Homework 4: Gaussian random variables

Solutions to this exercise sheet are to be handed in before the lecture on Friday, 16.11.12. Code (.m-file) and output (.mat-file, figures as .pdf-files) for the matlab-questions should be either submitted by email to `nicolas.ludolph@student.uni-tuebingen.de` (subject: [ML1] Exercise 4) or uploaded to Ilias. Use comments to explain your code. Please adhere to the file naming convention: `Homework4_<YourName>.<ext>`.

1. (10 points) **The Gaussian distribution**

   (a) Using the fact that $\sqrt{\frac{\beta}{2\pi}}\int_{-\infty}^{\infty}\exp\left(-\frac{\beta}{2}(s-\gamma)^2\right)ds = 1$, show that $\int_{-\infty}^{\infty}\exp(-as^2+bs)ds = \sqrt{\pi/a}\exp(\frac{b^2}{4a})$.

   (b) Suppose that $\mu \sim \mathcal{N}(0, 1/\alpha)$ and $x|\mu \sim \mathcal{N}(\mu, 1/\beta)$. By integrating out μ, show that the marginal distribution of x is given by $x \sim \mathcal{N}(0, 1/\beta + 1/\alpha)$ [Hint: Write down the joint distribution $p(x,\mu) = p(x|\mu)p(\mu)$. To perform the integral over μ, use the identity from part b)]

   (c) [optional] In the lecture, we showed that a product of two Gaussian probability density functions is *proportional* to a Gaussian density function, but we did not derive the proportionality-factor. Suppose that $p_1(x) = \mathcal{N}(x, \mu_1, 1/\beta_1)$ and $p_2(x) = \mathcal{N}(x, \mu_2, 1/\beta_2)$. Find Z such that

   $$p(x) = p_1(x)p_2(x) = \frac{1}{Z}\mathcal{N}(x, 1/\beta(\beta_1\mu_1 + \beta_2\mu_2), 1/\beta) \text{ where } \beta = \beta_1 + \beta_2. \tag{1}$$

   [Hint: Go through the calculations we did in the lecture again, but carefully keep track of the factors we had dropped.]

2. (20 points) **Multisensory integration.** Consider an experiment in which an observer has both visual and auditory information about the location of a target μ, and tries to combine both sources of information in order to increase her accuracy. Concretely, suppose that the observer has a visual $x_v|\mu \sim \mathcal{N}(\mu, \sigma_v^2)$ and an auditory measurement $x_a|\mu \sim \mathcal{N}(\mu, \sigma_a^2)$, where we assume the noise in the two modalities to be independent.

   (a) Calculate the posterior distribution of the location of the target, $p(\mu|x_v, x_a)$.

   (b) [matlab] Plot the posterior as well as the (visual and auditory) likelihood functions for $\mu_o = 0$, $\sigma_o = 100$, $x_v = -10$, $x_a = 10$, $\sigma_v = 5$, $\sigma_a = 20$. Explain why the posterior mean will always be closer to x_v than to x_a.

   (c) Suppose that the observer is not using the posterior mean as an estimate of where the target is, but the simple heuristic $x_h = \frac{1}{2}(x_v + x_a)$, i.e. the mean of the two measurements. As the sum of two Gaussians is Gaussian again, x_h has a Gaussian distribution. Calculate its mean and variance [Hint: the identities from the previous sheet might be useful].

   (d) Plot the variance of x_h as well as the posterior variance as a function of σ_a (for $\sigma_v = 5$ and $\sigma_a \in [5, 50]$). Observe that the posterior variance is always smaller than $\min(\sigma_v^2, \sigma_a^2)$, i.e. that there is a benefit from sensory integration even if the accuracy of the auditory signal is poor. For which values of σ_a is the estimate of x_h actually worse (i.e. has higher variance) than just using the visual signal alone?