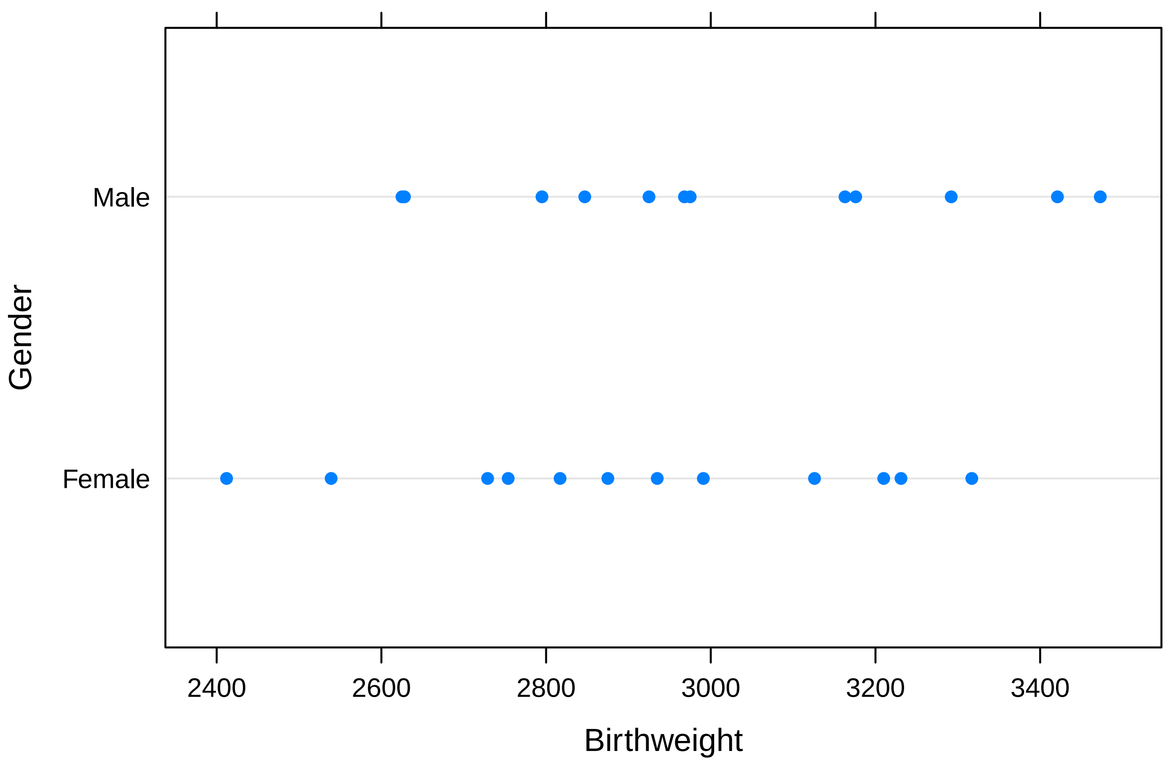
Final Project

Hierarchical Normal model for Baby Birthweight data

The birthweight data is from the R package **LearnBayes**. It is a dataframe with 24 observations on the following 3 variables.

- age : gestational age in weeks
- gender : gender of the baby where 0 (1) is male (female)
- weight : birthweight of baby in grams

```
library(LearnBayes)
data("birthweight")
library(lattice)
mygender <- with(birthweight, ifelse(birthweight$gender==0,"Male","Female"))
dotplot(mygender ~ weight, data = birthweight,
        xlab = "Birthweight", ylab = "Gender")
```



Suppose Y_{ij} are the birthweights for gender i, and assume

$$Y_{ij} \overset{ind}{\sim} N(\mu_i, \sigma^2), \;\; \mu_i \overset{ind}{\sim} N(\eta, \tau^2)$$

for $i = 1, 2$, $j = 1, \ldots, n_i$; $n = n_1 + n_2$, and prior

$$\pi(\eta, \tau^2, \sigma) \propto Ca^+(\tau; 0, b_\tau) \cdot Ca^+(\sigma; 0, b_\sigma),$$

where $Ca^+(x; 0, b)$ is the truncated Cauchy distribution with pdf

$$f(x|b) = \frac{2}{\pi} \frac{1}{b[1 + (x/b)^2]} I_{[x>0]}; \;\; b > 0.$$

The Gibbs sampling needs the full conditionals as following:

$$\begin{aligned}
p(\mu_i|\eta, \sigma^2, \tau^2, y) &= N\left(\left[\frac{1}{\sigma^2/n_i} + \frac{1}{\tau^2}\right]\left[\frac{\bar{y}_i}{\sigma^2/n_i} + \frac{\eta}{\tau^2}\right], \; \left[\frac{1}{\sigma^2/n_i} + \frac{1}{\tau^2}\right]^{-1}\right) \\
p(\eta|\mu_1, \mu_2, \sigma^2, \tau^2, y) &= N\left(\overline{\mu}, \tau^2/2\right) \\
p(\sigma|\mu_1, \mu_2, \eta, \tau^2, y) &\propto IG\left(\sigma^2; n/2, \sum_{i=1}^{2}\sum_{j=1}^{n_i}(y_{ij} - \mu_i)^2/2\right) \cdot Ca^+(\sigma; 0, b_\sigma) \\
p(\tau|\mu_1, \mu_2, \eta, \sigma^2, y) &\propto IG\left(\tau^2; 1, \sum_{i=1}^{2}(\mu_i - \eta)^2/2\right) \cdot Ca^+(\tau; 0, b_\tau),
\end{aligned}$$

where $\overline{y}_i = (\sum_{j=1}^{n_i} y_{ij})/n_i$, $\overline{\mu} = (\mu_1 + \mu_2)/2$, and $IG(x; a, b)$ is the inverted Gamma distribution with pdf

$$f(x|a, b) = \frac{b^a e^{-b/x} x^{-(a+1)}}{\Gamma(a)}.$$

The full conditionals for σ and τ are not a known density. Please compare the following three sampling methods to sample from $p(\tau|\mu_1, \mu_2, \eta, \sigma^2, y) \propto IG(\tau^2; a, b) \cdot Ca^+(\tau; 0, b_\tau)$ (or equivalently $p(\sigma|\mu_1, \mu_2, \eta, \tau^2, y)$)

1. Acceptance-Reject sampling with $(\tau^*)^2 \sim IG(a, b)$ and thus $C_{max} = Ca^+(0; 0, b_\tau)$ and the acceptance probability is $Ca^+(\tau^*; 0, b_\tau)/C_{max}$.
2. Independence Metropolis-Hastings with $(\tau^*)^2 \sim IG(a, b)$ and thus the acceptance probability is $\min\{Ca^+(\tau^*; 0, b_\tau)/Ca^+(\tau^{(t)}; 0, b_\tau), 1\}$.
3. Metroplis-Hastings with $\tau^* \sim \chi^2(\cdot|\tau^{(t)})$ and thus the acceptance probability is

   $$\min\left\{\frac{q(\tau^*) \cdot \chi^2(\tau^{(t)}|\tau^*)}{q(\tau^{(t)}) \cdot \chi^2(\tau^*|\tau^{(t)})}, 1\right\},$$

   where $q(\tau) = IG(\tau^2; a, b) \cdot Ca^+(\tau; 0, b_\tau)$.

Problem to solve

In this study, we use two Markov chain approaches to sample from a target posterior distribution:

- Metropolis-Hastings algorithm
- Gibbs sampling

Objective

- Diagnostic the convergence of each MCMC samples
- Plot the sampling distributions of $\mu_1, \mu_2, \eta, \sigma^2, \tau^2$.
- Discuss the efficiency of three sampling schemes.