Parts of the solutions for the 5th homework.

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Note: I only give the solutions for those questions some of you might have problems. If you have other problems, you can ask me directly.

Homework 6: Linear Classification

1(a).

$$\sigma'(s) = \frac{\exp(-s)}{(1+\exp(-s))^2};$$
$$\sigma''(s) = \frac{\exp(-s)\cdot(\exp(-s)-1)}{(1+\exp(-s))^3}.$$

1(c).

$$\begin{aligned}
p(t=1|x) &= \frac{p(x|t=1)p(t)}{p(x|t=1)p(t)+p(x|t=0)p(t)} \\
&= \frac{\pi_+\cdot\mathcal{N}(x;\mu_+,\Sigma)}{\pi_+\cdot\mathcal{N}(x;\mu_+,\Sigma)+\pi_-\cdot\mathcal{N}(x;\mu_-,\Sigma)} \\
&= \cdots; \\
&= \frac{1}{1+\exp\big(-(\mu_+^T\Sigma^{-1}-\mu_-^T\Sigma^{-1})x+(-\frac{1}{2}\mu_+^T\Sigma^{-1}\mu_+ + \frac{1}{2}\mu_-^T\Sigma^{-1}\mu_-)\big)}
\end{aligned}$$

Therefore:

$$w = \Sigma^{-1}(\mu_+ - \mu_-);$$
$$w_0 = -\frac{1}{2}\mu_+^T\Sigma^{-1}\mu_+ + \frac{1}{2}\mu_-^T\Sigma^{-1}\mu_-.$$

2(c).

$$\begin{aligned}
p(t=1|x) &= \frac{p(x|t=1)p(t)}{p(x|t=1)p(t)+p(x|t=0)p(t)} \\
&= \frac{\pi_+\cdot\mathcal{N}(x;\mu_+,\Sigma_+)}{\pi_+\cdot\mathcal{N}(x;\mu_+,\Sigma_+)+\pi_-\cdot\mathcal{N}(x;\mu_-,\Sigma_-)} \\
p(t=0|x) &= \frac{p(x|t=0)p(t)}{p(x|t=1)p(t)+p(x|t=0)p(t)} \\
&= \frac{\pi_-\cdot\mathcal{N}(x;\mu_-,\Sigma_-)}{\pi_+\cdot\mathcal{N}(x;\mu_+,\Sigma_+)+\pi_-\cdot\mathcal{N}(x;\mu_-,\Sigma_-)}
\end{aligned}$$

On the decision boundary, $p(t=1|x)$ should be equal to $p(t=0|x)$, so we can get

$$\pi_+ \cdot \mathcal{N}(x;\mu_+,\Sigma_+) = \pi_- \cdot \mathcal{N}(x;\mu_-,\Sigma_-).$$

$$\Rightarrow -\frac{1}{2}x^T(\Sigma_+^{-1}-\Sigma_-^{-1})x + (\mu_+^T\Sigma_+^{-1}-\mu_-^T\Sigma_-^{-1})x - \frac{1}{2}(\mu_+^T\Sigma^{-1}\mu_+ - \mu_-^T\Sigma^{-1}\mu_-) - \frac{1}{2}\log\frac{|\Sigma_+|}{|\Sigma_-|} = 0$$

$$\Rightarrow$$

$$A \propto -\frac{1}{2}(\Sigma_+^{-1}-\Sigma_-^{-1});$$

$$b^T \propto \mu_+^T\Sigma_+^{-1} - \mu_-^T\Sigma_-^{-1};$$

$$c \propto -\frac{1}{2}(\mu_+^T\Sigma^{-1}\mu_+ - \mu_-^T\Sigma^{-1}\mu_-) - \frac{1}{2}\log\frac{|\Sigma_+|}{|\Sigma_-|}$$