Parts of the solutions for the 3rd and 4th homework

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December 1, 2014

Note: I only give the solutions for those questions that some of you might have problems. If you have other problems, you can ask me directly.

Homework 5: Linear Regression revisited
1(b).
Describe two interpretation of this error functions in terms of i) replicated measurements.
In the weighted linear regression, less weight is given to the less precise measurements and more weight to more precise measurements when estimating the unknown parameters in the model. So if the replicated measurements are not precise, we should put less weight on those measurements.

Describe two interpretation of this error functions in terms of ii) a data-dependent noise-variance.
The standard statistical assumption for linear regression:

1. The mean of ϵ_i is 0 for all i;
2. The variance of ϵ_i is constant for all i, namely $sigma^2$;
3. ϵ_i and ϵ_j are independent of each other for all $i \neq j$.

Then what about when the variance of ϵ_i is not constant for all i. We can use weighted linear regression to deal with non-constant variance, where large w_i means that the i_{th} observation is of high quality/importance; more specially, we assume $Var(\epsilon_i) = \frac{\sigma^2}{w_i}$.

Homework 4: Gaussian random variables
1(b).
Hint: using the conclusion from 'Homework 3: Probability theory 1(e)' that

$$
\begin{aligned}
&E(E(X|Y)) = E(X);\\
&Var(X) = Var(E(X|Y)) + E(Var(X|Y)).
\end{aligned}
\tag{1}
$$

Since both μ and $x|\mu$ are Gaussian distributed, x is also Gaussian distributed (a good property of Gaussian distribution).Therefore, according to equation 1, we can derive the mean and variance of x.

$$
\begin{aligned}
E(x) &= E(E(x|\mu))\\
&= E(\mu)\\
&= 0;\\
Var(x) &= Var(E(x|\mu)) + E(Var(x|\mu))\\
&= Var(\mu) + E(1/\beta)\\
&= 1/\alpha + 1/\beta;
\end{aligned}
$$

So $x \sim \mathcal{N}(0, 1/\beta + 1/\alpha)$

1(c).

$$
\begin{aligned}
p(x) &= p_1(x) \cdot p_2(x)\\
&= \mathcal{N}(x; \mu_1, 1/\beta_1) \cdot \mathcal{N}(x; \mu_2, 1/\beta_2)\\
&= \frac{1}{Z} \cdot \mathcal{N}(x; \frac{\beta_1 * \mu_1 + \beta_2 * \mu_2}{\beta}, \frac{1}{\beta}),
\end{aligned}
$$

where

$$
Z = \sqrt{\frac{2\pi\beta}{\beta_1\beta_2}} \exp\big(\frac{1}{2}\frac{\beta_1\beta_2(\mu_2 - \mu_1)^2}{\beta}\big)
$$

2(a).Calculate the posterior distribution of the location of the target.

$$
\begin{aligned}
p(\mu|x_v, x_a) &= p(x_v, x_a|\mu)p(\mu)\\
&\text{the noise in the two modalities is independent}\\
&= p(x_v|\mu)p(x_a|\mu)p(\mu)\\
&= \mathcal{N}(x_v; \mu, \sigma_v^2) \cdot \mathcal{N}(x_a; \mu, \sigma_a^2) \cdot \mathcal{N}(\mu; \mu_0, \sigma_0^2)\\
&= \mathcal{N}(\mu; x_v, \sigma_v^2) \cdot \mathcal{N}(\mu; x_a, \sigma_a^2) \cdot \mathcal{N}(\mu; \mu_0, \sigma_0^2)\\
&\text{according to the conclusion from 1(c)}\\
&= \mathcal{N}(\mu; \frac{\beta_v x_v + \beta_a x_a + \beta_0\mu_0}{\beta_v + \beta_a + \beta_0}, \frac{1}{\beta_v + \beta_a + \beta_0}).
\end{aligned}
$$

2(c).
Hint: using the conclusion from 'Homework 3: Probability theory 1(a) and 1(b)'

$$E(x_h) = E(\frac{1}{2}(x_v + x_a)) = \frac{1}{2}\mu + \frac{1}{2}\mu = \mu;$$
$$Var(x_h) = Var(\frac{1}{2}(x_v + x_a)) = \frac{1}{4}Var(x_v) + \frac{1}{4}Var(x_v) = \frac{1}{4}(\sigma_v^2 + \sigma_a^2).$$