Dynamics of Neural Systems

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Exercise Sheet 6 due Dec 18^{th} 2014

1 Exercise 1 Linear dynamical system. Credits: 3

Assume a linear dynamical system of the form: $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{s}(t)$ with

$$\mathbf{A} = \begin{bmatrix} -0.5 & -0.5 & 0 \\ -0.5 & -0.5 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

For Ex. 1.1 - 1.4 assume $\mathbf{s}(t) = 0$.

1.1

Compute and sketch the solution for the initial conditions:

$$\mathbf{x}_{0,1} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{x}_{0,2} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{x}_{0,3} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{x}_{0,4} = \begin{bmatrix} 0 \\ 0 \\ 10^{-6} \end{bmatrix}$$

1.2

Compute the eigenvectors and the eigenvalues of $\mathbf{A}$ and explain the behavior observed in Ex 1.1.

1.3

Plot the vector field of the dynamics: $\mathbf{f}(\mathbf{x}) = \mathbf{A}\mathbf{x}$ as arrow/quiver plot in the following projection planes (take vectors $\mathbf{x}$ from this plane and project the three-dimensional function $\mathbf{f}$ to this plane):

1. plane defined by $x_3 = 0$.
2. plane defined by $x_2 = 0$.
3. plane defined by $x_1 = 0$.

Can you explain the result in terms of critical points / manifolds of the dynamics?

1.4

Plot the vectorfield of the dynamics in the projection on the plane defined by the basis:

$$\mathbf{e}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

How can this result be explained? projection on eigen space lead to decoupled solution on two axes.

1.5

Compute the stationary solution (for $t \to \infty$) for a constant input $\mathbf{s} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ with $\mathbf{x}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Project onto the same plane as before.

2 Exercise 2: Nonlinear 'decision' network. Credits: 4

Assume a network that consists of two linear threshold neurons that inhibit each other with the input signals (input currents) s_1 and s_2. The inhibition strength is given by the parameter c, and Θ is the firing threshold of the neurons. The network is described by the nonlinear differential equation:

$$\tau \dot{u}_1(t) = -u_1(t) - c[u_2 - \Theta]_+ + s_1 \tag{1}$$
$$\tau \dot{u}_2(t) = -u_2(t) - c[u_1 - \Theta]_+ + s_2 \tag{2}$$

The linear threshold function is defined by $[x]_+ = max(x, o)$.

2.1

Show that this network can be re-parameterized in the standard form:

$$\frac{\mathrm{d}\tilde{u}_1(\tilde{t})}{\mathrm{d}\tilde{t}} = -\tilde{u}_1(\tilde{t}) - c[\tilde{u}_2(\tilde{t})]_+ + \tilde{s}_1 \tag{3}$$
$$\frac{\mathrm{d}\tilde{u}_2(\tilde{t})}{\mathrm{d}\tilde{t}} = -\tilde{u}_2(\tilde{t}) - c[\tilde{u}_1(\tilde{t})]_+ + \tilde{s}_2 \tag{4}$$

In the following we drop the tilde and give the parameters directly in the standard form. Assume $c = 2$ and $s_1 = s_2 = 1$.

2.2

Plot the vector field of this dynamics as arrow/quiver plot, that is the function:

$$\mathbf{f}(\mathbf{u}) = -\mathbf{u} - c\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}[\mathbf{u}(\tilde{t})]_+ + \mathbf{s}$$

with $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\mathbf{s} = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}$.

What does this plot imply for the dynamics of the system? How many asymptotically stable fixpoints does it have?

2.3

Remark that this system is piecewise linear. Compute the fixpoints (if existent) in the four quadrants of the phase space that are defined by:

1. $u_1, u_2 > 0$.
2. $u_1 > 0$ and $u_2 < 0$.
3. $u_1 < 0$ and $u_2 > 0$.
4. $u_1, u_2 < 0$.

2.4

Analyze the stability of the solutions in those four quadrants of the phase space by computing the system matrix $\mathbf{A}$ and analyzing its eigenvalues. What does this explain about the behavior of the solutions that you have described in Ex 2.2?

2.5

Plot the vector fields of the DEQ for

1. $s_1 = 1.2, s_2 = 1$ and
2. $s_1 = 1, s_2 = 1.2$.

Compute the stationary solutions $\mathbf{u}(\infty)$ for the initial conditions

- $\mathbf{u}(0) = \mathbf{0}$,
- $\mathbf{u}(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$
- $\mathbf{u}(0) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

What does the result imply in terms of neural implementations of decision mechanisms that compare two input signals s_1 and s_2?

2.6

Choose the parameters $s_1 = s_2 = 1$ and $c = -2$. Plot again the vector field and investigate the solution for $t \to \infty$. How can this behavior be explained (eigenvalues!!)?

2.7

Replace the linear threshold function $[x]_+ = max(x, o)$ by a step threshold function:

$$\mathbf{1}(x) = \begin{cases} 1 & (x > 0) \\ 0 & (x \leq 0) \end{cases} \tag{5}$$

Plot again the vector field and analyze the behavior of the solutions for $t \to \infty$. Can you explain the difference?