Neural Dynamics. Ex 8. Po-Hsuan Huang

1-1. $\tau \dot{u}(x,t) = -u(x,t) + \int_{-\infty}^{\infty} w(x-x')u(x',t)\,dx' + s(x,t)$

stationary: $\dot{u}=0 \quad u(x,t)=u(x)$.

$$u^*(x) = \int_{-\infty}^{\infty} \frac{a}{\sqrt{2\pi}b} e^{-\frac{(x-x')^2}{4b^2}} \cos(k_0(x-x'))\,u(x')\,dx' + S(x).$$

1-2. $\tau \frac{d\tilde{u}_n}{dt} = -\tilde{u}_{n(t)} + \tilde{w}_n \tilde{u}_{n(t)} + \tilde{s}_{n(t)}$

stationary $\Rightarrow \tilde{u}_{n(t)} = \tilde{w}_n \tilde{u}_{n(t)} + \tilde{s}_{n(t)} \quad \tilde{u}_{n(t)} = \frac{\tilde{s}_{n(t)}}{1-\tilde{w}_n}$

$$\Rightarrow u(x,t) = \sum_{-\infty}^{\infty} \tilde{u}_{n(t)} \cdot e^{+in\phi} = \sum_{-\infty}^{\infty} \frac{\tilde{s}_n}{1-\tilde{w}_n} \cdot e^{in\phi} \; *$$

$$\tilde{S}_k = \int_{-\infty}^{\infty} s_{k(x)} \cdot e^{-ikx}\,dx = \exp\left(\frac{-\sigma^2 k^2}{2}\right) = \exp(-d^2k^2) \; *$$

$\sigma^2 = 2d^2$

$$\tilde{w}_k = \int_{-\infty}^{\infty} \frac{a}{\sqrt{2\pi}b} e^{-\frac{x^2}{4b^2}} \cos(k_0 x)\, e^{-ikx}\,dx$$

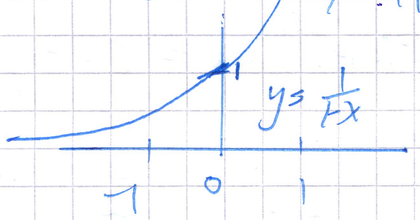
$$= \int_{-\infty}^{\infty} \frac{a}{\sqrt{2\pi}b} e^{-\frac{x^2}{4b^2}} \cdot \frac{1}{2}\left(e^{ik_0x} + e^{-ik_0x}\right) e^{-ikx}\,dx$$

$$= \int_{-\infty}^{\infty} \frac{a}{2\sqrt{2\pi}b} e^{-\frac{x^2}{4b^2}} \cdot e^{-i(k-k_0)x}\,dx + \int_{-\infty}^{\infty} \frac{a}{2\sqrt{2\pi}b} e^{-\frac{x^2}{4b^2}} e^{-i(k+k_0)x}\,dx.$$

$$= \int_{-\infty}^{\infty} a\left(\frac{1}{2\sqrt{2\pi}b} e^{-\frac{x^2}{4b^2}}\right) e^{-i(k+k_0)x}\,dx + \int_{-\infty}^{\infty} a \frac{1}{\sqrt{2\pi}b} e^{-\frac{x^2}{4b^2}} e^{-i(k+k_0)x}\,dx$$

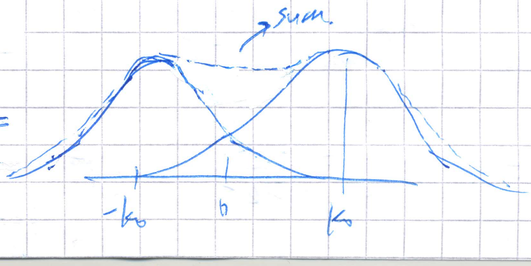
$$= a\exp(-b^2(k-k_0)^2) + a\exp(-b^2(k+k_0)^2) \; *$$

1-3. $$u^*(x,t) = \sum_{k=-\infty}^{\infty} \frac{e^{-d^2k^2}}{1-a\left(e^{-b^2(k-k_0)^2} + e^{-b^2(k+k_0)^2}\right)} e^{ikx}.$$



observing the plot of $y = \frac{1}{1-x}$, we see $y(x)$ approaches to infinity aymptotically as $x \to 1$, it makes sense because when a increase, it requires larger $u(x)$ to inhibit the positive feedback caused by $w(x)$.

$$e^{-b^2(k-k_0)^2} + e^{-b^2(k-(-k_0))^2} =$$



when $|k| < k_0$ $\quad u^*_{(x,t)} \sim \sum \frac{e^{-d^2k^2}}{1-a} e^{ikx}$

1.4. $\vec{U}_{(x)} = \sum_{k=-\infty}^{\infty} \frac{e^{-d^2k^2} e^{ikx}}{1-a\left(e^{-b^2(k-k_0)^2}+e^{-b^2(k+k_0)^2}\right)}$

1.5. $\delta(x)\delta(t) = L\cdot g(x,t) = \tau\dot{g}(x,t) + g(x,t) - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} w(x-x') g(x',t)\,dx'$

Fourier transform to k space, for each k mode.

$\delta(t)\tilde{\delta}_k = \tau\dot{\tilde{g}}_{(k)} + \tilde{g}_k(t) - \tilde{w}_k \tilde{g}_k(t).$

Fourier transform to ω space, for each ω mode.

$\tilde{\delta}_\omega \tilde{\delta}_k = i\omega\tau\cdot\tilde{g}_{k\omega} + \tilde{g}_{k,\omega} - \tilde{w}_k \tilde{g}_{k,\omega} = (1+i\omega\tau-\tilde{w}_k)\tilde{g}_{k\omega}$

$\tilde{g}_{k\omega} = \frac{\tilde{\delta}_\omega\tilde{\delta}_k}{1+i\omega-\tilde{w}_k}$ $\quad g(x,t) = \sum_{k=-\infty}^{\infty}\sum_{\omega=-\infty}^{\infty} \frac{\tilde{\delta}_\omega\tilde{\delta}_k}{1+i\omega-\tilde{w}_k} e^{i\omega t} e^{ikx}$

for $g(x-x', t-t')$ $\quad \tilde{\delta}_\omega = \int_{-\infty}^{\infty}\delta(t-t')e^{-i\omega t} = e^{-i\omega t'}$

$\tilde{\delta}_k = e^{-ikx'}$

$\tilde{w}_k = \int_{-\infty}^{\infty} w(x-x')e^{-ikx}dx = a\left(e^{-b^2(k-k_0)^2} + e^{-b^2(k+k_0)^2}\right)$

$g(x-x', t-t') = \sum_{k=-\infty}^{\infty}\sum_{\omega=-\infty}^{\infty} \frac{e^{i\omega(t-t')} e^{ik(x-x')}}{1+i\omega - a\left(e^{-b^2(k-k_0)^2} + e^{-b^2(k+k_0)^2}\right)}$

1.6. stationary solution. $\tilde{U}_{n(t)} = \frac{\tilde{S}_n}{1-\tilde{w}_n}$. $U^*_{(t)} = \sum_{-\infty}^{\infty}\tilde{U}_{n(t)}\cdot e^{ikx}$

$\tilde{S}_k = \int_{-\infty}^{\infty} \frac{c}{2\sqrt{\pi}d}\cdot\exp\left(\frac{-(x-vt)^2}{4d^2}\right)e^{-ikx}dx.$

$= c\int_{-\infty}^{\infty} N(x|vt, 2d^2)e^{-ikx}dx = C\cdot\exp(-d^2k^2)$

$\tilde{w}_k = \int_{-\infty}^{\infty} \text{sign}(x)e^{-c|x|}e^{-ikx}dx$ separate into $\int_{-\infty}^{0} f(x)dx + \int_0^\infty f(x)dx.$

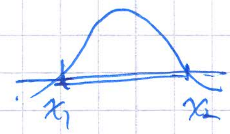
$\int_0^\infty e^{-cx}e^{-ikx}dx = \int_0^\infty e^{-(c+ik)x}dx = \frac{e^{-(c+ik)x}}{-(c+ik)}\Big]_0^\infty = \frac{1}{c+ik}$

$\int_{-\infty}^0 -e^{cx}e^{-ikx}dx = \int_{-\infty}^0 -e^{(c-ik)x}dx = \frac{-e^{(c-ik)x}}{(c-ik)}\Big]_{-\infty}^0 = \frac{-1}{c-ik}$

$\tilde{w}_k = \frac{1}{c+ik} - \frac{1}{c-ik} = \frac{-2ik}{c^2+k^2}$ $\qquad U^*_{(t)} = \sum_{k=-\infty}^{\infty} \frac{ce^{-d^2k^2}}{1+\frac{2ik}{c^2+k^2}} e^{ikx}$

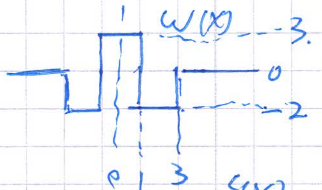
2.2. Stationary points $\frac{du(x,t)}{dt}=0$

$$U(x,t)=\int W(x')\,\mathbb{1}(U(x'))\,dx'+S(x)-h.$$

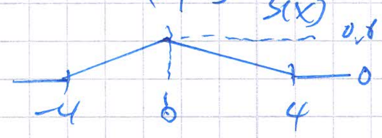


When stationary $U(x,t)>0$. when $|x|<|x^*|$

$$U(x,t)=\int_{x_1}^{x_2}W(x')dx'+S(x)-h.$$



when $1<|x^*|<3$



$$U(x,t)=\int_{-1}^{1}3\cdot dx+2\int_{1}^{x^*}-2\,dx+0.6\cdot\left(1-\frac{|x|}{4}\right)-1$$

Boundary: $U(x^*,t)=0.\ \Rightarrow$ for $x^*>0$. $x_2^*=\frac{9.6}{4+\frac{0.6}{4}}\sim 2.3$

symmetry $\Rightarrow x_1^*=-2.3$.

$$a^*=x_2^*-x_1^*=4.6..$$

Use criterion on slides $\frac{da}{dt}=\frac{1}{\tau}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)(W_{(a)}+S-h)=0.$

$6-4(x-1)+0.6\left(1-\frac{x}{4}\right)-1=0.$ as $x>0$.

$\Rightarrow x=\frac{9.6}{4.15}=x_2^*\Rightarrow$ the boundary is stable.

evaluate stability. $x_i(t)=x_i^*+\delta x_i(t).=x_0+\delta x_i$

$$U(x_i)=U(x_i^*)+U'(x_i)\,\delta x_i=U_0+U_0'\,\delta x_i$$

$U_0=0.$

$$\tau\cdot\frac{d\,u_0+U'\delta x}{dt}=-(U_0+U'\delta x)+\int W(x')\,\Theta(U(x+\delta x'))dx'+S(x_0+\delta x)-h$$

$$\because\ \tau\cdot\frac{du_0}{dt}=-u_0+\int w(x')\,\Theta(U(x'))dx'+S(x_0)-h.$$

$$\tau\frac{d\,U_0'\delta x}{dt}=-U_0'\,\delta x+\int w(x_0')\,\Theta'(U_{x})U'(x)\,\delta x.)dx'+S_0'\,\delta x.$$

~~set $\delta u(x,t)=U_0'\,\delta x(t)$~~

$$\tau\cdot\frac{d\,\delta x(x,t)}{dt}=-\delta x(x,t)+\int W(x')\,\Theta'(U(x'))\,\delta x(x',t)\,dx'+\frac{S_0'}{U_0'}\delta x.$$

Fourier Transform $\delta x=\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{\delta}_k(t)\,e^{ikx}\,dk.$

$$\frac{\tau}{2\pi}\int\frac{d\tilde{\delta}_k}{dt}e^{ikx}dk=\frac{-1}{2\pi}\int\tilde{\delta}_k e^{ikx}dk+\frac{1}{4\pi^2}\Theta'\iint\tilde{\omega}_k e^{i(k-k')x}\tilde{\delta}_k e^{ik'x}dk\,dk'+\frac{1}{2\pi}\int\frac{S_0'}{U_0'}\delta_k e^{ikx}$$

$$\frac{d\tilde{\delta}_k}{dt}=-\tilde{\delta}_k+\Theta'(U'(x_0))\,\tilde{\omega}_k\,\tilde{\delta}_k+\frac{S_0'}{U_0'}\tilde{\delta}_k$$

$$=\left(\Theta'(U'(x_0))+\frac{S_0'}{U_0'}-1\right)\tilde{\delta}_k$$

For $\frac{d\,\delta x}{dt}\to 0$. as $t\to\infty$ $\quad\Theta'(U_0)+\frac{S_0'}{U_0'}-1<0$

$S_0'=\frac{-0.6}{4}\text{sign}(x)\quad U_0'=\omega(x)\,\Theta(\underset{0}{U_0})+S'_0=S_0'$ marginally

$\Theta'(S_0')=0$ if $S_0'\neq 0\ \Rightarrow$ stable criterion: $1-1<0\Rightarrow$ stable.

2.3. $U(x,t) = \int w(x,x') \mathbb{1}(U(x')) dx' + S(x) - h$

if stable solution doesn't creat $U(x)>0$ region $\mathbb{1}(U(x))=0$

$U(x,t) = C \cdot (1 - \frac{|x|}{4}) - 1. = 0.$

$\Rightarrow : 1 - \frac{|x|}{4} = \frac{1}{C} = \frac{5}{3}$ x doesn't exist.

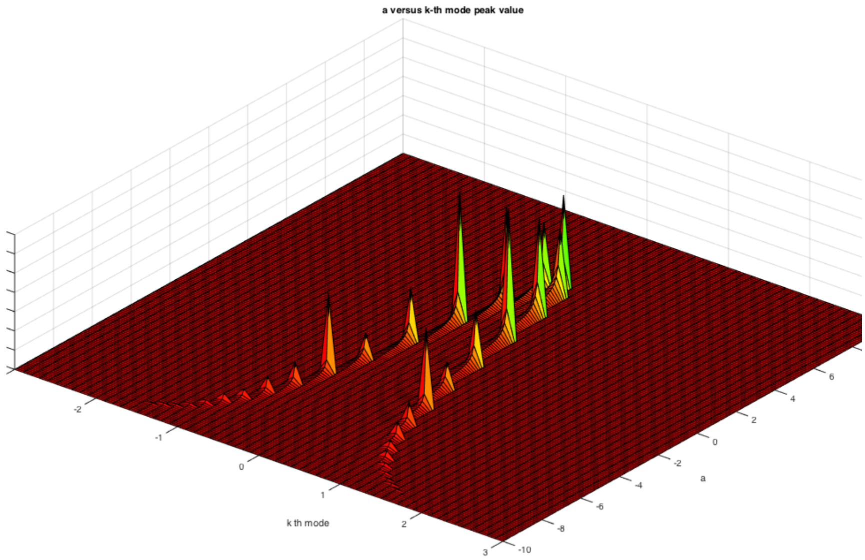
to realize such solution $4(1-\frac{1}{c}) = |x| > 0.$ $C > 1$ or $C < 0.$

when $0 < C < 1$, peak solution with an activated region emerges.

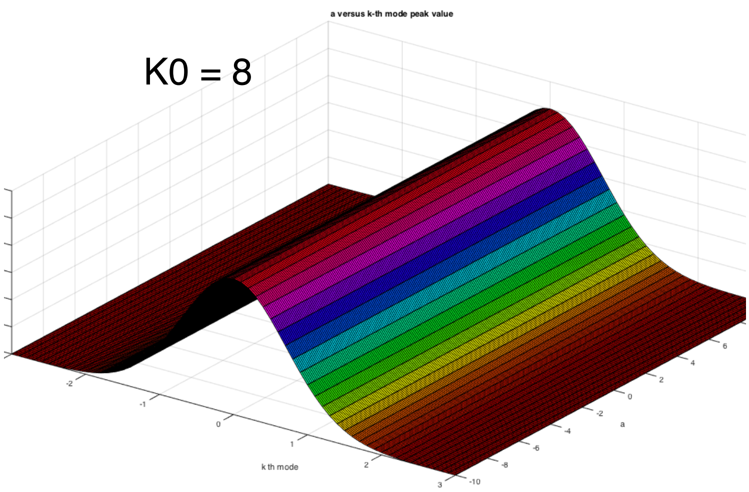
2.4 stationary solution with $U(x) > 0$ emerges.

Q1.3

K0 = 4;



From the above figure we can see the distribution is not very Gaussian since there are peaks when denominator of the weights in k space are close to zero at certain k values, leading to peaked distribution. The larger a, the smaller the denominator, even lead to minus weights.

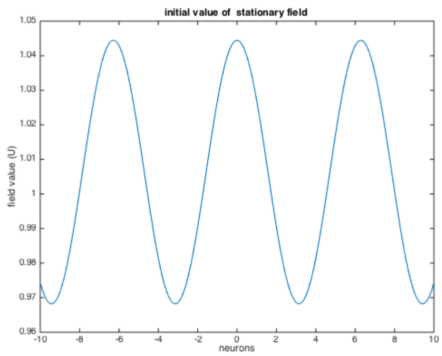


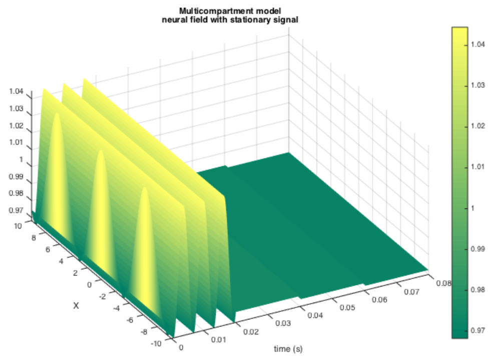
as a approaches 1the weight distribution of modes in k space becomes more concentrated at k=0, and becomes stronger. This

means the larger a, the larger the stationary value of neuron field potential,and the smaller the frequency in x space.
As K0 =8 , the whole denominator is close to 1, since the product of a and sum of exponential is very small inside interval l k l < k0, making the weight distribution determined by nominator, which is a Gaussian distribution.

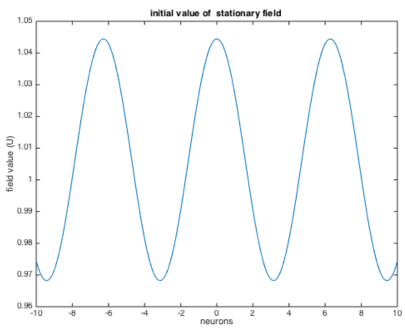
Q1.4

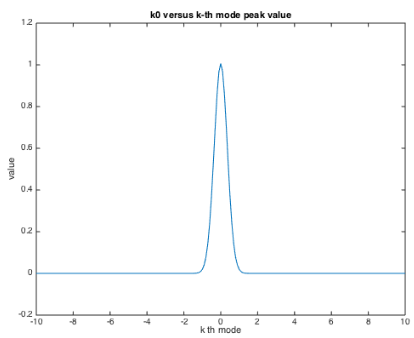
Initial conditions are stationary points. a =1, k0=4

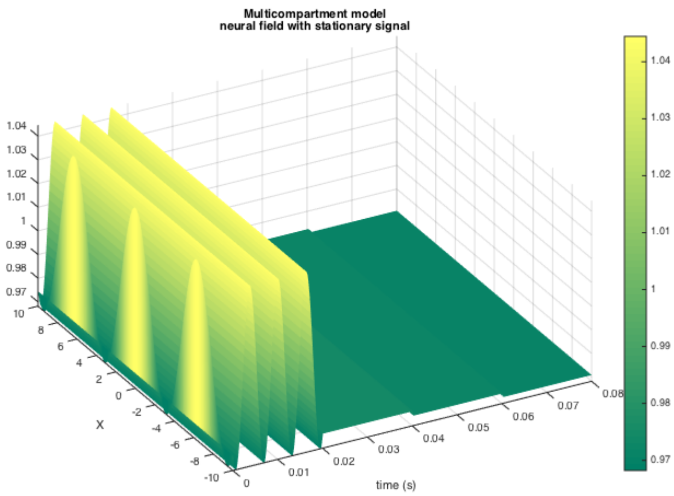




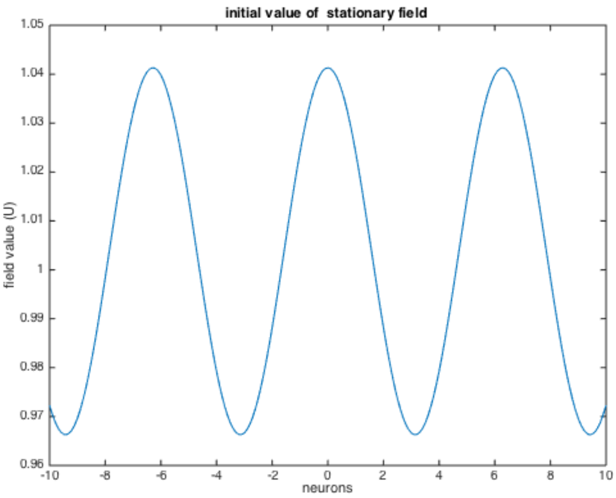
With White noise. a =1, k0=4, initial condition is the same as above, while the noise is filtered out since the weight distribution in k space is centralized at small k.Therefore, the output has no difference from the noise free one.

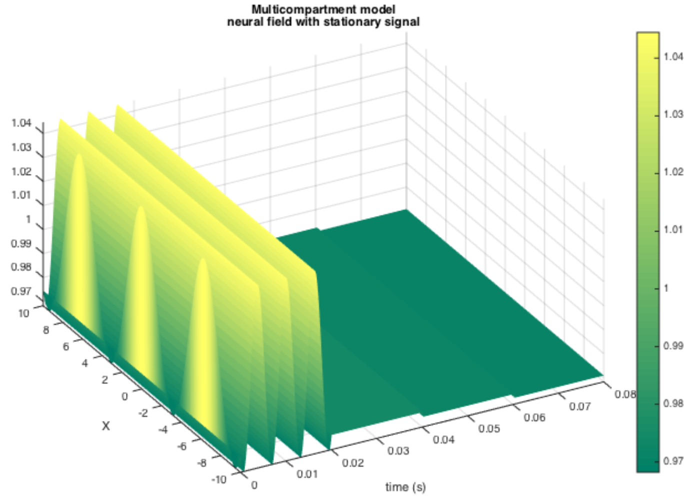




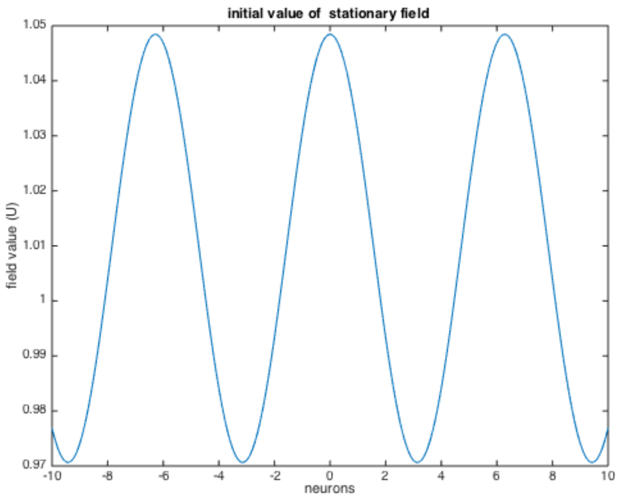


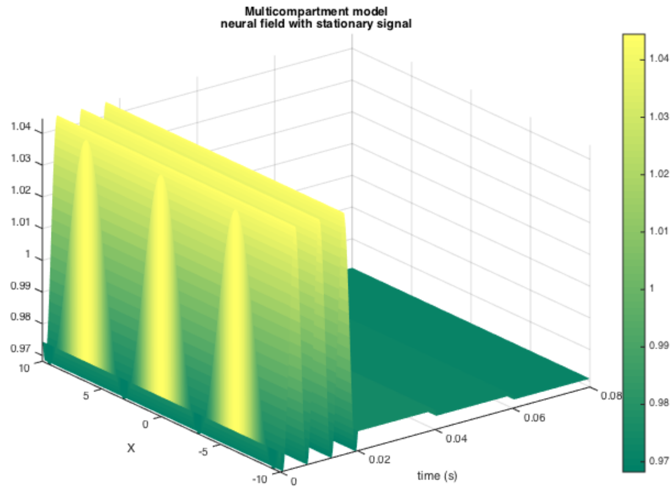
a= 0.7, k0=4, smaller a, weaker stationary field

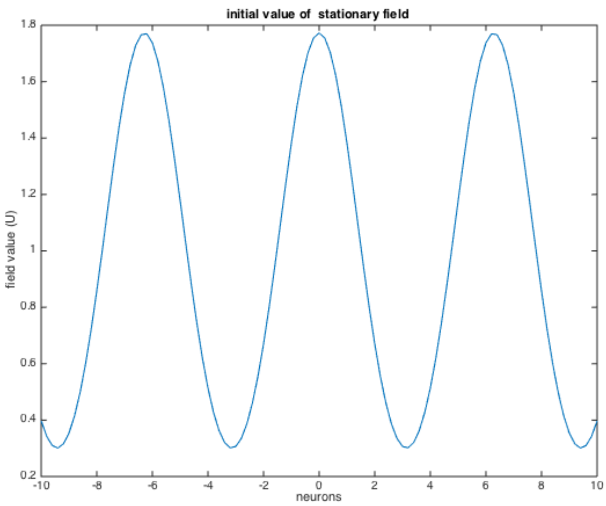


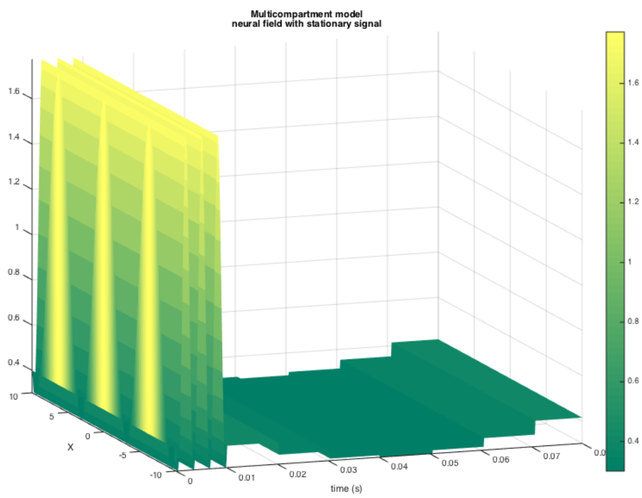


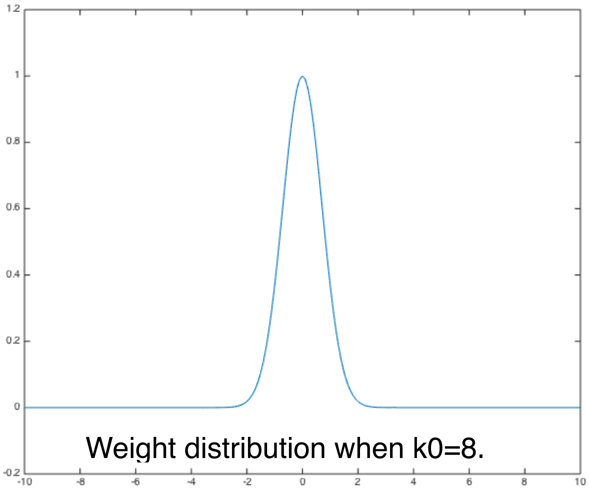
a = 1.5, k0=4 stronger stationary field









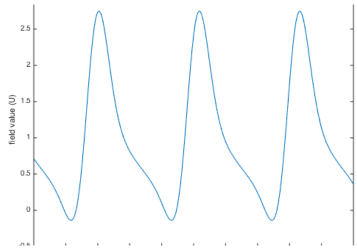


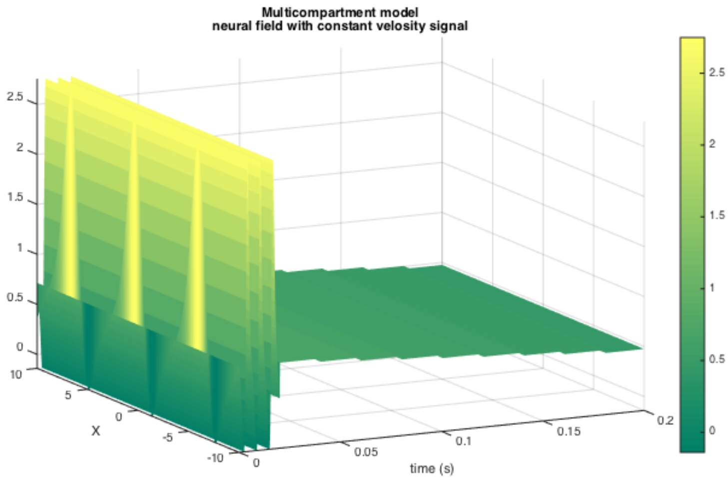
Weight distribution when k0=8.

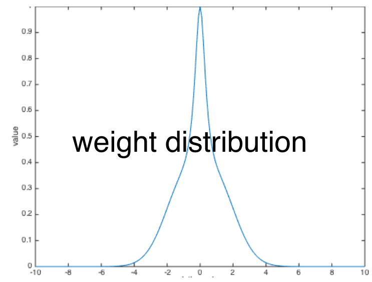
a =1, k0=8, larger coefficient in k space,wider distribution,and stronger stationary field.

Q1.6

Positive v

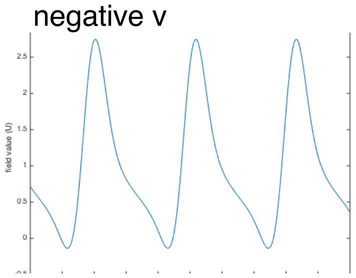


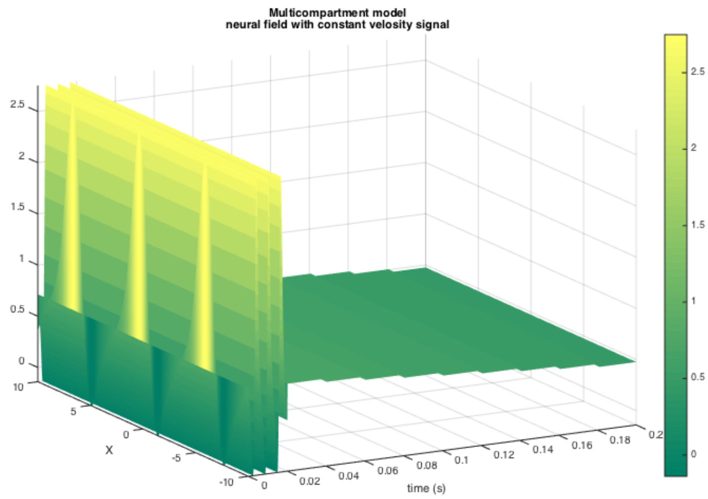




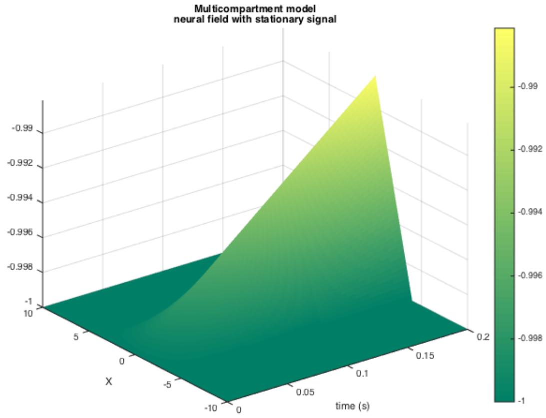
weight distribution

negative v

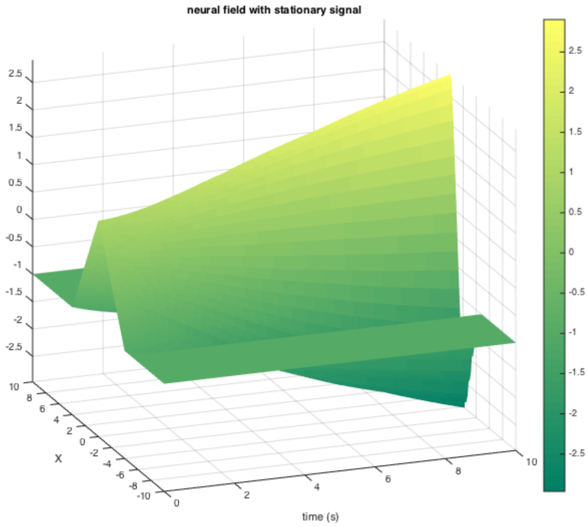




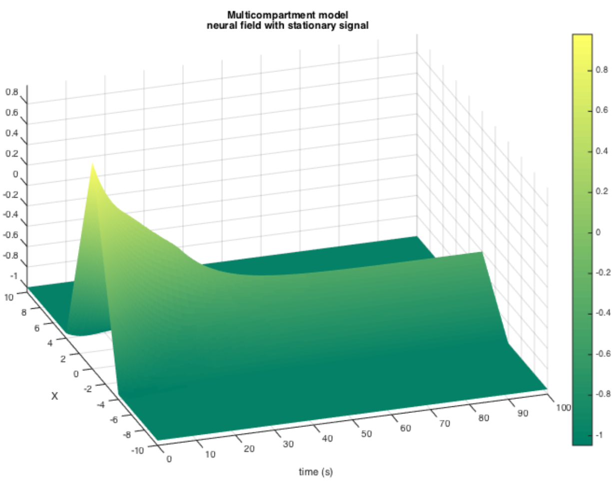
Q2.1



For homogeneous negative initial condition, U(x,0)= -h, not stationary. the neural field incrementally approaches its signal input.



For inhomogeneous initial condition, 3.3*s(x) -h, w*(x) = w(x), not stationary.



For inhomogeneous negative initial condition, 3.3*s(x) -h, w*(x) = 0.1w(x), stationary. We can observe that there are less negative potential on the boundary of x=-4 and x-4, due to smaller inhibitory field. w*.

Q2_2

For a stationary size of stable peak, it must fulfill the criteria W(a*)+h-s(x*) = 0, where a* is the stationary width of the solution.

Q2_4