# Homework 2

1

(a) $E(Y) = 1 \times f(1) + 0 \times f(0) = \theta$

$E(Y^2) = 1^2 \times f(1) + 0^2 \times f(0) = \theta$

$Var(Y) = E(Y^2) - [E(Y)]^2 = \theta - \theta^2 = \theta(1-\theta)$

(b) $$\prod_{i=1}^{n} f(Y_i;\theta) = f(Y_1;\theta) f(Y_2;\theta) \cdots f(Y_n;\theta)$$
$$= \theta^{Y_1}(1-\theta)^{(1-Y_1)}\ \theta^{Y_2}(1-\theta)^{(1-Y_2)} \cdots\ \theta^{Y_n}(1-\theta)^{(1-Y_n)}$$
$$= \theta^{\sum_{i=1}^{n} Y_i}\ (1-\theta)^{(n-\sum_{i=1}^{n} Y_i)}$$
$$= L$$

$$l = \ln L$$
$$= \sum_{i=1}^{n} Y_i \ln\theta + (n - \sum_{i=1}^{n} Y_i)\ln(1-\theta)$$

$$\frac{\partial l}{\partial \theta} = \frac{\sum_{i=1}^{n} Y_i}{\theta} - \frac{n - \sum_{i=1}^{n} Y_i}{1-\theta} = 0 \quad \Rightarrow \quad \hat{\theta} = \frac{\sum_{i=1}^{n} Y_i}{n}$$

(c) $$E(\hat{\theta}) = \frac{\sum_{i=1}^{n} E(Y_i)}{n} = \frac{n\theta}{n} = \theta$$

$$E(\hat{\theta}^2) = E\left[\left(\frac{\sum_{i=1}^{n} Y_i}{n}\right)^2\right] = \frac{n\theta + (n^2-n)\theta^2}{n^2} = \frac{\theta + (n-1)\theta^2}{n}$$

$$Var(\hat{\theta}) = E(\hat{\theta}^2) - [E(\hat{\theta})]^2 = \frac{\theta + (n-1)\theta^2}{n} - \theta^2 = \frac{\theta(1-\theta)}{n}$$

(d) $E(\hat{\theta}) = \theta \Rightarrow \hat{\theta}$ is unbiased

$\lim_{n\to\infty} E(\hat{\theta}) = \theta$ and $\lim_{n\to\infty} Var(\hat{\theta}) = 0 \Rightarrow \hat{\theta}$ is consistent

(e) $$[I(\theta)]^{-1} = \left[-E\left(\frac{\partial^2 l}{\partial \theta^2}\right)\right]^{-1} = \left[E\left(\frac{\sum_{i=1}^{n} Y_i}{\theta^2} + \frac{n - \sum_{i=1}^{n} Y_i}{(1-\theta)^2}\right)\right]^{-1} = \left[\frac{n}{\theta} + \frac{n}{1-\theta}\right]^{-1} = \frac{\theta(1-\theta)}{n}$$

(f) The asymptotic variance is the same as the exact finite sample variance.

2

$$f(X,Y) = \frac{1}{2\pi \sigma_X \sigma_Y \sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{X-\mu_X}{\sigma_X}\right)^2 - 2\rho\left(\frac{X-\mu_X}{\sigma_X}\right)\left(\frac{Y-\mu_Y}{\sigma_Y}\right) + \left(\frac{Y-\mu_Y}{\sigma_Y}\right)^2\right]\right\}$$

$$f(X) = \frac{1}{\sqrt{2\pi}\,\sigma_X} \exp\left[-\frac{1}{2}\left(\frac{X-\mu_X}{\sigma_X}\right)^2\right]$$

$$f(Y|X) = \frac{1}{\sqrt{2\pi}\,\sigma_Y\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{(X-\mu_X)^2}{\sigma_X^2} - 2\rho\frac{(X-\mu_X)(Y-\mu_Y)}{\sigma_X \sigma_Y} + \frac{(Y-\mu_Y)^2}{\sigma_Y^2} - (1-\rho^2)\frac{(X-\mu_X)^2}{\sigma_X^2}\right.\right.$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_Y\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)\sigma_Y^2}\left[\frac{\sigma_Y^2}{\sigma_X^2}(X-\mu_X)^2 - 2\rho\frac{\sigma_Y}{\sigma_X}(X-\mu_X)(Y-\mu_Y) + (Y-\mu_Y)^2 - (1-\rho^2)\frac{\sigma_Y^2}{\sigma_X^2}(X-\right.\right.$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_Y\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)\sigma_Y^2}\left[\rho^2\frac{\sigma_Y^2}{\sigma_X^2}(X-\mu_X)^2 - 2\rho\frac{\sigma_Y}{\sigma_X}(X-\mu_X)(Y-\mu_Y) + (Y-\mu_Y)^2\right]\right\}$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_Y\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2\sigma_Y^2(1-\rho^2)}\left[(Y-\mu_Y) - \rho\frac{\sigma_Y}{\sigma_X}(X-\mu_X)\right]^2\right\}$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_Y\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2}\left[\frac{Y-\mu_Y-\rho\frac{\sigma_Y}{\sigma_X}(X-\mu_X)}{\sigma_Y\sqrt{1-\rho^2}}\right]^2\right\}$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_{Y|X}} \exp\left\{-\frac{1}{2}\left[\frac{Y-\mu_{Y|X}}{\sigma_{Y|X}}\right]^2\right\}$$

$\mu_{Y|X} = \alpha + \beta X$, $\alpha = \mu_Y - \beta\mu_X$ and $\beta = \rho\frac{\sigma_Y}{\sigma_X}$

$\sigma_{Y|X}^2 = \sigma_Y^2(1-\rho^2)$

3

(a) $R = [0, 1, 1]$ , $\underline{r} = 1$ , $q = 1$ , $(R\hat{\beta} - \underline{r}) = \hat{\beta}_2 + \hat{\beta}_3 - 1$

(b) $s^2 R(X'X)^{-1}R' = s^2 [0, 1, 1] \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$$= s^2 c_{22} + s^2 c_{33} + 2 s^2 c_{23}$$

$$= \widehat{Var}(\hat{\beta}_2) + \widehat{Var}(\hat{\beta}_3) + 2\widehat{Cov}(\hat{\beta}_1, \hat{\beta}_2)$$

(c) $F = (\hat{\beta}_2 + \hat{\beta}_3 - 1)' \left[ \widehat{Var}(\hat{\beta}_2) + \widehat{Var}(\hat{\beta}_3) + 2\widehat{Cov}(\hat{\beta}_1, \hat{\beta}_2) \right]^{-1} (\hat{\beta}_2 + \hat{\beta}_3 - 1)$

$$= \left[ \frac{\hat{\beta}_2 + \hat{\beta}_3 - 1}{\sqrt{\widehat{Var}(\hat{\beta}_2) + \widehat{Var}(\hat{\beta}_3) + 2\widehat{Cov}(\hat{\beta}_2, \hat{\beta}_3)}} \right]^2 \sim F(1, n-3)$$

$$t = \frac{\hat{\beta}_2 + \hat{\beta}_3 - 1}{\sqrt{\widehat{Var}(\hat{\beta}_2) + \widehat{Var}(\hat{\beta}_3) + 2\widehat{Cov}(\hat{\beta}_2, \hat{\beta}_3)}} \sim t(n-3)$$