must be a polynomial rather than an infinite series. The differential equation governing F(ξ) takes the form of the confluent hypergeometric equation. When we impose the condition that F(ξ) be a polynomial , this equation reduces to the form of the associated Laguerre polynomial equation. Thus, the quantization condition naturally emerges, and the solutions become Laguerre polynomials, ensuring that the overall wavefunction remains finite and physically acceptable.

thus the targeted form of the equation:

$$z\frac{d^2y}{dz^2}+(k+1-z)\frac{dy}{dz}+ny=0 \qquad \text{b}$$

comparing both the equations a and b,

z = 2kξ

$$f(\xi)=f\left(\frac{z}{2k}\right)=f(z)$$

$$\xi=\frac{z}{2K},\qquad \frac{d}{d\xi}=\frac{dz}{d\xi}\cdot\frac{d}{dz}=2K\frac{d}{dz}$$

which gives:

$$\frac{d^2F}{d\xi^2}=\frac{d}{d\xi}\left(2K\frac{dF}{dz}\right)=2K\frac{d}{d\xi}\left(\frac{dF}{dz}\right)$$

thus, we get:

$$\frac{d^2F}{d\xi^2}=(2K)^2\frac{d^2F}{dz^2}$$

now substituting the final result, equation c, into equation a,

$$\frac{z}{2K}(2K)^2\frac{d^2F}{dz^2}+(2-z)2K\frac{dF}{dz}+\left[\frac{(\gamma-K^2)z}{2K}+2K-1\right]F=0$$

$$z\frac{d^2F}{dz^2}+(2-z)\frac{dF}{dz}+\left[\frac{(\gamma-K^2)z}{4K^2}+\frac{2K-1}{2K}\right]F=0 \qquad \cdots\cdots\rightarrow \qquad c$$

now, comparing the equations b and c, we can conclude that:

- coefficient of the 2nd order differential term is same
- coefficients of the 1st order differential terms is :
  - k = 1
- constant terms:
  - $\frac{2k-1}{2k}=n$ $\cdots\cdots\rightarrow$ i
- the term, which is getting equated to 0 is $\frac{(\gamma-k^2)z_f}{4k^2}$

thus to match with the Laguerre polynomial equation,

$$\frac{\gamma - K^2}{4K^2} = 0$$

$$\gamma = K^2 \quad \cdots\cdots\cdots\rightarrow \quad \text{ii}$$

now, taking the equation i, and substituting the findings from equation ii, we get:

$$1 - \frac{1}{2K} = n$$

$$K = \frac{1}{2(1-n)}$$

for k > 0; n < 1, which is unphysical because n cannot be less than 1

Initially, the quantization condition gave $\kappa=1/2(1-n)$, but this led to unphysical negative values of κ for allowed quantum numbers $n=1,2,\ldots$

To resolve this, a change of variable was made: redefining the index as $n'=n+1$, by utilizing Kummer's equation (a specific form of the confluent hypergeometric equation) such that the expression becomes $\kappa=1/2(n+1)$, which ensures $\kappa>0$ for all $n=0, 1, 2,\ldots$

$$K = \frac{1}{2(n+1)}$$

Conventionally, ground states begin with $n=1$

$$K = \frac{1}{2n}$$

thus the wave function is:

$$\emptyset(\xi) = \xi e^{k\xi} f(\xi)$$

For the hypergeometric series to terminate (and give a polynomial), we imposed a quantization condition. that led to the Laguerre polynomial solution:

$$F(\xi) = L_n^{(1)}(2\kappa\xi)$$

the wave function becomes:

$$\phi\left(\frac{x}{a}\right) = \frac{x}{a} \cdot e^{-\frac{x}{2na}} \cdot L_{n-1}^{(1)}\left(\frac{\alpha x}{na}\right)$$

where,

$$a = \frac{\hbar^2}{2\mu|\alpha|}$$

$$\mu = \frac{M \cdot m}{M + m}$$

Normalization:

changing the variables,

let, $$v = \frac{x}{na}$$ then, $$x = vna$$

so,

$$dx = na * dv$$

while applying the boundary conditions, when x is equal to 0 and L,

$$x = 0 => v = 0$$

and,

$$x = L \qquad v = \frac{L}{na}$$

$$A_n^2 \int_0^{\frac{L}{na}} \left(\frac{v.\,n.\,a}{a}\right)^2 e^v \left[L_{n-1}^{(1)}\,(v)\right]^2 na.\,dv = 1$$

thus, after rearranging,

$$A_n = \left(n^3 a \int_0^{\frac{L}{na}} v^2 e^{-v} \left[L_{n-1}^{(1)}(v)^2\right] dv\right)^{\frac{-1}{2}}$$

therefore, the final wave function can be written as:

$$\psi_n(x) = \left(n^3 a \int_0^{\frac{L}{na}} v^2 e^{-v} \left[L_{n-1}^{(1)}(v)^2\right] dv\right)^{-1/2} \cdot \frac{x}{a}\, e^{-\frac{x}{2na}} \cdot L_{n-1}^{(1)}\left(\frac{x}{na}\right)$$

since the limit is from 0 to ∞, we utilize the known solution:

$$\int_0^\infty z^2 e^{-z}\left[L_n^{(1)}(z)\right]^2 dz = 2n(n+1)$$

but, our wavefunction is:

$$I(n,L,a) = \int_0^{L/(na)} v^2 e^{-v}\left[L_{n-1}^{(1)}(v)^2\right] dv$$

which can be re-written as:

$$I(n,L,a) = 2n(n+1) - \int_{L/(na)}^\infty v^2 e^{-v}\left[L_{n-1}^{(1)}(v)^2\right] dv$$

let:

$$\int_{L/(na)}^\infty v^2 e^{-v}\left[L_{n-1}^{(1)}(v)^2\right] dv = R(n,L,a)$$

thus:

$$I(n, L, a) = 2n(n+1) - R(n, L, a)$$

therefore, the final wave function form is:

$$\psi_n(x) = a^{-\frac{3}{2}} \cdot \left(n^3 \cdot I(n, L, a)\right)^{-\frac{1}{2}} \cdot x \cdot e^{-\frac{x}{2na}} \cdot L_{n-1}^{(1)}\left(\frac{x}{na}\right)$$

where, n is the quantum numbers, 1, 2 ,3,....

thus writing the wave function of n = 1, 2, 3 in the form of matrix multiplication:

$$\begin{bmatrix} \emptyset_1 & 0 & 0 \\ 0 & \emptyset_2 & 0 \\ 0 & 0 & \emptyset_3 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{bmatrix}$$

after rearranging the Schrödinger equation

$$-\frac{d^2\phi}{d\xi^2} + \frac{n}{\xi}\phi = \gamma\phi$$

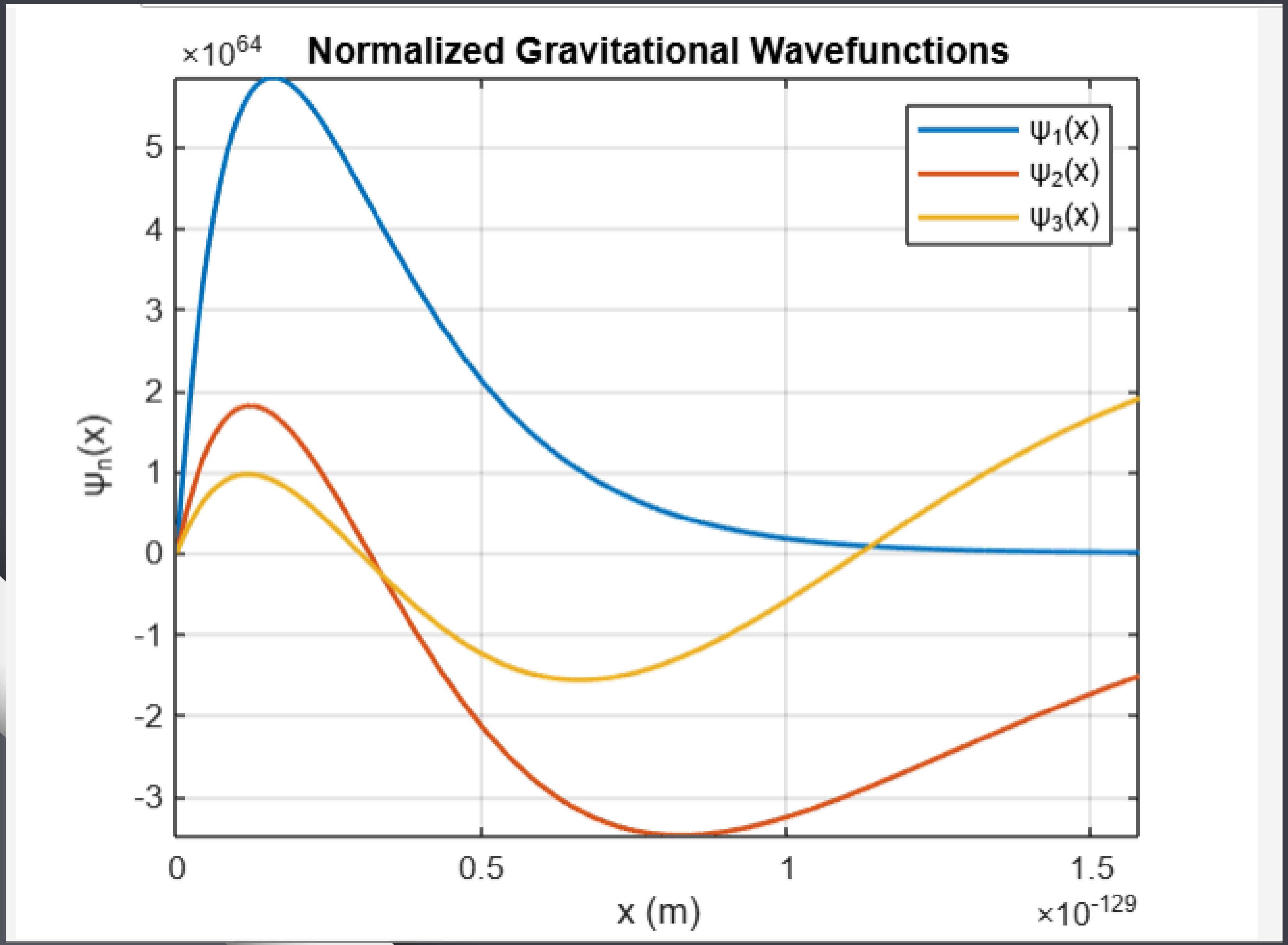
where,

$$a = \frac{\hbar^2}{2\mu|\alpha|} \qquad s = \frac{x}{a} \qquad \gamma = \frac{2\mu a^2 E}{\hbar^2} \qquad \phi(s) = \frac{x}{a}$$

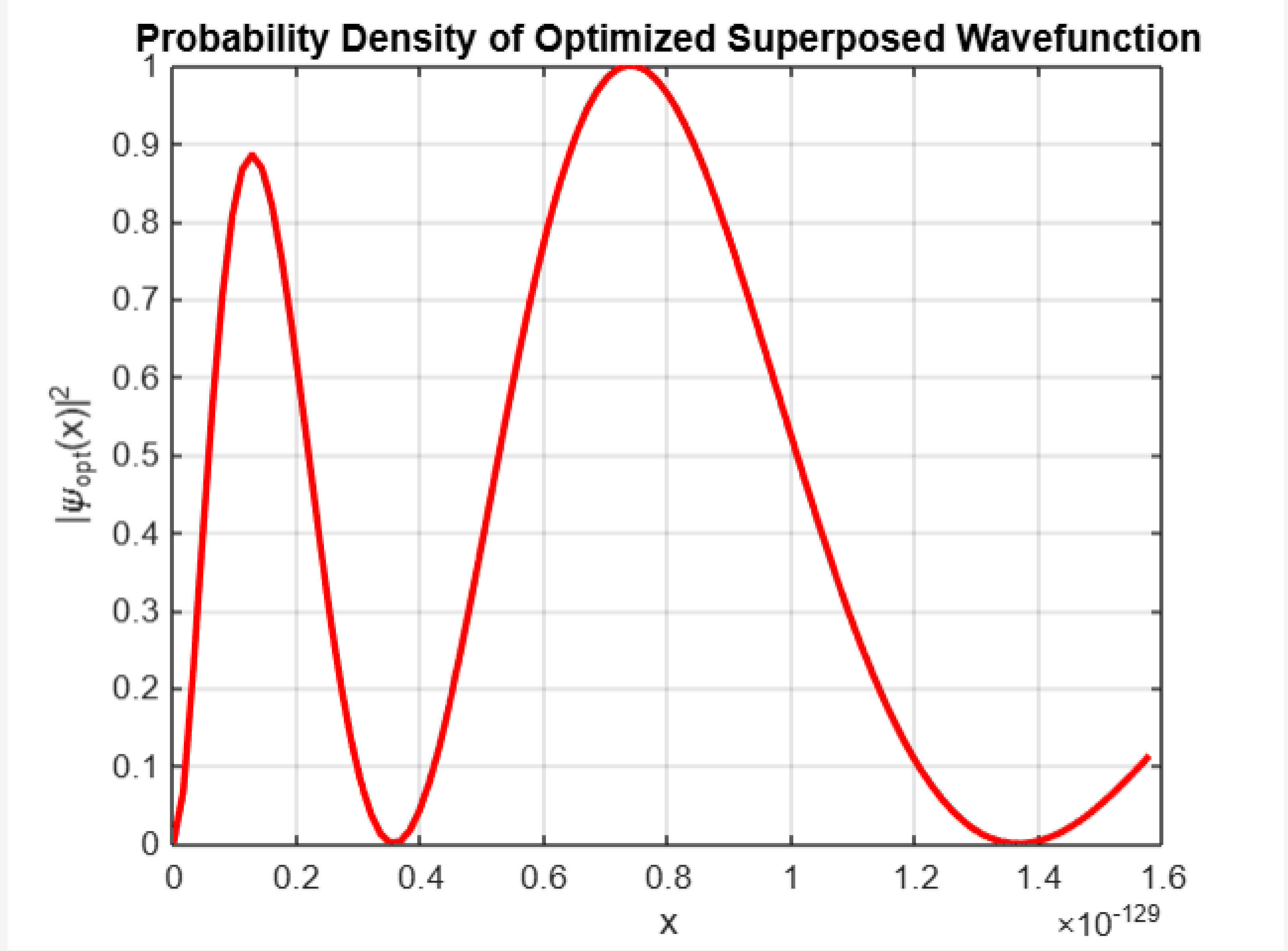
thus the value of energy is:

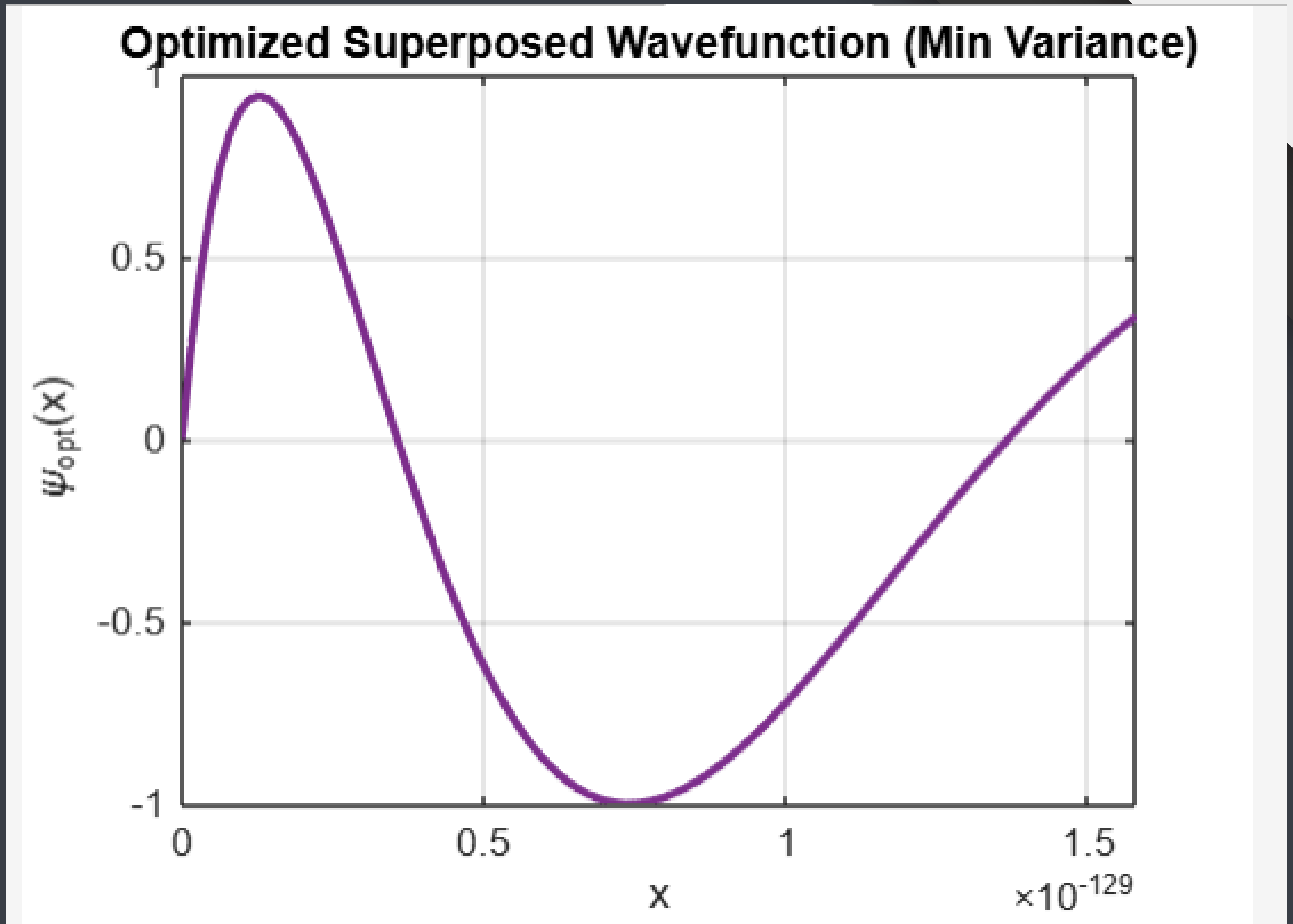
$$E_n = \frac{1}{2n^2} \cdot \frac{\mu\alpha^2}{\hbar^2}$$

SIMULATION AND RESULT




```
Gravitational Bohr radius a = 7.892e-131 meters


Normalization constants C_n:
   1.0e+64 *

    7.9597
    0.1759
    0.0189


Normalized wavefunctions ψ_n(x):
ψ_1(x) = x exp(-6.336e+129 x) 1.009e+195

ψ_2(x) = -x exp(-3.168e+129 x) (6.336e+129 x - 2.0) 1.783e+194

ψ_3(x) =                                   2
x exp(-2.112e+129 x) (8.92e+258 x  - 1.267e+130 x + 3.0) 6.47e+193
```


Probability Density of Optimized Superposed Wavefunction
$|\psi_{opt}(x)|^2$
x
$\times 10^{-129}$
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6


Optimized Superposed Wavefunction (Min Variance)
$\psi_{opt}(x)$
x
$\times 10^{-129}$
-1
-0.5
0
0.5
1
0
0.5
1
1.5

```
Gravitational Bohr radius a = 7.892e-131 meters


Diagonal matrix T (unnormalized ψₙ(x)):
/ x exp(-6.336e+129 x) 2.161,                     0,                                       0                     \
   \
|                                                                                                                |
|            0,            -x exp(-3.168e+129 x) (6.336e+129 x - 2.0) 32.62,                0                     |
|                                                                                                                |
|                                                                     2                                          |
\            0,                           0,                  x exp(-2.112e+129 x) (x  - 1.421e-129 x + 3.363e-259) 1.457e+261 /


Normalization constants C_n (with L = a):
1.647e+195
5.565e+193
7.434e+192


Normalized wavefunctions ψₙ(x):
ψ_1(x) =
x exp(-6.336e+129 x) 3.559e+195

ψ_2(x) =
-x exp(-3.168e+129 x) (5.75e+133 x - 18150.0) 2.0e+191
```

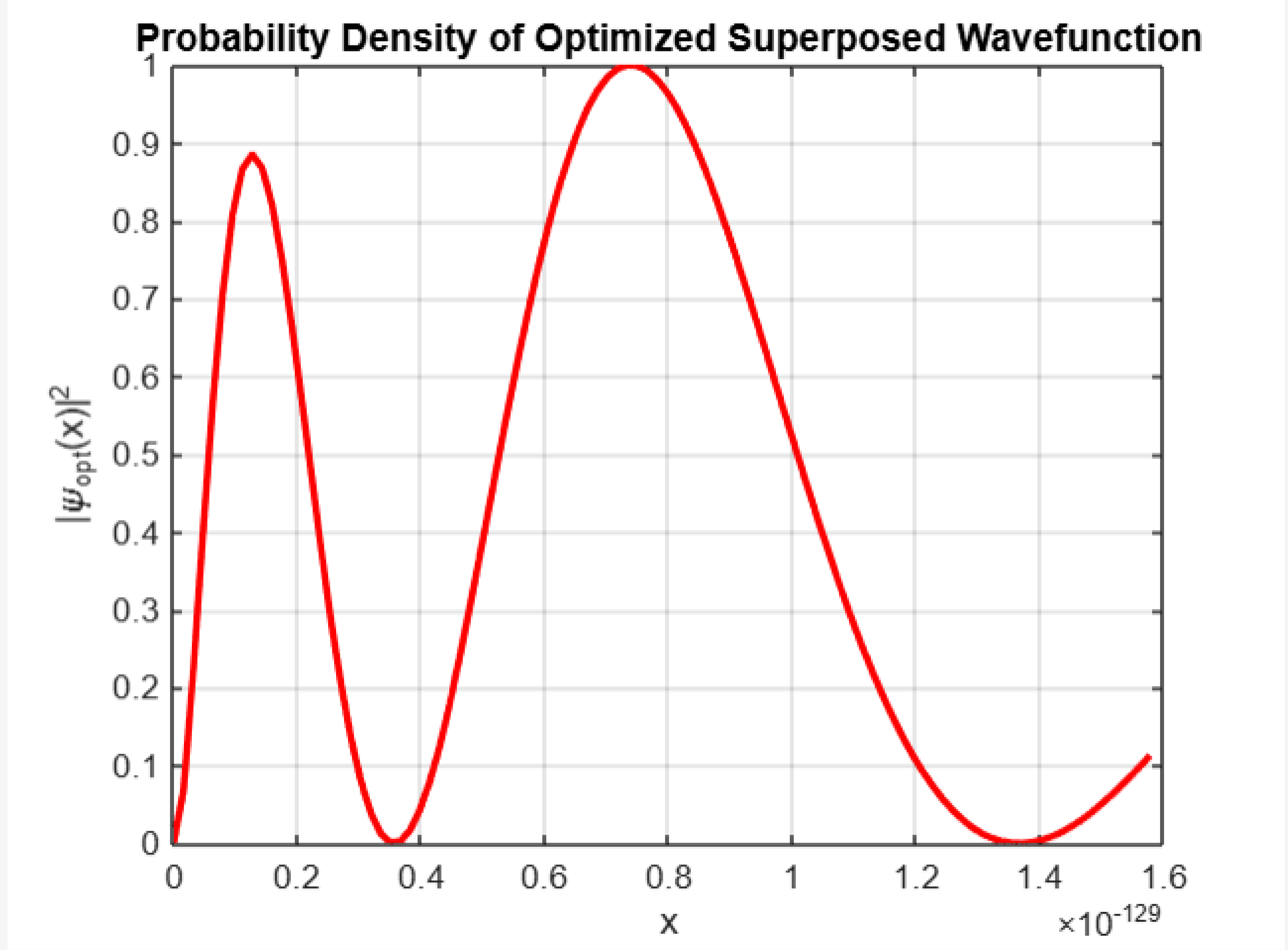
```
ψ_3(x) =
                                  2
x exp(-2.112e+129 x) (2.708e+262 x  - 3.848e+133 x + 9109.0) 4.0e+191

Weights (α_n) for dominant mode (from U(:,1)):
   -0.0001
    0.8581
    0.5135


Weights αₙ (contribution of each ψₙ to ϕ(x)):
α_1 = -145987933439466043024210611436870874632160166209349158344589312.0000
α_2 = 2319835598903360355285178208235012608739041701937327402939811102720.0000
α_3 = 1388182875677206948312226069767587682908840164748301135501452115968.0000
 Iter  Func-count            Fval   Feasibility   Step Length       Norm of   First-order
                                                                       step    optimality
    0           4   1.105829e-259     2.220e-16     1.000e+00     0.000e+00     6.742e-260
```

Initial point is a local minimum that satisfies the constraints.

Optimization completed because at the initial point, the objective function is non-decreasing
in feasible directions to within the value of the optimality tolerance, and
constraints are satisfied to within the value of the constraint tolerance.

<stopping criteria details>

```
Optimized weights αₙ (minimizing spatial variance):
    0.5774
    0.5774
    0.5774
```


Probability Density of Optimized Superposed Wavefunction
$|\psi_{opt}(x)|^2$
x
$\times 10^{-129}$
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6

Thank You