Dynamics of Neural Systems

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Exercise Sheet 7 due Jan 15^{th} 2015

1 Exercise 1 Nonlinear network with two divisive inhibition neurons. Credits: 3

Assume a network that consists of two nonlinear neurons with divisive feedback from the other neuron given by the pair of differential equations:

$$\tau \dot{u}_1(t) = -u_1(t) + \frac{s_1}{1 + u_2(t)} \tag{1}$$

$$\tau \dot{u}_2(t) = -u_2(t) + \frac{s_2}{1 + u_1(t)} \tag{2}$$

where the inputs are given by the nonnegative signals s_1 and s_2.

1.1

Prove that when the initial condition $\mathbf{u}(0)$ is within the first quadrant of the state space with $u_1, u_2 \geq 0$ the state never leaves the first quadrant.

Analyze the vector field of the dynamics on the boundaries of the positive quadrant. What needs to be fulfilled for the vector field in order to make the state leave the positive quadrant?

From now on we assume only states in the positive quadrant of the state space.

1.2

Compute the (relevant) stationary points for an input of the form $s_1 = s_2 = s \geq 0$. Compute these points explicitly for $s = 0$ and $s = \frac{3}{4}$.

1.3

Compute the linearized system for the obtained relevant fixed points and analyze its stability using eigenvalue analysis.

1.4

Plot the phase portraits for the two values of the stimulus signal and verify whether your computations are correct.

1.5

Using the technique discussed in the lecture, derive a Lyapunov function for this system. In which regions of the state space is the time derivative of this function negative? What does this imply for the global stability of the dynamical system? *Hint: Use the following theorems from the lecture:*

a) If the C^1 functions $F(x_1, x_2)$ and $G(x_1, x_2)$ have only a finite number of zeros (points with $F(x_1, x_2) = G(x_1, x_2) = 0$), then the function $E(\mathbf{x})$ with $|\epsilon| < 1$ is positive definite:

$$E(\mathbf{x}) = F^2(\mathbf{x}) + \epsilon F(\mathbf{x}) G(\mathbf{x}) + G^2(\mathbf{x})$$

b) Under the same conditions, the function $L(\mathbf{x})$ is negative definite, if $a, c < 0$ and $|b| < 2\sqrt{ac}$ with

$$L(\mathbf{x}) = aF^2(\mathbf{x}) + bF(\mathbf{x}) G(\mathbf{x}) + cG^2(\mathbf{x})$$

2 Exercise 2 Simple autoassociative memory. Credits: 3

Assume the very simple autoassociative memory network that is given by the dynamics:

$$\tau \dot{\mathbf{u}}(t) = -\mathbf{u}(t) + [\mathbf{M}\mathbf{u}(t)]_+ \tag{3}$$

where the matrix $\mathbf{M}$ is given by:

$$\mathbf{M} = \begin{bmatrix} 1 & -0.1 & -0.1 \\ -0.1 & 1 & -0.1 \\ -0.1 & -0.1 & -0.1 \end{bmatrix}$$

2.1

Which equation determines the fixed points of the network? Find the patterns stored in this memory network. Remark that the fixed points are degenerate with respect to their scale. This means if $\mathbf{u}^*$ is a fixed point then $\alpha\mathbf{u}^*$ with $\alpha > 0$ is also a fixed point.

Hint: E.g., try to iterate the nonlinear fixed point equation with random initial conditions, or try informed self-chosen initial conditions for finding the fixed points.

2.2

Prove the stability of the stable states. Remark that the fixed point equation implies the inequality $u_i \geq 0$.

Hint: Prove inequalities for the components of the fixed points for three cases: $u_1 > 0$, $u_2 > 0$ and $u_3 > 0$.

2.3

Prove the stability of the fixed points by deriving a Lyapunov function using the Cohen-Grossberg theorem.

Hint: Transform the system using the new state variable $\mathbf{v} = \mathbf{M}\mathbf{u}$ and remark that $\mathbf{M}$ is non-singular.

three different eigenvalues

3 Exercise 3: Conservative neural network dynamics. Credits: 3

Assume a two-neuron network with nonlinear synapses and voltage-dependent external input currents. Neurons are modelled by a single compartment model with membrane capacitance C and membrane resistance R. Neuron 1 has an additional conductance $g_1(V_2)$ that depends on the membrane potential of neuron 2. Likewise, Neuron 2 has an additional effective synaptic conductance $g_2(V_1)$ that depends on the membrane potential of neuron 1, and has a voltage-dependent external input current $I_{ext,2}(V_2)$. We assume for the following zero resting potential ($E_m = 0$) and drop physical units for simplicity with:

conductance

$$\tau = RC = 1 \qquad I_{ext,1}(V_1) = 2CV_1 \qquad g_1(V_2) = CV_2^2 \tag{4}$$

$$E_m = 0 \qquad I_{ext,2}(V_2) = \frac{1}{R}V_2 \qquad g_2(V_1) = CV_1 \tag{5}$$

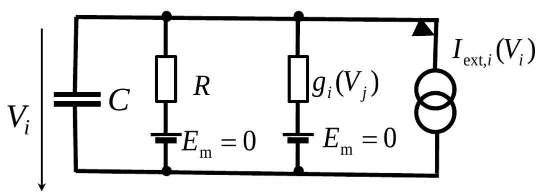

Figure 1: circuit diagram.

3.1

Write down the differential equation system that describes the system dynamics.

3.2

Derive a constant of the motion that is described by this dynamics, that is a function whose contour lines describe the trajectories of the dynamics.

Hint: Try to factorize the vector field of the dynamics in terms of factors that depend on V_1 and V_2.