Parts of the solutions for the 1st and 2nd homework.

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Note: I only give the solutions for those questions some of you might have problems. If you have other problems, you can ask me directly.

Homework 3: Probability theory
1(e). Show that $E(E(X|Y)) = E(X)$ and $Var(X) = E(Var(X)) + Var(E(X))$.

$$
\begin{aligned}
E(E(X|Y)) &= \int_y p(y) \int_x xp(x|y)dxdy \\
&= \int_y \int_x xp(x,y)dxdy \\
&= \int_x x \int_y p(x,y)dydx \\
&= \int_x xp(x)dx \\
&= E(X).
\end{aligned}
$$

$$
\begin{aligned}
Var(X) &= E(X^2) - E^2(X) \\
&= E\big(E(X^2|Y)\big) - E^2\big(E(X|Y)\big) \\
&= E\big(Var(X|Y) + E^2(X|Y)\big) - E^2\big(E(X|Y)\big) \\
&= E\big(Var(X|Y)\big) + E\big(E^2(X|Y)\big) - E^2\big(E(X|Y)\big) \\
&= E\big(Var(X|Y)\big) + Var\big(E(X|Y)\big).
\end{aligned}
$$

2(a). "Minty Hall" Problem:
Since there is a cash prize behind one of the three doors with equal probability, we can get $P(A) = 1/3, P(B) = 1/3, P(C) = 1/3$, where $P(A), P(B)$ and $P(C)$ represent the probability that the cash prize is behind $A, BandC$, respectively. We represent event D that the show master opens door B. Since you have already choose door A, the probability that the show master opening door B is $P(D) = 1/2$.

So if the cash prize is behind door A, the probability that the show master opening door B is $P(D|A) = 1/2$; similarly, we can get $P(D|B) = 0$; $P(D|C) = 1$. So with the Bayes theorem, we can derive the probability that the cash prize is behind A, given that the show master opens door B:

$$P(A|D) = \frac{P(D|A)P(A)}{P(D)} = \frac{\frac{1}{2} \cdot \frac{1}{3}}{\frac{1}{2}} = \frac{1}{3};$$

Similarly, we can get

$$P(C|D) = \frac{2}{3}.$$

Because $P(A|D) < P(C|D)$, the candidate should change his choice to door C.

3(c). Suppose that Y is an exponential random variable with parameter γ, and that X and Y are independent. Calculate $P(X < Y)$.

$$\begin{aligned} P(X < Y) &= \int_0^\infty \int_0^y p(x, y) dx dy \\ &= \int_0^\infty \int_0^y p(x)p(y) dx dy \\ &= \int_0^\infty \int_0^y (\lambda exp(-\lambda x)) \cdot (\gamma exp(-\gamma x)) dx dy \\ &= \frac{\lambda}{\lambda + \gamma}. \end{aligned}$$

Homework 2: Bayesian of Inference

1(b). The exponential distribution has the following form:

$$p(x|\theta) = f(x)g(\theta)exp(\phi^T(\theta)S(x); \tag{1}$$

The Poisson distribution has the following form:

$$\begin{aligned} p(x|\theta) &= \frac{1}{x!}\theta^x exp(-\theta) \\ &= \frac{1}{x!}exp(-\theta)exp(log(\theta^x)) \\ &= \frac{1}{x!}exp(-\theta)exp(xlog(\theta)) \end{aligned} \tag{2}$$

So, by comparing equation 2 with 1, we can get

$$g(\theta) = exp(-\theta), \qquad f(x) = \frac{1}{x!}, \qquad \phi(\theta) = log(\theta), \qquad S(x) = x.$$

2(d). Calculate the predictive distribution for the (n+1)th observation, and (numerically or analytically) calculate its mean and variance

$$\begin{aligned}
p(x_{new}|D) &= \int_\theta p(x_{new}|\theta, x)p(\theta|x)d\theta \\
&= \int_\theta p(x_{new}|\theta)p(\theta|x)d\theta \\
&= \int \frac{1}{x_{new}!}\theta^{x_{new}} exp(-\theta)\, Gamma(\alpha + \sum_{i=1}^n x_i, \beta + n)d\theta. \\
&= \cdots \\
&= \frac{1}{x_{new}!}\frac{(\beta+n)^{\alpha+\sum_{i=1}^n x_i}}{\Gamma(\alpha+\sum_{i=1}^n x_i)}\int_\theta \theta^{\alpha+\sum_{i=1}^n x_i + x_{new}-1} exp(-(\beta+n+1)\theta)d\theta \\
&= \frac{1}{x_{new}!}\frac{(\beta+n)^{\alpha+\sum_{i=1}^n x_i}}{\Gamma(\alpha+\sum_{i=1}^n x_i)}\frac{\Gamma(\alpha+\sum_{i=1}^n x_i + x_{new})}{(\beta+n+1)^{\alpha+\sum_{i=1}^n x_i + x_{new}}} \\
&= \frac{\Gamma(\alpha+\sum_{i=1}^n x_i + x_{new})}{\Gamma(x_{new}+1)\Gamma(\alpha+\sum_{i=1}^n x_i)}\left(\frac{\beta+n}{\beta+n+1}^{\alpha+\sum_{i=1}^n x_i}\right)\frac{1}{(\beta+n+1)^x_{new}} \\
&= \mathcal{NB}(r,p), \qquad \text{where } r = \alpha + \sum_{i=1}^n x_i, p = \frac{1}{\beta+n+1}, \\
&\quad (\mathcal{NB} \text{ means negative binomial distribution })
\end{aligned}$$

For negative binomial distribution $\mathcal{NB}(r,p)$, mean= $\frac{pr}{1-p}$, variance = $\frac{pr}{(1-p)^2}$, where $r = \alpha + \sum_{i=1}^n x_i, p = \frac{1}{\beta+n+1}$.