Dynamics of Neural Systems

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Exercise Sheet 1 due Nov 6^{th} 2014

1 Exercise 1. Nernst Equation. Credits: 2

1.1

Derive the Nernst equation, Eq. (2), from the Nernst-Planck equation, Eq. (1).*(Hint: In equilibrium holds $j_\mathrm{A} = 0$.)*

$$J_\mathrm{A} = J_{\mathrm{A,diff}} + J_{\mathrm{A,drift}} = -D_A \left(\frac{\mathrm{d}\,[\mathrm{A}]}{\mathrm{d}\,x} + \frac{z_\mathrm{A} F}{RT} [\mathrm{A}] \frac{\mathrm{d}\,V}{\mathrm{d}\,x} \right) \tag{1}$$

$$E_m = \frac{RT}{z_\mathrm{A} F} \ln \left(\frac{[\mathrm{A}]_{\mathrm{out}}}{[\mathrm{A}]_{\mathrm{in}}} \right) \tag{2}$$

Variables:

- J_A = The flux of ion A within electric field, sum of diffusion and drift flux.
- D_A = The diffusion constant of ion A.
- $[\mathrm{A}]$ = The concentration of ion A at current location.
- x = The current position, from x_{inside} =0 to x_{outside} =thickness of membrane.
- z_A = The signed valency of ion A.
- F = Faraday's constant (Coulombs per mole).
- R = The ideal gas constant (Joules per Kelvin per mole).
- T = The temperature (in Kelvin).
- V = The membrane potential (in Volt).
- E_m = The membrane equilibrium potential (in Volt).
- $[\mathrm{A}]_{\mathrm{out}}$ = The extracellular concentration of ion A.
- $[\mathrm{A}]_{\mathrm{in}}$ = The intracellular concentration of ion A.

2 Exercise 2. GHK Equations. Credits: 3

2.1

Given the Goldman Hodgkin Katz (GHK) voltage equation, Eq. (3), assume the membrane of a neuron is selectively permeable for only **one monovalent anion** type A with permeability P_A (all other permeabilities are zero). Derive the Nernst equation, Eq. (2).

$$E_m = \frac{RT}{F} \ln \left(\frac{\sum_i^N P_{\mathrm{C}_i^+} \left[\mathrm{C}_i^+\right]_{\mathrm{out}} + \sum_j^M P_{\mathrm{A}_j^-} \left[\mathrm{A}_j^-\right]_{\mathrm{in}}}{\sum_i^N P_{\mathrm{C}_i^+} \left[\mathrm{C}_i^+\right]_{\mathrm{in}} + \sum_j^M P_{\mathrm{A}_j^-} \left[\mathrm{A}_j^-\right]_{\mathrm{out}}} \right) \tag{3}$$

Variables:

- C_i^+ = One type of monovalent cation (positive charge, e.g. K^+).
- N = Number of relevant cations.
- A_j^- = One type of monovalent anion (negative charge, e.g. Cl^-).
- M = Number of relevant anions.
- P_{ion} = The permeability of an ion type (meters per second).

2.2

We assume a homogeneous electric field across the membrane with $\frac{\mathrm{d}V}{\mathrm{d}x} = -\frac{V_0}{L}$. Write down the Nernst-Planck equation for this case.

2.3

From the previous result, derive a differential equation for $[A]$ as a function of x. Assume a constant current density j_A and take into account $j_\mathrm{A} = z_\mathrm{A} F J_\mathrm{A}$.

2.4

Solve this DEQ. *(Hint: Use Eq.* (4)*.)*

$$\int \frac{\mathrm{d}x}{ax+b} = \frac{\ln(ax+b)}{a} \tag{4}$$

2.5

Using the previous solution, derive a relationship between $[\mathrm{A}]_\mathrm{in}$ and $[\mathrm{A}]_\mathrm{out}$ in a mathematical form of Eq. (5)

$$k = \frac{a + b\,[\mathrm{A}]_\mathrm{out}}{c + d\,[\mathrm{A}]_\mathrm{in}} \tag{5}$$

2.6

Using this relationship, derive the GHK current equation, Eq. (6), for j_A as a function of V_0.

$$j_\mathrm{A} = P_\mathrm{A} z_\mathrm{A} F \frac{z_\mathrm{A} F V_0}{RT} \left(\frac{[\mathrm{A}]_\mathrm{in} - [\mathrm{A}]_\mathrm{out}\, e^{-\frac{z_\mathrm{A} F V_0}{RT}}}{1 - e^{-\frac{z_\mathrm{A} F V_0}{RT}}} \right) \tag{6}$$

Variables:

- j_A = The current density of ion A.
- L = The thickness of the membrane.
- $P_\mathrm{ion} = \frac{D_\mathrm{ion}}{L}$

2.7

From the GHK current equation, derive the GHK voltage equation for one ion type A, Eq. (3), by setting the membrane current to zero, i.e. $I = aJ_\mathrm{A} = 0$.

- a = The membrane area.

3 Exercise 3. Linear Membrane Model. Credits: 4

3.1 Equivalent simplified electrical circuit.

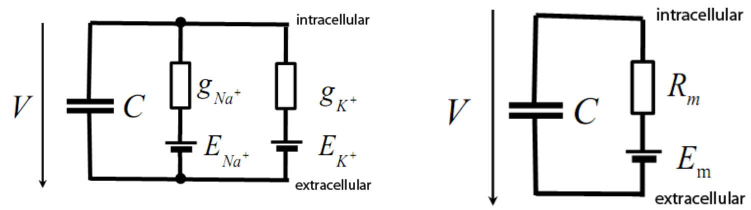

Figure 1: *Left)* Detailed circuit with single channel conductances and reversal potentials. *Right)* Equivalent simplified circuit.

Given Fig. 1, compute E_m as a function of g_{K^+}, g_{Na^+}, E_{K^+}, E_{Na^+}.

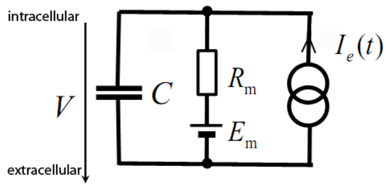

Figure 2: Equivalent circuit with external current I_e.

3.2

Given the equivalent circuit of a membrane in Fig. 2, derive the response of the membrane potential $V(t)$ to a step input current $I_e(t)$, assuming $V(t) = E_m$ for $t \leq t_0$:

$$I_e(t) = \begin{cases} 0 & t \leq t_0 \\ I_0 & t > t_0 \end{cases} \tag{7}$$

where I_0 is a constant.

3.3

Given the equivalent circuit of a membrane in Fig. 2, derive the response (of the membrane potential V) to a rectangular input current $I_e(t)$. *(Hint: Use previous result and exploit that the network is time-invariant.)*

$$I_e(t) = \begin{cases} 0 & t \leq 0 \\ I_0 & 0 < t \leq t_e \\ 0 & t_e < t \end{cases} \tag{8}$$