Homework 1

1

$$y = \hat{y} + \hat{u}$$

$$(\underline{i}'\underline{i})^{-1}\underline{i}'y = (\underline{i}'\underline{i})^{-1}\underline{i}'\hat{y} + (\underline{i}'\underline{i})^{-1}\underline{i}'\hat{u}$$

$$\bar{y} = \bar{\hat{y}}$$

2

$$\begin{bmatrix} \underline{i}' \\ \underline{x}_2' \end{bmatrix} [\underline{i}\ ,\ \underline{x}_2] \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix} = \begin{bmatrix} \underline{i}' \\ \underline{x}_2' \end{bmatrix} y$$

$$\begin{bmatrix} \underline{i}'\underline{i} & \underline{i}'\underline{x}_2 \\ \underline{x}_2'\underline{i} & \underline{x}_2'\underline{x}_2 \end{bmatrix} \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix} = \begin{bmatrix} \underline{i}'y \\ \underline{x}_2'y \end{bmatrix}$$

$$\begin{bmatrix} n & \sum_i x_{i2} \\ \sum_i x_{i2} & \sum_i x_{i2}^2 \end{bmatrix} \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix} = \begin{bmatrix} \sum_i y_i \\ \sum_i x_{i2} y_i \end{bmatrix}$$

$$\begin{cases} n\hat{\beta}_1 + \hat{\beta}_2 \sum_i x_{i2} = \sum_i y_i \\ \hat{\beta}_1 \sum_i x_{i2} + \hat{\beta}_2 \sum_i x_{i2}^2 = \sum_i x_{i2} y_i \end{cases}$$

3

(1) $m^0 \underline{i} = [I_n - \underline{i}(\underline{i}'\underline{i})^{-1}\underline{i}']\underline{i} = \underline{i} - \underline{i}(\underline{i}'\underline{i})^{-1}\underline{i}'\underline{i} = \underline{0}$

(2) Step 1 : $\hat{y}' m^0 \hat{y} = \hat{y}' m_0 (y - \hat{u})$

$$= \hat{y}' m_0 y - \hat{y}' m_0 \hat{u}$$

$$= \hat{y}' m_0 y - \hat{y}' \hat{u}$$

$$= \hat{y}' m_0 y - \hat{\beta}' X' \hat{u}$$

$$= \hat{y}' m_0 y$$

Step 2 : $R^2 = \dfrac{\hat{y}' m_0 \hat{y}}{y' m_0 y}$

$$= \frac{\hat{y}' m_0 y}{y' m_0 y} \frac{\hat{y}' m_0 y}{\hat{y}' m_0 \hat{y}}$$

$$\Rightarrow \quad r = \frac{\hat{y}' m_0 y}{\sqrt{y' m_0 y}\sqrt{\hat{y}' m_0 \hat{y}}}$$

4 $\sum_i (y_i - \bar{y})^2 = \sum_i (y_i - \hat{y}_i + \hat{y}_i - \bar{y})^2$

$$= \sum_i (y_i - \hat{y}_i)^2 + \sum_i (\hat{y}_i - \bar{y})^2 + 2 \sum_i (y_i - \hat{y}_i)(\hat{y}_i - \bar{y})$$

$$TSS = RSS + ESS + 2 \left[\sum_i \hat{u}_i \hat{y}_i - \bar{y} \sum_i \hat{u}_i \right]$$

In a model with an intercept, $\begin{bmatrix} \sum_i \hat{u}_i \\ \sum_i x_{i2} \hat{u}_i \\ \vdots \\ \sum_i x_{ik} \hat{u}_i \end{bmatrix} = \underline{0}$. However,

in a model without an intercept, $\begin{bmatrix} \sum_i x_{i1} \hat{u}_i \\ \sum_i x_{i2} \hat{u}_i \\ \vdots \\ \sum_i x_{ik} \hat{u}_i \end{bmatrix} = \underline{0}$,

which does not imply $\sum_i \hat{u}_i = 0$. Therefore,

$$TSS = RSS + ESS - 2 \bar{y} \sum_i \hat{u}_i$$

5

Based on Question 2, we know $X'X = \begin{bmatrix} n & \sum_i X_{i2} \\ \sum_i X_{i2} & \sum_i X_{i2}^2 \end{bmatrix}$,

and hence $(X'X)^{-1} = \dfrac{1}{n\sum_i X_{i2}^2 - (\sum_i X_{i2})^2} \begin{bmatrix} \sum_i X_{i2}^2 & -\sum_i X_{i2} \\ -\sum_i X_{i2} & n \end{bmatrix}$.

Let's arrange the following matrix first.

$$[1, X_{f2}] \begin{bmatrix} \sum_i X_{i2}^2 & -\sum_i X_{i2} \\ -\sum_i X_{i2} & n \end{bmatrix} \begin{bmatrix} 1 \\ X_{f2} \end{bmatrix}$$

$$= [\sum_i X_{i2}^2 - X_{f2}\sum_i X_{i2} , \; nX_{f2} - \sum_i X_{i2}] \begin{bmatrix} 1 \\ X_{f2} \end{bmatrix}$$

$$= \sum_i X_{i2}^2 - X_{f2}\sum_i X_{i2} + nX_{f2}^2 - X_{f2}\sum_i X_{i2}$$

$$= \sum_i X_{i2}^2 - 2X_{f2}\sum_i X_{i2} + nX_{f2}^2$$

Now we can arrange the following matrix.

$$\frac{\sum_i X_{i2}^2 - 2X_{f2}\sum_i X_{i2} + nX_{f2}^2}{n\sum_i X_{i2}^2 - (\sum_i X_{i2})^2}$$

$$= \frac{\sum_i X_{i2}^2 - \frac{1}{n}(\sum_i X_{i2})^2 + \frac{1}{n}(\sum_i X_{i2})^2 - 2X_{f2}\sum_i X_{i2} + nX_{f2}^2}{n[\sum_i X_{i2}^2 - \frac{1}{n}(\sum_i X_{i2})^2]}$$

$$= \frac{1}{n} + \frac{X_{f2}^2 - 2X_{f2}\bar{X}_2 + \bar{X}_2^2}{\sum_i X_{i2}^2 - 2\frac{\sum_i X_{i2}}{n}\sum_i X_{i2} + n\frac{\sum_i X_{i2}}{n}\frac{\sum_i X_{i2}}{n}}$$

$$= \frac{1}{n} + \frac{(X_{f2} - \bar{X}_2)^2}{\sum_i (X_{i2} - \bar{X}_2)^2}$$