Primjena računara u biologiji

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Interval Estimation, Hypothesis Testing and Type II Error

Elementary Statistics with R

Interval Estimation

Interval Estimation

It is a common requirement to efficiently estimate population parameters based on simple random sample data. In the R tutorials of this section, we demonstrate how to compute the estimates. The steps are to be illustrated with a built-in data frame named **survey**. It is the outcome of a Statistics student survey in an Australian university.

The data set belongs to the MASS package, which has to be pre-loaded into the R workspace prior to use.

```
> library(MASS)      # load the MASS package
> head(survey)
     Sex Wr.Hnd NW.Hnd  ...
1 Female   18.5   18.0  ...
2   Male   19.5   20.5  ...
3   Male   18.0   13.3  ...
       .................
```

Interval Estimation (2)

For further details of the survey data set, please consult the R documentation.

```
> help(survey)
```

- Point Estimate of Population Mean
- Interval Estimate of Population Mean with Known Variance
- Interval Estimate of Population Mean with Unknown Variance
- Sampling Size of Population Mean
- Point Estimate of Population Proportion
- Interval Estimate of Population Proportion
- Sampling Size of Population Proportion

Point Estimate of Population Mean

For any particular random sample, we can always compute its sample mean. Although most often it is not the actual population mean, it does serve as a good **point estimate**. For example, in the data set survey, the survey is performed on a sample of the student population. We can compute the sample mean and use it as an estimate of the corresponding population parameter.

Problem

Find a point estimate of mean university student height with the sample data from survey.

Point Estimate of Population Mean (2)

Solution

For convenience, we begin with saving the survey data of student heights in a variable height.survey.

```
> library(MASS)                  # load the MASS package
> height.survey = survey$Height
```

It turns out not all students have answered the question, and we must filter out the missing values. Hence we apply the mean function with the "na.rm" argument as TRUE.

```
> mean(height.survey, na.rm=TRUE)  # skip missing values
[1] 172.38
```

Answer

A point estimate of the mean student height is 172.38 centimeters.

Interval Estimate of Population Mean with Known Variance

After we found a point estimate of the population mean, we would need a way to quantify its accuracy. Here, we discuss the case where the population variance σ^2 is assumed known.

Let us denote the $100(1 - \alpha/2)$ percentile of the standard normal distribution as $z_{\alpha/2}$. For random sample of sufficiently large size, the end points of the **interval estimate** at $(1 - \alpha)$ confidence level is given as follows:

$$\bar{x} \pm z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$

Problem

Assume the population standard deviation σ of the student height in survey is 9.48. Find the margin of error and interval estimate at 95% confidence level.

Interval Estimate of Population Mean with Known Variance (2)

Solution

We first filter out missing values in survey$Height with the na.omit function, and save it in height.response.

```
> library(MASS)                    # load the MASS package
> height.response = na.omit(survey$Height)
```

Then we compute the standard error of the mean.

```
> n = length(height.response)
> sigma = 9.48                     # population standard deviation
> sem = sigma/sqrt(n); sem         # standard error of the mean
[1] 0.65575
```

Interval Estimate of Population Mean with Known Variance (3)

Since there are two tails of the normal distribution, the 95% confidence level would imply the 97.5^{th} percentile of the normal distribution at the upper tail. Therefore, $z_{\alpha/2}$ is given by qnorm(.975). We multiply it with the standard error of the mean sem and get the margin of error.

```
> E = qnorm(.975)*sem; E          # margin of error
[1] 1.2852
```

We then add it up with the sample mean, and find the confidence interval as told.

```
> xbar = mean(height.response)   # sample mean
> xbar + c(-E, E)
[1] 171.10 173.67
```

Interval Estimate of Population Mean with Known Variance (4)

Since there are two tails of the normal distribution, the 95% confidence level would imply the 97.5^{th} percentile of the normal distribution at the upper tail. Therefore, $z_{\alpha/2}$ is given by qnorm(.975). We multiply it with the standard error of the mean sem and get the margin of error.

```
> E = qnorm(.975)*sem; E          # margin of error
[1] 1.2852
```

We then add it up with the sample mean, and find the confidence interval as told.

```
> xbar = mean(height.response)   # sample mean
> xbar + c(-E, E)
[1] 171.10 173.67
```

Answer

Assuming the population standard deviation σ being 9.48, the margin of error for the student height survey at 95% confidence level is 1.2852 centimeters. The confidence interval is between 171.10 and 173.67 centimeters.

Interval Estimate of Population Mean with Known Variance (5)

Alternative Solution

Instead of using the textbook formula, we can apply the ztest function in the TeachingDemos package. It is not a core R package, and must be installed and loaded into the workspace beforehand.

```
> library(TeachingDemos)          # load TeachingDemos package
> z.test(height.response, sd=sigma)

       One Sample z-test

data:  height.response
z = 262.88, n = 209.000, Std. Dev. = 9.480,
Std. Dev. of the sample mean = 0.656, p-value < 2.2e-16
alternative hypothesis: true mean is not equal to 0
95 percent confidence interval:
 171.10 173.67
sample estimates:
mean of height.response
                 172.38
```

Interval Estimate of Population Mean with Unknown Variance

After we found a point estimate of the population mean, we would need a way to quantify its accuracy. Here, we discuss the case where the population variance is not assumed.

Let us denote the 100(1 − α/2) percentile of the Student t distribution with $n-1$ degrees of freedom as $t_{\alpha/2}$. For random samples of sufficiently large size, and with standard deviation s, the end points of the **interval estimate** at (1 − α) confidence level is given as follows:

$$\bar{x} \pm t_{\alpha/2} \frac{s}{\sqrt{n}}$$

Problem

Without assuming the population standard deviation of the student height in survey, find the margin of error and interval estimate at 95% confidence level.

Interval Estimate of Population Mean with Unknown Variance (2)

Solution

We first filter out missing values in survey$Height with the na.omit function, and save it in height.response.

```
> library(MASS)                    # load the MASS package
> height.response = na.omit(survey$Height)
```

Then we compute the sample standard deviation.

```
> n = length(height.response)
> s = sd(height.response)          # sample standard deviation
> SE = s/sqrt(n); SE               # standard error estimate
[1] 0.68117
```

Interval Estimate of Population Mean with Unknown Variance (3)

Since there are two tails of the Student t distribution, the 95% confidence level would imply the 97.5th percentile of the Student t distribution at the upper tail. Therefore, $t_{\alpha/2}$ is given by qt(.975, df=n-1). We multiply it with the standard error estimate SE and get the margin of error.

```
> E = qt(.975, df=n−1)∗SE; E     # margin of error
[1] 1.3429
```

We then add it up with the sample mean, and find the confidence interval.

```
> xbar = mean(height.response)   # sample mean
> xbar + c(−E, E)
[1] 171.04 173.72
```

Answer

Without assumption on the population standard deviation, the margin of error for the student height survey at 95% confidence level is 1.3429 centimeters. The confidence interval is between 171.04 and 173.72 centimeters.

Interval Estimate of Population Mean with Unknown Variance (4)

Alternative Solution

Instead of using the textbook formula, we can apply the t.test function in the built-in stats package.

```
> t.test(height.response)

        One Sample t-test

data:  height.response
t = 253.07, df = 208, p-value < 2.2e-16
alternative hypothesis: true mean is not equal to 0
95 percent confidence interval:
 171.04 173.72
sample estimates:
mean of x
   172.38
```

Sampling Size of Population Mean

The quality of a sample survey can be improved by increasing the sample size. The formula below provide the sample size needed under the requirement of population mean interval estimate at $(1 - \alpha)$ confidence level, margin of error E, and population variance σ^2. Here, $z_{\alpha/2}$ is the $100(1 - \alpha/2)$ percentile of the standard normal distribution.

$$\boxed{n = \frac{(z_{\alpha/2})^2 \sigma^2}{E^2}}$$

Problem

Assume the population standard deviation σ of the student height in survey is 9.48. Find the sample size needed to achieve a 1.2 centimeters margin of error at 95% confidence level.

Sampling Size of Population Mean (2)

Solution

Since there are two tails of the normal distribution, the 95% confidence level would imply the 97.5th percentile of the normal distribution at the upper tail. Therefore, $z_{\alpha/2}$ is given by qnorm(.975).

```
> zstar = qnorm(.975)
> sigma = 9.48
> E = 1.2
> zstar^2 * sigma^2/ E^2
[1] 239.75
```

Answer

Based on the assumption of population standard deviation being 9.48, it needs a sample size of 240 to achieve a 1.2 centimeters margin of error at 95% confidence level.

Point Estimate of Population Proportion

Multiple choice questionnaires in a survey are often used to determine the the proportion of a population with certain characteristic. For example, we can estimate the proportion of female students in the university based on the result in the sample data set survey.

Problem

Find a point estimate of the female student proportion from survey.

Solution

We first filter out missing values in survey$Sex with the na.omit function, and save it in gender.response.

```
> library(MASS)                   # load the MASS package
> gender.response = na.omit(survey$Sex)
> n = length(gender.response)     # valid responses count
```

Point Estimate of Population Proportion (2)

To find out the number of female students, we compare gender.response with the factor 'Female', and compute the sum. Dividing it by n gives the female student proportion in the sample survey.

```
> k = sum(gender.response == "Female")
> pbar = k/n; pbar
[1] 0.5
```

Answer

The point estimate of the female student proportion in survey is 50%.

Interval Estimate of Population Proportion

After we found a point sample estimate of the population proportion, we would need to estimate its confidence interval.

Let us denote the 100(1 −α/2) percentile of the standard normal distribution as $z_{\alpha/2}$. If the samples size n and population proportion p satisfy the condition that $np \geq 5$ and $n(1 - p) \geq 5$, than the end points of the interval estimate at $(1 - \alpha)$ confidence level is defined in terms of the sample proportion as follows.

$$\bar{p} \pm z_{\alpha/2}\sqrt{\frac{\bar{p}(1-\bar{p})}{n}}$$

Problem

Compute the margin of error and estimate interval for the female students proportion in survey at 95% confidence level.

Interval Estimate of Population Proportion (2)

Solution

We first determine the proportion point estimate. Further details can be found in the previous tutorial.

```
> library(MASS)                  # load the MASS package
> gender.response = na.omit(survey$Sex)
> n = length(gender.response)    # valid responses count
> k = sum(gender.response == "Female")
> pbar = k/n; pbar
[1] 0.5
```

Then we estimate the standard error.

```
> SE = sqrt(pbar*(1-pbar)/n); SE     # standard error
[1] 0.032547
```

Interval Estimate of Population Proportion (3)

Since there are two tails of the normal distribution, the 95% confidence level would imply the 97.5^{th} percentile of the normal distribution at the upper tail. Therefore, $z_{\alpha/2}$ is given by qnorm(.975). Hence we multiply it with the standard error estimate SE and compute the margin of error.

```
> E = qnorm(.975)*SE; E                  # margin of error
[1] 0.063791
```

Combining it with the sample proportion, we obtain the confidence interval.

```
> pbar + c(-E, E)
[1] 0.43621 0.56379
```

Answer

At 95% confidence level, between 43.6% and 56.3% of the university students are female, and the margin of error is 6.4%.

Interval Estimate of Population Proportion (4)

Alternative Solution

Instead of using the textbook formula, we can apply the prop.test function in the built-in stats package.

```
> prop.test(k, n)

       1-sample proportions test without continuity
           correction

data:  k out of n, null probability 0.5
X-squared = 0, df = 1, p-value = 1
alternative hypothesis: true p is not equal to 0.5
95 percent confidence interval:
 0.43672 0.56328
sample estimates:
  p
0.5
```

Sampling Size of Population Proportion

The quality of a sample survey can be improved by increasing the sample size. The formula below provide the sample size needed under the requirement of population proportion interval estimate at $(1 - \alpha)$ confidence level, margin of error E, and *planned* proportion estimate p. Here, $z_{\alpha/2}$ is the $100(1 - \alpha/2)$ percentile of the standard normal distribution.

$$\boxed{n = \frac{(z_{\alpha/2})^2 p(1-p)}{E^2}}$$

Problem

Using a 50% planned proportion estimate, find the sample size needed to achieve 5% margin of error for the female student survey at 95% confidence level.

Sampling Size of Population Proportion (2)

Solution

Since there are two tails of the normal distribution, the 95% confidence level would imply the 97.5th percentile of the normal distribution at the upper tail. Therefore, $z_{\alpha/2}$ is given by qnorm(.975).

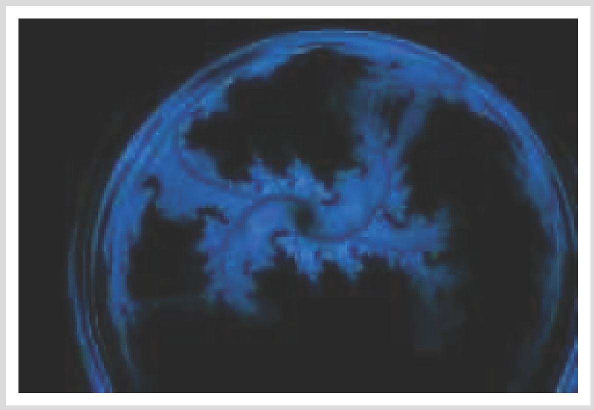
```
> zstar = qnorm(.975)
> p = 0.5
> E = 0.05
> zstar^2 * p * (1-p) / E^2
[1] 384.15
```

Answer

With a planned proportion estimate of 50% at 95% confidence level, it needs a sample size of 385 to achieve a 5% margin of error for the survey of female student proportion.

Hypothesis Testing

Hypothesis Testing



Researchers retain or reject hypothesis based on measurements of observed samples. The decision is often based on a statistical mechanism called **hypothesis testing**. A **type I error** is the mishap of falsely rejecting a null hypothesis when the null hypothesis is true. The probability of committing a type I error is called the **significance level** of the hypothesis testing, and is denoted by the Greek letter α .

In the following tutorials, we demonstrate the procedure of hypothesis testing in R first with the intuitive critical value approach. Then we discuss the popular p-value approach as alternative.

- Lower Tail Test of Population Mean with Known Variance
- Upper Tail Test of Population Mean with Known Variance
- Two-Tailed Test of Population Mean with Known Variance
- Lower Tail Test of Population Mean with Unknown Variance
- Upper Tail Test of Population Mean with Unknown Variance
- Two-Tailed Test of Population Mean with Unknown Variance
- Lower Tail Test of Population Proportion
- Upper Tail Test of Population Proportion
- Two-Tailed Test of Population Proportion

Lower Tail Test of Population Mean with Known Variance

The null hypothesis of the **lower tail test of the population mean** can be expressed as follows:

$$\boxed{\mu \geq \mu_0}$$

where μ_0 is a hypothesized lower bound of the true population mean μ.

Let us define the test statistic z in terms of the sample mean, the sample size and the population standard deviation σ :

$$\boxed{z = \frac{\bar{x} - \mu_0}{\sigma / \sqrt{n}}}$$

Then the null hypothesis of the lower tail test is to be *rejected* if $z \leq -z_\alpha$, where z_α is the $100(1 - \alpha)$ percentile of the standard normal distribution.

Lower Tail Test of Population Mean with Known Variance (2)

Problem

Suppose the manufacturer claims that the mean lifetime of a light bulb is more than 10,000 hours. In a sample of 30 light bulbs, it was found that they only last 9,900 hours on average. Assume the population standard deviation is 120 hours. At .05 significance level, can we reject the claim by the manufacturer?

Solution

The null hypothesis is that $\mu \geq 10000$. We begin with computing the test statistic.

```
> xbar = 9900            # sample mean
> mu0 = 10000            # hypothesized value
> sigma = 120            # population standard deviation
> n = 30                 # sample size
> z = (xbar−mu0)/(sigma/sqrt(n))
> z                      # test statistic
[1] −4.5644
```

Lower Tail Test of Population Mean with Known Variance (3)

We then compute the critical value at .05 significance level.

```
> alpha = .05
> z.alpha = qnorm(1−alpha)
> −z.alpha                # critical value
[1] −1.6449
```

Answer

The test statistic -4.5644 is less than the critical value of -1.6449. Hence, at .05 significance level, we reject the claim that mean lifetime of a light bulb is above 10,000 hours.

Lower Tail Test of Population Mean with Known Variance (4)

Alternative Solution

Instead of using the critical value, we apply the pnorm function to compute the lower tail **p-value** of the test statistic. As it turns out to be less than the .05 significance level, we reject the null hypothesis that $\mu \geq 10000$.

```
> pval = pnorm(z)
```

```
> pval                    # lower tail p−value
[1] 2.5052e−06
```

Upper Tail Test of Population Mean with Known Variance

The null hypothesis of the **upper tail test of the population mean** can be expressed as follows:

$$\mu \le \mu_0$$

where μ_0 is a hypothesized upper bound of the true population mean μ.

Let us define the test statistic z in terms of the sample mean, the sample size and the population standard deviation σ :

$$z = \frac{\bar{x} - \mu_0}{\sigma / \sqrt{n}}$$

Then the null hypothesis of the upper tail test is to be *rejected* if $z \ge z_\alpha$, where z_α is the $100(1-\alpha)$ percentile of the standard normal distribution.

Upper Tail Test of Population Mean with Known Variance (2)

Problem

Suppose the food label on a cookie bag states that there is at most 2 grams of saturated fat in a single cookie. In a sample of 35 cookies, it is found that the mean amount of saturated fat per cookie is 2.1 grams. Assume that the population standard deviation is 0.25 grams. At .05 significance level, can we reject the claim on food label?

Solution

The null hypothesis is that $\mu \leq 2$. We begin with computing the test statistic.

```
> xbar = 2.1                # sample mean
> mu0 = 2                   # hypothesized value
> sigma = 0.25              # population standard deviation
> n = 35                    # sample size
> z = (xbar−mu0)/(sigma/sqrt(n))
> z                         # test statistic
[1] 2.3664
```

Upper Tail Test of Population Mean with Known Variance (3)

We then compute the critical value at .05 significance level.

```
> alpha = .05
> z.alpha = qnorm(1−alpha)
> z.alpha                 # critical value
[1] 1.6449
```

Answer

The test statistic 2.3664 is greater than the critical value of 1.6449. Hence, at .05 significance level, we reject the claim that there is at most 2 grams of saturated fat in a cookie.

Upper Tail Test of Population Mean with Known Variance (4)

Alternative Solution

Instead of using the critical value, we apply the pnorm function to compute the upper tail **p-value** of the test statistic. As it turns out to be less than the .05 significance level, we reject the null hypothesis that $\mu \le 2$.

```
> pval = pnorm(z, lower.tail=FALSE)
```

```
> pval                    # upper tail p−value
[1] 0.0089802
```

Two-Tailed Test of Population Mean with Known Variance

The null hypothesis of the **two-tailed test of the population mean** can be expressed as follows:

$$\mu = \mu_0$$

where μ_0 is a hypothesized value of the true population mean μ.

Let us define the test statistic z in terms of the sample mean, the sample size and the population standard deviation σ :

$$z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$$

Then the null hypothesis of the two-tailed test is to be *rejected* if $z \le -z_{\alpha/2}$ or $z \ge z_{\alpha/2}$, where $z_{\alpha/2}$ is the $100(1 - \alpha/2)$ percentile of the standard normal distribution.

Two-Tailed Test of Population Mean with Known Variance (2)

Problem

Suppose the mean weight of King Penguins found in an Antarctic colony last year was 15.4 kg. In a sample of 35 penguins same time this year in the same colony, the mean penguin weight is 14.6 kg. Assume the population standard deviation is 2.5 kg. At .05 significance level, can we reject the null hypothesis that the mean penguin weight does not differ from last year?

Solution

The null hypothesis is that $\mu = 15.4$. We begin with computing the test statistic.

```
> xbar = 14.6            # sample mean
> mu0 = 15.4             # hypothesized value
> sigma = 2.5            # population standard deviation
> n = 35                 # sample size
> z = (xbar−mu0)/(sigma/sqrt(n))
> z                      # test statistic
[1] −1.8931
```

Two-Tailed Test of Population Mean with Known Variance (3)

We then compute the critical values at .05 significance level.

```
> alpha = .05
> z.half.alpha = qnorm(1−alpha/2)
> c(−z.half.alpha, z.half.alpha)
[1] −1.9600  1.9600
```

Answer

The test statistic -1.8931 lies between the critical values -1.9600 and 1.9600. Hence, at .05 significance level, we do *not* reject the null hypothesis that the mean penguin weight does not differ from last year.

Two-Tailed Test of Population Mean with Known Variance (4)

Alternative Solution

Instead of using the critical value, we apply the pnorm function to compute the two-tailed **p-value** of the test statistic. It doubles the *lower* tail p-value as the sample mean is *less* than the hypothesized value. Since it turns out to be greater than the .05 significance level, we do *not* reject the null hypothesis that $\mu = 15.4$.

```
> pval = 2 * pnorm(z)    # lower tail
> pval                   # two−tailed p−value
[1] 0.058339
```

Lower Tail Test of Population Mean with Unknown Variance

The null hypothesis of the **lower tail test of the population mean** can be expressed as follows:

$$\boxed{\mu \geq \mu_0}$$

where μ_0 is a hypothesized lower bound of the true population mean μ.

Let us define the test statistic t in terms of the sample mean, the sample size and the sample standard deviation s :

$$\boxed{t = \frac{\bar{x} - \mu_0}{s / \sqrt{n}}}$$

Then the null hypothesis of the lower tail test is to be *rejected* if $t \leq -t_\alpha$, where t_α is the $100(1 - \alpha)$ percentile of the Student t distribution with $n - 1$ degrees of freedom.

Lower Tail Test of Population Mean with Unknown Variance (2)

Problem

Suppose the manufacturer claims that the mean lifetime of a light bulb is more than 10,000 hours. In a sample of 30 light bulbs, it was found that they only last 9,900 hours on average. Assume the sample standard deviation is 125 hours. At .05 significance level, can we reject the claim by the manufacturer?

Solution

The null hypothesis is that $\mu \geq 10000$. We begin with computing the test statistic.

```
> xbar = 9900            # sample mean
> mu0 = 10000            # hypothesized value
> s = 125                # sample standard deviation
> n = 30                 # sample size
> t = (xbar−mu0)/(s/sqrt(n))
> t                      # test statistic
[1] −4.3818
```

Lower Tail Test of Population Mean with Unknown Variance (3)

We then compute the critical value at .05 significance level.

```
> alpha = .05
> t.alpha = qt(1−alpha, df=n−1)
> −t.alpha                # critical value
[1] −1.6991
```

Answer

The test statistic -4.3818 is less than the critical value of -1.6991. Hence, at .05 significance level, we can reject the claim that mean lifetime of a light bulb is above 10,000 hours.

Lower Tail Test of Population Mean with Unknown Variance (4)

Alternative Solution

Instead of using the critical value, we apply the pt function to compute the lower tail **p-value** of the test statistic. As it turns out to be less than the .05 significance level, we reject the null hypothesis that $\mu \geq 10000$.

```
> pval = pt(t, df=n-1)
> pval                     # lower tail p-value
[1] 7.035e-05
```

Upper Tail Test of Population Mean with Unknown Variance

The null hypothesis of the **upper tail test of the population mean** can be expressed as follows:

$$\boxed{\mu \le \mu_0}$$

where μ_0 is a hypothesized upper bound of the true population mean μ.

Let us define the test statistic t in terms of the sample mean, the sample size and the sample standard deviation s :

$$\boxed{t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}}$$

Then the null hypothesis of the upper tail test is to be *rejected* if $t \ge t_\alpha$, where t_α is the $100(1-\alpha)$ percentile of the Student t distribution with $n-1$ degrees of freedom.

Upper Tail Test of Population Mean with Unknown Variance (2)

Problem

Suppose the food label on a cookie bag states that there is at most 2 grams of saturated fat in a single cookie. In a sample of 35 cookies, it is found that the mean amount of saturated fat per cookie is 2.1 grams. Assume that the sample standard deviation is 0.3 gram. At .05 significance level, can we reject the claim on food label?

Solution

The null hypothesis is that $\mu \leq 2$. We begin with computing the test statistic.

```
> xbar = 2.1               # sample mean
> mu0 = 2                  # hypothesized value
> s = 0.3                  # sample standard deviation
> n = 35                   # sample size
> t = (xbar−mu0)/(s/sqrt(n))
> t                        # test statistic
[1] 1.9720
```

Upper Tail Test of Population Mean with Unknown Variance (3)

We then compute the critical value at .05 significance level.

```
> alpha = .05
> t.alpha = qt(1−alpha, df=n−1)
> t.alpha                # critical value
[1] 1.6991
```

Answer

The test statistic 1.9720 is greater than the critical value of 1.6991. Hence, at .05 significance level, we can reject the claim that there is at most 2 grams of saturated fat in a cookie.

Upper Tail Test of Population Mean with Unknown Variance (3)

Alternative Solution

Instead of using the critical value, we apply the pt function to compute the upper tail **p-value** of the test statistic. As it turns out to be less than the .05 significance level, we reject the null hypothesis that $\mu \leq 2$.

```
> pval = pt(t, df=n−1, lower.tail=FALSE)
> pval                          # upper tail p−value
[1] 0.028393
```

Two-Tailed Test of Population Mean with Unknown Variance

The null hypothesis of the **two-tailed test of the population mean** can be expressed as follows:

$$\mu = \mu_0$$

where μ_0 is a hypothesized value of the true population mean μ.

Let us define the test statistic t in terms of the sample mean, the sample size and the sample standard deviation s :

$$t = \frac{\bar{x} - \mu_0}{s / \sqrt{n}}$$

Then the null hypothesis of the two-tailed test is to be *rejected* if $t \le -t_{\alpha/2}$ or $t \ge t_{\alpha/2}$, where $t_{\alpha/2}$ is the $100(1 - \alpha)$ percentile of the Student t distribution with $n - 1$ degrees of freedom.

Two-Tailed Test of Population Mean with Unknown Variance (2)

Problem

Suppose the mean weight of King Penguins found in an Antarctic colony last year was 15.4 kg. In a sample of 35 penguins same time this year in the same colony, the mean penguin weight is 14.6 kg. Assume the sample standard deviation is 2.5 kg. At .05 significance level, can we reject the null hypothesis that the mean penguin weight does not differ from last year?

Solution

The null hypothesis is that $\mu = 15.4$. We begin with computing the test statistic.

```
> xbar = 14.6              # sample mean
> mu0 = 15.4               # hypothesized value
> s = 2.5                  # sample standard deviation
> n = 35                   # sample size
> t = (xbar−mu0)/(s/sqrt(n))
> t                        # test statistic
[1] −1.8931
```

Two-Tailed Test of Population Mean with Unknown Variance (3)

We then compute the critical values at .05 significance level.

```
> alpha = .05
> t.half.alpha = qt(1−alpha/2, df=n−1)
> c(−t.half.alpha, t.half.alpha)
[1] −2.0322  2.0322
```

Answer

The test statistic -1.8931 lies between the critical values -2.0322, and 2.0322. Hence, at .05 significance level, we do *not* reject the null hypothesis that the mean penguin weight does not differ from last year.

Two-Tailed Test of Population Mean with Unknown Variance (4)

Alternative Solution

Instead of using the critical value, we apply the pt function to compute the two-tailed **p-value** of the test statistic. It doubles the *lower* tail p-value as the sample mean is *less* than the hypothesized value. Since it turns out to be greater than the .05 significance level, we do *not* reject the null hypothesis that $\mu = 15.4$.

```
> pval = 2 * pt(t, df=n-1)  # lower tail
> pval                      # two-tailed p-value
[1] 0.066876
```

Lower Tail Test of Population Proportion

The null hypothesis of the **lower tail test about population proportion** can be expressed as follows:

$$\boxed{p \geq p_0}$$

where p_0 is a hypothesized lower bound of the true population proportion p.

Let us define the test statistic z in terms of the sample proportion and the sample size:

$$\boxed{z = \frac{\bar{p} - p_0}{\sqrt{p_0(1 - p_0)/n}}}$$

Then the null hypothesis of the lower tail test is to be *rejected* if $z \leq -z_\alpha$, where z_α is the $100(1 - \alpha)$ percentile of the standard normal distribution.

Lower Tail Test of Population Proportion (2)

Problem

Suppose 60% of citizens voted in last election. 85 out of 148 people in a telephone survey said that they voted in current election. At 0.5 significance level, can we reject the null hypothesis that the proportion of voters in the population is above 60% this year?

Solution

The null hypothesis is that $p \geq 0.6$. We begin with computing the test statistic.

```
> pbar = 85/148          # sample proportion
> p0 = .6                # hypothesized value
> n = 148                # sample size
> z = (pbar-p0)/sqrt(p0*(1-p0)/n)
> z                      # test statistic
[1] -0.6376
```

Lower Tail Test of Population Proportion (3)

We then compute the critical value at .05 significance level.

```
> alpha = .05
> z.alpha = qnorm(1−alpha)
> −z.alpha                # critical value
[1] −1.6449
```

Answer

The test statistic -0.6376 is *not* less than the critical value of -1.6449. Hence, at .05 significance level, we do *not* reject the null hypothesis that the proportion of voters in the population is above 60% this year.

Lower Tail Test of Population Proportion (4)

Alternative Solution 1

Instead of using the critical value, we apply the pnorm function to compute the lower tail **p-value** of the test statistic. As it turns out to be greater than the .05 significance level, we do not reject the null hypothesis that $p \geq 0.6$.

```
> pval = pnorm(z)
> pval                    # lower tail p−value
[1] 0.26187
```

Lower Tail Test of Population Proportion (5)

Alternative Solution 2

We apply the prop.test function to compute the p-value directly. The Yates continuity correction is disabled for pedagogical reasons.

```
> prop.test(85, 148, p=.6, alt="less", correct=FALSE)

        1-sample proportions test without continuity
        correction

data:  85 out of 148, null probability 0.6
X-squared = 0.4065, df = 1, p-value = 0.2619
alternative hypothesis: true p is less than 0.6
95 percent confidence interval:
 0.0000 0.63925
sample estimates:
      p
0.57432
```

Lower Tail Test of Population Proportion (5)

Alternative Solution 2

We apply the prop.test function to compute the p-value directly. The Yates continuity correction is disabled for pedagogical reasons.

```
> prop.test(85, 148, p=.6, alt="less", correct=FALSE)

        1-sample proportions test without continuity
        correction

data:  85 out of 148, null probability 0.6
X-squared = 0.4065, df = 1, p-value = 0.2619
alternative hypothesis: true p is less than 0.6
95 percent confidence interval:
 0.0000 0.63925
sample estimates:
      p
0.57432
```

Upper Tail Test of Population Proportion

The null hypothesis of the **upper tail test about population proportion** can be expressed as follows:

$$\boxed{p \leq p_0}$$

where p_0 is a hypothesized upper bound of the true population proportion p.

Let us define the test statistic z in terms of the sample proportion and the sample size:

$$\boxed{z = \frac{\bar{p} - p_0}{\sqrt{p_0(1 - p_0)/n}}}$$

Then the null hypothesis of the upper tail test is to be *rejected* if $z \geq z_\alpha$, where z_α is the $100(1 - \alpha)$ percentile of the standard normal distribution.

Upper Tail Test of Population Proportion (2)

Problem

Suppose that 12% of apples harvested in an orchard last year was rotten. 30 out of 214 apples in a harvest sample this year turns out to be rotten. At .05 significance level, can we reject the null hypothesis that the proportion of rotten apples in harvest stays below 12% this year?

Solution

The null hypothesis is that $p \le 0.12$. We begin with computing the test statistic.

```
> pbar = 30/214          # sample proportion
> p0 = .12               # hypothesized value
> n = 214                # sample size
> z = (pbar-p0)/sqrt(p0*(1-p0)/n)
> z                      # test statistic
[1] 0.90875
```

Upper Tail Test of Population Proportion (3)

We then compute the critical value at .05 significance level.

```
> alpha = .05
> z.alpha = qnorm(1−alpha)
> z.alpha                # critical value
[1] 1.6449
```

Answer

The test statistic 0.90875 is *not* greater than the critical value of 1.6449. Hence, at .05 significance level, we do *not* reject the null hypothesis that the proportion of rotten apples in harvest stays below 12% this year.

Upper Tail Test of Population Proportion (4)

Alternative Solution 1

Instead of using the critical value, we apply the pnorm function to compute the upper tail **p-value** of the test statistic. As it turns out to be greater than the .05 significance level, we do not reject the null hypothesis that $p \leq 0.12$.

```
> pval = pnorm(z, lower.tail=FALSE)
> pval                    # upper tail p−value
[1] 0.18174
```

Upper Tail Test of Population Proportion (5)

Alternative Solution 2

We apply the prop.test function to compute the p-value directly. The Yates continuity correction is disabled for pedagogical reasons.

```
> prop.test(30, 214, p=.12, alt="greater", correct=FALSE)

        1-sample proportions test without continuity
        correction

data:  30 out of 214, null probability 0.12
X-squared = 0.8258, df = 1, p-value = 0.1817
alternative hypothesis: true p is greater than 0.12
95 percent confidence interval:
 0.10563 1.00000
sample estimates:
      p
0.14019
```

Upper Tail Test of Population Proportion (5)

Alternative Solution 2

We apply the prop.test function to compute the p-value directly. The Yates continuity correction is disabled for pedagogical reasons.

```
> prop.test(30, 214, p=.12, alt="greater", correct=FALSE)

        1-sample proportions test without continuity
        correction

data:  30 out of 214, null probability 0.12
X-squared = 0.8258, df = 1, p-value = 0.1817
alternative hypothesis: true p is greater than 0.12
95 percent confidence interval:
 0.10563 1.00000
sample estimates:
      p
0.14019
```

Two-Tailed Test of Population Proportion

The null hypothesis of the **two-tailed test about population proportion** can be expressed as follows:

$$\boxed{p = p_0}$$

where p_0 is a hypothesized value of the true population proportion p.

Let us define the test statistic z in terms of the sample proportion and the sample size:

$$\boxed{z = \frac{\bar{p} - p_0}{\sqrt{p_0(1-p_0)/n}}}$$

Then the null hypothesis of the two-tailed test is to be *rejected* if $z \le -z_{\alpha/2}$ or $z \ge z_{\alpha/2}$, where $z_{\alpha/2}$ is the $100(1 - \alpha)$ percentile of the standard normal distribution.

Two-Tailed Test of Population Proportion (2)

Problem

Suppose a coin toss turns up 12 heads out of 20 trials. At .05 significance level, can one reject the null hypothesis that the coin toss is fair?

Solution

The null hypothesis is that $p = 0.5$. We begin with computing the test statistic.

```
> pbar = 12/20           # sample proportion
> p0 = .5                # hypothesized value
> n = 20                 # sample size
> z = (pbar-p0)/sqrt(p0*(1-p0)/n)
> z                      # test statistic
[1] 0.89443
```

Two-Tailed Test of Population Proportion (3)

We then compute the critical values at .05 significance level.

```
> alpha = .05
> z.half.alpha = qnorm(1−alpha/2)
> c(−z.half.alpha, z.half.alpha)
[1] −1.9600  1.9600
```

Answer

The test statistic 0.89443 lies between the critical values -1.9600 and 1.9600. Hence, at .05 significance level, we do *not* reject the null hypothesis that the coin toss is fair.

Two-Tailed Test of Population Proportion (4)

Alternative Solution 1

Instead of using the critical value, we apply the pnorm function to compute the two-tailed **p-value** of the test statistic. It doubles the *upper* tail p-value as the sample proportion is *greater* than the hypothesized value. Since it turns out to be greater than the .05 significance level, we do not reject the null hypothesis that $p = 0.5$.

```
> pval = 2 * pnorm(z, lower.tail=FALSE)  # upper tail
> pval                   # two−tailed p−value
[1] 0.37109
```

Two-Tailed Test of Population Proportion (5)

Alternative Solution 2

We apply the prop.test function to compute the p-value directly. The Yates continuity correction is disabled for pedagogical reasons.

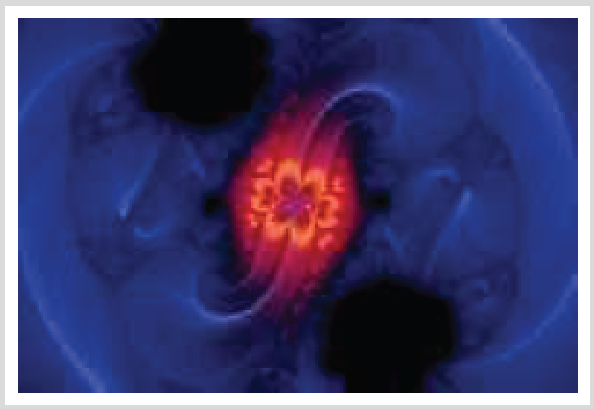
```
> prop.test(12, 20, p=0.5, correct=FALSE)

        1-sample proportions test without continuity
        correction

data:  12 out of 20, null probability 0.5
X-squared = 0.8, df = 1, p-value = 0.3711
alternative hypothesis: true p is not equal to 0.5
95 percent confidence interval:
  0.38658 0.78119
sample estimates:
  p
0.6
```

Type II Error

Type II Error



In hypothesis testing, a **type II error** is due to a failure of rejecting an invalid null hypothesis. The probability of avoiding a type II error is called the **power** of the hypothesis test, and is denoted by the quantity $1 - \beta$.

In the following tutorials, we demonstrate how to compute the power of a hypothesis test based on scenarios from our previous discussions on hypothesis testing. The approach is based on a parametric estimate of the region where the null hypothesis would not be rejected. The probability of a type II error is then derived based on a hypothetical true value.

- Type II Error in Lower Tail Test of Population Mean with Known Variance
- Type II Error in Upper Tail Test of Population Mean with Known Variance
- Type II Error in Two-Tailed Test of Population Mean with Known Variance
- Type II Error in Lower Tail Test of Population Mean with Unknown Variance
- Type II Error in Upper Tail Test of Population Mean with Unknown Variance
- Type II Error in Two-Tailed Test of Population Mean with Unknown Variance

Type II Error in Lower Tail Test of Population Mean with Known Variance

In a lower tail test of the population mean, the null hypothesis claims that the true population mean μ is greater than a given hypothetical value μ_0.

$$\mu \geq \mu_0$$

A **type II error** occurs if the hypothesis test based on a random sample fails to reject the null hypothesis even when the true population mean μ is in fact less than μ_0.

Assume that the population has a known variance σ^2. By the Central Limit Theorem, the population of all possible means of samples of sufficiently large size n approximately follows the normal distribution. Hence we can compute the range of sample means for which the null hypothesis will *not* be rejected, and then obtain an estimate of the probability of type II error.

We demonstrate the procedure with the following:

Type II Error in Lower Tail Test of Population Mean with Known Variance (2)

Problem

Suppose the manufacturer claims that the mean lifetime of a light bulb is more than 10,000 hours. Assume actual mean light bulb lifetime is 9,950 hours and the population standard deviation is 120 hours. At .05 significance level, what is the probability of having type II error for a sample size of 30 light bulb?

Solution

We begin with computing the standard deviation of the mean, sem.

```
> n = 30                # sample size
> sigma = 120           # population standard deviation
> sem = sigma/sqrt(n); sem   # standard error
[1] 21.909
```

Type II Error in Lower Tail Test of Population Mean with Known Variance (3)

We next compute the lower bound of sample means for which the null hypothesis $\mu \geq 10000$ would not be rejected.

```
> alpha = .05              # significance level
> mu0 = 10000              # hypothetical lower bound
> q = qnorm(alpha, mean=mu0, sd=sem); q
[1] 9964
```

Therefore, so long as the sample mean is greater than 9964 in a hypothesis test, the null hypothesis will not be rejected. Since we assume that the actual population mean is 9950, we can compute the probability of the sample mean above 9964, and thus found the probability of type II error.

```
> mu = 9950                # assumed actual mean
> pnorm(q, mean=mu, sd=sem, lower.tail=FALSE)
[1] 0.26196
```

Type II Error in Lower Tail Test of Population Mean with Known Variance (4)

Answer

If the light bulbs sample size is 30, the actual mean light bulb lifetime is 9,950 hours and the population standard deviation is 120 hours, then the probability of type II error for testing the null hypothesis $\mu \geq 10000$ at .05 significance level is 26.2%, and the power of the hypothesis test is 73.8%.

Exercise

Under same assumptions as above, if the actual mean light bulb lifetime is 9,965 hours, what is the probability of type II error at .05 significance level? What is the power of the hypothesis test?

Type II Error in Upper Tail Test of Population Mean with Known Variance

In a upper tail test of the population mean, the null hypothesis claims that the true population mean μ is less than a given hypothetical value μ_0.

$$\boxed{\mu \le \mu_0}$$

A **type II error** occurs if the hypothesis test based on a random sample fails to reject the null hypothesis even when the true population mean μ is in fact greater than μ_0.

Assume that the population has a known variance σ^2. By the Central Limit Theorem, the population of all possible means of samples of sufficiently large size n approximately follows the normal distribution. Hence we can compute the range of sample means for which the null hypothesis will *not* be rejected, and then obtain an estimate of the probability of type II error.

We demonstrate the procedure with the following:

Type II Error in Upper Tail Test of Population Mean with Known Variance (2)

Problem

Suppose the food label on a cookie bag states that there is at most 2 grams of saturated fat in a single cookie. Assume the actual mean amount of saturated fat per cookie is 2.09 grams, and the population standard deviation is 0.25 grams. At .05 significance level, what is the probability of having type II error for a sample size of 35 cookies?

Solution

We begin with computing the standard deviation of the mean, sem.

```
> n = 35                    # sample size
> sigma = 0.25              # population standard deviation
> sem = sigma/sqrt(n); sem   # standard error
[1] 0.042258
```

Type II Error in Upper Tail Test of Population Mean with Known Variance (3)

We next compute the upper bound of sample means for which the null hypothesis $\mu \le 2$ would not be rejected.

```
> alpha = .05              # significance level
> mu0 = 2                  # hypothetical upper bound
> q = qnorm(alpha, mean=mu0, sd=sem, lower.tail=FALSE); q
[1] 2.0695
```

Therefore, so long as the sample mean is less than 2.0695 in a hypothesis test, the null hypothesis will not be rejected. Since we assume that the actual population mean is 2.09, we can compute the probability of the sample mean below 2.0695, and thus found the probability of type II error.

```
> mu = 2.09                # assumed actual mean
> pnorm(q, mean=mu, sd=sem)
[1] 0.31386
```

Type II Error in Upper Tail Test of Population Mean with Known Variance (4)

Answer

If the cookies sample size is 35, the actual mean amount of saturated fat per cookie is 2.09 grams and the population standard deviation is 0.25 grams, then the probability of type II error for testing the null hypothesis $\mu \leq 2$ at .05 significance level is 31.4%, and the power of the hypothesis test is 68.6%.

Exercise

Under same assumptions as above, if the actual mean amount of saturated fat per cookie is 2.075 grams, what is the probability of type II errors? What is the power of the hypothesis test?

Type II Error in Two-Tailed Test of Population Mean with Known Variance

In a two-tailed test of the population mean, the null hypothesis claims that the true population mean μ is equal to a given hypothetical value μ_0.

$$\boxed{\mu = \mu_0}$$

A **type II error** occurs if the hypothesis test based on a random sample fails to reject the null hypothesis even when the true population mean μ is in fact different from μ_0.

Assume that the population has a known variance σ^2. By the Central Limit Theorem, the population of all possible means of samples of sufficiently large size n approximately follows the normal distribution. Hence we can compute the range of sample means for which the null hypothesis will *not* be rejected, and then obtain an estimate of the probability of type II error.

We demonstrate the procedure with the following:

Type II Error in Two-Tailed Test of Population Mean with Known Variance (2)

Problem

Suppose the mean weight of King Penguins found in an Antarctic colony last year was 15.4 kg. Assume the actual mean population weight is 15.1 kg, and the population standard deviation is 2.5 kg. At .05 significance level, what is the probability of having type II error for a sample size of 35 penguins?

Solution

We begin with computing the standard deviation of the mean, sem.

```
> n = 35                    # sample size
> sigma = 2.5               # population standard deviation
> sem = sigma/sqrt(n); sem   # standard error
[1] 0.42258
```

Type II Error in Two-Tailed Test of Population Mean with Known Variance (3)

We next compute the lower and upper bounds of sample means for which the null hypothesis $\mu = 15.4$ would not be rejected.

```
> alpha = .05             # significance level
> mu0 = 15.4              # hypothetical mean
> I = c(alpha/2, 1-alpha/2)
> q = qnorm(I, mean=mu0, sd=sem); q
[1] 14.572 16.228
```

Therefore, so long as the sample mean is between 14.572 and 16.228 in a hypothesis test, the null hypothesis will not be rejected. Since we assume that the actual population mean is 15.1, we can compute the lower tail probabilities of both end points.

```
> mu = 15.1               # assumed actual mean
> p = pnorm(q, mean=mu, sd=sem); p
[1] 0.10564 0.99621
```

Type II Error in Two-Tailed Test of Population Mean with Known Variance (4)

Finally, the probability of type II error is the probability between the two end points.

```
> diff(p)                 # p[2]-p[1]
[1] 0.89056
```

Answer

If the penguin sample size is 35, the actual mean population weight is 15.1 kg and the population standard deviation is 2.5 kg, then the probability of type II error for testing the null hypothesis $\mu = 15.4$ at .05 significance level is 89.1%, and the power of the hypothesis test is 10.9%.

Exercise

Under same assumptions as above, if actual mean population weight is 14.9 kg, what is the probability of type II errors? What is the power of the hypothesis test?

Type II Error in Lower Tail Test of Population Mean with Unknown Variance

In a lower tail test of the population mean, the null hypothesis claims that the true population mean μ is greater than a given hypothetical value μ_0.

$$\boxed{\mu \geq \mu_0}$$

A **type II error** occurs if the hypothesis test based on a random sample fails to reject the null hypothesis even when the true population mean μ is in fact less than μ_0.

Let s^2 be the sample variance. For sufficiently large n, the population of the following statistics of all possible samples of size n is approximately a Student t distribution with n - 1 degrees of freedom.

$$\boxed{\frac{\bar{x} - \mu}{s / \sqrt{n}}}$$

This allows us to compute the range of sample means for which the null hypothesis will *not* be rejected, and to obtain the probability of type II error. We demonstrate the procedure with the following:

Type II Error in Lower Tail Test of Population Mean with Unknown Variance (2)

Problem

Suppose the manufacturer claims that the mean lifetime of a light bulb is more than 10,000 hours. Assume in a random sample of 30 light bulbs, the standard deviation of the lifetime is 125 hours. If actual mean light bulb lifetime is 9,950 hours, what is the probability of type II error for a hypothesis test at .05 significance level?

Solution

We begin with computing the standard error estimate, SE.

```
> n = 30                    # sample size
> s = 125                   # sample standard deviation
> SE = s/sqrt(n); SE        # standard error estimate
[1] 22.822
```

Type II Error in Lower Tail Test of Population Mean with Unknown Variance (3)

We next compute the lower bound of sample means for which the null hypothesis $\mu \geq 10000$ would not be rejected.

```
> alpha = .05            # significance level
> mu0 = 10000            # hypothetical lower bound
> q = mu0 + qt(alpha, df=n-1) * SE; q
[1] 9961.2
```

Therefore, so long as the sample mean is greater than 9961.2 in a hypothesis test, the null hypothesis will not be rejected. Since we assume that the actual population mean is 9950, we can compute the probability of the sample mean above 9961.2, and thus found the probability of type II error.

```
> mu = 9950              # assumed actual mean
> pt((q - mu)/SE, df=n-1, lower.tail=FALSE)
[1] 0.31329
```

Type II Error in Lower Tail Test of Population Mean with Unknown Variance (4)

Answer

If the light bulbs sample size is 30, the sample standard variance is 125 hours and the actual mean light bulb lifetime is 9,950 hours, then the probability of type II error for testing the null hypothesis $\mu \geq 10000$ at .05 significance level is 31.3%, and the power of the hypothesis test is 68.7%.

Exercise

Under same assumptions as above, if the actual mean light bulb lifetime is 9,965 hours, what is the probability of type II errors? What is the power of the hypothesis test?

Type II Error in Upper Tail Test of Population Mean with Unknown Variance

In a upper tail test of the population mean, the null hypothesis claims that the true population mean μ is less than a given hypothetical value μ_0.

$$\mu \le \mu_0$$

A **type II error** occurs if the hypothesis test based on a random sample fails to reject the null hypothesis even when the true population mean μ is in fact greater than μ_0.

Let s^2 be the sample variance. For sufficiently large n, the population of the following statistics of all possible samples of size n is approximately a Student t distribution with n - 1 degrees of freedom.

$$\frac{\bar{x} - \mu}{s/\sqrt{n}}$$

This allows us to compute the range of sample means for which the null hypothesis will *not* be rejected, and to obtain the probability of type II error. We demonstrate the procedure with the following:

Type II Error in Upper Tail Test of Population Mean with Unknown Variance (2)

Problem

Suppose the food label on a cookie bag states that there is at most 2 grams of saturated fat in a single cookie. Assume in a random sample of 35 cookies, the standard deviation of saturated fat is 0.3 grams. If actual mean amount of saturated fat per cookie is 2.09 grams, what is the probability of type II error for a hypothesis test at .05 significance level?

Solution

We begin with computing the standard error estimate, SE.

```
> n = 35                    # sample size
> s = 0.3                   # sample standard deviation
> SE = s/sqrt(n); SE        # standard error estimate
[1] 0.050709
```

Type II Error in Upper Tail Test of Population Mean with Unknown Variance (3)

We next compute the upper bound of sample means for which the null hypothesis $\mu \leq 2$ would not be rejected.

```
> alpha = .05              # significance level
> mu0 = 2                  # hypothetical upper bound
> q = mu0 + qt(alpha, df=n-1, lower.tail=FALSE) * SE; q
[1] 2.0857
```

Therefore, so long as the sample mean is less than 2.0857 in a hypothesis test, the null hypothesis will not be rejected. Since we assume that the actual population mean is 2.09, we can compute the probability of the sample mean below 2.0857, and thus found the probability of type II error.

```
> mu = 2.09                # assumed actual mean
> pt((q - mu)/SE, df=n-1)
[1] 0.46681
```

Type II Error in Upper Tail Test of Population Mean with Unknown Variance (4)

Answer

If the cookies sample size is 35, the sample standard deviation of saturated fat per cookie is 0.3 grams and the actual mean amount of saturated fat per cookie is 2.09 grams, then the probability of type II error for testing the null hypothesis $\mu \leq 2$ at .05 significance level is 46.7%, and the power of the hypothesis test is 53.3%.

Exercise

Under same assumptions as above, if the actual mean saturated fat per cookie is 2.075 grams, what is the probability of type II errors? What is the power of the hypothesis test?

Type II Error in Two-Tailed Test of Population Mean with Unknown Variance

In a two-tailed test of the population mean, the null hypothesis claims that the true population mean μ is equal to a given hypothetical value μ_0.

$$\mu = \mu_0$$

A **type II error** occurs if the hypothesis test based on a random sample fails to reject the null hypothesis even when the true population mean μ is in fact different from μ_0.

Let s^2 be the sample variance. For sufficiently large n, the population of the following statistics of all possible samples of size n is approximately a Student t distribution with n - 1 degrees of freedom.

$$\frac{\bar{x} - \mu}{s/\sqrt{n}}$$

This allows us to compute the range of sample means for which the null hypothesis will *not* be rejected, and to obtain the probability of type II error. We demonstrate the procedure with the following:

Type II Error in Two-Tailed Test of Population Mean with Unknown Variance (2)

Problem

Suppose the mean weight of King Penguins found in an Antarctic colony last year was 15.4 kg. Assume in a random sample 35 penguins, the standard deviation of the weight is 2.5 kg. If actual mean penguin weight is 15.1 kg, what is the probability of type II error for a hypothesis test at .05 significance level?

Solution

We begin with computing the standard error estimate, SE.

```
> n = 35                    # sample size
> s = 2.5                   # sample standard deviation
> SE = s/sqrt(n); SE        # standard error estimate
[1] 0.42258
```

Type II Error in Two-Tailed Test of Population Mean with Unknown Variance (2)

We next compute the lower and upper bounds of sample means for which the null hypothesis $\mu = 15.4$ would not be rejected.

```
> alpha = .05              # significance level
> mu0 = 15.4               # hypothetical mean
> I = c(alpha/2, 1-alpha/2)
> q = mu0 + qt(I, df=n-1) * SE; q
[1] 14.541 16.259
```

Therefore, so long as the sample mean is between 14.541 and 16.259 in a hypothesis test, the null hypothesis will not be rejected. Since we assume that the actual population mean is 15.1, we can compute the lower tail probabilities of both end points.

```
> mu = 15.1                # assumed actual mean
> p = pt((q - mu)/SE, df=n-1); p
[1] 0.097445 0.995168
```

Type II Error in Two-Tailed Test of Population Mean with Unknown Variance (3)

Finally, the probability of type II error is the probability between the two end points.

```
> diff(p)                 # p[2]-p[1]
[1] 0.89772
```

Answer

If the penguin sample size is 35, the sample standard deviation of penguin weight is 2.5 kg and the actual mean population weight is 15.1 kg, then the probability of type II error for testing the null hypothesis $\mu = 15.4$ at .05 significance level is 89.8%, and the power of the hypothesis test is 10.2%.

Exercise

Under same assumptions as above, if actual mean population weight is 14.9 kg, what is the probability of type II errors? What is the power of the hypothesis test?

Acknowlegments

Some parts of the material in this presentation is taken from http://www.r-tutor.com/

Deo materijala je preuzet sa sajta http://www.e-statistika.rs